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# INVARIANCE OF TOTAL NONNEGATIVITY OF A TRIDIAGONAL MATRIX UNDER ELEMENT–WISE PERTURBATION

MOHAMMAD ADM AND JÜRGEN GARLOFF

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*Abstract.* Tridiagonal matrices are considered which are totally nonnegative, i. e., all their minors are nonnegative. The largest amount is given by which the single entries of such a matrix can be perturbed without losing the property of total nonnegativity.

## 1. Introduction

In this paper we consider tridiagonal matrices which are totally nonnegative, i. e., all their minors are nonnegative. We are interested in the largest amount by which the single entries of such a matrix can be varied without losing the property of total nonnegativity. For the properties of totally nonnegative matrices the reader is referred to [2] and to the two recent monographs [3, 8]. The question by which amount single entries of a general, not necessarily tridiagonal, matrix can be perturbed without losing the property of total nonnegativity is answered for a few specific entries in [3, Section 9.5]. A related question is the conjecture by the second author about the totally nonnegative matrix interval [4], see [3, Section 3.2] and [8, Section 3.2] and for related results [5, 6]. This conjecture was positively answered for the tridiagonal case in [4] and can be stated as follows: Assume that we are given three $n$-by-$n$ tridiagonal matrices $A=(a_{ij})$, $B=(b_{ij})$, and $C=(c_{ij})$, and assume that $a_{ii}\leqslant c_{ii}\leqslant b_{ii}$, $i=1,\dots,n$, and $b_{i,i+1}\leqslant c_{i,i+1}\leqslant a_{i,i+1}$, $b_{i+1,i}\leqslant c_{i+1,i}\leqslant a_{i+1,i}$, $i=1,\dots,n-1$. Then, if $A$ and $B$ are nonsingular and totally nonnegative, then matrix $C$ is nonsingular and totally nonnegative, too. The problem of finding the largest amount by which the single entries of a totally positive matrix, i. e., a matrix having all its minors positive, can be perturbed without losing the property of totally positivity was solved in [1].

The organization of our paper is as follows. In the next section we explain our notation and collect some auxiliary results in Section 3. In Section 4 we present our results in the nonsingular case, and in Section 5 in the singular case.

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## 2. Notation

We now introduce the notation used in our paper.

For $k,n \in \mathbb{N},\ 1 \leqslant k \leqslant n$, we denote by $Q_{k,n}$ the set of all strictly increasing sequences of $k$ integers chosen from $\{1,2,\ldots,n\}$. Let $A$ be a real $n\times n$ matrix. For $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_k), \beta = (\beta_1,\beta_2,\ldots,\beta_k) \in Q_{k,n}$ we denote by $A[\alpha|\beta]$ the $k\times k$ submatrix of $A$ contained in the rows indexed by $\alpha_1,\alpha_2,\ldots,\alpha_k$ and columns indexed by $\beta_1,\beta_2,\ldots,\beta_k$. We suppress the brackets when we enumerate the indices explicitly. When $\alpha = \beta$, the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$ and $\det A[\alpha]$ is called a *principal minor*. In the special case where $\alpha = (1,2,\ldots,k)$, we refer to the principal submatrix $A[\alpha]$ as a *leading principal submatrix* (and to $\det A[\alpha]$ as a *leading principal minor*) *of order* $k$. By $A(\alpha|\beta)$ we denote the $(n-k)\times(n-k)$ submatrix of $A$ contained in the rows indexed by the elements of $\{1,2,\ldots,n\}\setminus\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$, and columns indexed by $\{1,2,\ldots,n\}\setminus\{\beta_1,\beta_2,\ldots,\beta_k\}$ (where both sequences are ordered strictly increasingly) with the similar notation $A(\alpha)$ for the complementary principal submatrix.

In the sequel we put $\det A[\alpha_1,\alpha_2] := 1$ if $\alpha_1 > \alpha_2$ (possibly $\alpha_2 = 0$).

A minor $\det A[\alpha|\beta]$ is called *quasi-initial* if either $\alpha = (1,2,\ldots,k)$ and $\beta \in Q_{k,n}$ is arbitrary or $\alpha \in Q_{k,n}$ is arbitrary, while $\beta = (1,2,\ldots,k)$.

