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LINEAR PRESERVERS ON STRICTLY UPPER TRIANGULAR MATRIX ALGEBRAS

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Abstract. Let $\mathscr{S}_n(\mathbb{F})$ be the algebra of all $n\times n$ strictly upper triangular matrices over a filed $\mathbb{F}$. In this note, we characterize linear maps $\varphi : \mathscr{S}_n(\mathbb{F}) \to \mathscr{S}_n(\mathbb{F})$, with $|\mathbb{F}| \geqslant 3$, that preserve the adjugate function; i.e., $adj(\varphi(A)) = \varphi(adj(A))$. Also, some results about rank-1 linear/additive preservers on $\mathscr{S}_n(\mathbb{F})$ and, more generally, on block strictly upper triangular algebras are obtained.

1. Introduction

Throughout $\mathbb{F}$ will denote an arbitrary field and $\mathscr{M}_n(\mathbb{F})$ the algebra of all $n\times n$ matrices over $\mathbb{F}$. Also, $\mathscr{T}_n(\mathbb{F})$ (respectively, $\mathscr{S}_n(\mathbb{F})$) denotes the algebra of all $n\times n$ *upper triangular* (respectively, *strictly upper triangular*) matrices over $\mathbb{F}$. For an $n\times n$ matrix A, $adj(A)$ will denote the adjugate (or the classical adjoint) of A. Let $\mathscr{S}$ be any of $\mathscr{M}_n(\mathbb{F})$, $\mathscr{T}_n(\mathbb{F})$, or $\mathscr{S}_n(\mathbb{F})$.

DEFINITION 1.1. We say that a linear map $\varphi : \mathscr{S} \to \mathscr{S}$ preserves the adjugate function if $adj(\varphi(A)) = \varphi(adj(A))$, for any A in $\mathscr{S}$.

Adjugate preserving linear maps on $\mathscr{M}_n(\mathbb{F})$ were first studied by R. Sinkhorn [5] for $\mathbb{F} = \mathbb{C}$. The author used the classical result of Frobenius [4] for determinant preservers. In [2], Chan et al. use the structure of rank-n preservers to generalize Sinkhorn's result for an arbitrary infinite field. In [3], Chooi and Lim first determined the structure of rank-1 linear preservers on $\mathscr{T}_n(\mathbb{F})$ and then used this structure to characterize nonsingular adjugate preserving linear maps on $\mathscr{T}_n(\mathbb{F})$, where $\mathbb{F}$ is an arbitrary field. In section 2, we extend the result of Chooi and Lim to adjugate preserving linear maps on $\mathscr{S}_n(\mathbb{F})$. We do this by using a characterization of linear maps on $\mathscr{T}_n(\mathbb{F})$ preserving singular matrices and nonsingular matrices [3]. As it will be seen, the form of a general linear adjugate preserver on $\mathscr{S}_n(\mathbb{F})$ is completely different from that of such a preserver on $\mathscr{T}_n(\mathbb{F})$.

In Section 3, first we consider linear rank-1 preservers on $\mathscr{S}_n(\mathbb{F})$. And a characterization similar to that of Chooi-Lim [3] is obtained for such rank-1 preservers. Then

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we consider linear rank-1 preservers on certain block strictly upper triangular matrix algebras and arrive at some results, similar to those of Bell and Sourour [1]. Finally, in Section 4, *additive* rank-1 preserver maps on such algebras will be considered. As it will be seen, some results of Bell and Sourour [1] are valid for this case.

The author would like to acknowledge his inspiration by results of [1] and [3].

For easy reference, here we quote a needed result of [3].

THEOREM 1.2. *Let $|\mathbb{F}|\geqslant 3$, and $\varphi:\mathscr{T}_n(\mathbb{F})\to\mathscr{T}_n(\mathbb{F})$ be linear. Then φ preserves both singular and nonsingular matrices if and only if there exist a permutation σ of $\{1,\dots,n\}$ and nonzero numbers $\lambda_1,\dots,\lambda_n$ in $\mathbb{F}$ such that*

$$[\varphi([a_{ij}])]_{kk}=\lambda_k a_{\sigma(k)\sigma(k)},\quad 1\leqslant k\leqslant n.$$

2. Adjugate preservers

In the following, E_{ij} denotes the $n\times n$ matrix which has a 1 at the ij-th position and 0 everywhere else. Our main result is

