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EXISTENCE OF MAXIMAL SEMIDEFINITE INVARIANT SUBSPACES AND SEMIGROUP PROPERTIES OF SOME CLASSES OF ORDINARY DIFFERENTIAL OPERATORS

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Abstract. We describe sufficient conditions for the operator $Lu = \frac{1}{g(x)}L_0u$, with L_0 an ordinary differential operator dissipative on its domain and a function g changing its sign, to have maximal semidefinite invariant subspaces in the Krein space $L_{2,g}(a,b)$ with the indefinite inner product $[u,v] = \int_a^b g(x)u(x)\overline{v(x)}\,dx$. The semigroup properties of the restrictions of an operator to these subspaces are studied. The similarity problem of L to a selfadjoint operator is discussed.

1. Introduction

We examine the differential operators of the form

$$Lu = \frac{1}{g(x)}L_0u, \ \ x \in (a,b), \tag{1}$$

where L_0 is an ordinary differential operator of order $2m$ defined by the differential expression

$$L_0u = \sum_{i,j=0}^{m} \frac{d^i}{dx^i}a_{ik}\frac{d^ju}{dx^j} \ \ (x \in (a,b),\ -\infty \leqslant a < b \leqslant \infty)$$

and some boundary conditions. The real valued function $g(x)$ in (1) changes its sign on (a,b). Let $(u,v) = \int_a^b u(x)\overline{v(x)}\,dx$. We consider the case of a J-dissipative operator L in the Krein space $F_0 = L_{2,g}(a,b)$ endowed with the following inner product and indefinite inner product:

$$(u,v)_0 = (|g(x)|u(x),v(x)), \ \ [u,v]_0 = (g(x)u(x),v(x)) \ \ (\text{thus } \|u\|_0^2 = (u,u)_0).$$

By the definition, we have that $\mathrm{Re}\,[-Lu,u]_0 \geqslant 0$ for all $u \in D(L)$, with $D(L)$ the domain of L. The spectral problems for the operator L with a weight function $g(x)$ changing its sign on (a,b) (and elliptic problems of this type) were the subject on many investigations. These problems arise in many fields of physics and applied mathematics (see, for instance, [35, Ch. 5,6]).

The most interesting questions are the completeness questions and the Riesz basis property of eigenfunctions and associated functions of L in F_0. Probably, the first

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advances in the study of the Riesz basis property were made in [7, 40, 41]. Now it is possible to say that there is a significant number of the articles devoted to this question and we can refer to [11, 44, 45, 16, 30, 42], [36]–[39], where necessary bibliography and the latest results can be found. Some other spectral properties are studied also in [8, 9].

One more interesting question is the question on existence of maximal semidefinite invariant subspaces of L in the Krein space F_0 and the corresponding question on similarity of L to a selfadjoint or normal operator (see the definitions below and in [4]). The last two questions are closely connected and in the case of the J-selfadjoint operator L are equivalent ([24, Prop. 2.2]). If the operator L_0 is selfadjoint in $L_2(a,b)$ and the operator $R: L_{2,g}(a,b) \to L_{2,g}(a,b)$, $Ru = L_0^{-1}g(x)u$, is compact then this question is equivalent to the Riesz basis property of eigenfunctions and associated function of L in F_0. The most difficult case is the case of $0 \notin \rho(L_0)$. The first results devoted to existence of maximal semidefinite invariant subspaces for operators acting in some Krein (Pontryagin) space belong to L. S. Pontryagin.

The most significant generalizations of his results were obtained in the articles by Krein M. G., Langer H., Azizov T. Ya., Shkalikov A. A. (see the results and the bibliography in [5, 6, 48, 49]). We refer also to the author results in [47, 46].

If maximal semidefinite invariant subspaces exist then the operator admits a decomposition into the sum of two commuting operators defined on these subspaces; under certain conditions, these operators to within a multiplication by -1 are generators of analytic semigroups. The last fact allows us to study the solvability questions for different equations involving this operator.

The basic aim of the present article is to consider the question of existence of maximal semidefinite invariant subspaces of the operator L and to study the respective semigroup properties of L. Together with the conventional case, we also examine the case when the domain of L comprises discontinuous functions satisfying certain gluing conditions at the discontinuity points. The operators of this type (among them are the SchrÖdinger operators with δ and δ'-interactions) arise in physics and applied mathematics (see [1, 2]). These discontinuity points are called interaction points. Second order operators with δ and δ'-interactions and the function g changing its sign are studied in [22]. In the present article, we mostly pay attention to the most difficult case when the point 0 is the limit point of a continuous spectrum of L and use the results of [47]. Simpler case $0 \in \rho(L)$ was considered recently in [33], where more stringent conditions were used on the operator L itself and on the weight function g. The results of this article concerning with similarity are partially known in the case of $m = 1$. We refer to the articles [26, 32] which contains the most essential results in this direction. Among the other articles devoted to the similarity questions, we note the articles [12, 14, 15, 19, 20, 21, 23, 27, 28, 29].

2. Definitions and auxiliary statements

2.1. Basic definitions.

Recall that a Krein space (see [4]) is a Hilbert space H with an inner product $(\cdot,\cdot)$ in addition endowed with an indefinite inner product of the form $[x,y]=(Jx,y)$, where $J=P^{+}-P^{-}$ ($P^{\pm}$ are orthoprojections in H, $P^{+}+P^{-}=I$). We put $H^{\pm}=R(P^{\pm})$. In what follows, the symbol I stands for the identity and the symbols $D(A)$, $R(A)$, and $N(A)$ designate the domain, the range, and the kernel of an operator A. The operator J is called a fundamental symmetry. A subspace M in H is said to be nonnegative (positive, uniformly positive) if the inequality $[x,x]\geqslant 0$ (respectively, $[x,x]>0$, $[x,x]\geqslant\delta\|x\|^{2}$ $(\delta>0)$) holds for all $x\in M$. Nonpositive, negative, uniformly negative subspaces in H are defined in a similar way. If a nonnegative subspace M admits no nontrivial nonnegative extensions, then it is called a maximal nonnegative subspace. Maximal nonpositive (positive, negative, nonnegative, etc.) subspaces in H are defined by analogy. A densely defined operator A is said to be dissipative (strictly dissipative, uniformly dissipative) in H if $-\mathrm{Re}\,(Ax,x)\geqslant 0$ for all $x\in D(A)$ ($-\mathrm{Re}\,(Ax,x)>0$ for all $x\in D(A)$ or $-\mathrm{Re}\,(Ax,x)\geqslant\delta\|u\|^{2}$ $(\delta>0)$ for all $x\in D(A)$). Similarly, a densely defined operator A is called a J-dissipative (strictly J-dissipative or uniformly J-dissipative) whenever the operator JA is dissipative (strictly dissipative or uniformly dissipative). A dissipative (J-dissipative) operator is said to be maximal dissipative (maximal J-dissipative) if it admits no nontrivial dissipative (J-dissipative) extensions. Let $A:H\to H$ be a J-dissipative operator. We say that a subspace $M\subset H$ is invariant under A if $D(A)\cap M$ is dense in M and $Ax\in M$ for all $x\in D(A)\cap M$. An operator A such that $-A$ is dissipative (maximal dissipative) is called accretive (maximal accretive). Hence, taking the sign into account we can say that all statements valid for an accretive operator are true for a dissipative operator as well. In what follows, we replace the word "maximal" with the letter m and thus we write m-dissipative rather than maximal dissipative. If A is an operator in a Krein space H then we denote by A^{*} and A^{c} the adjoint operators with respect to the inner product and the indefinite inner product in H, respectively. The latter operator possesses the usual properties of an adjoint operator (see [4]). Let A_0 and A_1 be two Banach spaces continuously embedded into a topological linear space E: $A_0\subset E$, $A_1\subset E$. Such a pair $\{A_0,A_1\}$ is called compatible or an interpolation couple. The definition of the interpolation space $(A_0,A_1)_{\theta,q}$ can be found in [50]. As conventionally, the symbols $W_p^s(a,b)$ and $B_{p,q}^s(a,b)$ $(1\leqslant p,q\leqslant\infty$, $s>0)$ stand for the Sobolev space and Besov space of functions defined on (a,b) (see [50]). The space $W_{p,loc}^s(a,b)$ comprises the functions $f(x)$ such that $f\in W_p^s(c,d)$ for every bounded interval (c,d) with $[c,d]\subset(a,b)$. The Lebesgue space $L_{p,loc}(a,b)$ is defined similarly.

