Operators and Matrices
Volume 8, Number 1 (2014), 255–277 doi:10.7153/oam-08-14

ANALYTICAL ASPECTS OF ISOSPECTRAL DRUMS

W. ARENDT, A. F. M. TER ELST AND J. B. KENNEDY

Abstract. We reexamine the proofs of isospectrality of the counterexample domains to Kac' question 'Can one hear the shape of a drum?' from an analytical viewpoint. We reformulate isospectrality in a more abstract setting as the existence of a similarity transform intertwining two operators associated with elliptic forms, and give several equivalent characterizations of this property as intertwining the forms and form domains, the associated operators and operator domains, and the semigroups they generate. On a representative pair of counterexample domains, we use these criteria to show that the similarity transform intertwines not only the Laplacians with Neumann (or Dirichlet) boundary conditions but also any two appropriately defined elliptic operators on these domains, even if they are not self-adjoint. However, no such transform can intertwine these operators if Robin boundary conditions are imposed instead of Neumann or Dirichlet. We also remark on various operator-theoretic properties of such intertwining similarity transforms.

1. Introduction

It took 30 years for Kac' famous question 'Can one hear the shape of a drum?' [13] to find an answer. Gordon, Webb and Wolpert [12] constructed two non-congruent planar domains whose Laplacians with Dirichlet (or Neumann) boundary conditions are isospectral, that is, they have the same sequence of eigenvalues, counted with multiplicities. The standard counterexample takes the form of two polygons obtained by stitching together seven copies of a given non-equilateral triangle in two different ways. These domains are manifestations in the plane of a general principle first enunciated by Sunada [16] and developed by Bérard [8], which to the best of our knowledge accounts for all known isospectral pairs, and which was used in [12]. Namely, if H and K are two subgroups of a finite group G, then a unitary intertwining operator between the spaces $L_2(H \setminus G)$ and $L_2(K \setminus G)$ induces an isometry between appropriate subspaces of any Hilbert space on which G acts unitarily. Subsequent to the publication of [12], several mathematicians, for example Bérard [9], Buser–Conway–Doyle–Semmler [10] and Chapman [11], gave simplified and more accessible proofs of the isospectrality of such domains. The argument in all these expository proofs consists in showing that an eigenfunction on the first polygon can be transposed to an eigenfunction on the second by taking particular linear combinations of its values on the seven equal constituent triangles, and vice versa.

Following the approach taken by Bérard [9], if we consider $L_2(\Omega_1)$ and $L_2(\Omega_2)$ rather as $L_2(T)^7$, where T is the basic triangle ('brique fondamentale') and Ω_1 and Ω_2 the polygons (see Figure 1), then we can construct an isometry Φ on $L_2(T)^7$ induced

Mathematics subject classification (2010): 58J53, 35P05, 47F05, 35R30.
Keywords and phrases: Isospectral domains, Laplacian, elliptic differential operator, form method, Dirichlet boundary conditions, Neumann boundary conditions, Robin boundary conditions.

Paper OaM-08-14

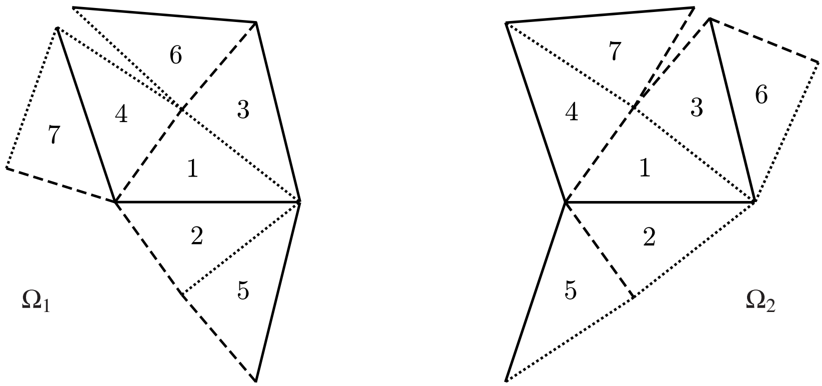

Figure 1: *Two isospectral domains composed of seven isometric triangles. These are based on the 'warped propeller' domains of* [10].

by a 7×7 matrix B of scalars acting as a family of Euclidean isometries superimposing the seven triangles. The core of the argument is that Φ restricts to an isometry mapping the Sobolev space $H_0^1(\Omega_1)$ onto $H_0^1(\Omega_2)$, whilst its adjoint Φ^* maps $H_0^1(\Omega_2)$ onto $H_0^1(\Omega_1)$. Isospectrality of the Laplacians then follows from the variational characterization of the eigenvalues. The later works of Buser *et al* [10] and Chapman [11] motivate and describe in lay terms how Φ acts purely as a map between eigenfunctions, without touching upon the concept of isometric Sobolev spaces.

The aim of this paper is to reconsider these arguments from a more analytical perspective. Rather than transposing eigenfunctions, we construct Φ as a similarity transform intertwining the realizations of the Laplacian with Neumann (or Dirichlet) boundary conditions, or equivalently, the semigroups generated by these realizations, on the respective polygons. Moreover, we consider the operator-theoretic properties of such a transform Φ more carefully. We will give a general characterization of maps Φ that intertwine any two operators associated with elliptic forms. In light of this characterization it looks like a miracle that there exists a matrix which fulfils the criterion, but the key point is rather that Φ and its adjoint respect the form domains $H^1(\Omega_1)$ and $H^1(\Omega_2)$. That the transform Φ intertwines the elliptic forms implies that it also intertwines the associated operators and semigroups. In the case of the Laplacian, this is then equivalent to the isospectral property. But it is not necessary that we consider only the Laplacian: since the Sobolev spaces are intertwined, any elliptic operator on $L_2(T)$, even if it is not self-adjoint, will yield two operators on $L_2(\Omega_1)$ and $L_2(\Omega_2)$ which are similar. In place of isospectrality, the correct setting is now that of similarity, a stronger property in the non-self-adjoint case.

We can consider the Laplacian with Robin boundary conditions in this setting. The question as to whether there exist isospectral pairs for the third boundary condition just as for the first and second seems to be a natural one, and was briefly mentioned in the survey article [15]; but otherwise it appears to have received little attention, and no answer. We will show that any operator acting as a family of superimposing isometries that intertwines the Robin Laplacians on Ω_1 and Ω_2 must also simultaneously intertwine the Dirichlet *and* Neumann Laplacians, which is easily shown to be impossible. Thus there is no reason to suppose that any known pairs of domains which are Dirichlet or Neumann isospectral are also Robin isospectral, and it is an open question as to whether there exists *any* noncongruent pair of Robin isospectral domains. The striking implication is that isospectrality could be essentially related to the boundary conditions and not the coefficients of the operators being intertwined. Thus it may well be the case that one can hear the shape of a drum after all, if one loosens the membrane before striking it.

There is another motivation for studying similarity transforms within our framework. It has been shown (see [2], and cf. also [4] and [5] for the case of Riemannian manifolds) that two (Lipschitz) domains are necessarily congruent if there exists an order isomorphism intertwining the Laplacians. Thus, in our case, the similarity transform Φ is not an order isomorphism, even though, at least in the case of Neumann boundary conditions, Φ may be taken as a positive linear map. What goes wrong is that Φ is no longer disjointness preserving, as on each triangle it adds (the function values on) several distinct triangles together; thus Φ may be written as a finite sum of order isomorphisms, and due to this 'mixing' property, Φ^{-1} is not positive. Understanding and seeking to narrow the operator-theoretic gap between the characterization of such positive results as in [2] and the negative counterexamples may help us to understand Kac' problem better, as well as offering an alternative approach to the standard one via heat and wave traces.

In fact, a version of Kac' question is still open. The results here, just as those of [12] and the other expositions, can be interpreted as saying that these seven triangles can be put together in two different ways to induce isomorphic Sobolev spaces, which is of course the essential idea behind Bérard's version of Sunada's Theorem. With trivial and obvious modifications, the same is true for all other known counterexamples as presented in [10] (which are all based on Sunada's Theorem in the same way, and which can all be analyzed within our framework in the obvious way). But the point is that the phenomenon exhibited by these domains is somehow exceptional and does not really answer Kac' question. If we interpret the 'correct' setting for Kac' question as being C^∞-domains in the plane, then the question is still wide open; there is no known counterexample among C^1 or convex planar domains. In four dimensions there is a counterexample of two non-congruent convex domains, given by Urakawa [17] in 1982, which was in fact the first Euclidean example. However, the issue of regularity of the boundary seems to be far more than a technicality, a point also made in the survey article of Protter [15]. While it is certainly clear that any counterexamples generated via the principle of Sunada's Theorem must have corners, there are also remarkable and profound positive results obtained by Zelditch [20], [21], who proves that, within a certain class of domains in $\mathbb{R}^2$ with analytic boundary and certain symmetry conditions, any two isospectral domains are congruent. The presence of corners in a domain also has significant consequences for the asymptotic behaviour of the eigenvalues; for example, the curvature of the boundary appears in all terms of the asymptotic expansion of the heat kernel about $t = 0$ (see, e.g., [18]).