THEOREM 2.1. *Let $n\geqslant 3$ be an integer, $|\mathbb{F}|\geqslant 3$, and $\varphi:\mathscr{S}_n(\mathbb{F})\to\mathscr{S}_n(\mathbb{F})$ be a linear map. Then φ preserves the adjugate function if and only if either*

(a) $\varphi(E_{1n})=0$ and rank $\varphi(A)\leqslant n-2$ for all A in $\mathscr{S}_n(\mathbb{F})$, or

(b) there exist a permutation σ of $\{1,\dots,n-1\}$ and nonzero numbers $\lambda_1,\dots,\lambda_{n-1}$ in $\mathbb{F}$ such that $\varphi(E_{1n})=\lambda_1\cdots\lambda_{n-1}E_{1n}$ and

$$[\varphi([a_{ij}])]_{k,k+1}=\lambda_k a_{\sigma(k),\sigma(k)+1},\tag{2.1}$$

for all $k=1,\dots,n-1$.

Proof. The sufficiency part is clear. We will prove the necessity part. Define the map $\psi:\mathscr{S}_n(\mathbb{F})\to\mathscr{T}_{n-1}(\mathbb{F})$ by

$$\psi(A)=\begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n}\\ 0 & a_{23} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a_{n-1,n}\end{bmatrix},\quad \textit{for every } A=[a_{ij}]\in\mathscr{S}_n(\mathbb{F}).\tag{2.2}$$

It is clear that ψ is linear and bijective. Let $\phi:=\psi\varphi\psi^{-1}$. Then $\phi:\mathscr{T}_{n-1}(\mathbb{F})\to\mathscr{T}_{n-1}(\mathbb{F})$ is linear. Note that for any $A\in\mathscr{S}_n(\mathbb{F})$, the matrix $adjA$ is an $n\times n$ matrix whose entries are zero except possibly the one at $(1,n)$ position with $\det(\psi(A))$, i.e., $\operatorname{adj}(A)=det(\psi(A))E_{1n}$. Since φ preserves the adjugate function, we have

$$\begin{aligned}(det\,\psi(A))\varphi(E_{1n}) &= \varphi((det\,\psi(A))E_{1n})=\varphi(adjA)=adj\,\varphi(A)\\ &=(det\,\psi(\varphi(A)))E_{1n}.\end{aligned}\tag{2.3}$$

Now we consider two cases:

Case I. rank $\varphi(A) \leqslant n-2$ for all rank-$(n-1)$ matrices $A \in \mathscr{S}_n(\mathbb{F})$.

Let $B \in \mathscr{S}_n(\mathbb{F})$ with rank $B \leqslant n-2$. Then adj $\varphi(B) = \varphi(\text{adj } B) = 0$, and so rank $\varphi(B) \leqslant n-2$. We thus conclude that rank $\varphi(A) \leqslant n-2$ for all matrices $A \in \mathscr{S}_n(\mathbb{F})$. Let $A \in \mathscr{S}_n(\mathbb{F})$ be of rank $n-1$. Then det $\psi(\varphi(A)) = 0$. It follows from (2.3) that $\varphi(E_{1n}) = 0$. This proves that (a) is true.

Case II. rank $\varphi(A_0) = n-1$ for some rank-$(n-1)$ matrix $A_0 \in \mathscr{S}_n(\mathbb{F})$.

So both det $\psi(A_0)$ and det $\psi(\varphi(A_0))$ are nonzero. By (2.3), we have

$$\varphi(E_{1n}) = \lambda E_{1n}, \tag{2.4}$$

for some nonzero $\lambda \in \mathbb{F}$. We thus conclude from (2.3)and (2.4) that rank $\varphi(A) = n-1$ for all rank-$(n-1)$ matrices $A \in \mathscr{S}_n(\mathbb{F})$. Also, since adj $\varphi(A) = 0$ for all matrices $A \in \mathscr{S}_n(\mathbb{F})$ with rank $A \leqslant n-2$, we conclude that ϕ preserves singular matrices and nonsingular matrices. By Theorem 1.2, there are a permutation σ of $\{1,\ldots,n-1\}$ and nonzero numbers $\lambda_1,\ldots,\lambda_{n-1}$ in $\mathbb{F}$ such that for every $A \in \mathscr{S}_n(\mathbb{F})$ we have

$$[\varphi(A)]_{k,k+1} = [\phi(\psi(A))]_{kk} = \lambda_k a_{\sigma(k),\sigma(k)+1}; \;\; 1 \leqslant k \leqslant n-1, \tag{2.5}$$

which establishes (2.1). Now we use (2.4) and (2.5) to prove $\lambda = \lambda_1 \cdots \lambda_{n-1}$. For