2.2. Some general results.

We present here general results which are the base for our further considerations.

LEMMA 1. *Let H be a Hilbert space (a Krein space). A maximal dissipative (J-dissipative) operator $A:H\to H$ is always closed and a closed operator A is m-*

dissipative if and only if $\rho(A)\cap\{z:\operatorname{Re}z\geqslant 0\}\neq\emptyset$, *in this case* $\mathbb{C}^+=\{z\in\mathbb{C}:\ \operatorname{Re}z>0\}\subseteq\rho(A)$ (*see* [4, Lemma 2.8]).

Let H be a complex Hilbert space with the norm $\|\cdot\|$ and the inner product $(\cdot,\cdot)$ and let $L:H\to H$ be a closed densely defined operator. Assign $S_\theta=\{z:|\arg z|<\theta\}$ for $\theta\in(0,\pi]$ and $S_\theta=(0,\infty)$ for $\theta=0$. Recall that $L:H\to H$ is a sectorial operator if there exists $\theta\in[0,\pi)$ such that $\sigma(L)\subseteq\overline{S_\theta}$, $\mathbb{C}\setminus\overline{S_\theta}\subseteq\rho(L)$, and, for every $\omega>\theta$, there exists a constant $c(\omega)$ such that

$$\|(L-\lambda I)^{-1}\|\leqslant c/|\lambda|\ \ \forall\lambda\in\mathbb{C}\setminus S_\omega. \tag{2}$$

The minimal quantity θ with the above property is called the sectoriality angle of L. Let L be sectorial and injective (we do not require that $0\in\rho(L)$). In this case, completing the subspaces $D(L^k)$ and $R(L^k)$ ($k\in\mathbb{N}$, $\mathbb{N}$ is the set of positive integers) with respect to the norms $\|u\|_{D_L}=\|L^ku\|$ and $\|u\|_{R_{L^k}}=\|L^{-k}u\|$, we obtain new spaces denoted by D_{L^k} and R_{L^k}, respectively (see [18],[3],[13]). Interpolation properties of these spaces can be found in [18]. In the case of $0\in\rho(L)$, these spaces and their interpolation properties are described in [17] (see also [50, Sect. 1.14.3]).

Let H_1,H be a pair of compatible Hilbert spaces and let $H_1\cap H$ be densely embedded into H and H_1. The symbol $(\cdot,\cdot)$ stands for the inner product in H. Define the negative space H_1' constructed on this pair as the completion of $H\cap H_1$ with respect to the norm

$$\|u\|_{H_1'}=\sup_{v\in H_1\cap H}|(u,v)|/\|v\|_{H_1}.$$

The following lemma can be found in [47, Prop. 2.4].

LEMMA 2. *The spaces* H_1, H_1' *are dual to each other with respect to the pairing* $(\cdot,\cdot)$ *and*

$$(H_1,H_1')_{1/2,2}=H. \tag{3}$$

Thus the space of antilinear continuous functionals over H_1 *can be identified with* H_1' *and the norm in* H_1 *is equivalent to the norm* $\sup_{v\in H_1'}|(v,u)|/\|v\|_{H_1'}$.

Let $L:H\to H$ be a strictly m-J-dissipative operator in a Krein space H with the indefinite inner product $[\cdot,\cdot]=(J\cdot,\cdot)$ and the norm $\|\cdot\|$, where J is the fundamental symmetry and the symbol $(\cdot,\cdot)$ designates the inner product in H. Define the quantities

$$\|u\|_{F_1}^2=-\operatorname{Re}[Lu,u]\ \ (u\in D(L)),\ \ \|u\|_{F_{-1}}^2=-\operatorname{Re}[L^{-1}u,u]\ \ (u\in R(L)).$$

Suppose also that

$$\exists c>0:\ |[Lu,v]|\leqslant c\|u\|_{F_1}\|v\|_{F_1}\ \ \forall u,v\in D(L). \tag{4}$$

Obviously, the quantity $\|u\|_{F_1}$ is a norm on $D(L)$.

Define the spaces F_1 and F_{-1} as completions of $D(L)\cap R(L)$ with respect to the norms $\|\cdot\|_{F_1}$ and $\|\cdot\|_{F_{-1}}$, respectively.

The following lemma results from [47, Prop. 2.7, Remark 2.2].

LEMMA 3. *Let $L: H \to H$ be a strictly m-J-dissipative operator satisfying* (4) *with a nonempty resolvent set. Then the spaces F_{-1}, F_1, and H are compatible, the subspaces $F_1 \cap H$ and $F_{-1} \cap H$ are dense in H and F_1, in H and F_{-1}, respectively. The operators $J : H \cap F_{\mp 1} \to H$ admit extensions to isomorphisms of F_{-1} onto F_1' and F_1 onto F_{-1}', respectively, where F_1' and F_{-1}' are negative spaces constructed on the pairs (F_1, H) and (F_{-1}, H). The spaces F_{-1} and F_1 are dual to each other. The norms*

$$\|u\|_{F_{-1}} = \sup_{v \in D(L) \cap R(L)} |[u,v]|/\|v\|_{F_1}, \quad \|u\|_{F_1} = \sup_{v \in D(L) \cap R(L)} |[u,v]|/\|v\|_{F_{-1}}$$

are equivalent to the above norms in F_{-1} and F_1, respectively. Moreover, the expression $[u,v]$ is defined for $u \in F_1$ and $v \in F_{-1}$ and we have the inequality

$$|[u,v]| \leqslant c_1 \|u\|_{F_{-1}} \|v\|_{F_1},$$

with c_1 a constant independent of u, v.

Given a strictly m-J-dissipative operator L satisfying the condition (4), we can construct the spaces D_L and R_L using the same definition as in the case of a sectorial operator. Assign $F_s = (F_1, H)_{1-s,2}$ $(s \in (0,1))$. Let $S_\theta^0 = S_\theta \cup (-S_\theta)$ $(-S_\theta = \{z : -z \in S_\theta\})$.