This article is organized as follows. We start in Section 2 by characterizing operators which intertwine two semigroups generated by sectorial forms, and relating this to the isospectral property. We phrase many of our results in the language of semigroups, as this allows us to work on L_2-spaces in place of the more abstruse operator domains. In Section 3 we recast the arguments given in [9] and [10] within this framework, showing first how we can decompose the large domains Ω_1 and Ω_2 into their constituent triangles, and give conditions allowing us to merge the associated Sobolev spaces together. In this setting we then prove that realizations of the Neumann Laplacian on the two non-congruent polygons in Figure 1 are similar. We work principally with the Neumann case as there are fewer conditions on the Sobolev spaces involved, and as the similarity transform and associated matrix have the particularly nice property that they may be taken to be positive. In Section 4 we discuss properties of the intertwining operator Φ constructed in Section 3 from a more analytical, operator-theoretic perspective. The results in Section 3 are extended to more general elliptic operators in Section 5. We then consider Dirichlet boundary conditions in Section 6. The underlying ideas are the same, but the details of the construction turn out to be a little more complicated than in the Neumann case. We therefore limit ourselves to indicating the differences vis-à-vis the Neumann Laplacian. Finally, in Section 7, we show that these arguments cannot be extended to Robin boundary conditions.

2. Forms and intertwining operators

We start by introducing some basic terms and results from the theory of sectorial forms. The idea is to consider equivalent formulations of isospectrality for the Dirichlet and Neumann Laplacians which are more suitable for adaptation to more general operators. To that end, let H and V be complex Hilbert spaces such that V is densely embedded in H. Let $a\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form. Assume that a is *elliptic*, that is, there exist $\omega \in \mathbb{R}$ and $\mu > 0$ such that

$$\operatorname{Re} a(u,u) + \omega \|u\|_H^2 \geqslant \mu \|u\|_V^2 \tag{1}$$

for all $u \in V$. Denote by A the operator associated with a. That is, the domain of A is given by

$$D(A) = \{u \in V : \text{there exists an } f \in H \text{ such that } a(u,v) = (f,v)_H \text{ for all } v \in V\},$$

and $Au = f$ for all $u \in D(A)$ and $f \in H$ such that $a(u,v) = (f,v)_H$ for all $v \in V$. Then $-A$ generates a holomorphic semigroup on H.

We are of course particularly interested in the Dirichlet and Neumann Laplacians on $H = L_2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open set with finite measure. These are self-adjoint operators with compact resolvent, and can be characterized as follows. We omit the standard proof.

PROPOSITION 2.1. *Let A be an operator in a separable, infinite dimensional Hilbert space H. The following are equivalent.*

(i) *A is self-adjoint, bounded from below and has compact resolvent.*

(ii) *There exist a Hilbert space V which is densely and compactly embedded in H and a symmetric, continuous elliptic form $a\colon V\times V\to\mathbb{C}$ such that A is associated with a.*

(iii) *There exist an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H and an increasing sequence of real numbers $(\lambda_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\lambda_n=\infty$ such that*

$$D(A)=\{u\in H : \sum_{n=1}^{\infty}|\lambda_n\,(u,e_n)_H|^2<\infty\}$$

and $Au=\sum_{n=1}^{\infty}\lambda_n\,(u,e_n)_H\,e_n$ for all $u\in D(A)$.

If A satisfies these equivalent conditions, we call λ_n the *n-th eigenvalue of A* and $(\lambda_n)_{n\in\mathbb{N}}$ the sequence of eigenvalues of A, where repetition is possible.

Let us now assume that we have two forms a_1 and a_2, with dense form domains V_1 and V_2 in Hilbert spaces H_1 and H_2, respectively. We assume throughout that both a_1 and a_2 are continuous and elliptic. Let A_1 and A_2 be the operators associated with a_1 and a_2, which are automatically bounded from below thanks to the ellipticity assumption. Denote by S^1 and S^2 the semigroups generated by $-A_1$ and $-A_2$. If A_1 and A_2 also are self-adjoint and have compact resolvent then we call them *isospectral* if they have the same sequence of eigenvalues. In this case we will denote by $(e_n)_{n\in\mathbb{N}}$ the sequence of (normalized) eigenfunctions of A_1 on H_1 and by $(f_n)_{n\in\mathbb{N}}$ the similarly normalized eigenfunctions of A_2 on H_2. It is then immediate that there exists a unitary operator $U\in\mathscr{L}(H_1,H_2)$ such that

$$U^{-1}S_t^2U=S_t^1, \tag{2}$$

for all $t>0$; we may simply choose U such that $Ue_n=f_n$ for all $n\in\mathbb{N}$. If we assume $H_i=L_2(\Omega_i)$ for some open $\Omega_i\subset\mathbb{R}^d$ and A_i is the Dirichlet (or Neumann) Laplacian on Ω_i for both $i\in\{1,2\}$, then Kac' question may be phrased as asking whether the existence of an intertwining operator as in (2) implies the existence of an isometry $\tau\colon\Omega_1\to\Omega_2$.

However, we wish to consider more general operator-theoretic notions than isospectrality, in particular allowing for non-self-adjoint operators. Moreover, the similarity transform Φ that we construct in Section 3 is, in general, not unitary. See also Section 4. (This assertion is also true for the equivalent constructions in [9] and [10], where the mechanism is of course the same.) The next proposition gives a general characterization of an operator $\Phi\colon H_1\to H_2$ that intertwines A_1 and A_2 in terms of the forms a_1 and a_2. Of particular interest to us in what follows is Condition (iii). We do not require A_1 or A_2 to be self-adjoint or have compact resolvent.

PROPOSITION 2.2. *Let $\Phi\in\mathscr{L}(H_1,H_2)$. Consider the following conditions.*

(i) *$S_t^2\Phi=\Phi S_t^1$ for all $t>0$.*

(ii) *$\Phi(D(A_1))\subset D(A_2)$ and $A_2\Phi u=\Phi A_1u$ for all $u\in D(A_1)$.*

(iii) $\Phi(V_1) \subset V_2$, $\Phi^*(V_2) \subset V_1$ *and* $a_2(\Phi u, v) = a_1(u, \Phi^* v)$ *for all* $u \in V_1$ *and* $v \in V_2$.

Then (i)$\Leftrightarrow$(ii)$\Leftarrow$(iii).

Proof. '(i)$\Rightarrow$(ii)'. Let $u \in D(A_1)$. Then

$$\Phi A_1 u = \lim_{t \downarrow 0} t^{-1} \Phi (I - S^1_t) u = \lim_{t \downarrow 0} t^{-1} (I - S^2_t) \Phi u = A_2 \Phi u.$$

So $\Phi u \in D(A_2)$ and $A_2 \Phi u = \Phi A_1 u$.

'(ii)$\Rightarrow$(i)'. Replacing A_k by $\omega I + A_k$, we may assume that both S^1 and S^2 are exponentially decreasing. Let $u \in D(A_1)$ and $\lambda > 0$. Then $(\lambda I + A_2) \Phi u = \Phi (\lambda I + A_1) u$. Since $\lambda I + A_1$ is surjective, it follows that $\Phi (\lambda I + A_1)^{-1} v = (\lambda I + A_2)^{-1} \Phi v$ for all $v \in H_1$. Hence by iteration, $\Phi (\lambda I + A_1)^{-n} = (\lambda I + A_2)^{-n} \Phi$ for all $n \in \mathbb{N}$. Then by (7) in [19] Section IX.7 one deduces (i).

'(iii)$\Rightarrow$(ii)'. Let $u \in D(A_1)$. Then for all $v \in V_2$ one has $\Phi v \in V_1$ and

$$a_2(\Phi u, v) = a_1(u, \Phi^* v) = (A_1 u, \Phi^* v)_{H_1} = (\Phi A_1 u, v)_{H_2}.$$

Hence $\Phi u \in D(A_2)$ and $A_2 \Phi u = \Phi A_1 u$. $\square$

We remark that under additional assumptions on the operators A_1 and A_2 it can be proved that all three statements in Proposition 2.2 are equivalent. It suffices that a_1 and a_2 have the *square root property* on H, which means that for the square root operator $(\omega_k I + A_k)^{1/2}$ of $\omega_k I + A_k$, defined as in [3], Section 3.8, where ω_k is the constant in (1), we have $D((\omega_k I + A_k)^{1/2}) = V_k$, for both $k \in \{1, 2\}$. This is not always the case; a counterexample has been given by McIntosh [14], although it is always true if A_1 and A_2 are self-adjoint. As we do not need this equivalence in the sequel, we do not go into details.

We next assume that the intertwining operator $\Phi \in \mathcal{L}(H_1, H_2)$ is invertible and thus an isomorphism between H_1 and H_2.

COROLLARY 2.3. *Let* $\Phi \in \mathcal{L}(H_1, H_2)$ *be invertible. Consider the following statements.*

(i) $\Phi(D(A_1)) \subset D(A_2)$ *and* $A_2 \Phi u = \Phi A_1 u$ *for all* $u \in D(A_1)$.

(ii) $\Phi(D(A_1)) = D(A_2)$ *and* $A_2 \Phi u = \Phi A_1 u$ *for all* $u \in D(A_1)$.

(iii) $\Phi^{-1} S^2_t \Phi = S^1_t$ *for all* $t > 0$.

