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ASYMPTOTIC GENERALIZED VALUE DISTRIBUTION OF SOLUTIONS OF THE SCHRÖDINGER EQUATION

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Abstract. The theory of generalized value distribution for boundary values of Herglotz functions is applied to the Weyl-Titchmarsh m-function in Sturm-Liouville theory, and leads to a description of generalized value distribution of the logarithmic derivative $-\frac{v'}{v}$, where v is a basic solution of the Schrödinger equation.

1. Introduction

Given a Lebesgue measurable function f, the distribution of values of f may be described by a mapping $\mathscr{M} : (A,S) \to \mathbb{R}$, defined for a pair of Borel sets A, S by $\mathscr{M}(A,S) = |\lambda \in A : f(\lambda) \in S|$. Here, $|.|$ denotes Lebesgue measure, and $\mathscr{M}(A,S)$ is the Lebesgue measure of the points λ in A such that $f(\lambda)$ is in S.

A case of particular interest is when f is the (real) boundary value function of a Herglotz function F. In this case, the mapping $\mathscr{M}$ is defined in terms of a family of measures $\{\mu_y\}$ ($y \in \mathbb{R}$), corresponding to a family of Herglotz functions F_y generated from F. A theory of value distribution for boundary values of Herglotz functions has been developed in recent years [7]. This theory applies to Herglotz functions quite generally, even when they attain boundary values with strictly positive imaginary part. Of special importance is the case when the Herglotz function F is taken to be the Weyl-Titchmarsh m-function associated with the Schrödinger equation. It is well known that the boundary behaviour of the m-function is closely linked with spectral properties of the corresponding operator. Even in the case that the m-function exhibits highly irregular boundary behaviour, its value distribution may be quite regular, and is therefore an important tool in spectral analysis. Results about the asymptotic value distribution of solutions of the Schrödinger equation have also been obtained, and have been used for spectral analysis [2].

The theory of value distribution for boundary values of Herglotz functions has been generalized [3, 4], in order to allow for a description of value distribution in terms of measures other than Lebesgue measure. An interesting feature of this generalized theory is that it is closely connected with compositions of Herglotz functions. In this paper we apply the generalized theory of value distribution for Herglotz functions, and obtain a result regarding the asymptotic generalized value distribution of solutions of the Schrödinger equation.

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The paper is organized as follows. In Section 2 we state some basic results about Herglotz functions, in particular their integral representation. In Section 3 we define the generalized value distribution of a Herglotz function and present some related results. Finally, in Section 4 we apply this theory to the Weyl-Titchmarsh m-function associated with the Schrödinger equation on the half-line.

2. Herglotz function preliminaries

Let F be a Herglotz function, that is, analytic with positive imaginary part in the upper half-plane $\mathbb{C}_+ = \{z : \mathrm{Im} z > 0\}$. Then, F admits the integral representation [6, 1]

$$F(z) = c_1 + c_2 z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\rho(t), \tag{1}$$

where c_1, c_2 are real constants ($c_2 \geqslant 0$), and the function $\rho(t)$ is non-decreasing, right-continuous, and determined up to an additive constant. For a given Herglotz function F, the constants c_1, c_2 are specified by

$$c_1 = \mathrm{Re} F(i), \qquad c_2 = \lim_{s \to +\infty} \frac{1}{s} \mathrm{Im} F(is).$$

The function $\rho(t)$ gives rise to a measure μ, defined for finite intervals $(a,b]$ by $\mu((a,b]) = \rho(b) - \rho(a)$, and μ extends to Borel sets. The measure μ is referred to as the 'spectral measure' corresponding to the Herglotz function F, and satisfies the condition

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu(t) < \infty, \tag{2}$$

which is sufficient for the integral in (1) to converge absolutely.

The decomposition of μ into an absolutely continuous part $\mu_{a.c.}$, and a singular part μ_s, with respect to Lebesgue measure, is determined by the boundary behaviour of F near the real axis [8]. The boundary value $F_+(\lambda)$ of F at the point $\lambda \in \mathbb{R}$, is defined by $F_+(\lambda) = \lim_{\varepsilon \to 0^+} F(\lambda + i\varepsilon)$, and exists as a finite number Lebesgue almost everywhere. Then, the support of $\mu_{a.c.}$ is the set $\{\lambda \in \mathbb{R} : 0 < \mathrm{Im} F_+(\lambda) < +\infty\}$, and the density function f of $\mu_{a.c.}$ is given by $f(\lambda) = \frac{1}{\pi} \mathrm{Im} F_+(\lambda)$, whereas the support of μ_s is the set $\{\lambda \in \mathbb{R} : \mathrm{Im} F_+(\lambda) = +\infty\}$.