LEMMA 4. (see [47, Prop. 2.6, Lemmas 2.1, 2.2, 2.4]) *Let $L: H \to H$ be a strictly m-J-dissipative operator satisfying* (4) *with a nonempty resolvent set. Then so is the operator L^{-1} and $D(L^m) \cap R(L^m)$ is dense in H, $D(L)$, and $R(L)$ for every $m = 1, 2, \ldots$. Moreover, the following equalities are equivalent:*

$$(D_L, R_L)_{1/2,2} = H, \tag{5}$$

$$(F_1, F_{-1})_{1/2,2} = H; \tag{6}$$

if (6) *holds then there exists $\theta \in [0, \pi/2)$ such that $\sigma(L) \subseteq \overline{S_\theta^0}$, $\mathbb{C} \setminus \overline{S_\theta^0} \subseteq \rho(L)$, and, for every $\omega \in (\theta, \pi/2)$, there exists a constant $c(\omega)$ such that*

$$\|(L - \lambda I)^{-1}\| \leqslant c/|\lambda| \ \ \forall \lambda \in \mathbb{C} \setminus S_\omega^0; \tag{7}$$

if there exists $s \in (0,1)$ such that $J \in L(F_s)$ then the equality (6) *holds.*

By the above membership $J \in L(F_s)$, we mean that the operator $J|_{F_1 \cap F_0}$ admits an extension of the class $L(F_s)$. Let $L: H \to H$ be a strictly m-J-dissipative operator satisfying (4). Define the space H_1 as the completion of $D(L)$ with respect to the norm $\|u\|_{H_1}^2 = -\mathrm{Re}[Lu,u] + \|u\|_H^2$ and the space H_{-1} as the completion of H with respect to the norm $\|u\|_{H_{-1}} = \sup_{v \in H_1} |[u,v]|/\|v\|_{H_1}$. Assign $H_s = (H_1, H)_{1-s,2}$.

LEMMA 5. (see [47, Lemma 2.5]) *Let $L: H \to H$ be a strictly m-J-dissipative operator satisfying* (4). *Suppose that there exists a constant $m > 0$ such that*

$$\|u\|^2 \leqslant m(-\mathrm{Re}\,[Lu,u] + \|u\|_{H_{-1}}^2) \ \ \forall u \in D(L). \tag{8}$$

Then there exists a number $\omega_0 \geqslant 0$ *such that* $I_{\omega_0} = \{i\omega : |\omega| \geqslant \omega_0\} \subset \rho(L)\}$, *i. e.,* $\rho(L) \neq \emptyset$. *The inequality* (8) *certainly holds whenever*

$$(H_1, H_{-1})_{1/2,2} = H. \tag{9}$$

If there exists $s \in (0,1)$ *such that* $J \in L(H_s)$ *then the equality* (9) *holds.*

We heavily rely on the following theorem [47, Theorem 3.1].

THEOREM 1. *Let* $L : H \to H$ *be a strictly* m-J*-dissipative operator in a Krein space* H *satisfying* (4) *with a nonempty resolvent set. If the equality* (5) *holds then there exist maximal uniformly positive and uniformly negative* L*-invariant subspaces* H^+ *and* H^- *such that* $H = H^+ + H^-$, $D(L) = D(L) \cap H^+ + D(L) \cap H^-$ *(both sums are direct),* $\sigma(L|_{H^\pm}) \subset \mathbb{C}^\mp$, *and the operators* $\pm L|_{H^\pm}$ *are generators of analytic semigroups. If there exist uniformly positive and uniformly negative* L*-invariant subspaces* H^+ *and* H^- *such that* $H = H^+ + H^-$ *and* $D(L) = D(L) \cap H^+ + D(L) \cap H^-$ *(the sums are direct) then the equality* (5) *holds.*

2.3. Conditions on the data and auxiliary statements.

We assume that the function g satisfies the condition

$$g(x) \in L_1(c,d), \ \ \forall (c,d) \subseteq (a,b), \ d - c < \infty. \tag{10}$$

and there exist open subsets G^+ and G^- of $G = (a,b)$ consisting of finitely many disjoint intervals such that $g(x) > 0$ a.e. (almost everywhere) on G^+, $g(x) < 0$ a.e. on G^- and $\overline{G^+} \cup \overline{G^-} = [a,b]$. A point $x_0 \in \partial G^+ \cap \partial G^-$ is called a turning point. Let $\{x_i\}_{i=1}^s$ be all turning points of g. The fundamental symmetry J in $L_{2,g}(a,b) = F_0$ is given as $J = \chi_{G^+} - \chi_{G^-}$, with $\chi_{G^\pm}$ the characteristic functions of the corresponding sets. Let $\{y_k\}_{k=1}^N$ $(N \leqslant \infty)$ be a collection of points in (a,b). It is possible that some of these points coincide with turning points. We assume that the sequence $\{y_k\}_{i=1}^N$ is increasing and either $N < \infty$ or $\lim_{n\to\infty} y_n = b$ (the latter condition for the interaction points is exposed in [31]). In what follows, the symbols $f(x+)$ and $f(x-)$ stand for the right and left limits of a function $f(x)$ at a point x, respectively. Let $y_0 = a$. Consider a sesquilinear form

$$a(u,v) = \sum_{j,k=0}^{m} (a_{jk}(x)u^{(i)}, v^{(k)}) + \sum_{i=1}^{H} \sum_{j=0}^{m-1} (u^{(j)}(y_i+)\overline{U_j^i v} - u^{(j)}(y_i-)\overline{V_j^i v}),$$

where the operators $U_j^i v$, $V_j^i v$ are of the form

$$U_j^i v = \sum_{k=0}^{m-1} (a_{ji}^k v^{(k)}(y_i+) + b_{ji}^k v^{(k)}(y_i-)), \ \ V_j^i v = \sum_{k=0}^{m-1} (c_{ji}^k v^{(k)}(y_i+) + d_{ji}^k v^{(k)}(y_i-)),$$

with $a_{ji}^k, b_{ji}^k, c_{ji}^k, d_{ji}^k$ complex constants and the functions a_{ik} satisfy (10) for all i,k. Define the quantity

$$\|u\|_1^2 = \int_a^b \sum_{i=0}^{m} p_i(x)|u^{(i)}(x)|^2\,dx$$

where $p_i(x)$ $(i \leqslant m)$, $1/p_m(x)$ are nonnegative functions satisfying the condition (10). To define the domain of the form $a(u,v)$, we introduce the boundary and gluing conditions of the form

$$B_k u = \sum_{i=0}^{m-1} (\alpha_{ik} u^{(i)}(a) + \beta_{ik} u^{(i)}(b)) = 0 \ (k = 1,2,\dots,m_1 \ ,m_1 \leqslant 2m), \tag{11}$$

$$B_k u = \sum_{i=0}^{m-1} \alpha_{ik} u^{(i)}(a) = 0, \ \ (k = 1,2,\dots,m_1, \ m_1 \leqslant m), \tag{12}$$

$$B_k u = \sum_{i=0}^{m-1} \beta_{ik} u^{(i)}(b) = 0 \ (k = 1,2,\dots,m_1, \ m_1 \leqslant m), \tag{13}$$

$$B_j^k u = \sum_{i=1}^{m} (\alpha_{ji}^k u^{(i-1)}(y_k+) - \beta_{ji}^k u^{(i-1)}(y_k-)) = 0, \tag{14}$$

where $j = 1,2,\dots,m_0$ $(m_0 \leqslant m)$, $k = 1,2,\dots,N$, and the vectors

$$(\alpha_{j1}^k, \alpha_{j2}^k, \dots, \alpha_{jm}^k, -\beta_{j1}^k, \dots, -\beta_{jm}^k) \ \ (j = 1,2,\dots,m_0)$$

are linearly independent for every $k = 1,2,\dots,N$. If the expression $B_k u$ contains the values of u and its derivatives only in one of the points a or b then we call the boundary condition $B_k u = 0$ local. If all boundary condition are local (it is always true for an unbounded interval (a,b)) then we say that the boundary conditions are local. Otherwise, the boundary conditions are called nonlocal.