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathscr{S}_n(\mathbb{F})$$

we have

$$\begin{aligned} \lambda E_{1n} &= \varphi(E_{1n}) = \varphi(adjA) = adj(\varphi(A)) = adj(\begin{bmatrix} 0 & \lambda_1 & * & \cdots & * \\ 0 & 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \vdots & \ldots & \lambda_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}) \\ &= (\lambda_1 \cdots \lambda_{n-1}) E_{1n}, \end{aligned}$$

which proves that λ is as claimed. This proves that (b) holds. □

REMARK 1. While a map φ of the form given in (a) is always singular, a map φ of the form given in (b) could be singular or nonsingular. This is shown in below example.

EXAMPLE. Let $n \geqslant 3$ and $|\mathbb{F}| \geqslant 3$. Also, let φ_1 and φ_2 be linear maps on $\mathscr{S}_n(\mathbb{F})$ defined by $\varphi_1(A) = A$ and

$$\varphi_2(A) = \begin{bmatrix} 0 & a_{12} & 0 & 0 & \cdots & a_{1n} \\ 0 & 0 & a_{23} & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_{34} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1,n} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

for every $A = [a_{ij}] \in \mathscr{S}_n(\mathbb{F})$. We see that each φ_i is a linear adjugate preserver on $\mathscr{S}_n(\mathbb{F})$ that is of the form (b) of Theorem 2.1. Obviously φ_1 is nonsingular but φ_2 is singular when $n \geqslant 4$. When $n = 3$, a φ of the form given in (b) of Theorem 2.1 is necessarily nonsingular.

REMARK 2. For $n = 2$, it is easy to see that φ is adjugate preserving if and only if it is the identity map.

3. Linear rank-1 preservers

For an $n \times n$ matrix A, we denote by A^+ the matrix obtained from A by taking the symmetric image of its entries with respect to the minor diagonal (i.e., the one through $(n,1)$ and $(1,n)$ positions) of A. Also, let $\psi : \mathscr{S}_n(\mathbb{F}) \to \mathscr{T}_{n-1}(\mathbb{F})$ be the bijective linear map defined as in (2.2).

THEOREM 3.1. *Let $n \geqslant 2$ and $\varphi : \mathscr{S}_n(\mathbb{F}) \to \mathscr{S}_n(\mathbb{F})$ be linear. Then φ is a rank-1 preserver if and only if either*

1. *Im φ is an $(n-1)$-dimensional rank-1 subspace of $\mathscr{S}_n(\mathbb{F})$, or*
2. *There are invertible matrices P and Q in $\mathscr{T}_{n-1}(\mathbb{F})$ such that either*
   - *(a) $\varphi(A) = \psi^{-1}(P\psi(A)Q)$ for all $A \in \mathscr{S}_n(\mathbb{F})$ or*
   - *(b) $\varphi(A) = \psi^{-1}(P(\psi(A))^+Q)$ for all $A \in \mathscr{S}_n(\mathbb{F})$.*

Proof. For $n = 2$, a linear map $\varphi : \mathscr{S}_2(\mathbb{F}) \to \mathscr{S}_2(\mathbb{F})$ is rank-1 preserver if and only if φ is not the zero map, and the theorem becomes trivially true. Let $n \geqslant 3$ and assume that φ is a rank-1 preserver. Note that ψ preserves rank and hence $\phi = \psi\varphi\psi^{-1}$ is a rank-1 preserving linear map on $\mathscr{T}_{n-1}(\mathbb{F})$. Now the result follows from Theorem 2.3 of [3]. The sufficiency of the conditions is clear. □

REMARK 3. Let k be a positive integer $\leqslant n$ and $\mathscr{J}_k(\mathbb{F})$ be the set of all $n \times n$ matrices $[a_{ij}]$ for which a_{ij} is 0 if $1 \leqslant j < k+i-1 \leqslant n$. Note that $\mathscr{J}_2(\mathbb{F}) = \mathscr{S}_n(\mathbb{F})$. A similar version of Theorem 3.1 is true for $\mathscr{J}_k(\mathbb{F})$ in place of $\mathscr{S}_n(\mathbb{F})$.