(iv) *If* $u \in D(A_1)$ *and* $\lambda \in \mathbb{R}$ *are such that* $A_1 u = \lambda u$, *then* $\Phi u \in D(A_2)$ *and* $A_2 \Phi u = \lambda \Phi u$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv). *If in addition* A_1 *is self-adjoint and has compact resolvent, then all four statements are equivalent.*

We say that A_1 and A_2 are *similar*, or equivalently, that the semigroups S^1 and S^2 are *similar*, if the equivalent statements (i)–(iii) hold. In the case where A_1 and A_2 are self-adjoint and have compact resolvent, we may replace (ii) with the statement '$\Phi(D(A_1)) = D(A_2)$ and the spectra of A_1 and A_2 coincide'. Thus we may regard similarity as a more general property than isospectrality.

The next result was stated in [2] Lemma 1.3 for self-adjoint operators, but we note that it is also a direct consequence of Proposition 2.2 and Corollary 2.3 without requiring this assumption.

COROLLARY 2.4. *Let $\Phi \in \mathscr{L}(H_1, H_2)$ be unitary. Then the following are equivalent.*

(i) $S_t^2 \Phi = \Phi S_t^1$ *for all* $t > 0$.

(ii) $\Phi(V_1) = V_2$ *and* $a_2(\Phi u, \Phi v) = a_1(u, v)$ *for all* $u, v \in V_1$.

We finish this section by pointing out that the existence of a unitary similarity transform is guaranteed by self-adjointness of the operators alone, and compactness of the resolvents is not needed.

PROPOSITION 2.5. *Let A_1 and A_2 be two self-adjoint operators on H_1 and H_2, respectively. Assume that the semigroups S^1 and S^2 are similar. Then there exists a unitary operator $U \in \mathscr{L}(H_1, H_2)$ such that*

$$U^{-1} S_t^1 U = S_t^2$$

for $t > 0$.

Proof. We consider the polar decomposition $\Phi = U|\Phi|$, where $U \in \mathscr{L}(H_1, H_2)$ is unitary and $|\Phi| = (\Phi^*\Phi)^{1/2} \in \mathscr{L}(H_1)$ is invertible and self-adjoint. Since

$$\Phi^* S_t^2 = (S_t^2 \Phi)^* = (\Phi S_t^1)^* = S^1 \Phi^*$$

for all $t > 0$, we see that Φ^* is also an intertwining operator. Thus $|\Phi|$ commutes with S_t^1 for all $t > 0$, and so

$$U S_t^1 = U|\Phi|\,|\Phi|^{-1} S_t^1 = \Phi S_t^1 |\Phi|^{-1} = S_t^2 \Phi |\Phi|^{-1} = S_t^2 U$$

for all $t > 0$. □

3. Isospectral domains for the Neumann Laplacian

For an open polygon Ω in $\mathbb{R}^2$ we denote by Δ_Ω^N the Neumann Laplacian on $L_2(\Omega)$. This realization of the Laplacian with Neumann boundary conditions is self-adjoint, has compact resolvent and its negative is bounded from below, with a sequence of eigenvalues $0 = \lambda_0 \leqslant \lambda_1 \leqslant \ldots \to \infty$. We will consider the two (very) warped propeller-like domains from Figure 1 and show that $\Delta_{\Omega_1}^N$ and $\Delta_{\Omega_2}^N$ are similar. This will be done with the help of our form criterion established in Proposition 2.2. As a corollary we deduce that $\Delta^N_{\Omega_1}$ and $\Delta^N_{\Omega_2}$ are isospectral even though Ω_1 and Ω_2 are obviously not congruent. Note that Ω_1 and Ω_2 look like propellers if the constituent triangles are equilateral.

Since we wish to decompose our polygons into their constituent triangles, we need to start with some basic facts about traces and integration by parts. We let $\Omega \subset \mathbb{R}^2$ be an arbitrary open polygon, although in practice we only need the following results for our warped propellers. On the boundary Γ of Ω we let σ denote the usual surface measure; on each straight line segment, σ is simply one-dimensional Lebesgue measure. The Trace Theorem states that there exists a unique bounded operator $\mathrm{Tr} : H^1(\Omega) \to L_2(\Gamma)$ such that $\mathrm{Tr}\,(u) = u|_\Gamma$ for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$. Observe that, since Ω is Lipschitz, the space $H^1(\Omega) \cap C(\overline{\Omega})$ is dense in $H^1(\Omega)$. By $\nu(z) = (\nu_1(z), \nu_2(z))$ we denote the outer unit normal to Ω at $z \in \Gamma$. Then $\nu(z)$ is constant on each straight line segment of the boundary. The integration by parts formula states that

$$-\int_\Omega (\partial_j u)\, v = \int_\Omega u\, \partial_j v - \int_\Gamma \nu_j\, u\, v$$

for all $u, v \in H^1(\Omega)$ and $j \in \{1,2\}$. Here the integral over Γ is with respect to σ, and we have omitted the trace to simplify notation.

The *Neumann Laplacian* is by definition the operator Δ^N_Ω on $L_2(\Omega)$ such that $-\Delta^N_\Omega$ is associated with the form $a \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ given by

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}. \tag{3}$$

We denote by S the semigroup generated by Δ^N_Ω.

If $u \in H^1(\Omega)$ is such that the distributional Laplacian $\Delta u \in L_2(\Omega)$, and if $h \in L_2(\Gamma)$, then we say that $\partial_\nu u = h$ if

$$\int_\Omega (\Delta u)\, v + \int_\Omega \nabla u \cdot \nabla v = \int_\Gamma h\, v \tag{4}$$

for all $v \in H^1(\Omega)$. That is, we define the normal derivative via Green's formula. Based on this definition, the operator Δ^N_Ω has the domain

$$D(\Delta^N_\Omega) = \{u \in H^1(\Omega) : \Delta u \in L_2(\Omega) \text{ and } \partial_\nu u = 0\}.$$

This is valid whenever Ω is a Lipschitz domain.

Now let T be a fixed scalene triangle whose three different sides are labelled Γ_1, Γ_2 and Γ_3 as in Figure 2. Thus if Ω_1 and Ω_2 are broken into their seven constituent triangles, then each is congruent to T.

Now let Ω be either one of Ω_1 and Ω_2 and consider the seven open disjoint triangles $T_1, \ldots, T_7$ such that

$$\overline{\Omega} = \bigcup_{k=1}^{7} \overline{T}_k.$$

Two triangles $\overline{T}_k$ and $\overline{T}_l$ may have a common side; there are six such sides inside Ω.

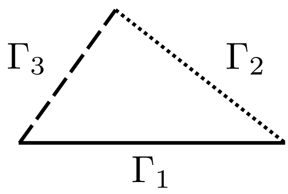

Figure 2: *The triangle T.*

If $u \in H^1(\Omega)$, then $u_k := u|_{T_k} \in H^1(T_k)$ for all $k \in \{1,\dots,7\}$. Conversely, the following basic result holds.

LEMMA 3.1. *Let $u \in L_2(\Omega)$ be such that $u_k := u|_{T_k} \in H^1(T_k)$ for all $k \in \{1,\dots,7\}$. Then $u \in H^1(\Omega)$ if and only if u_k and u_l have the same trace on common sides of T_k and T_l for all $k,l \in \{1,\dots,7\}$ with $k \neq l$. Moreover, if $u \in H^1(\Omega)$ then $(\partial_j u)|_{T_k} = \partial_j u_k$ on T_k for all $k \in \{1,\dots,7\}$ and $j \in \{1,2\}$.*

Proof. Since $H^1(\Omega) \cap C(\overline{\Omega})$ is dense in $H^1(\Omega)$ the condition on the traces is clearly necessary. Assume now that u satisfies this trace condition. Let $\varphi \in C_c^1(\Omega)$ and $j \in \{1,2\}$. Then

$$
\begin{aligned}
-\int_\Omega u\,\partial_j\varphi &= -\sum_{k=1}^{7}\int_{T_k} u_k\,\partial_j\varphi \\
&= \sum_{k=1}^{7}\Big(\int_{T_k}(\partial_j u_k)\,\varphi - \int_{\partial T_k}\nu_{k,j}\,u_k\,\varphi\Big) \\
&= \sum_{k=1}^{7}\int_{T_k}(\partial_j u_k)\,\varphi \qquad (5)\\
&= \int_\Omega w\,\varphi,
\end{aligned}
$$

where $w \in L_2(\Omega)$ is such that $w|_{T_k} = \partial_j u_k$. Thus $\partial_j u = w$ in Ω by definition of the weak derivative of a function. Here we denote by ν_k the outer unit normal to T_k on ∂T_k with its two components $\nu_{k,1}$ and $\nu_{k,2}$. Since $\nu_k = -\nu_l$ on $\overline{T}_k \cap \overline{T}_l$ whenever $k \neq l$, we have

$$\sum_{k=1}^{7}\int_{\partial T_k}\nu_{k,j}\,u\,\varphi = 0,$$

which we used in (5). $\square$

If $\tau\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, then the map $U\colon L_2(\tau(\Omega)) \to L_2(\Omega)$ given by $u \mapsto u \circ \tau$ is a unitary operator, $U(H^1(\tau(\Omega))) = H^1(\Omega)$ and $U|_{H^1(\tau(\Omega))}$ is also unitary and isometric.