Let $y_\infty = \lim_{n\to\infty} y_n \leqslant +\infty$ in the case of $N = \infty$ and $y_\infty = y_N$ otherwise. Denote by S the class of functions $u \in L_{1,loc}(a,b)$ such that $u \in W_\infty^m(c,d)$ for every finite interval $(c,d) \subseteq (a,b)$ with $y_i \notin (c,d)$ for all i, the support supp u is bounded, $b \notin \operatorname{supp} u$ in the case of $b = y_\infty$, and u satisfies the gluing conditions (14) and the boundary conditions (11) if $a \neq -\infty$, $b \neq +\infty$, and $b \neq y_\infty$, the boundary conditions (12) if $a \neq -\infty$ and $b = +\infty$ or $b = y_\infty$, and the boundary conditions (13) if $a = -\infty$ and $b \neq +\infty$ and $b \neq y_\infty$. Define the spaces H_1 and F_1 as the completions of S with respect to the norms $\|u\|_{H_1} = (\|u\|_1^2 + \|u\|_0^2)^{1/2}$ and $\|\cdot\|_1$. By the symbol $\mu(\cdot)$, we mean the Lebesgue measure.

LEMMA 6. *Assume that either there exists a neighborhood U about the set $\{y_k\}_{k=1}^N$ such that $p_0(x) > 0$ almost everywhere on U or $m_0 = m$, the matrices $\{\alpha_{ji}^k\}_{i,j=1}^m$ and $\{\beta_{ji}^k\}_{i,j=1}^m$ are nondegenerate for every k, and*

$$\mu(\{x \in (a,b) : \ p_0(x) \neq 0\}) > 0. \tag{15}$$

Then the space F_1 can be identified with a subspace of $L_{1,loc}(a,b)$ comprising the functions u which, after a possible modification on a set of zero measure, possess the following properties: for every finite interval $(c,d) \subseteq (a,b)$ such that $y_i \notin (c,d)$ for all i, u has the generalized derivatives $u^{(i)}$ for $i \leqslant m$ on (c,d), $u^{(i)} \in L_\infty(c,d)$ for $i \leqslant m-1$ and these derivatives are absolutely continuous on (c,d), and $u^{(m)} \in L_1(c,d)$.

If $(c,d)\subseteq(a,b)$ is an arbitrary finite interval and $d<b$ or $b\neq y_\infty$ then there exists a constant c such that

$$\sum_{i=0}^{m-1}\|u^{(i)}\|_{L_\infty(c,d)}+\|u^{(m)}\|_{L_1(c,d)}\leqslant c\|u\|_1\ \ \forall u\in F_1. \tag{16}$$

There exist $\lim_{x\to y_k\pm 0}u^{(i)}(x)$, $\lim_{x\to a+}u^{(i)}(x)$ (if $a\neq-\infty$), and $\lim_{x\to b-}u^{(i)}(x)$ (if $b\neq+\infty$ and $b\neq y_\infty$), where $i\leqslant m-1$ and $k=1,2,\dots,N$. Thereby, all traces in (11)-(14) are defined in the usual sense. The space $F_1\cap F_0$ is dense in F_1 and F_0. The space H_1 being the subspace of F_0 is dense in F_0.

Proof. We present the proof in the latter case, i.e., $s=m$, the corresponding matrices are nondegenerate and the relation (15) holds. The proof in the former case uses the same ideas and much simpler. Let $u\in S$ and let (c,d) be a finite interval such that $y_i\notin(c,d)$ for all i. Since $u\in W_2^m(c,d)$, we can change u on a set of zero measure so that the new function u together with its derivatives $u^{(i)}$ $(i\leqslant m-1)$ is absolutely continuous on every such interval (c,d). We begin with the proof of the estimate (16) which ensures all statements of the lemma except for the density claims. There exists an interval (c,d) of finite length such that $\mu(\{x\in(a,b):\ p_0(x)\neq0\}\cap(c,d))>0$ and $y_i\notin[c,d]$ for all i. Let $u\in S$. Write out the Taylor formula

$$u(x)=\sum_{j=0}^{m-1}u^{(j)}(c)\frac{(x-c)^j}{j!}+\int_c^x u^{(m)}(\tau)\frac{(x-\tau)^{m-1}}{(m-1)!}\,d\tau. \tag{17}$$

Involving this equality and the Hölder inequality, we infer

$$\Big\|\sqrt{p_0}\sum_{j=0}^{m-1}u^{(j)}(c)\frac{(x-c)^j}{j!}\Big\|_{L_2(c,d)}\leqslant\big\|\sqrt{p_0}u\big\|_{L_2(c,d)}+$$

$$\Big\|\sqrt{p_0}\int_c^x u^{(m)}(\tau)\frac{(x-\tau)^{m-1}}{(m-1)!}\,d\tau\Big\|_{L_2(c,d)}\leqslant\big\|\sqrt{p_0}u\big\|_{L_2(c,d)}+$$

$$(\|p_0\|_{L_1(c,d)}\|(p_m)^{-1}\|_{L_1(c,d)})^{1/2}\Big(\int_c^d p_m|u^{(m)}(\tau)|^2\,d\tau\Big)^{1/2}\frac{(d-c)^{m-1}}{(m-1)!}.$$

Due to the linear independence of the functions $(x-c)^j$ and the choice of the interval (c,d), the left-hand side of this inequality is estimated from below by the quantity $\delta_0\sum_{j=0}^{m-1}|u^{(j)}(c)|$, where δ_0 is a positive constant independent of the numbers $u^{(j)}(c)$. The right-hand side is estimated from above by $c\|u\|_1$ (c is some positive constant). Hence, we have the estimate

$$\sum_{j=0}^{m-1}|u^{(j)}(c)|\leqslant c_1\|u\|_1\ \ \forall u\in S, \tag{18}$$

where the constant c_1 is independent of u. With this estimate in hand, we use the equality (17) one more time and arrive at the estimate

$$\|u\|_{W_\infty^{m-1}(c,d)}+\|u^{(m)}\|_{L_1(c,d)}\leqslant c_2\|u\|_1\ \ \forall u\in S. \tag{19}$$

Consider an arbitrary finite interval (c_1,d_1) such that $(c,d) \subseteq (c_1,d_1) \subseteq (a,b)$ and $d_1 < b$ in the case of $y_\infty = b$. There exist at most finitely many points $y_i \in (c_1,d_1)$. Let y_{j_0} be the nearest to the interval (c,d). Let, for example, $y_{j_0} < c$. Again the Taylor formula allows us to say that the estimate

$$\|u\|_{W_\infty^{m-1}(y_{j_0},d)} + \|u^{(m)}\|_{L_1(y_{j_0},d)} \leqslant c_3\|u\|_1 \ \ \forall u \in S. \tag{20}$$

is valid. Since the functions $u^{(i)}$ $(i \leqslant m-1)$ are continuous on $[y_{j_0},d)$ if we take $u^{(i)}(y_{j_0}) = u^{(i)}(y_{j_0}+)$ (the existence of limits on the right and on the left results from the definition of the absolute continuity), the estimate (20) implies that

$$\sum_{j=0}^{m-1} |u^{(j)}(y_{j_0}+)| \leqslant c_2\|u\|_1 \ \ \forall u \in S, \tag{21}$$

In view of the conditions (14), we conclude that

$$\sum_{j=0}^{m-1} |u^{(j)}(y_{j_0}-)| \leqslant c_3\|u\|_1 \ \ \forall u \in S, \tag{22}$$

where c_3 is a constant. Next, we can use the Taylor formula (17) of an arbitrary finite semiinterval $(c_2,y_{j_0}]$ such that $y_i \notin (c_2,y_{j_0})$ for all i, where we take take $c = y_{j_0}$ and put $u^{(i)}(y_{j_0}) = u^{(i)}(y_{j_0}-)$. The above arguments validate the estimate of the form (19) on (c_2,y_{j_0}). This estimate and the estimate (20) give the estimate

$$\|u\|_{W_\infty^{m-1}(c_2,d)} + \|u^{(m)}\|_{L_1(c,d)} \leqslant c_2\|u\|_1 \ \ \forall u \in S. \tag{23}$$

Repeating the arguments if necessary, we can obtain the estimate (23) on the whole interval (c_1,d_1). This estimates implies that any Cauchy sequence $u_n \in S$ in the norm of the space F_1 has the limit which is the function possessing the properties from the claim of the lemma.