The $n$-by-$n$ matrix whose only nonzero entry is in the $(i,j)^{th}$ position and this entry is a one, is denoted by $E_{ij}$. An $n$-by-$n$ matrix $A=(a_{ij})$ is referred to as a *tridiagonal (or Jacobi)* matrix if $a_{ij}=0$ whenever $|i-j|>1$. An $n$-by-$n$ matrix $A$ is called *totally nonnegative* (abbreviated $TN$ henceforth) if $\det A[\alpha|\beta] \geqslant 0$ for all $\alpha,\beta \in Q_{k,n}, k=1,2,\ldots,n$. If in addition, $A$ is nonsingular we say A is an $NsTN$ matrix.

## 3. Auxiliary results

We start with some basic fact on tridiagonal matrices.

The determinant of an $n$-by-$n$ tridiagonal matrix $A=(a_{ij})$ can be evaluated by using the following recursion equations:

$$\det A = a_{11}\det A[2,\ldots,n] - a_{12}a_{21}\det A[3,\ldots,n] \tag{1}$$
$$= a_{n,n}\det A[1,\ldots,n-1] - a_{n-1,n}a_{n,n-1}\det A[1,\ldots,n-2]. \tag{2}$$

The following proposition extends both relations.

PROPOSITION 1. [8, Formula (4.1)] *For an $n$-by-$n$ tridiagonal matrix $A=(a_{ij})$ the following relation holds true*

$$\begin{aligned}\det A =& \det A[1,\ldots,i-1]\det A[i,\ldots,n] \\ &- a_{i-1,i}a_{i,i-1}\det A[1,\ldots,i-2]\det A[i+1,\ldots,n], \quad i=2,\ldots,n.\end{aligned} \tag{3}$$

We will make use of the following properties of $NsTN$ matrices.

PROPOSITION 2. [2, Corollary 3.8], [8, Theorem 1.13] *All principal minors of an NsTN matrix are positive.*

PROPOSITION 3. [3, Theorem 3.3.5] *If $A$ is nonsingular, then $A$ is NsTN if and only if the leading principal minors of $A$ are positive and all its quasi-initial minors are nonnegative.*

PROPOSITION 4. [2, Theorem 2.3], [8, Theorem 4.3 and p. 100] *Let $A$ be a tridiagonal, entry-wise nonnegative matrix.*

*a) $A$ is TN if and only if all its principal minors formed from consecutive rows and columns are nonnegative.*

*b) $A$ is NsTN if and only if all its leading principal minors are positive.*

PROPOSITION 5. [7, Lemma 6] *Let $n > 2$ and $A$ be an $n$-by-$n$ tridiagonal, entry-wise nonnegative matrix. Then $A$ is TN if*

*(i)* $\det A \geqslant 0$,

*(ii)* $\det A[1,\ldots,n-1] \geqslant 0$,

*(iii)* $\det A[1,\ldots,k] > 0,\ k = 1,\ldots,n-2$.

## 4. The nonsingular case

In this section we consider the variation of single entries of a tridiagonal $NsTN$ matrix $A = (a_{ij})$ such that the resulting matrix remains $NsTN$. We may restrict the discussion of the off-diagonal entries to the entries which are lying above the main diagonal since the related statements for the entries below the main diagonal follow by consideration of the transposed matrix.

The (zero) entries $a_{ij}$ with $j > i+2$ cannot be (strictly) increased because the resulting matrix will not be $TN$. This can be seen by considering the minor

$$\det A[i,i+1|i+1,i+3] = -a_{i+1,i+1}a_{i,i+3}$$

which is zero by $a_{i,i+3} = 0$. If $a_{i,i+3} + t > 0$ the minor becomes negative because $a_{i+1,i+1} > 0$ by Proposition 2.

In the sequel we treat the variation of the diagonal entries of $A$ and of its entries in the first and second upper diagonal. The following Lemma is a special case of Koteljanskiĭ's inequality, e. g., [3, Formula (6.4)]. We give the proof here since the proof of Lemma 8 will refer to it.