Now consider the first warped propeller Ω_1 from Figure 1. Denote by $T_1,\ldots,T_7$ the seven disjoint triangles isomorphic to T such that $\overline{\Omega}_1 = \bigcup_{k=1}^{7} \overline{T}_k$, and for all $k \in \{1,\ldots,7\}$ denote by τ_k the isometry mapping T onto T_k. If we define a map

$$\Phi_1(w) = (w|_{T_1} \circ \tau_1, \ldots, w|_{T_7} \circ \tau_7),$$

then $\Phi_1 \colon L_2(\Omega_1) \to L_2(T)^7$ is unitary and $\Phi_1(H^1(\Omega_1)) = V_1$, where

$$\begin{aligned} V_1 = \{(u_1,\ldots,u_7) \in H^1(T)^7 \ : \ & u_1 = u_2 \text{ and } u_4 = u_7 \text{ on } \Gamma_1 \\ & u_1 = u_3 \text{ and } u_2 = u_5 \text{ on } \Gamma_2 \\ & u_1 = u_4 \text{ and } u_3 = u_6 \text{ on } \Gamma_3\}. \end{aligned}$$

Here we mean more precisely that u_1 and u_2 have the same trace on Γ_1, and so on. Since the trace is a continuous mapping from $H^1(T)$ into $L_2(\partial T)$, the space V_1 is closed in $H^1(T)^7$.

We now define a form $\tilde{a}_1 \colon V_1 \times V_1 \to \mathbb{C}$ by

$$\tilde{a}_1(u,v) := \sum_{k=1}^{7} \int_T \nabla u_k \cdot \overline{\nabla v_k}, \tag{6}$$

where we have written $u = (u_1,\ldots,u_7)$ and $v = (v_1,\ldots,v_7)$. Then $\tilde{a}_1$ is continuous, symmetric and elliptic with respect to $L_2(T)^7$. We denote by $\widetilde{A}_1$ the self-adjoint operator on $L_2(T)^7$ associated with $\tilde{a}_1$ and by $\widetilde{S}^1$ the semigroup generated by $-\widetilde{A}_1$. We next show that the operators $-\widetilde{A}_1$ and $\Delta^N_{\Omega_1}$ (and the semigroups they generate) are similar. Let S^1 be the semigroup generated by the Neumann Laplacian on Ω_1. We denote by a_1 the form associated with the semigroup S^1 on Ω_1, cf. (3).

PROPOSITION 3.2. *If $t > 0$ then*

$$\Phi_1^{-1}\, \widetilde{S}^1_t\, \Phi_1 = S^1_t.$$

Proof. Let $u,v \in H^1(\Omega_1)$ and write $\Phi_1(u) = (u_1,\ldots,u_7)$ and $\Phi_1(v) = (v_1,\ldots,v_7)$. Then

$$\begin{aligned} \tilde{a}_1(\Phi_1 u, \Phi_1 v) &= \sum_{k=1}^{7} \int_T \nabla u_k \cdot \overline{\nabla v_k} \\ &= \sum_{k=1}^{7} \int_T \nabla(u \circ \tau_k) \cdot \overline{\nabla(v \circ \tau_k)} \\ &= \sum_{k=1}^{7} \int_T (\nabla u) \circ \tau_k \cdot \overline{(\nabla v) \circ \tau_k} \\ &= \sum_{k=1}^{7} \int_{T_k} \nabla u \cdot \overline{\nabla v} \\ &= \int_{\Omega_1} \nabla u \cdot \overline{\nabla v}. \end{aligned} \tag{7}$$

Now the claim follows from Corollary 2.4. $\square$

An obvious analogue holds for Ω_2. Namely, define a unitary map $\Phi_2\colon L_2(\Omega_2) \to L_2(T)^7$ in the obvious way, and let

$$\begin{aligned} V_2 = \{(u_1,\dots,u_7) \in H^1(T)^7 \,:\, & u_1 = u_2 \text{ and } u_3 = u_6 \text{ on } \Gamma_1 \\ & u_1 = u_3 \text{ and } u_4 = u_7 \text{ on } \Gamma_2 \\ & u_1 = u_4 \text{ and } u_2 = u_5 \text{ on } \Gamma_3\}, \end{aligned}$$

so that $\Phi_2(H^1(\Omega_2)) = V_2$. We define $\tilde{a}_2\colon V_2 \times V_2 \to \mathbb{C}$ by

$$\tilde{a}_2(\Phi_2 u, \Phi_2 v) := \sum_{k=1}^{7} \int_T \nabla u_k \cdot \overline{\nabla v_k},$$

where $\Phi_2 u = (u_1,\dots,u_7)$ for all $u \in H^1(\Omega_2)$, etc., and denote by $\widetilde{A}_2$ the self-adjoint operator on $L_2(T)^7$ associated with $\tilde{a}_2$. Let $\widetilde{S}^2$ be the semigroup generated by $-\widetilde{A}_2$ and S^2 the semigroup generated by $\Delta^N_{\Omega_2}$.

PROPOSITION 3.3. *The operators $\widetilde{A}_2$ on $L_2(T)^7$ and $-\Delta^N_{\Omega_2}$ on $L_2(\Omega_2)$ are similar. Precisely,*

$$\Phi_2^{-1}\,\widetilde{S}^2_t\,\Phi_2 = S^2_t$$

for all $t > 0$.

The similarities established so far are quite simple; analogous results hold for any polygon decomposed into triangles. But the attraction of this approach is that, to show that $\Delta^N_{\Omega_1}$ and $\Delta^N_{\Omega_2}$ are similar, it suffices to prove the statement for $\widetilde{A}_1$ and $\widetilde{A}_2$, which are both defined as operators on $L_2(T)^7$. This is exactly what we shall now do, and it is here that the special combinatorial relations defining V_1 and V_2 are crucial.

We define a map $B\colon \mathbb{R}^7 \to \mathbb{R}^7$ by

$$B = \begin{pmatrix} 0&1&1&1&0&0&0 \\ 1&0&1&0&0&0&1 \\ 1&0&0&1&1&0&0 \\ 1&1&0&0&0&1&0 \\ 0&0&0&1&0&1&1 \\ 0&1&0&0&1&0&1 \\ 0&0&1&0&1&1&0 \end{pmatrix}.$$

This is an analogue for our domains of the matrix T^N considered in [9] (with $a = 0$ and $b = 1$; see the discussion in Section 4). Moreover, define $\Phi\colon L_2(T)^7 \to L_2(T)^7$ by

$$(\Phi u)_k = \sum_{l=1}^{7} b_{kl}\, u_l.$$

The adjoint Φ^* may also be defined directly with respect to B^* by

$$(\Phi^* u)_l = \sum_{k=1}^{7} b_{kl}\, u_k.$$

It is a simple (but central) calculation to show that

$$\Phi(V_1) \subset V_2 \quad \text{and} \quad \Phi^*(V_2) \subset V_1.$$

Moreover, for all $u \in V_1$ and $v \in V_2$ one has

$$\begin{aligned}\tilde{a}_2(\Phi u, v) = \sum_{k=1}^{7} \int_T \nabla(\Phi u)_k \cdot \overline{\nabla v_k} &= \sum_{k=1}^{7}\sum_{l=1}^{7} b_{kl} \int_T \nabla u_l \cdot \overline{\nabla v_k} \\ &= \sum_{l=1}^{7} \int_T \nabla u_l \cdot \overline{\nabla (\Phi^* v)_l} = \tilde{a}_1(u, \Phi^* v). \qquad (8)\end{aligned}$$

Using Proposition 2.2 it follows that $\Phi \widetilde{S}^1_t = \widetilde{S}^2_t \Phi$ for all $t > 0$. It is easy to verify that the matrix B is invertible, implying that Φ is invertible as an operator on $L_2(T)^7$, and therefore

$$\widetilde{S}^1_t = \Phi^{-1} \widetilde{S}^2_t \Phi$$

for all $t > 0$. So the semigroups are similar. If we define

$$U = \Phi_2^{-1} \Phi \Phi_1,$$

then $U \colon L_2(\Omega_1) \to L_2(\Omega_2)$ is an isomorphism such that

$$S^1_t = U^{-1} S^2_t U$$

for all $t > 0$. We have proved the following result.

THEOREM 3.4. *The semigroups S^1 and S^2 are similar. In particular, $\Delta^N_{\Omega_1}$ and $\Delta^N_{\Omega_2}$ are isospectral, i.e. they have the same sequence of eigenvalues, even though Ω_1 and Ω_2 are not congruent.*

4. Order properties of the similarity transform

We keep the notation of the previous section and consider the intertwining isomorphism $U \colon L_2(\Omega_1) \to L_2(\Omega_2)$ more closely. Recall that a linear map $R \colon L_2(\Omega_1) \to L_2(\Omega_2)$ is called *positive* if $f \geqslant 0$ implies $Rf \geqslant 0$ for all $f \in L_2(\Omega_1)$. We then write $R \geqslant 0$. One calls R a *lattice homomorphism* if $R(f \vee g) = Rf \vee Rg$ for all $f, g \in L_2(\Omega_1, \mathbb{R})$. The map R is called *disjointness preserving* if $f \cdot g = 0$ a.e. implies $(Rf) \cdot (Rg) = 0$ a.e. for all $f, g \in L_2(\Omega_1)$. It is well known that R is a lattice homomorphism if and only if R is positive and disjointness preserving. Finally, we call R an *order isomorphism* or a *lattice isomorphism* if R is bijective and both R and R^{-1} are positive. This is equivalent to R being a bijective lattice homomorphism.

We recall from [2] Theorem 3.20 the following result.