Proceed with the second part of the proof. By construction, the class S is dense in F_1. Obviously, the class $C_0^\infty((a,b)\setminus\{y_i:\ i=1,\dots,N\}) \subset S$ is dense in F_0. We can just assume the contrary and apply the Du Bois-Reymond lemma. Thus, the space $F_1 \cap F_0$ is dense in F_1 and F_0 and H_1 is dense in F_0 as well. □

Obviously, the form $a(u,v)$ is defined on functions $u,v \in S$. The main conditions on a is as follows.

$$\exists M,m>0: \ \ \mathrm{Re}\, a(u,u) \geqslant m\|u\|_1^2, \ \ |a(u,v)| \leqslant M\|u\|_1\|v\|_1 \ \forall u,v \in S. \tag{24}$$

The previous condition implies that the form $a(u,v)$ defined on S admits an extension and thus is defined for $u,v \in H_1$ and $u,v \in F_1$. We have that $H_1 \subset F_0$. Construct the negative space H_1' endowed with the norm $\|u\|_{H_1'} = \sup_{v\in H_1} |(u,v)_0|/\|v\|_{H_1}$. The inner product $(u,v)_0$ admits an extension to a duality relation between H_1 and H_1' and thus, for every antilinear continuous functional f over H_1, there exists $u \in H_1'$ such that $f(v) = (u,v)_0$. Given $u \in F_1$, the form $a(u,v)$ defines an antilinear continuous functional over H_1 and thereby there exists an element $Au \in H_1'$ such that $a(u,v) = (-Au,v)_0$. This procedure of construction of an operator using a sesquilinear form with certain properties is conventional (see, for instance, [18]).

LEMMA 7. *The operator $A : F_0 \to F_0$ with the domain $D(A) = \{u \in H_1 : Au \in F_0\}$ is an m-dissipative operator and the operator $L = JA$ with $D(L) = D(A)$ is m-J-dissipative. If the form $a(u,v)$ is symmetric on S, i.e., $a(u,v) = \overline{a(v,u)}$ for all $u,v \in S$, then A is selfadjoint (respectively, L is J-selfadjoint).*

Proof. Let $f \in F_0$. Consider the equality $a(u,v) + \lambda_0(u,v)_0 = (f,v)_0$ $(\lambda_0 > 0)$. In view of (24), we have $\operatorname{Re}(a(u,u) + \lambda_0(u,u)_0) \geqslant c_0\|u\|_{H_1}^2$ and $|a(u,v) + \lambda_0(u,v)_0| \leqslant c_1\|u\|_{H_1}\|v\|_{H_1}$ for all $u,v \in H_1$, where c_1, c_0 are positive constants. The Lax-Milgram theorem (see [18, Theorem C.5.3]) implies that there exists a function $u \in H_1$ such that $a(u,v) + \lambda_0(u,v)_0 = (f,v)_0$ for all $v \in H_1$. By definition, $u \in D(A)$ and $-Au + \lambda_0 u = f$. Thus, positive reals belongs to $\rho(A)$. Due to the density of F_0 in H_1', $D(A)$ is dense in H_1 and thereby in F_0. Moreover, it easy to justify that A is a closed operator. Thus, A is a densely defined closed dissipative operator, and thereby L is an m-J-dissipative (see Lemma 1). The next claim is a consequence of the definition. Let the form $a(u,v)$ be symmetric. We have that

$$-a(u,v) = (Au,v)_0 = -\overline{a(v,u)} = \overline{(Av,u)} = (u,Av), \ \ \forall u,v \in D(A),$$

i.e., the operator A is symmetric and $\{\lambda : \lambda > 0\} \subseteq \rho(A)$. Hence, A is selfadjoint and thereby $L = JA$ is J-selfadjoint. □

We present some examples. Let the interaction points be absent. Assume that the form $a(u,v)$ is representable as $a(u,v) = \int_a^b \sum_{i=0}^m p_i(x)u^{(i)}v^{(i)}$, with $p_i \in L_{1,loc}(a,b)$ and $1/p_m \in L_1(c,d)$ for every bounded segment $[c,d] \subseteq [a,b]$. In this case the differential expression $Lu = \sum_{i=1}^m (-1)^i (p_i u^{(i)})^{(i)}$ can be written with the use of quasiderivatives (see [34]). If all coefficients a_{ij} of the form a are sufficiently smooth, for example, $a_{ij} \in W_{1,loc}^{max(i,j)}$ and the higher order coefficient a_{mm} is strictly positive and bounded, then the domain of L includes the functions in $W_{2,loc}^{2m}(a,b)$, the operator L is defined by the differential expression

$$Lu = -\sum_{i,j=0}^{m} (-1)^i \frac{d^i}{dx^i} a_{ij} \frac{d^j u}{dx^j} \ \ (x \in (a,b))$$

and some boundary conditions

$$B_k u = \sum_{i=0}^{2m-1} (\alpha_{ik}u^{(i)}(a) + \beta_{ik}u^{(i)}(b)) = 0 \ (k = 1,2,\ldots,2m),$$
$$B_k u = \sum_{i=0}^{2m-1} \alpha_{ik}u^{(i)}(a) = 0, \ \ (k = 1,2,\ldots,m),$$
$$B_k u = \sum_{i=0}^{2m-1} \beta_{ik}u^{(i)}(b) = 0 \ (k = 1,2,\ldots,m),$$

Here the first condition is used if $a \neq -\infty$, $b \neq +\infty$, the second if $a \neq -\infty$ and $b = +\infty$, and the third if $a = -\infty$ and $b \neq +\infty$. In this case the differential expression is understood in the usual sense. Similar statements can be obtained in the presence of interaction points as well. In this case the description of L is slightly more complicated.

2.4. Regularity conditions of the turning points.

We employ regularity conditions of the turning points those of [36]–[39]). Note that the condition (I) below was used earlier in some other situations as a necessary and sufficient condition ensuring some integral inequalities (see [10]). So, it is sometimes called the Bennewitz condition (see [30]).

(I) A point $x_k \in \partial G^+ \cap \partial G^-$ is regular if there exists a right neighborhood $(x_k, x_k + \delta) = I$ or a left neighborhood $(x_k - \delta, x_k) = I$ about this point such that $I \subset G^+ \cup G^-$ and, for some $\omega \in (0,1)$, we have

$$\Big|\int_{x_k}^{x_k+\omega\eta} |g(\tau)|\,d\tau\Big| \leqslant \frac{1}{2}\Big|\int_{x_k}^{x_k+\eta} |g(\tau)|\,d\tau\Big| \ \ \forall\eta \in (0,\delta),\ (\text{or } \forall\eta \in (-\delta,0))\ . \tag{25}$$

In some cases, we need an additional regularity condition of a boundary point.