3. Herglotz functions and value distribution

Given a Herglotz function F, we define a one-parameter family of Herglotz functions F_y $(y \in \mathbb{R})$ by

$$F_y(z) = \frac{1}{y - F(z)}. \tag{3}$$

Let $\{\mu_y\}$ be the measures corresponding to F_y through the integral representation (1). The generalized value distribution associated with the Herglotz function F is defined by

$$\nu_S(A) = \int_S \mu_y(A) d\sigma(y), \tag{4}$$

for any Borel sets A, S, where the measure σ corresponds to a Herglotz function ϕ, with integral representation

$$\phi(z) = a_\phi + b_\phi z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\sigma(t). \tag{5}$$

(We note that in the case of the standard theory of value distribution of Herglotz functions, the integral in (4) takes place with respect to Lebesgue measure). In the special case when the boundary values of F are real almost everywhere, then the measures μ_y are purely singular, and we have [3]

$$\nu_S(A) = \nu_S\big(A \cap F_+^{-1}(S)\big) = \nu_{\mathbb{R}}\big(A \cap F_+^{-1}(S)\big). \tag{6}$$

Thus the measure ν_S of the set A is concentrated on the points λ in A at which the boundary value of F is in S, and also it agrees on this set with the measure $\nu_{\mathbb{R}}$ (for which the integral in (4) takes place over $\mathbb{R}$).

The measure ν_S is closely related with compositions of Herglotz functions. For any Borel set B, we have [3]

$$\nu_S(B) = \mu_{(\phi_S \circ F)}(B) - b_\phi \mu(B), \tag{7}$$

where $\mu_{(\phi_S \circ F)}$ is the measure corresponding to the composed Herglotz function $\phi_S \circ F$, and ϕ_S is the Herglotz function having the same representation as ϕ, except that integration takes place over the set S instead of $\mathbb{R}$. Thus, if $b_\phi = 0$, then ν_S is precisely the measure corresponding to the function $\phi_S \circ F$.

A key result in the description of asymptotic value distribution of solutions of the Schrödinger equation, in the case of Lebesgue measure, was an estimate of value distribution for a family of Herglotz functions translated by an increment $i\delta$ off the real axis, defined by

$$F_y^\delta(z) = \frac{1}{y - F(z+i\delta)}, \qquad y \in \mathbb{R}, \;\; z \in \mathbb{C}_+. \tag{8}$$

Let μ_y^δ denote the measures corresponding to the Herglotz functions F_y^δ, and A be a bounded Borel set. Then, we have [2]

$$\left| \int_S \mu_y^\delta(A)dy - \int_S \mu_y(A)dy \right| \leqslant E_A(\delta), \tag{9}$$

where $E_A(\delta)$ is a non-decreasing function of δ, with $\lim_{\delta \to 0^+} E_A(\delta) = 0$. The estimate is uniform for arbitrary Herglotz function F and Borel set S.

An analogous result holds in the case of generalized value distribution. If the measure σ is absolutely continuous, then for any $\varepsilon > 0$ we have

$$\left| \int_S \mu_y^\delta(A)d\sigma(y) - \int_S \mu_y(A)d\sigma(y) \right| \leqslant CE_A(\delta) + \varepsilon, \tag{10}$$

which holds uniformly for all Herglotz functions F such that $F(i)$ lies in a compact set K, and the constant C depends on K and ε. This result is obtained from the relation

$$\int_S \mu_y^\delta(A)d\sigma(y) = \frac{1}{\pi}\int_{\mathbb{R}} \theta(t+i\delta,A)d\nu_s(t), \tag{11}$$

where θ is the angle subtended at the point $z\in\mathbb{C}_+$ by the set S on the real axis, defined by

$$\theta(z,S) = \int_S \operatorname{Im}\left[\frac{1}{t-z}\right]dt. \tag{12}$$

From (11), by considering measures ν_0, ν_1, such that $\mu_{(\phi_S\circ F)}(B) = \nu_0(B)+\nu_1(B)$ for any Borel set B (and hence $\nu_S(B) = \nu_0(B)+\nu_1(B)-b_\phi\mu(B)$ by (7)), where ν_0 is bounded by Lebesgue measure, and ν_1 can be made arbitrarily small, we obtain (10). A detailed proof of (10) will be published elsewhere.

4. Asymptotic value distribution and the Schrödinger equation

We consider the Schrödinger equation on the half-line $0\leqslant x<\infty$, at complex spectral parameter z,

$$-\frac{d^2f(x,z)}{dx^2}+V(x)f(x,z) = zf(x,z), \tag{13}$$

where the potential function V is real-valued and integrable over bounded subintervals of $[0,+\infty)$. We make no special assumptions about V in the limit as $x\to+\infty$. We are assuming the limit-point case at infinity [5], in which case no boundary conditions are required at infinity to define the associated operator $T=-\frac{d^2}{dx^2}+V$ as a self-adjoint operator (with Dirichlet boundary condition at $x=0$).