THEOREM 4.1. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be two Lipschitz domains. If there exists an order isomorphism $U\colon L_2(\Omega_1) \to L_2(\Omega_2)$ such that*

$$U S_t^1 = S_t^2 U$$

for all $t > 0$, then Ω_1 and Ω_2 are congruent.

Here, as before, S^1 and S^2 are the semigroups generated by $\Delta^N_{\Omega_1}$ and $\Delta^N_{\Omega_2}$, respectively. Using Theorem 4.1 it follows that the similarity transform U in Theorem 3.4 is not an order isomorphism. In fact, $U = \Phi_2^{-1}\Phi\Phi_1$. Recall that $\Phi\colon L_2(T)^7 \to L_2(T)^7$ is given by the matrix B, which is clearly positive. Thus Φ is a positive map. Since Φ_1 and Φ_2 are order isomorphisms, U is also positive. Hence Φ^{-1}, and equivalently also U^{-1}, is not positive.

It is easy to see that a map from $L_2(T)^7$ into $L_2(T)^7$ given by a matrix as above is disjointness preserving if and only if each row in the matrix has at most one nonzero entry. By way of contrast, our matrix B has three nonzero entries in each row. It follows that our Φ is the sum of three lattice homomorphisms. This shows directly that Φ and U are not disjointness preserving.

Finally, we mention that the intertwining isomorphism U and the matrix B that induces it are not unique. If we let $\mathbb{1}$ denote the 7×7 matrix whose (k,l)th-entry is 1 for all $k,l \in \{1,\dots,7\}$, and define $\widehat{B} := \alpha(\mathbb{1} - B) + \gamma B$, then it may be verified that $\widehat{B}$ gives rise in the same way to another intertwining isomorphism $\widehat{\Phi}$, provided that the coefficients $\alpha, \gamma \in \mathbb{R}$ satisfy basic non-degeneracy conditions. Our original matrix B and the similarity transform Φ that it induces are easily seen to be normal but not unitary. We know from Section 2 that in such a case one can always find a unitary transform related to Φ, for example, via the polar decomposition $\Phi = U|\Phi|$. However, if we choose the coefficients α and γ appropriately, namely, as a pair of simultaneous solutions to $4\alpha^2 + 3\gamma^2 = 1$ and $2\alpha^2 + 4\alpha\gamma + \gamma^2 = 0$, then it is easy to check that $(\widehat{B})^* \widehat{B} = I$, that is, the matrix $\widehat{B}$ is unitary. In this case, the similarity transform associated with $\widehat{B}$ is also unitary, and one may check that one of the operators thus obtained coincides with the U obtained from the polar decomposition of our original transform Φ.

We note that Bérard [9] assumes from the beginning of his construction that his matrix is orthogonal by imposing a restriction equivalent to the one just stated for α and γ. The cases $\alpha = 0$, $\gamma = 1$ and $\alpha = \gamma = 1$ (for which the respective matrices are not orthogonal) correspond respectively to the mappings T_3 and T_4 considered in [10] Section 2.

5. Isospectral domains for general elliptic operators

We will now generalize our construction from Section 3 to allow for general non-self-adjoint elliptic operators on $L_2(\Omega_1)$ and $L_2(\Omega_2)$. These operators still have compact resolvent, but are in general not self-adjoint. Hence the original formulation involving isospectrality is not strong enough. It turns out, however, that the machinery of the previous sections works in exactly the same fashion to give the desired similarity of the operators in the general case. We start with a basic lemma describing how elliptic differential sectorial forms transform under isometries.

LEMMA 5.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, $C = (c_{ij})_{i,j\in\{1,\dots,d\}} : \Omega \to M_{d\times d}(\mathbb{C})$ a bounded measurable map and τ an isometry. Define $a\colon H^1(\Omega)\times H^1(\Omega) \to \mathbb{C}$ by*

$$a(u,v) = \int_\Omega \sum_{i,j=1}^d c_{ij}\,(\partial_i u)\,\overline{\partial_j v} \tag{9}$$

and $\widehat{\Omega} = \tau(\Omega)$. Next, define the bounded measurable map $C_\tau = \widehat{C} = (\hat{c}_{ij})_{i,j\in\{1,\dots,d\}} : \widehat{\Omega} \to M_{d\times d}(\mathbb{C})$ by

$$C_\tau(y) = \widehat{C}(y) = (D\tau)\,C(\tau^{-1}(y))\,(D\tau)^{-1}, \tag{10}$$

where $D\tau$ denotes the derivative of τ. Finally, define the form $\hat{a}\colon H^1(\widehat{\Omega})\times H^1(\widehat{\Omega}) \to \mathbb{C}$ by

$$\hat{a}(u,v) = \int_{\widehat{\Omega}} \sum_{i,j=1}^d \hat{c}_{ij}\,(\partial_i u)\,\overline{\partial_j v}.$$

Then $\hat{a}(u,v) = a(u\circ\tau, v\circ\tau)$ for all $u,v \in H^1(\widehat{\Omega})$.

Proof. Denote by $\langle\cdot,\cdot\rangle$ the inner product on $\mathbb{C}^d$. Then

$$\begin{aligned}
a(u\circ\tau, v\circ\tau) &= \int_\Omega \langle C^t\,\nabla(u\circ\tau), \nabla(v\circ\tau)\rangle \\
&= \int_\Omega \langle C^t\,(D\tau)^t\,((\nabla u)\circ\tau), (D\tau)^t\,((\nabla v)\circ\tau)\rangle \\
&= \int_\Omega \langle (D\tau)\,C^t\,(D\tau)^t\,((\nabla u)\circ\tau), ((\nabla v)\circ\tau)\rangle \\
&= \int_{\widehat{\Omega}} \langle (D\tau)\,(C^t\circ\tau^{-1})\,(D\tau)^t\,\nabla u, \nabla v\rangle \\
&= \hat{a}(u,v)
\end{aligned}$$

as required. □

Let $\Omega \subset \mathbb{R}^d$ be an open set and $C = (c_{ij})_{i,j\in\{1,\dots,d\}} : \Omega \to M_{d\times d}(\mathbb{C})$ a bounded measurable map. Define $a\colon H^1(\Omega)\times H^1(\Omega) \to \mathbb{C}$ by

$$a(u,v) = \int_\Omega \sum_{i,j=1}^d c_{ij}\,(\partial_i u)\,\overline{\partial_j v}.$$

Suppose that there exists a $\mu > 0$ such that

$$\operatorname{Re} \sum_{i,j=1}^d c_{ij}\,\xi_i\,\overline{\xi_j} \geqslant \mu\,|\xi|^2 \qquad \text{for all } \xi \in \mathbb{C}^d \tag{11}$$

almost everywhere on Ω. Then the form a is elliptic. Let A be the operator associated with the form a on $L_2(\Omega)$. Note that A is self-adjoint if $c_{ij} = \overline{c_{ji}}$ a.e. for all $i,j \in$

$\{1,\dots,d\}$. We emphasize that we do not assume this. If Ω is bounded and Lipschitz, then by a result of Auscher–Tchamitchian [7] the form a has the square root property on $L_2(\Omega)$. Also note that if C is the identity matrix, then A is the Neumann Laplacian. In this case, if τ is an isometry and $\widehat{C}$ is as in Lemma 5.1, then also $\widehat{C}$ is the identity matrix. So the Neumann Laplacian is transformed into the Neumann Laplacian, and the proof of (7) is a special case of the previous lemma. This is one of the remarkable properties of the Laplacian. However, if we consider an elliptic operator, then we have to take into account the conjugation with the derivative of the isometry.

Next, let C be a bounded measurable elliptic matrix valued function on our reference triangle T. Thus

$$C = (c_{ij})_{i,j\in\{1,2\}} \colon T \to M_{2\times 2}(\mathbb{C})$$

is a bounded measurable map satisfying the ellipticity condition (11). Let Ω_1 and Ω_2 be the two propellers as before. Define the form $a_1 \colon H^1(\Omega_1)\times H^1(\Omega_1)\to\mathbb{C}$ by

$$a_1(u,v) = \sum_{k=1}^{7} \int_{T_k} (C_{\tau_k})_{ij}\,(\partial_i u)\,\overline{\partial_j v}, \tag{12}$$

where for all $k\in\{1,\dots,7\}$ the isometry τ_k is as in Section 3 and C_{τ_k} is defined in (10). We define the form $a_2 \colon H^1(\Omega_2)\times H^1(\Omega_2)\to\mathbb{C}$ analogously. Let $n\in\{1,2\}$. Then a_n is elliptic. Let A_n be the operator associated with a_n on $L_2(\Omega_n)$ and let S^n be the semigroup generated by $-A_n$ on $L_2(\Omega_n)$. Next define the form $\tilde{a}\colon H^1(T)\times H^1(T)\to\mathbb{C}$ by

$$\tilde{a}(u,v) = \sum_{i,j=1}^{2} \int_T c_{ij}\,(\partial_i u)\,\overline{\partial_j v}. \tag{13}$$

Moreover, define the form $\tilde{a}_n\colon V_n\times V_n\to\mathbb{C}$ by

$$\tilde{a}_n(u,v) = \sum_{k=1}^{7} \tilde{a}(u_k,v_k). \tag{14}$$

Let $\widetilde{A}_n$ be the operator associated with $\tilde{a}_n$ on $L_2(T)^7$ and let $\widetilde{S}^n$ be the semigroup generated by $-\widetilde{A}_n$ on $L_2(T)^7$.