LEMMA 6. *The following inequality holds true*

$$\frac{\det A}{\det A(i)} \leqslant \frac{\det A(n)}{\det A(i,n)}, \quad i = 1,\ldots,n-1. \tag{4}$$

*Proof.* We have

$$\det A(i) = a \det A[i+1,\ldots,n] \text{ with } a = \det A[1,\ldots,i-1] > 0,$$

and similarly,

$$\det A(i,n) = a \det A[i+1,\ldots,n-1], \quad i = 1,\ldots,n-2.$$

By Proposition 1 there exist $b_1, b_2 \geqslant 0$ with

$$\det A = b_1 \det A[i,\ldots,n] - b_2 \det A[i+1,\ldots,n]$$

and similarly,

$$\det A(n) = b_1 \det A[i,\ldots,n-1] - b_2 \det A[i+1,\ldots,n-1].$$

It follows that

$$\begin{aligned} d &:= \det A \det A(i,n) - \det A(n) \det A(i) \\ &= ab_1(\det A[i,\ldots,n] \det A[i+1,\ldots,n-1] - \det A[i+1,\ldots,n] \det A[i,\ldots,n-1]). \end{aligned}$$

Application of (1) to $A[i,\ldots,n]$ and $A[i,\ldots,n-1]$ yields

$$\begin{aligned} d = {} & ab_1 a_{i,i+1} a_{i+1,i} (\det A[i+1,\ldots,n] \det A[i+2,\ldots,n-1] \\ & - \det A[i+2,\ldots,n] \det A[i+1,\ldots,n-1]). \end{aligned}$$

Repeated application of (1) results in (where $c$ is a nonnegative constant)

$$\begin{aligned} d &= c(\det A[n-2,n-1,n] \det A[n-1] - \det A[n-1,n] \det A[n-2,n-1]) \\ &= c[(a_{n-2,n-2}(a_{n-1,n-1}a_{n,n} - a_{n-1,n}a_{n,n-1}) - a_{n-2,n-1}a_{n-1,n-2}a_{n,n})a_{n-1,n-1} \\ &\quad -(a_{n-1,n-1}a_{n,n} - a_{n-1,n}a_{n,n-1})(a_{n-2,n-2}a_{n-1,n-1} - a_{n-2,n-1}a_{n-1,n-2})] \\ &= -ca_{n-1,n}a_{n,n-1}a_{n-2,n-1}a_{n-1,n-2} \leqslant 0 \end{aligned}$$

from which inequality (4) follows. $\square$

THEOREM 7. *Let $i \in \{1,\ldots,n\}$. Then the matrix $A + tE_{ii}$ is NsTN if and only if*

$$-\frac{\det A}{\det A(i)} < t. \tag{5}$$

*Proof.* By Proposition 4(b), it suffices to show that condition (5) is equivalent to $a_{i,i} + t \geqslant 0$ and all leading principal minors of $A_t := A + tE_{ii}$ are positive. Expansion of $\det A_t$ along its $i^{th}$ row (or column) yields

$$\det A_t = \det A + t \det A(i) \tag{6}$$

which is required to be positive for all nonnegative $t$. Therefore condition (5) follows from $\det A_t > 0$. Conversely, since $\det A_t(n) = \det A(n) + t \det A(i,n)$, Lemma 6 assures that the leading principal minor of order $n-1$ of $A_t$ is positive under condition (5). By application of Lemma 6 to $A(n), A(n-1,n), \ldots$ the positivity of the remaining leading principal minors follows. Finally, $a_{i,i} + t \geqslant 0$ is guaranteed by Proposition 1 because for $i \geqslant 2$

$$a_{i,i} + t = \frac{\det A_t[1,\ldots,i] + a_{i-1,i}a_{i,i-1} \det A[1,\ldots,i-2]}{\det A[1,\ldots,i-1]} \geqslant 0. \quad \square$$

REMARK 1. The statement for nonnegative $t$ can be found in [2, Corollary 2.4].

LEMMA 8. *If $a_{i+1,i} > 0$, then the following inequality holds true*

$$\frac{\det A}{\det A(i|i+1)} \leqslant \frac{\det A(n)}{\det A(i,n|i+1,n)}, \quad i = 1,\dots,n-2. \tag{7}$$

*Proof.* The proof parallels the one of Lemma 6. Expansion of $\det A(i|i+1)$ along its $i^{th}$ row yields

$$\det A(i|i+1) = a \det A[i+2,\dots,n], \text{ where } a = a_{i+1,i} \det A[1,\dots,i-1] > 0 \tag{8}$$

and similarly,

$$\det A(i,n|i+1,n) = a \det A[i+2,\dots,n-1].$$

As in the proof of Lemma 6, we apply Proposition 1 (with $i$ replaced by $i+1$) and obtain

$$\begin{aligned}&\det A \det A(i,n|i+1,n) - \det A(n) \det A(i|i+1)\\ &= ab_1(\det A[i+1,\dots,n] \det A[i+2,\dots,n-1] - \det A[i+2,\dots,n] \det A[i+1,\dots,n-1]).\end{aligned}$$