REMARK 4. If the map φ of Theorem 3.1 is injective or if it preserves rank-1 matrices in both directions (i.e., $\varphi(A)$ is a rank-1 matrix if and only if A is a rank-1 matrix) then the alternative 1 will not occur and φ is in one of the forms given in (a) or (b).

Now we consider a special class of *block* strictly upper triangular matrices. Let $n = mk$, where $k \geqslant 2, m \geqslant 1$, and $A \in \mathscr{M}_n(\mathbb{F})$ be of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{bmatrix},$$

where A_{ij}'s are $m\times m$ matrices. The algebra of all such block upper triangular matrices will be denoted by $\mathscr{T}_{m,k}(\mathbb{F})$. By $\mathscr{S}_{m,k}(\mathbb{F})$ we denote the algebra of all *block strictly upper triangular* matrices obtained from $\mathscr{T}_{m,k}(\mathbb{F})$ by setting $A_{ii}=0$, $1\leqslant i\leqslant k$. Note that when $m=1$ we have $k=n$ and $\mathscr{T}_{1,n}(\mathbb{F})$ and $\mathscr{S}_{1,n}(\mathbb{F})$ become $\mathscr{T}_n(\mathbb{F})$ and $\mathscr{S}_n(\mathbb{F})$ respectively. Define

$$\pi:\mathscr{S}_{m,k}(\mathbb{F})\to\mathscr{T}_{m,k-1}(\mathbb{F}) \tag{3.1}$$

by

$$\begin{bmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1k}\\ 0 & 0 & A_{23} & \cdots & A_{2k}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & A_{k-1,k}\\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \longmapsto \begin{bmatrix} A_{12} & A_{13} & \cdots & A_{1k}\\ 0 & A_{23} & \cdots & A_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_{k-1,k} \end{bmatrix}.$$

Clearly, the map π is linear, bijective, and preserves rank.

THEOREM 3.2. *Let k and m be integers such that $k,m\geqslant 2$, and $\varphi:\mathscr{S}_{m,k}(\mathbb{F})\to\mathscr{S}_{m,k}(\mathbb{F})$ be linear and injective. Then φ is a rank-1 preserver if and only if there are invertible matrices P and Q in $\mathscr{T}_{m,k-1}(\mathbb{F})$ such that either*

1. *$\varphi(A)=\pi^{-1}(P\psi(A)Q)$ for all $A\in\mathscr{S}_{m,k}(\mathbb{F})$, or*

2. *$\varphi(A)=\pi^{-1}(P(\psi(A))^{+}Q)$ for all $A\in\mathscr{S}_{m,k}(\mathbb{F})$,*

where π is the bijective linear map defined in (3.1). Thus φ preserves every rank.

Proof. Let $\phi=\pi\varphi\pi^{-1}$. Then $\phi:\mathscr{T}_{m,k-1}(\mathbb{F})\to\mathscr{T}_{m,k-1}(\mathbb{F})$ is linear and bijective. If φ preserves rank-1 matrices, then so does ϕ. Now the necessity of the conditions follows from Theorem 4.4 of [1]. The sufficiency of the conditions is clear. □

4. Surjective additive rank-1 preservers

Now we consider surjective *additive* maps on $\mathscr{S}_{m,k}(\mathbb{F})$ preserving rank-1 matrices. Every automorphism θ of the field $\mathbb{F}$ induces a map Θ on $\mathscr{M}_n(\mathbb{F})$ defined by $\Theta(A)=[\theta(a_{ij})]$ for every $A=[a_{ij}]\in\mathscr{M}_n(\mathbb{F})$. Obviously Θ preserves rank and is additive. Let us first consider the simple case that $m=1$; so $k=n$ and $\mathscr{S}_{m,k}(\mathbb{F})=\mathscr{S}_n(\mathbb{F})$. Here, we borrow the notation and some definitions from [1]. For additive maps $f_1,f_2,\ldots,f_n$ from $\mathbb{F}$ to $\mathbb{F}$, where f_1 is bijective, let $\mathbf{f}=(f_1,\ldots,f_n)$, and define $\hat{\mathbf{f}}$ on $\mathscr{T}_n(\mathbb{F})$ by

$$\hat{\mathbf{f}}\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ 0 & a_{22} & \cdots & a_{2n}\\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}\right)=\begin{bmatrix} f_1(a_{11}) & f_2(a_{11})+a_{12} & \cdots & f_n(a_{11})+a_{1n}\\ 0 & a_{22} & \cdots & a_{2n}\\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