Using Lemma 5.1 it follows as in Section 3 that

$$\Phi_n\, S^n_t\,(\Phi_n)^{-1} = \widetilde{S}^n_t$$

for all $t>0$, where Φ_n is the *same* transform as in Section 3. Arguing as in Section 3 one has

$$\Phi\,\widetilde{S}^1_t = \widetilde{S}^2_t\,\Phi$$

for all $t>0$, where surprisingly Φ is, again, the same transform as in Section 3.

Therefore we have proved the following theorem.

THEOREM 5.2. *Let $U=\Phi_2^{-1}\Phi\Phi_1$. Then*

$$S^1_t = U^{-1}\,S^2_t\,U$$

for all $t>0$. In particular, the operators A_1 on $L_2(\Omega_1)$ and A_2 on $L_2(\Omega_2)$ are similar even though Ω_1 and Ω_2 are not congruent.

6. Isospectral elliptic operators with Dirichlet boundary conditions

In this section we wish to extend Theorem 5.2 to the case of Dirichlet boundary conditions. All the arguments are the same as before, but now we have to impose more boundary conditions on the Sobolev spaces. Let $\Omega \subset \mathbb{R}^d$ be open and $C = (c_{ij}) \colon \Omega \to M_{d\times d}(\mathbb{C})$ a bounded measurable map. Assume that C satisfies the ellipticity condition (11). Let a be as in (9) and let $a^D = a|_{H_0^1(\Omega)\times H_0^1(\Omega)}$, where $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. Then the operator associated with a^D in $L_2(\Omega)$ is the corresponding elliptic differential operator with Dirichlet boundary conditions. For domains with Lipschitz boundary there is a useful characterization of the Sobolev space $H_0^1(\Omega)$.

LEMMA 6.1. *Suppose $\Omega \subset \mathbb{R}^d$ is open and Ω has a Lipschitz boundary. Then*

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \operatorname{Tr} u = 0 \ \sigma\text{-a.e.}\}.$$

Proof. See [1] Lemma A.6.10. □

Now we return to the two propellers. Let

$$C = (c_{ij})_{i,j\in\{1,2\}} \colon T \to M_{2\times 2}(\mathbb{C})$$

be a bounded measurable map satisfying the ellipticity condition (11). Let $n \in \{1,2\}$. Let $a_n \colon H^1(\Omega_n) \times H^1(\Omega_n) \to \mathbb{C}$ be as in (12) and set

$$a_n^D = a_n|_{H_0^1(\Omega_n)\times H_0^1(\Omega_n)}.$$

Let A_n^D be the operator associated with a_n^D on $L_2(\Omega_n)$ and let $S^{D,n}$ be the semigroup generated by $-A_n^D$. Let $\Phi_n \colon L_2(\Omega_n) \to L_2(T)^7$ be as in Section 3. Define $V_n^D = \Phi_n(H_0^1(\Omega_n))$. Let $\tilde{a} \colon H^1(T) \times H^1(T) \to \mathbb{C}$ be as in (13). Define $\tilde{a}_n^D \colon V_n^D \times V_n^D \to \mathbb{C}$ by

$$\tilde{a}_n^D(u,v) = \sum_{k=1}^{7} \tilde{a}(u_k, v_k).$$

So $\tilde{a}_n^D = \tilde{a}_n|_{V_n^D \times V_n^D}$, where $\tilde{a}_n$ is as in (14). Let $\widetilde{A}_n^D$ be the operator associated with $\tilde{a}_n^D$ on $L_2(T)^7$ and let $\widetilde{S}^{D,n}$ be the semigroup generated by $-\widetilde{A}_n^D$ on $L_2(T)^7$. Then as before one has

$$\Phi_n S_t^{D,n} \Phi_n^{-1} = \widetilde{S}_t^{D,n}$$

for all $t > 0$.

Next we determine V_n^D. Since Lemma 6.1 imposes boundary conditions on the 9 parts of the boundary of Ω_n, one has

$$\begin{aligned} V_1^D = \{(u_1,\ldots,u_7) \in H^1(T)^7 \ : \ & u_1 = u_2 \text{ and } u_4 = u_7 \text{ and } u_3 = u_5 = u_6 = 0 \text{ on } \Gamma_1 \\ & u_1 = u_3 \text{ and } u_2 = u_5 \text{ and } u_4 = u_6 = u_7 = 0 \text{ on } \Gamma_2 \\ & u_1 = u_4 \text{ and } u_3 = u_6 \text{ and } u_2 = u_5 = u_7 = 0 \text{ on } \Gamma_3\} \end{aligned}$$

and

$$\begin{aligned} V_2^D = \{(u_1,\dots,u_7) \in H^1(T)^7 \ : \ & u_1 = u_2 \text{ and } u_3 = u_6 \text{ and } u_4 = u_5 = u_7 = 0 \text{ on } \Gamma_1 \\ & u_1 = u_3 \text{ and } u_4 = u_7 \text{ and } u_2 = u_5 = u_6 = 0 \text{ on } \Gamma_2 \\ & u_1 = u_4 \text{ and } u_2 = u_5 \text{ and } u_3 = u_6 = u_7 = 0 \text{ on } \Gamma_3\}. \end{aligned}$$

Now define $B^D = (b_{kl}^D)_{k,l\in\{1,\dots,7\}} \colon \mathbb{R}^7 \to \mathbb{R}^7$ by

$$B^D = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

and define $\Phi^D \colon L_2(T)^7 \to L_2(T)^7$ by

$$(\Phi^D u)_k = \sum_{l=1}^{7} b_{kl}^D u_l.$$

A surprising but simple calculation shows that

$$\Phi^D(V_1^D) \subset V_2^D \quad \text{and} \quad (\Phi^D)^*(V_2^D) \subset V_1^D.$$

Literally the same argument as in (8) gives

$$\tilde{a}_2^D(\Phi^D u, v) = \tilde{a}_1^D(u, (\Phi^D)^* v)$$

for all $u \in V_1^D$ and $v \in V_2^D$. Therefore Proposition 2.2 implies that

$$\Phi^D \widetilde{S}_t^{D,1} = \widetilde{S}_t^{D,2} \Phi^D$$

for all $t > 0$. Hence we have extended everything to Dirichlet boundary conditions.

THEOREM 6.2. *Let $U^D = \Phi_2^{-1} \Phi^D \Phi_1$. Then*

$$S_t^{D,1} = (U^D)^{-1} S_t^{D,2} U^D$$

for all $t > 0$. In particular, the operators A_1^D on $L_2(\Omega_1)$ and A_2^D on $L_2(\Omega_2)$ are isospectral even though Ω_1 and Ω_2 are not congruent.

7. Operators with Robin boundary conditions

If we again let $\Omega \subset \mathbb{R}^2$ be an arbitrary polygon, or more generally Lipschitz planar domain, with boundary Γ, then for all $\beta \in \mathbb{R}$ we define a new form $a^\beta \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$a^\beta(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \beta \int_\Gamma u \overline{v}. \tag{15}$$

It follows from the Trace Inequality that the form a^β is continuous and $L_2(\Omega)$-elliptic for all $\beta \in \mathbb{R}$. We denote by $-\Delta_\Omega^\beta$ the operator on $L_2(\Omega)$ associated with a^β and call Δ_Ω^β the *Robin Laplacian* with boundary coefficient β, which has domain given by

$$D(\Delta_\Omega^\beta) = \{u \in H^1(\Omega) : \Delta u \in L_2(\Omega) \text{ and } \partial_\nu u + \beta u = 0 \text{ on } \Gamma\}.$$

In the boundary condition $\partial_\nu u + \beta u = 0$, the normal derivative $\partial_\nu u$ is defined by (4), as in the case of the Neumann Laplacian, and by u we mean the trace of u on Γ. As is true of its Dirichlet and Neumann counterparts, the Robin Laplacian is self-adjoint and has compact resolvent, and its negative is bounded from below. We denote by S^β the semigroup generated by Δ_Ω^β. When $\beta = 0$ we recover the Neumann Laplacian, and for $\beta \in (0,\infty)$, the Robin Laplacian 'interpolates' between the Dirichlet and Neumann Laplacians in a strong sense [6].

The boundary condition $\partial_\nu u + \beta u = 0$ corresponds to an 'elastically supported membrane'. So if we interpret the Dirichlet boundary condition as representing a drum with a taut membrane and the Neumann condition naïvely as representing a gong, then the Robin condition describes a drum whose membrane is not properly attached to the body of the drum, but rather allowed to move a little as the membrane vibrates.