The claim follows now by proceeding as in the proof of Lemma 6. $\square$

THEOREM 9. *Let $i \in \{1,\dots,n-1\}$. If $a_{i+1,i} > 0$, the matrix $A + tE_{i,i+1}$ is NsTN if and only if*

$$-a_{i,i+1} \leqslant t < \frac{\det A}{\det A(i|i+1)}.$$

*If $a_{i+1,i} = 0$, only the restriction $-a_{i,i+1} \leqslant t$ is required.*

*Proof.* Let $a_{i+1,i} > 0$. Expansion of the determinant of $A_t := A + tE_{i,i+1}$ along its $i^{th}$ row yields.

$$\det A_t = \det A - t \det A(i|i+1). \tag{9}$$

To show the inequality on the right-hand side, we continue similarly as in the proof of Theorem 7 with the application of Lemma 8.

If $a_{i+1,i} = 0$, each leading principal minor of $A_t$ is independent of $t$.

For nonpositive $t$, $\det A_t$ is positive. However, we have to assure that $a_{i,i+1} + t$ is nonnegative. $\square$

THEOREM 10. *For $i = 1,\dots,n-2$ the matrix $A + tE_{i,i+2}$ is NsTN if*

$$0 \leqslant t \leqslant \frac{a_{i,i+1} a_{i+1,i+2}}{a_{i+1,i+1}}. \tag{10}$$

*Proof.* Since $a_{i,i+2} = 0$, $t$ must be nonnegative if $A_t := A + tE_{i,i+2}$ is $TN$. All leading principal minors of order $k$ of $A_t$ with $k \geqslant i+2$ are monotonically increasing with respect to $t$ and the remaining ones are leading principal minors of $A$. Therefore by Proposition 3, it remains to consider the quasi-initial minors (note that $A_t$ is no longer tridiagonal if $t>0$).

Let $\alpha=(1,\ldots,i+k)$ and $\beta\in Q_{i+k,n}$ arbitrary. If $k=0$ it suffices to treat the case $\beta_i=i+2$ since if $\beta_i>i+2$, $A_t[\alpha|\beta]$ contains a zero column. If $\beta_{i-1}=i+1$, then $\det A_t[\alpha|\beta]=0$ because all entries of $A[1,\ldots,i-1|i+1,i+2]$ are zero. If $\beta_{i-1}\leqslant i$, then

$$\det A_t[\alpha|\beta]=t\det A[1,\ldots,i-1|\beta_1,\ldots,\beta_{i-1}].$$

Therefore, $\det A_t[\alpha|\beta]\geqslant 0$ for all $t\geqslant 0$.

Now let $k>1$. We can restrict the discussion to $\beta_{i+k}=i+k+1$, because if $\beta_{i+k}>i+k+1$, then $A_t[\alpha|\beta]$ contains a zero column and if $\beta_{i+k}=i+k$, then $\det A_t[\alpha|\beta]$ is a leading principal minor.

Since in the last column of $A_t[\alpha|\beta]$ the only possibly non-zero entry is in the last position it suffices to consider $\alpha=(1,\ldots,i+k-1)$. Continuing in this way, we arrive at $\alpha=(1,\ldots,i+1)$ and $\beta_{i+1}=i+2$.

If $\beta_i=i+1$, we have

$$\begin{aligned}\det A_t[\alpha|\beta]&=\det A[1,\ldots,i-1|\beta_1,\ldots,\beta_{i-1}]\det A_t[i,i+1|i+1,i+2]\\&=\det A[1,\ldots,i-1|\beta_1,\ldots,\beta_{i-1}](a_{i,i+1}a_{i+1,i+2}-ta_{i+1,i+1}).\end{aligned}\tag{11}$$

Therefore condition (10) guarantees $\det A_t[\alpha|\beta]\geqslant 0$.