(II) The boundary point $a \neq -\infty$ ($b \neq \infty$) is regular in the following cases:

a) there exists a right neighborhood $I = (a, a+\delta)$ about a (a left neighborhood $I = (b-\delta, b)$ about b) such that, for some $\omega \in (0,1)$, we have

$$\int_{a}^{a+\omega\eta} |g(\tau)|\,d\tau \leqslant \frac{1}{2}\int_{a}^{a+\eta} |g(\tau)|\,d\tau \ \ \forall\eta \in (0,\delta); \tag{26}$$

$$(\text{respectively, } \int_{b-\omega\eta}^{b} |g(\tau)|\,d\tau \leqslant \frac{1}{2}\int_{b-\eta}^{b} |g(\tau)|\,d\tau \ \ \forall\eta \in (0,\delta);) \tag{27}$$

In the following statement, we describe an equivalent regularity conditions. To simplify the exposition, we state them for a point $x_0 = 0$ and the interval $(0,1)$. The condition (25) is actually the condition (b) below with $f(\eta) = \int_0^\eta |g(\tau)|\,d\tau$ stated for an arbitrary point.

THEOREM 2. *The following conditions for a nondecreasing function* $f : (0,1) \to R_+$ *are equivalent:*

(a) $\forall\gamma \in (0,1)\ \exists\omega \in (0,1)$ *such that* $\forall\varepsilon \in (0,1)\ f(\omega\varepsilon) \leqslant \gamma f(\varepsilon)$*;*

(b) $\exists\omega \in (0,1)$ *such that* $f(\omega\varepsilon) \leqslant f(\varepsilon)/2\ \forall\varepsilon \in (0,1)$*;*

(c) $\exists\beta \in (0,1)\ \exists\omega \in (0,1)$ *such that* $\forall\varepsilon \in (0,1)\ f(\omega\varepsilon) \leqslant \beta f(\varepsilon)$*;*

(d) *there exist constants* $c, d > 0$ *such that*

$$f(\eta) \leqslant c\Big(\frac{\eta}{\xi}\Big)^d f(\xi)\ \ \forall 0 < \eta \leqslant \xi < 1.$$

(e) *there are no sequences* a_n, b_n *such that* $0 < a_n < b_n < 1$ *and*

$$a_n/b_n \to 0,\ \ f(a_n)/f(b_n) \to 1\ \ as\ n \to \infty.$$

The equivalence of (b), (d), and (e) is proven in Theorem 6 of [36]. The complete proof can be found in [39].

Let $g(x) \in L_1(0,1)$ and $g(x) > 0$ a.e. on $(0,1)$. Denote by $\tilde{W}_2^m(0,1)$ the subspace of $W_2^m(0,1)$ comprising functions $u(x)$ such that $u^{(i)}(0) = 0$ for $0 \leqslant i \leqslant m-1$.

THEOREM 3. (see Theorem 18 in [38]) *Each of the conditions* (a)-(e) *stated for the function* $f(\eta)=\int_0^{\eta}|g(\tau)|\,d\tau$ *is equivalent to the following claim: there exists* $\theta\in(0,1)$ *such that* $(W_2^m(0,1),L_{2,g}(0,1))_{1-\theta,2}=(\tilde{W}_2^m(0,1),L_{2,g}(0,1))_{1-\theta,2}$.

COROLLARY 1. *Under the conditions of Theorem 3, there exists* $\theta\in(0,1)$ *such that* $(\overset{\circ}{W}{}_2^m(0,1),L_{2,g}(0,1))_{1-\theta,2}=(\bar{W}_2^m(0,1),L_{2,g}(0,1))_{1-\theta,2}$*, where* $\bar{W}_2^m(0,1)=\{u\in W_2^m(0,1):\ u^{(i)}(1)=0,\ i=0,1,\ldots,m-1\}$.

3. The main results

Our main result is the following theorem.

THEOREM 4. *Assume that the conditions* (10)*,* (24)*, and the conditions of Lemma 6 hold and there exists a neighborhood* U *about the set* $\{x_k\}_{k=1}^s$ *such that* p_m*,* $1/p_m\in L_\infty(U)$*. Let one of the following conditions be valid:*

a) *the boundary conditions are local and every of the point* x_k $(k=1,2,\ldots,s)$ *is regular;*

b) *the boundary conditions are nonlocal and one of the boundary points and all points* x_k $(k=1,2,\ldots,s)$ *are regular; moreover,* $p_m,1/p_m\in L_\infty(U\cap(a,b))$ *for some neighborhood* U *about this boundary point.*

Then there exist maximal uniformly positive and uniformly negative subspaces $M^{\pm}$ *of the space* F_0 *invariant under* L *such that*

$$\overline{C^{\pm}}\subset\rho(L|_{M^{\pm}}),\ \ F_0=M^{+}+M^{-},\ D(L)=D(L)\cap M^{+}+D(L)\cap M^{-},$$

where the sums are direct and the operators $\pm L|_{M^{\pm}}$ *are generators of analytic semigroups.*

Proof. To apply Theorem 1, we look for sufficient conditions ensuring the equality (6), where $H=F_0$. As before in Subsection 2.2, we introduce the space H_{-1} as the completion of the space F_0 with respect to the norm $\|u\|_{H_{-1}}=\sup_{v\in H_1}|[u,v]_0|/\|v\|_{H_1}$. Define also the spaces $H_s=(H_1,F_0)_{1-s,2}$. First, we prove that $\rho(L)\neq\emptyset$ using Lemma 5. We would like to check the condition (9). The conditions of the theorem and Lemma 6 imply that, for functions with support lying in a sufficiently small open neighborhood U about the set $\{x_k\}_{k=1}^s$, the norms in the spaces F_1 and H_1 are equivalent to the norm of the space $W_2^m(U)$. First, we examine the case of local boundary conditions. Let $x_k\in\partial G^+\cap\partial G^-$. In this case, either on the interval $O_k^-=(x_k-\varepsilon,x_k)$ or on $O_k^+=(x_k,x_k+\varepsilon)$, the inequality (25) holds (we assume that ε is smaller than δ that of (25)). For example, examine the former interval. In the latter case the arguments are the same. Decreasing ε if necessary, we may assume that $[x_k-\varepsilon,x_k)\in G^+\cap U$ or $[x_k-\varepsilon,x_k)\in G^-\cap U$, $(x_k,x_k+\varepsilon]\in G^-\cap U$ or $(x_k,x_k+\varepsilon]\in G^+\cap U$, and $y_i\notin[x_k-\varepsilon,x_k)\cup(x_k,x_k+\varepsilon]$ for all i. Assign $O_k=O_k^+\cup O_k^-\cup\{x_k\}$ and define the space W_1 as the space comprising the functions $u\in W_2^m(O_k)$ if x_k is not an interaction point and the functions $u\in W_2^m(O_k^+)\cap W_2^m(O_k^-)$ satisfying (14) at x_k otherwise. The norm in W_1 coincides with that in $W_2^m(O_k)$. Obviously, there exists $v\in H_1$ with the property