Our goal is to show that no operator formed as a sum of superimposing isometries between component triangles of Ω_1 and Ω_2 (as in the Neumann and Dirichlet cases) can intertwine the Robin Laplacians $\Delta_{\Omega_1}^\beta$ and $\Delta_{\Omega_2}^\beta$ for any $\beta \neq 0$. We make this statement precise by recalling some notation from Section 3. If we first consider Ω_1, we recall that $\Phi_1 \colon L_2(\Omega_1) \to L_2(T)^7$ is the unitary operator associated with the family of isometries $\tau_k \colon T \to T_k$, such that

$$\Phi_1(w) = (w|_{T_1} \circ \tau_1, \ldots, w|_{T_7} \circ \tau_7)$$

for all $w \in L_2(\Omega_1)$, and moreover $\Phi_1(H^1(\Omega_1)) = V_1$. Note that since the Robin Laplacian has the same form domain as the Neumann Laplacian, Φ_1 is still the correct operator to use in this case. However, we now wish to consider the image of $\partial\Omega_1$, the boundary of Ω_1, under the isometries τ_k. We write

$$\Gamma_k^1 := \tau_k^{-1}(\partial\Omega_1 \cap \overline{T}_k)$$

for all $k \in \{1,\ldots,7\}$. Then $\Gamma_k^1 \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$; for example, $\Gamma_1^1 = \emptyset$ and $\Gamma_5^1 = \Gamma_1 \cup \Gamma_3$. (Cf. Figures 1 and 2.) For fixed $\beta \in \mathbb{R}$ we now define a form $\tilde{a}_1^\beta \colon V_1 \times V_1 \to \mathbb{C}$ by

$$\tilde{a}_1^\beta(\Phi_1 u, \Phi_1 v) = a_1^\beta(u,v)$$

for $u,v \in H^1(\Omega_1)$. Then

$$\tilde{a}_1^\beta(u,v) = \sum_{k=1}^{7} \int_T \nabla u_k \cdot \overline{\nabla v_k} + \beta \int_{\Gamma_k^1} u_k \overline{v_k}$$

for all $u = (u_1,\dots,u_7), v = (v_1,\dots,v_7) \in V_1$. Note that $\tilde{a}_1^0$ coincides with the form $\tilde{a}_1$ introduced in (6). We do the same for Ω_2, so that the unitary operator $\Phi_2 \colon L_2(\Omega_2) \to L_2(T)^7$ intertwines the forms a_2^β and $\tilde{a}_2^\beta : V_2 \times V_2 \to \mathbb{C}$ given by

$$\tilde{a}_2^\beta(u,v) = \sum_{k=1}^{7} \int_T \nabla u_k \cdot \overline{\nabla v_k} + \beta \int_{\Gamma_k^2} u_k \overline{v_k}.$$

For an arbitrary invertible matrix $P \colon \mathbb{R}^7 \to \mathbb{R}^7$ given by $P = (p_{kl})$ we construct an associated operator $\Phi \colon L_2(T)^7 \to L_2(T)^7$ by setting

$$(\Phi u)_k = \sum_{l=1}^{7} p_{kl}\, u_l \qquad \text{and} \qquad (\Phi^* u)_l = \sum_{k=1}^{7} p_{kl}\, u_k, \tag{16}$$

where $(u_1,\dots,u_7) \in L_2(T)^7$. We will prove that, for any $\beta \neq 0$, there is no matrix P such that the associated operator $\Phi \colon L_2(T)^7 \to L_2(T)^7$ satisfies $\Phi(V_1) \subset V_2$, $\Phi^*(V_2) \subset V_1$ and

$$\tilde{a}_2^\beta(\Phi u, v) = \tilde{a}_1^\beta(u, \Phi^* v) \tag{17}$$

for all $u \in V_1$ and $v \in V_2$. Since this is equivalent to the non-existence of an operator $U = \Phi_2^{-1} \Phi \Phi_1 \colon L_2(\Omega_1) \to L_2(\Omega_2)$ intertwining a_1^β and a_2^β, the impossibility of (17) then implies via Proposition 2.2 that the Robin Laplacians cannot be intertwined by an operator expressible as a sum of isometries between the triangles. To that end, we first show that the question as to whether (17) holds is independent of the coefficient $\beta \neq 0$.

PROPOSITION 7.1. *Let $\Phi \in \mathscr{L}(L_2(T)^7, L_2(T)^7)$ be defined by* (16) *and satisfy $\Phi(V_1) \subset V_2$ and $\Phi^*(V_2) \subset V_1$. If* (17) *holds for some $\beta \in \mathbb{R} \setminus \{0\}$, then the same is true for all $\beta \in \mathbb{R}$.*

Proof. This follows easily from the definition (16) of the operator Φ. Just as in the Neumann case (cf. (8)), we have

$$\tilde{a}_2^\beta(\Phi u, v) = \sum_{k=1}^{7} \sum_{l=1}^{7} p_{kl} \Big(\int_T \nabla u_l \cdot \overline{\nabla v_k} + \beta \int_{\Gamma_k^2} u_l \overline{v_k} \Big), \tag{18}$$

while

$$\tilde{a}_1^\beta(u, \Phi^* v) = \sum_{k=1}^{7} \sum_{l=1}^{7} p_{kl} \Big(\int_T \nabla u_l \cdot \overline{\nabla v_k} + \beta \int_{\Gamma_l^1} u_l \overline{v_k} \Big). \tag{19}$$

By assumption, the two are equal, and so

$$\beta \sum_{k=1}^{7} \sum_{l=1}^{7} p_{kl} \Big(\int_{\Gamma_k^2} u_l \overline{v_k} - \int_{\Gamma_l^1} u_l \overline{v_k} \Big) = 0. \tag{20}$$

Since $\beta \neq 0$, if we take any other $\beta_0 \in \mathbb{R}$ and multiply (20) by β_0/β, we see from (18) and (19) applied to β_0 that (17) must hold for β_0. $\square$

Our next result, which applies to any bounded Lipschitz domains ω_1 and ω_2 in $\mathbb{R}^d$, states that if a unitary operator U intertwines two Robin Laplacians for two separate values of $\beta \in \mathbb{R}$, then the *same* operator intertwines the Robin Laplacians for *all* values of $\beta \in \mathbb{R}$, including the Neumann Laplacians, as well as the Dirichlet Laplacians, and also acts isometrically on the traces of functions in $H^1(\omega_1)$.

PROPOSITION 7.2. *Let ω_1 and ω_2 be bounded Lipschitz domains in $\mathbb{R}^d$. For all $\beta \in \mathbb{R}$ denote by a_1^β and a_2^β the forms given by (15) on ω_1 and ω_2. Suppose $U \in \mathscr{L}(L_2(\omega_1), L_2(\omega_2))$ is unitary, with $U(H^1(\omega_1)) = H^1(\omega_2)$. The following statements are equivalent.*

(i) *There exist $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta_1 \neq \beta_2$ such that*

$$a_1^{\beta_n}(u,v) = a_2^{\beta_n}(Uu, Uv)$$

for all $u, v \in H^1(\omega_1)$ and $n \in \{1,2\}$.

(ii) *The operator U intertwines the Neumann Laplacians on ω_1 and ω_2. Moreover, if $u, v \in H^1(\omega_1)$, then*

$$\int_{\partial\omega_1} u\overline{v} = \int_{\partial\omega_2} (Uu)\overline{(Uv)}, \tag{21}$$

where by u we mean the trace of u, etc.

Moreover, if these equivalent conditions are satisfied, then $U(H_0^1(\omega_1)) = H_0^1(\omega_2)$ and U intertwines the Dirichlet Laplacians on ω_1 and ω_2.

Proof. '(i)⇒(ii)'. By writing out the form condition in (i) for β_1 and β_2 and taking the difference of the two expressions, we obtain directly that

$$\int_{\omega_1} \nabla u \cdot \overline{\nabla v} = \int_{\omega_2} \nabla(Uu) \cdot \overline{\nabla(Uv)} \quad \text{and} \quad \int_{\partial\omega_1} u\overline{v} = \int_{\partial\omega_2} (Uu)\overline{(Uv)} \tag{22}$$

for all $u, v \in H^1(\omega_1)$. It follows immediately from Corollary 2.4 that U intertwines the Neumann Laplacians on ω_1 and ω_2.

'(ii)⇒(i)'. Fix $\beta \in \mathbb{R}$ and $u, v \in H^1(\omega_1)$. Since U intertwines the Neumann Laplacians, by Corollary 2.4 it intertwines the associated forms. Therefore

$$\int_{\omega_1} \nabla u \cdot \overline{\nabla v} = \int_{\omega_2} \nabla(Uu) \cdot \overline{\nabla(Uv)}. \tag{23}$$

Moreover, since by assumption (21) holds, it follows directly from the definition (15) of a^{β_n} that (i) holds for all $\beta_1, \beta_2 \in \mathbb{R}$.

Finally, to prove the last assertion, suppose that $w \in H_0^1(\omega_1)$. Then (22) applied to $u = v = w$ implies that

$$\int_{\partial\omega_2} |Uw|^2 = \int_{\partial\omega_1} |w|^2 = 0.$$

By Lemma 6.1, it follows that $Uw \in H_0^1(\Omega_2)$. Thus $U(H_0^1(\omega_1)) \subset H_0^1(\omega_2)$. Since $U^{-1} = U^*$ has exactly the same properties as U, an identical argument shows that $U^{-1}(H_0^1(\omega_2)) \subset H_0^1(\omega_1)$ and therefore $U(H_0^1(\omega_1)) = H_0^1(\omega_2)$. Moreover, it is clear that $U|_{H_0^1(\omega_1)}$ is still a continuous linear bijection, and since (23) holds for all $u, v \in H_0^1(\omega_1) \subset H^1(\omega_1)$, by Corollary 2.4 this means U intertwines the Dirichlet Laplacians. $\square$

We now show that no one operator Φ of the form (16) can simultaneously intertwine the Dirichlet and Neumann Laplacians, which is a noteworthy observation in its own right. It is also worth noting that it can be proved by observing that the families of matrices $\widehat{B} = \alpha\mathbb{1} - \gamma B$ and $\widehat{B}^D := \alpha\mathbb{1} - \gamma B^D$, for nontrivial combinations of α and γ, are the only ones giving rise to operators intertwining the Neumann and Dirichlet Laplacians, respectively, and they have no matrix in common. However, we give a different proof based on reflections. The principle is that, if one reflects a triangle T along one of its sides, and wishes to reflect functions in $H^1(T)$ across to the larger domain, one does so by taking even reflections along the common line. But to preserve $H_0^1(T)$ the reflection should be odd.