Let u, v, be solutions of (13) which satisfy at $x=0$, for $z\in\mathbb{C}_+$,

$$\left.\begin{array}{ll} u(0,z)=1 & v(0,z)=0 \\ u'(0,z)=0 & v'(0,z)=1. \end{array}\right\} \tag{14}$$

Then, the Weyl-Titchmarsh m-function is defined by the condition

$$u(.,z)+m(z)v(.,z)\in L^2(0,\infty). \tag{15}$$

The m-function m^N for the truncated interval $[N,\infty)$ is defined in a similar way. If u^N, v^N are solutions of (13), with V defined on the interval $[N,\infty)$, and satisfy the conditions (14) at $x=N$, then m^N is defined by the condition

$$u^N(.,z)+m^N(z)v^N(.,z)\in L^2(N,\infty).$$

In terms of $m(z)$, $m^N(z)$ is given by [2]

$$m^N(z) = \frac{u'(N,z)+m(z)v'(N,z)}{u(N,z)+m(z)v(N,z)}. \tag{16}$$

In Theorem 1 below we give an expression for the asymptotic generalized value distribution of the Herglotz function $-\frac{v'}{v}$, where v is the solution of (13) which satisfies the boundary conditions (14). This expression involves an integral of the generalized angle subtended θ_σ and the boundary values of the m-function m^N; θ_σ is defined by (12) except that, integration takes place with respect to the measure σ instead of Lebesgue measure. If the measure σ is absolutely continuous, and z is restricted on a compact subset of $\mathbb{C}_+$, then $\theta_\sigma(z,S)$ is bounded.

Before we state Theorem 1, we introduce the 'distance of separation' γ, defined for z_1, $z_2 \in \mathbb{C}_+$ by

$$\gamma(z_1,z_2) = \frac{|z_1 - z_2|}{\sqrt{\operatorname{Im} z_1}\sqrt{\operatorname{Im} z_2}}. \tag{17}$$

γ is invariant under Möbius transformations [2]. Also, for any two points z_1, $z_2 \in \mathbb{C}_+$, and Borel set S, the following inequality holds, relating the generalized angle subtended θ_σ and γ:

$$\left|\theta_\sigma(z_1,S) - \theta_\sigma(z_2,S)\right| \leqslant \gamma(z_1,z_2)\sqrt{\theta_\sigma(z_1,S)}\sqrt{\theta_\sigma(z_2,S)}. \tag{18}$$

THEOREM 1. *Suppose that the measure σ is absolutely continuous, with density function h_σ, and let A be a bounded Borel subset of an essential support of the absolutely continuous part $\mu_{a.c.}$ of the spectral measure μ of the Dirichlet Schrödinger operator $T = -\frac{d^2}{dx^2} + V$ acting in $L^2(0,\infty)$. Moreover, we make the following assumptions:*

(i) For any fixed $z \in \mathbb{C}_+$, there exists a compact subset K_z of $\mathbb{C}_+$ such that for all N sufficiently large we have $-\frac{v'(N,z)}{v(N,z)} \in K_z$,

(ii) There exists a compact set K_1 of $\mathbb{C}_+$ such that for all $\lambda \in A$ and N sufficiently large, we have $m_+^N(\lambda) \in K_1$, and $-\frac{v'(N,i)}{v(N,i)} \in K_1$,

(iii) For $z = \lambda + i\delta$, $\lambda \in A$ and any $\delta > 0$ fixed, we have $K_z \subset K_1$.
Then, for any Borel subset S of $\mathbb{R}$ we have

$$\lim_{N\to\infty}\left|\nu_{-S}^N(A) - \frac{1}{\pi}\int_A \theta_{\sigma_r}\left(m_+^N(\lambda),S\right)d\lambda\right| = 0, \tag{19}$$

where the measure ν_S^N is defined by $\nu_S^N(B) = \int_S \mu_y^N(B)\,d\sigma(y)$ for any Borel set B, the measures μ_y^N correspond to the family of Herglotz functions

$$F_y^N(z) = \frac{1}{y + \frac{v'(N,z)}{v(N,z)}}, \qquad y \in \mathbb{R},$$

the set $-S$ is defined by $-S = \{\lambda \in \mathbb{R} : -\lambda \in S\}$, and the measure σ_r has density function h_{σ_r} given by $h_{\sigma_r}(t) = h_\sigma(-t)$.