$v|_{O_k} = u$. Let $W_s = (W_1, W_0)_{1-s,2}$. Demonstrate that there exists $s_0 > 0$ such that the operators $S_k^{\pm}$

$$S_k^{\pm} u = \begin{cases} u, \ x \in G^{\pm} \cap O_k \\ 0, \ x \in G^{\mp} \cap O_k \end{cases}$$

are continuous as operators from W_s into W_s for all $s \in [0, s_0]$. Define some auxiliary spaces. Assign $A_1 = W_2^m(O_k^-)$, $A_0 = L_{2,g}(O_k^-)$, $A_1^0 = \{u \in A_1 : u^{(l)}(x_k) = 0 \ (l = 0, 1, \ldots, m-1)\}$. Theorem 3 implies that there exists $s_0 > 0$ such that

$$A_s = (A_1, A_0)_{1-s,2} = A_s^0 = (A_1^0, A_0)_{1-s,2}.$$

Define an operator $P_0 : W_s \to A_s$, $P_0 u = u|_{O_k^-}$. Obviously, $P_0 \in L(W_s, A_s)$ for all s. Introduce also the operator

$$P_1 : A_s^0 \to W_s, \ P_1 u = \begin{cases} u, \ x \in O_k^- \\ 0, \ x \in O_k^+ \end{cases}.$$

It is immediate that $P_1 \in L(A_s^0, W_s)$ for all $s \in [0,1]$. In this case, for $s < s_0$, we obtain that $P_1 P_0 \in L(W_s)$. By construction, $P_1 P_0 u = S_k^- u$. Hence, the operator S_k^- and thereby the operator S_k^+ belong to the class $L(W_s)$ for all $s < s_0$.

Given an arbitrary point $x_k \in \partial G^+ \cap \partial G^-$, construct a neighborhood O_k with the above properties. Next, define functions $\varphi_k \in C_0^\infty(O_k)$ such that $\varphi_k = 1$ in some neighborhood about x_k and $\operatorname{supp} \varphi_k' \in G^+ \cup G^-$. In is possible in view of the definition of the neighborhoods O_k. Without loss of generality, we may assume that the neighborhoods O_k are disjoint and the closures does not contain the boundary points a and b. Demonstrate that there exists $s_0 > 0$ such that the operator

$$S : u \to \begin{cases} u, \ \ x \in G^+ \\ 0, \ \ x \in G^- \end{cases}$$

is continuous as an operator from H_s into H_s for all $s < s_0$. Take as s_0 the minimal of the constants s_0 defined in the above proof for every point x_k. Fix $s < s_0$. Consider the operators $S_k u = \varphi_k u$. Obviously, $S_k \in L(H_1) \cap L(F_0)$ and, therefore, $S_k \in L(H_s)$ for all $s \in [0,1]$. Moreover, the supports of $S_k u$ lie in the corresponding neighborhoods O_k. Hence, $S_k \in L(H_s, W_s)$ (the spaces W_s are different for different neighborhoods). In this case $S_k^+ S_k \in L(H_s, W_s)$. Constructs functions $\tilde{\varphi}_k \in C_0^\infty(O_k)$ such that $\tilde{\varphi}_k = 1$ in some neighborhoods about $\operatorname{supp} \varphi_k$. As is easily seen, the operators $\tilde{S}_k : u \to \tilde{\varphi}_k u$, where the functions $\tilde{\varphi}_k u$ are extended by zero on the whole (a,b), possess the property $\tilde{S}_k \in L(W_s, H_s)$. Thus, we infer $\tilde{S}_k S_k^+ S_k \in L(H_s)$ for all $s < s_0$.

Examine the operator

$$P : u \to (1 - \sum_{k=1}^{N} \varphi_k) u(x).$$

By definition, it is not difficult to establish that $P \in L(H_s)$ for all $s \in [0,1]$. By construction, $SP \in L(H_s)$ for all $s \in [0,1]$. In this case the operator

$$S = SP + \sum_{k=1}^{N} \tilde{S}_k S_k^+ S_k$$

satisfies the condition $S \in L(H_s)$ for all $s < s_0$. As a consequence, $J \in L(H_s)$ for all $s < s_0$. Lemma 5 validates the equality (9) and the relation $\rho(L) \neq \emptyset$.

Consider the case of nonlocal boundary conditions. Recall that in this case $a \neq -\infty$, $b \neq +\infty$, and $y_\infty \neq b$, i.e., the number of interaction points is finite. For example, let the point $x = a$ be regular. Let $(a, a+\delta)$ be the neighborhood that of the definition of the regularity such that $p_m, 1/p_m \in L_\infty(a, a+\delta)$. Without loss of generality, we may assume that $(a, a+\delta] \subset G^+$ or $(a, a+\delta] \subset G^-$ and $y_i \notin (a, a+\delta]$ for all i. Take a function $\varphi(x) \in C^\infty[a, a+\delta]$ such that $\operatorname{supp}\varphi(x) \subset [a, a+\delta)$ and $\varphi(x)$ is equal to 1 on some set of the form $[a, a+\delta_1]$ $(\delta_1 < \delta)$. Demonstrate that there exists $s_1 > 0$ such that the operator $S_0 u = \varphi(x)u(x)$ is continuous as the mapping from H_s into H_s for $s < s_1$. Construct an operator P_1 taking a function $u \in H_1$ into the function $\varphi(x)u \in \bar{W}_2^m(a, a+\delta) = \{u \in W_2^m(a, a+\delta) : \; u^{(i)}(a+\delta) = 0, \; i = 0, 1, \ldots, m-1\}$. Obviously, $P_1 \in L(H_1, \bar{W}_2^m(a, a+\delta)) \cap L(L_{2,g}(a,b), L_{2,g}(a, a+\delta))$. Hence, $P_1 \in L(H_s, \bar{W}_2^{ms}(a, a+\delta))$, where $\bar{W}_2^{ms}(a, a+\delta)) = (\bar{W}_2^m(a, a+\delta)), L_{2,g}(a, a+\delta))_{1-s,2}$. In view of the regularity of a, there exists $s_1 > 0$ such that $\bar{W}_2^{sm}(a, a+\delta)) = \overset{\circ}{W}{}_2^{sm}(a, a+\delta)$ for $s < s_1$. The operator P_2 taking a function $u \in \overset{\circ}{W}{}_2^{sm}(a, a+\delta)$ into its zero extension on the whole (a,b) possesses the property $P_2 \in L(\overset{\circ}{W}{}_2^{sm}(a, a+\delta), H_s)$ for all s. In this case the operator $P_2 P_1 u = S_0 u$ is such that $S_0 \in L(H_s)$ for $s < s_1$. Since the operator of multiplication by a function $\varphi \in C_0^\infty(a,b)$ such that $y_i \notin \operatorname{supp} \varphi'(x)$ and $x_k \notin \operatorname{supp} \varphi'(x)$ for all i, k belongs to $L(H_s)$ for an arbitrary $s \in [0,1]$, we can conclude that the operator of multiplication by a function $\varphi(x) \in C^\infty[a,b]$ such that set $\operatorname{supp} \varphi'(x)$ does not contain the points x_k and y_i for all i, k, $\varphi(x) = 1$ in some neighborhood about a, and $\varphi(x) = 0$ in some neighborhood about b belongs to $L(H_s)$ for $s < s_1$. The multiplication operators by functions representable as $(1 - \varphi(x))$ for these functions $\varphi(x)$ possess this property as well. Next, we repeat the above arguments. As before, we can establish that there exists a parameter $s_0 \leqslant s_1$ such that the above operators P, $\sum_{k=1}^{N} \tilde{S}_k S_k^+ S_k$ belong to the class $L(H_s)$ for $s < s_0$. Next, we can construct a function $\psi(x) \in C^\infty(\mathbb{R})$ such that $\psi(x) = 1$ on $\overline{G^+}$, $\psi(x) = 0$ beyond some small neighborhood V about $\overline{G^+}$, and $\operatorname{supp} \varphi'(x)$ does not contain the points x_k anf y_i for all i, k. This neighborhood can be chosen so small that $SPu(x) = \psi(x)Pu(x)$. The operator of multiplication by the function $\psi(x)$ belongs to $L(H_s)$ for $s < s_0$ and thus so is the operator SP. Hence, the operator

$$S = SP + \sum_{k=1}^{N} \tilde{S}_k S_k^+ S_k$$

satisfies the condition $S \in L(H_s)$ for for $s < s_0$. Again, Lemma 5 ensures the relation $\rho(L) \neq \emptyset$. The next our aim is to prove the equality (6). As before, it suffices to prove that there exists a number $s_0 > 0$ such that $J \in L(F_s)$ for all $s < s_0$. Since the norms in H_1 and F_1 are equivalent for functions whose supports lie in some neighborhood about the set $\{x_k\}_{k=1}^{s}$, it is easy to see that the arguments in this case coincide with those used in the proof of the equality (9). So we can state that the equality (6) holds. Now the claim results from Lemma 4 and Theorem 1. □