If $\beta_i=i$, i.e., $\beta=(1,\ldots,i,i+2)$, we have

$$\det A_t[\alpha|\beta]=\det A[\alpha|\beta]-t\det A[1,\ldots,i-1,i+1|1,\ldots,i].$$

Then if $\det A[1,\ldots,i-1,i+1|1,\ldots,i]=0$, then $\det A_t[\alpha|\beta]=\det A[\alpha|\beta]\geqslant 0$. When

$$\det A[1,\ldots,i-1,i+1|1,\ldots,i]>0,$$

then $\det A_t[\alpha|\beta]\geqslant 0$ if and only if

$$t\leqslant\frac{\det A[\alpha|\beta]}{\det A[1,\ldots,i-1,i+1|1,\ldots,i]}.\tag{12}$$

Therefore to see that (10) implies $\det A_t[\alpha|\beta]\geqslant 0$ it remains to show that the right-hand side of inequality (12) is not smaller than the right-hand side of (10). Since

$$\det A[\alpha|\beta]=a_{i+1,i+2}\det A[1,\ldots,i]$$

and

$$\det A[1,\ldots,i-1,i+1|1,\ldots,i]=a_{i+1,i}\det A[1,\ldots,i-1],$$

we obtain using (2)

$$\begin{aligned}0&\leqslant a_{i+1,i+2}\det A[1,\ldots,i+1]\\&=a_{i+1,i+2}(a_{i+1,i+1}\det A[1,\ldots,i]-a_{i,i+1}a_{i+1,i}\det A[1,\ldots,i-1])\\&=a_{i+1,i+1}a_{i+1,i+2}\det A[1,\ldots,i]-a_{i,i+1}a_{i+1,i+2}a_{i+1,i}\det A[1,\ldots,i-1]\\&=a_{i+1,i+1}\det A[\alpha|\beta]-a_{i,i+1}a_{i+1,i+2}\det A[1,\ldots,i-1,i+1|1,\ldots,i]\end{aligned}$$

from which it follows that

$$\frac{a_{i,i+1}a_{i+1,i+2}}{a_{i+1,i+1}} \leqslant \frac{\det A[\alpha|\beta]}{\det A[1,\ldots,i-1,i+1|1,\ldots,i]}.$$

Now let $\beta = (1,\ldots,i+k)$ and $\alpha \in Q_{i+k,n}$ be arbitrary.

If $k=2$ and $\alpha_{i+2} = i+2$, then $\det A_t[\alpha|\beta]$ is a leading principal minor and if $\alpha_{i+2} = i+3$, then

$$\det A_t[\alpha|\beta] = a_{i+3,i+2}\det A[\alpha_1,\ldots,\alpha_{i+1}|1,\ldots,i+1] \geqslant 0.$$

If $\alpha_{i+2} > i+3$, then $\det A_t[\alpha|\beta] = 0$ because $A_t[\alpha|\beta]$ contains a zero row.

If $k>2$ it suffices to treat only the case $\alpha_{i+k} = i+k+1$ since a submatrix $A_t[\alpha|\beta]$ with $\alpha_{i+k} > i+k+1$ contains a zero row and if $\alpha_{i+k} = i+k$ it is a leading principal submatrix. Since for $\alpha_{i+k} = i+k+1$

$$\det A_t[\alpha|\beta] = a_{i+k+1,i+k}\det A_t[\alpha_1,\ldots,\alpha_{i+k-1}|1,\ldots,i+k-1],$$

this case reduces to the case of $\beta = (1,\ldots,i+k-1)$. Continuing in this way, we arrive at the case $k=2$ already treated above. □

REMARK 2. If $A$ is irreducible, i. e., all the entries in its super- and subdiagonal are positive, the determinant in the second row of (11) is positive, so that $\det A_t[\alpha|\beta] \geqslant 0$ implies inequality (10). Therefore, condition (10) is also necessary.