Proof. We sketch the proof, which is based on the proof of Theorem 1 in [2]. For any positive number $p > 0$, we divide the set A into a finite partition $A = A_0 \cup A_1 \cup$

$\ldots \cup A_n$ of $(n+1)$ disjoint sets such that

$$\gamma\big(m_+(\lambda), m^{(j)}\big) \leqslant p \qquad \text{all } \lambda \in A_j,\ j = 1,2,\ldots,n, \tag{20}$$

where $m^{(j)} = m_+(\lambda_j)$ for some fixed $\lambda_j \in A_j$, and the set A_0 has arbitrarily small measure. From (16) we obtain an expression for the boundary values of m^N, and since γ is invariant under Möbius transformations, (20) implies

$$\gamma\left(m_+^N(\lambda), \frac{u'(N,\lambda) + m^{(j)} v'(N,\lambda)}{u(N,\lambda) + m^{(j)} v(N,\lambda)}\right) \leqslant p, \qquad \text{all } \lambda \in A_j,\ j = 1,2,\ldots,n. \tag{21}$$

Then, by using (18) we obtain an estimate in terms of θ_{σ_r}, and integrating with respect to λ over A_j leads to the bound

$$\left| \frac{1}{\pi}\int_{A_j} \theta_{\sigma_r}\left(m_+^N(\lambda), S\right) d\lambda - \frac{1}{\pi}\int_{A_j} \theta_{\sigma_r}\left(\frac{u'(N,\lambda) + m^{(j)} v'(N,\lambda)}{u(N,\lambda) + m^{(j)} v(N,\lambda)}, S\right) d\lambda \right| \leqslant Cp|A_j|, \tag{22}$$

valid for all $\lambda \in A_j$, $j = 1,\ldots,n$ and $N > 0$ (C is a constant depending on the compact set K_1).

Now, for $j = 1,\ldots,n$, we define the set $A_j^{\delta_0} = \{z : z = \lambda + i\delta_0, \lambda \in A_j\}$, for some $\delta_0 > 0$. We have [2]

$$\gamma\left(-\frac{v'(N,z)}{v(N,z)}, -\frac{u'(N,z) + \overline{m}^{(j)} v'(N,z)}{u(N,z) + \overline{m}^{(j)} v(N,z)}\right) \to 0 \tag{23}$$

uniformly in $\overline{m}^{(j)}$, and for all $z \in A_j^{\delta_0}$, $j = 1,\ldots,n$, as $N \to \infty$. Thus, as before we may obtain an estimate of the generalized angle subtended. We have

$$\left| \int_{A_j} \frac{1}{\pi} \theta_\sigma\left(-\frac{v'(N,\lambda+i\delta_0)}{v(N,\lambda+i\delta_0)}, -S\right) d\lambda \right.$$

$$\left. - \int_{A_j} \frac{1}{\pi} \theta_\sigma\left(-\frac{u'(N,\lambda+i\delta_0) + \overline{m}^{(j)} v'(N,\lambda+i\delta_0)}{u(N,\lambda+i\delta_0) + \overline{m}^{(j)} v(N,\lambda+i\delta_0)}, -S\right) d\lambda \right| \leqslant \frac{1}{\pi} p|A_j| \tag{24}$$

for all $\lambda \in A_j$, $j = 1,\ldots,n$, and N sufficiently large.

Each of the two integrals in (24) is the generalized value distribution (in a different, but equivalent form) of the Herglotz function in the integrand, which is translated by an increment $i\delta_0$ off the real axis. Therefore, in each case we may use (10) to compare the difference between the integrals in (24) with the corresponding integrals in the limit as $\delta \to 0^+$. We can make this difference arbitrarily small by our choice of δ_0. Combining with (22), and adding over all j, then yields (19).

We note the identity $\theta_{\sigma_r}(-\bar{z},-S)=\theta_\sigma(z,S)$, and that

$$\lim_{\delta\to 0^+}\frac{1}{\pi}\int_{A_j}\theta_\sigma\left(-\frac{v'(N,\lambda+i\delta)}{v(N,\lambda+i\delta)},-S\right)d\lambda$$

$$=\lim_{\delta\to 0^+}\int_{A_j}\left\{\frac{1}{\pi}\int_{-S}\mathrm{Im}\left[\frac{1}{y+\frac{v'(N,\lambda+i\delta)}{v(N,\lambda+i\delta)}}\right]d\sigma(y)\right\}d\lambda$$

$$=\lim_{\delta\to 0^+}\int_{-S}\left\{\frac{1}{\pi}\int_{A_j}\mathrm{Im}F_y^N(\lambda+i\delta)d\lambda\right\}d\sigma(y)=\int_{-S}\mu_y^N(A_j)d\sigma(y)=v_{-S}^N(A_j). \quad \square$$

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