Recall that the operators $A_1, A_2 : H \to H$ (H is a Hilbert space) are called similar if there exists a bounded and boundedly invertible operator T such that $T(D(A_1)) = D(A_2)$ and $A_2 = TA_1T^{-1}$.

THEOREM 5. *Assume that the conditions of Theorem 4 hold and the form $a(u,v)$ is symmetric on the class S. Then the operator L is similar to a selfadjoint operator in F_0.*

Proof. The result follows from Theorem 4, Lemma 7, and Prop. 2.2 in [24]. □

REMARK 1. The conditions on the coefficients in the form $a(u,v)$ insuring the fulfillment of the conditions of Theorem 4 and on the constants in boundary and gluing conditions can be specified in particular situations.

REMARK 2. The conditions of Theorem 4 on p_m can be weakened. But in this case, we need the corresponding refinements of Theorem 3.

REMARK 3. The proof of Theorem 4 is a modification of the corresponding proof of the Riesz basis property in indefinite spectral Sturm-Liouville problems (see [42] or [45]).

REMARK 4. All considerations become much more complicated in the case of $\ker L \neq \{0\}$ or even in the case when the condition of Lemma 6 are violated. These cases require a separate discussion.

EXAMPLE 1. We examine the simplest case. We take $(a,b) = (-\infty, \infty)$ and

$$a(u,v) = (u', v') + (qu, v) +$$

$$\sum_{k=1}^{N} [(\alpha_{11}^k u(y_k+) + \alpha_{12}^k u(y_k-))\overline{v(y_k+)} + (\alpha_{21}^k u(y_k+) + \alpha_{22}^k u(y_k-))\overline{v(y_k-)}],$$

where the function q is nonnegative and real-valued, $\{y_k\}_{k=1}^N$ is the set of interaction points, and α_{ij}^k are complex numbers. The gluing conditions (14) at the points y_k are written as

$$u(y_k+) = \alpha_k u(y_k-), \ \ \alpha_k \neq 0, \ \ k = 1, 2, \ldots, N. \tag{28}$$

Let $g(x) = \operatorname{sgn} x \, g_0(x)$, with $g_0(x) > 0$ a.e. The corresponding operator L is written as

$$Lu = \frac{\operatorname{sgn} x}{g_0(x)}(u_{xx} - q(x)u), \ x \in \mathbb{R},$$

and the domain of L consists of the functions $u \in L_{2,g_0+q}(\mathbb{R})$ such that $u_x \in L_2(\mathbb{R})$, $u_{xx} \in L_{1,loc}(\mathbb{R})$, $Lu \in L_{2,g_0}(\mathbb{R})$ which satisfy (28) and the conditions

$$\alpha_k u_x(y_k+) - u_x(y_k-) - \Delta_k u(y_k-) = 0, \ \Delta_k = |\alpha_k|^2 \alpha_{11}^k + \alpha_{12}^k \overline{\alpha_k} + \alpha_{21}^k \alpha_k + \alpha_{22}^k,$$

where $k = 1,2\dots,N$ and the symbols u_x, u_{xx} stand for the generalized derivatives in the Sobolev sense. We suppose that

(A) $g_0, q \in L_{1,loc}(\mathbb{R})$, $q \geqslant 0$ a.e. on $\mathbb{R}$, and $\mu\{x \in (a,b):\ q(x) \neq 0\} > 0$.

(B) $\sum_{k=1}^N \Delta_k(y_k - y_1) < \infty$, $|\alpha_k| \leqslant 1$, $\Delta_k \in \mathbb{R}$, and $\Delta_k \geqslant 0$ for all k.

The fundamental symmetry J in the space $F_0 = L_{2,g_0}(\mathbb{R})$ is given by the equality $Ju = \operatorname{sgn} x u$. Obviously, F_0 is a Krein space. The norm in F_1 is defined as $\|u\|_{F_1}^2 = \int_{-\infty}^{\infty} |u_x|^2 + q(x)|u|^2\, dx$. The last two conditions in (B), the equality

$$a(u,v) = (u',v') + (qu,v) + \sum_{k=1}^N \Delta_k u(y_k-)\overline{v(y_k-)}$$

resulting from (28), and Lemma 7 ensure that L is m-J-dissipative and J-selfadjoint in F_0. The first two conditions in (B) imply (24). Indeed, the first inequality in (24) is obvious. Next, employing the representation

$$u(y_k-) = \int_{y_{k-1}}^{y_k} u'\, dx + \sum_{j=2}^{k-1} \int_{y_{j-1}}^{y_j} u'\, dx \prod_{s=j}^{k-1} \alpha_s + u(y_1-) \prod_{s=1}^{k-1} \alpha_s,$$

the Hölder inequality, and the second condition in (B), we infer

$$|u(y_k-)|^2 \leqslant 2(y_k - y_1) \int_{y_1}^{y_k} |u'|^2\, dx + 2|u(y_1-)|^2.$$

In view of Lemma 6, the last summand and the whole right-hand side are estimated by the quantity $c(y_k - y_1)\|u\|_{F_1}^2$, with c a positive constant. In this case, we have

$$\Big|\sum_{k=1}^N \Delta_k u(y_k-)\overline{v(y_k-)}\Big| \leqslant \Big(\sum_{k=1}^N \Delta_k |u(y_k-)|^2\Big)^{1/2} \sum_{k=1}^N \Delta_k |v(y_k-)|^2\Big)^{1/2},$$

$$\sum_{k=1}^N \Delta_k |u(y_k-)|^2 \leqslant c\|u\|_{F_1}^2 \sum_{k=1}^N \Delta_k(y_k - y_1),\ \sum_{k=1}^N \Delta_k |v(y_k-)|^2 \leqslant c\|v\|_{F_1}^2 \sum_{k=1}^N \Delta_k(y_k - y_1).$$

The last two inequalities and the definition of the form $a(u,v)$ ensure the last inequality in (24). We note that the first two conditions in (B) are not optimal; moreover, it is possible that $y_k = 0$ for some k.

THEOREM 6. *Let the condition* (A) *hold. Assume also that* 0 *is a regular point of* g. *Then the claim of Theorem 4 holds and, moreover,* $L : F_0 \to F_0$ *is similar to a selfadjoint operator.*

REMARK 5. In the case of the absence of the interaction points, this theorem is a slight generalization of Kostenko's result [32], where it assumed additionally that $g_0 \notin L_1(a,b)$ and g_0 is an even function.

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