It is easy to see that $\hat{\mathbf{f}}$ is a surjective additive rank-1 preserver on $\mathscr{T}_n(\mathbb{F})$. Also, define $\check{\mathbf{f}}$ by

$$\check{\mathbf{f}}\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{n-1,n} \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & f_n(a_{nn}) + a_{1n} \\ \vdots & & & \\ 0 & 0 & \cdots & f_2(a_{nn}) + a_{n-1,n} \\ 0 & 0 & \cdots & f_1(a_{nn}) \end{bmatrix}.$$

Again, $\check{\mathbf{f}}$ is a surjective additive rank-1 preserver. Now we are ready to state the first result. Note that ψ is the bijective linear map defined in (2.2).

THEOREM 4.1. *Suppose $\varphi : \mathscr{S}_n(\mathbb{F}) \to \mathscr{S}_n(\mathbb{F})$ is additive, surjective, and rank-1 preserver, where $n \geqslant 4$. Then $\varphi = \psi^{-1}\phi\psi$, where the map $\phi : \mathscr{T}_{n-1}(\mathbb{F}) \to \mathscr{T}_{n-1}(\mathbb{F})$ can be expressed as a composition of some or all of the following maps:*

1. *Multiplication from left by an invertible matrix in $\mathscr{T}_{n-1}(\mathbb{F})$.*
2. *Multiplication from right by an invertible matrix in $\mathscr{T}_{n-1}(\mathbb{F})$.*
3. *The induced map Θ.*
4. *The map $\hat{f}$.*
5. *The map $\check{f}$.*
6. *The map $A \mapsto A^+$.*

Proof. The proof follows from properties of the map ψ and use of Theorem 5.5 of [1] for ϕ. □

The following corollary follows immediately.

COROLLARY 4.2. *Let φ be as in Theorem 4.1. Then φ is injective and preserves any rank.*

To state the next result, we need a terminology. A map L from a vector space V to vector space W is called *semilinear* if it is additive and there is an automorphism θ of $\mathbb{F}$ such that $L(cv) = \theta(c)L(v)$, for all c in $\mathbb{F}$ and v in V. Note that in Theorem 4.3 given below, π is the bijective linear map defined in (3.1).

THEOREM 4.3. *Suppose $\varphi : \mathscr{S}_{m,k}(\mathbb{F}) \to \mathscr{S}_{m,k}(\mathbb{F})$ is additive, surjective, and rank-1 preserver, where $k \geqslant 2, m \geqslant 2$. Then $\varphi = \pi^{-1}\phi\pi$, where the map $\phi : \mathscr{T}_{m,k-1}(\mathbb{F}) \to \mathscr{T}_{m,k-1}(\mathbb{F})$ is semilinear and is expressible as a composition of some or all of the following maps:*

1. *Multiplication from the left by an invertible matrix in $\mathscr{T}_{m,k-1}(\mathbb{F})$.*
2. *Multiplication from the right by an invertible matrix in $\mathscr{T}_{m,k-1}(\mathbb{F})$.*

3. The induced map Θ.

4. The map $A \mapsto A^+$.

Proof. The proof follows from properties of π and Theorem 5.5 and Corollary 5.7 of [1]. □

REMARK 5. If $\mathbb{F} = \mathbb{R}$, then the map ϕ, and consequently φ, becomes linear.

REMARK 6. Using/adapting examples given in Section 6 of [1], it can be shown that: (a) Surjectivity condition in both theorems of this section is necessary; (b) Theorem 4.1 is not true for $n = 3$; and (c) In general the property "φ preserves rank-1 matrices in both directions", instead of surjectivity for φ, is not enough.

REMARK 7. For a field $\mathbb{F}$ that does not have a proper isomorphic subfield, it follows from Theorem 7.2 of [1] that: Theorems 4.1 and 4.3 remain valid if the "surjectivity of φ" is replaced with "φ preserves rank-1 matrices in both directions". Moreover, in Theorem 4.1 it is enough to assume that the map f_1 is injective.

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