PROPOSITION 7.3. *No invertible operator $\Phi\colon L_2(T)^7 \to L_2(T)^7$ of the form (16) simultaneously satisfies the Neumann condition*

$$\Phi(V_1) \subset V_2 \quad \textit{and} \quad \Phi^*(V_2) \subset V_1$$

and the Dirichlet condition

$$\Phi(V_1^D) \subset V_2^D \quad \textit{and} \quad \Phi^*(V_2^D) \subset V_1^D.$$

Proof. Assume Φ is associated with $P = (p_{kl})\colon \mathbb{R}^7 \to \mathbb{R}^7$.

Consider $m := p_{12}$. Let $w \in C_c^\infty(T \cup \Gamma_3)$ be such that w does not vanish identically on Γ_3. If we define $u = (0, w, 0, \dots, 0)$, then it is easily checked that $u \in V_1$. Moreover, we have $(\Phi u)_1 = mw$ and $(\Phi u)_4 = p_{42} w$, using the definition (16) of Φ. But since $\Phi u \in V_2$, we must have $(\Phi u)_1 = (\Phi u)_4$ on Γ_3 in the sense of traces. Since $w \not\equiv 0$ on Γ_3, this means $p_{42} = m$. Alternatively, choose $v = (w, 0, 0, w, 0, 0, 0)$. Then $v \in V_2^D$. Moreover, $\Phi^* v = (0, 2mw, 0, 0, 0, 0, 0)$. But $\Phi^* v \in V_1^D$ by assumption. So $2mw$ vanishes on Γ_3. This implies that $p_{12} = m = 0$.

Arguing similarly, it follows that $p_{kl} = 0$ for all $(k,l) \in \{1, \dots, 7\}^2 \setminus S$, where $S = \{(1,1), (2,4), (3,2), (4,3), (5,6), (6,7), (7,5)\}$. Since P is invertible, one has $p_{kl} \neq 0$ for all $(k,l) \in S$. Then

$$\Phi u = (p_{11} u_1, p_{24} u_4, p_{32} u_2, p_{43} u_3, p_{56} u_6, p_{67} u_7, p_{75} u_5).$$

If w is as above, but one chooses this time $u = (w,0,0,w,0,0,0)$, then $u \in V_1$. So $\Phi u \in V_2$ by assumption. Hence $(\Phi u)_1 = (\Phi u)_4$ on Γ_3, which implies that $p_{11} w = 0$ on Γ_3. This is a contradiction. □

Note that the same proof also works if P has complex coefficients. Our main result, that the Robin Laplacians on Ω_1 and Ω_2 are not intertwined by any operator acting as a linear combination of isometries between triangles, now follows easily.

THEOREM 7.4. *Suppose $\beta \neq 0$. Then there does not exist an invertible operator $\Psi\colon L_2(\Omega_1) \to L_2(\Omega_2)$ of the form $\Psi = \Phi_2^{-1} \Phi \Phi_1$, where $\Phi\colon L_2(T)^7 \to L_2(T)^7$ is of the form* (16)*, which intertwines $\Delta^{\beta}_{\Omega_1}$ and $\Delta^{\beta}_{\Omega_2}$.*

Proof. Suppose that there does exist such a Ψ, and therefore a Φ associated with some invertible operator $P\colon \mathbb{R}^7 \to \mathbb{R}^7$. By using the polar decomposition of P (see Section 4), we may assume without loss of generality that P and therefore also Φ and Ψ are unitary. By Proposition 7.1, the map Φ satisfies (17) for all $\beta \in \mathbb{R}$ and therefore Ψ intertwines both the Neumann and Dirichlet Laplacians on Ω_1 and Ω_2 by Proposition 7.2. But this contradicts Proposition 7.3. □

It is clear that the same method of proof works not only for more general elliptic operators, but also for all known planar counterexamples, and indeed, should still be true for all pairs of (Dirichlet or Neumann) isospectral domains for which Sunada's principle applies. In particular, there are no known pairs of noncongruent domains for which the Robin Laplacians are isospectral (for any $\beta \neq 0$), and there is no reason to suppose that any known Dirichlet or Neumann counterexamples have this property.

Acknowledgements. The authors wish to thank Moritz Gerlach for providing the pictures. The second named author is most grateful for the hospitality extended to him during a fruitful stay at the University of Ulm. He wishes to thank the University of Ulm for financial support. Part of this work is supported by the Marsden Fund Council from Government funding, administered by the Royal Society of New Zealand. The third named author is supported by a fellowship of the Alexander von Humboldt Foundation, Germany. Part of this work is supported by the EU Marie Curie IRSES program, project 'AOS', No. 318910.

REFERENCES

[1] H. W. ALT, *Lineare Funktionalanalysis*, Springer-Verlag, Berlin etc., 1985.
[2] W. ARENDT, *Does diffusion determine the body?* J. Reine Angew. Math. **550** (2002), 97–123.
[3] W. ARENDT, C. BATTY, M. HIEBER, AND F. NEUBRANDER, *Vector-valued Laplace transforms and Cauchy problems*, vol. 96 of Monographs in Mathematics. Birkhäuser, Basel, 2001.
[4] W. ARENDT, M. BIEGERT, AND A. F. M. TER ELST, *Diffusion determines the manifold*, J. Reine Angew. Math. **667** (2012), 1–25.
[5] W. ARENDT, AND A. F. M. TER ELST, *Diffusion determines the compact manifold*, Banach Center Publ. (2014). In press, arXiv: 1104.1012.
[6] W. ARENDT, AND M. WARMA, *Dirichlet and Neumann boundary conditions: What is in between?*, J. Evol. Equ. **3** (2003), 119–135.

[7] P. AUSCHER, AND P. TCHAMITCHIAN, *Square roots of elliptic second order divergence operators on strongly Lipschitz domains: L^2 theory*, J. Anal. Math. **90** (2003), 1–12.
[8] P. BÉRARD, *Transplantation et isospectralité, I*, Math. Ann. **292** (1992), 547–559.
[9] P. BÉRARD, *Domaines plans isospectraux à la Gordon-Webb-Wolpert: une preuve élémentaire*, Afrika Mat. (3) **1** (1993), 135–146.
[10] P. BUSER, J. CONWAY, P. DOYLE, AND K.-D. SEMMLER, *Some planar isospectral domains*, Internat. Math. Res. Notices **9** (1994), 391–400.
[11] S. J. CHAPMAN, *Drums that sound the same*, Amer. Math. Monthly **102** (1995), 124–138.
[12] C. GORDON, D. L. WEBB, AND S. WOLPERT, *One cannot hear the shape of a drum*, Bull. Amer. Math. Soc. **27** (1992), 134–138.
[13] M. KAC, *Can one hear the shape of a drum?*, Amer. Math. Monthly **73** (1966), 1–23.
[14] A. MCINTOSH, *On the comparability of $A^{1/2}$ and $A^{*1/2}$*, Proc. Amer. Math. Soc. **32** (1972), 430–434.
[15] M. H. PROTTER, *Can one hear the shape of a drum? Revisited*, SIAM Rev. **29** (1987), 185–197.
[16] T. SUNADA, *Riemannian coverings and isospectral manifolds*, Ann. of Math. **121** (1985), 169–186.
[17] H. URAKAWA, *Bounded domains which are isospectral but not congruent*, Ann. Sci. École Norm. Sup. (4) **15** (1982), 441–456.
[18] K. WATANABE, *Plane domains which are spectrally determined*, Ann. Global Anal. Geom. **18** (2000), 447–475.
[19] K. YOSIDA, *Functional Analysis*, Sixth edition, Grundlehren der mathematischen Wissenschaften 123. Springer-Verlag, New York etc., 1980.
[20] S. ZELDITCH, *Spectral determination of analytic bi-axisymmetric plane domains*, Geom. Funct. Anal. **10** (2000), 628–677.
[21] S. ZELDITCH, *Inverse spectral problem for analytic domains, II, $\mathbb{Z}_2$-symmetric domains*, Ann. of Math. **170** (2009), 205–269.

(Received December 29, 2013)

W. Arendt
Institute of Applied Analysis, University of Ulm
Helmholtzstr. 18, D-89069 Ulm, Germany
e-mail: wolfgang.arendt@uni-ulm.de

A. F. M. ter Elst
Department of Mathematics, University of Auckland
Private bag 92019, Auckland 1142, New Zealand
e-mail: terelst@math.auckland.ac.nz

J. B. Kennedy
Institute of Applied Analysis
University of Ulm
Helmholtzstr. 18, D-89069 Ulm, Germany
current address *Institute of Analysis, Dynamics and Modelling*
University of Stuttgart
Pfaffenwaldring 57, D-70569 Stuttgart, Germany
e-mail: james.kennedy@mathematik.uni-stuttgart.de