EXAMPLE 1. We choose $A$ as

$$A := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Then $A$ is $NsTN$ $(\det A = 5)$. In the following we give the largest interval from which $t$ can be chosen such that the matrix $A(t) := A + tE_{ij}$ is $NsTN$, $i,j = 1,2,3,4$. The intervals are given in the $(i,j)$ position $(i \leqslant j)$ of the respective entry. If $t$ is chosen as the left and right endpoint of the interval for the entries on the diagonal and superdiagonal, respectively, the matrix $A(t)$ is singular.

$$\begin{array}{cccc} (-\frac{5}{4},\infty) & [-1,\frac{5}{3}) & [0,\frac{1}{2}] & [0,0] \\ & (-\frac{5}{6},\infty) & [-1,\frac{5}{4}) & [0,\frac{1}{2}] \\ & & (-\frac{5}{6},\infty) & [-1,\frac{5}{3}) \\ & & & (-\frac{5}{4},\infty) \end{array}$$

## 5. The singular case

In this section we consider the variation of single entries of a tridiagonal $TN$ matrix such that the resulting matrix remains $TN$. By Proposition 4(a), we can restrict the discussion to irreducible tridiagonal $TN$ matrices. By considering principal minors of order 2 we see that then not only the entries in the super- and subdiagonal are positive, but also the entries on the main diagonal must be positive.

LEMMA 11. *Let $A = (a_{ij})$ be an $n$-by-$n$ irreducible, tridiagonal, entry-wise nonnegative matrix. Then $A$ is $TN$ if and only if*
*(i)* $\det A \geqslant 0$,
*(ii)* $\det A[1,\ldots,k] > 0,\ k = 1,\ldots,n-1$.

*Proof.* By Proposition 5 it suffices to show that the total nonnegativity of $A$ implies (ii). We proceed by induction on $k = 1,\ldots,n-1$.

For $k = 1$, we have $a_{11} > 0$ (see above). Suppose that we have already shown that $\det A[1,\ldots,k-1] > 0$. Then we obtain by application of (2) to $\det A[1,\ldots,k+1]$

$$\det A[1,\ldots,k+1] = a_{k+1,k+1} \det A[1,\ldots,k] - a_{k,k+1} a_{k+1,k} \det A[1,\ldots,k-1].$$

Since $a_{k+1,k+1}$, $a_{k,k+1}$, $a_{k+1,k}$, $\det A[1,\ldots,k-1] > 0$, and $\det A[1,\ldots,k+1] \geqslant 0$, it follows that $\det A[1,\ldots,k] > 0$. $\square$

LEMMA 12. *Let $A$ be an $n$-by-$n$ irreducible, tridiagonal $TN$ matrix. Then $\det A(i) > 0,\ i = 1,\ldots,n$.*

*Proof.* By Proposition 2, we have only to consider the case $\det A = 0$. Suppose that $\det A(i) = 0$ for $i \in \{1,\ldots,n\}$. Then we have

$$0 = \det A(i) = \det A[1,\ldots,i-1] \det A[i+1,\ldots,n].$$

By Lemma 11 it follows that $\det A[i+1,\ldots,n] = 0$, whence by (3) $\det A[i,\ldots,n] = 0$. Continuing in this way, we arrive at $a_{11} = 0$, a contradiction. $\square$

Now let $A$ be an $n$-by-$n$ irreducible, tridiagonal $TN$ matrix. From (6) and Lemma 12 we obtain that $A_t := A + tE_{ii}$, $i = 1,\ldots,n$, is not $TN$ if $t < 0$. On the other hand, $A_t$ is $NsTN$ for all $t > 0$, see [2, Corollary 2.4].

By the proof of Lemma 12, we have that $\det A[i,\ldots,n] > 0$, $i = 2,\ldots,n$. Therefore, it follows from (8) that $\det A(i|i+1) > 0$, $i = 1,\ldots,n-1$, and by (9) we see that $A_t := A + tE_{i,i+1}$ is not $TN$ if $t > 0$. On the other hand, as in the proof of Theorem 9 we obtain that $A_t$ is $NsTN$ if and only if $-a_{i,i+1} \leqslant t < 0$.

Finally, we extend Theorem 10 to the singular case. We add $\varepsilon > 0$ to $a_{11}$. Then the resulting matrix $B_\varepsilon$ becomes $NsTN$ and we apply the perturbation result of Theorem 10 to $B_\varepsilon$. The bound in (10) remains in force when $\varepsilon$ tends to 0.

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*Mohammad Adm*
*Department of Mathematics and Statistics*
*University of Konstanz*
*D-78464 Konstanz, Germany*
*e-mail:* mjamathe@yahoo.com

*Jürgen Garloff*
*Faculty of Computer Science*
*University of Applied Sciences / HTWG Konstanz*
*D-78405 Konstanz, Germany*
*e-mail:* garloff@htwg-konstanz.de