ESSENTIAL NORM OF GENERALIZED COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

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Abstract. Upper and lower bounds for the essential norm of generalized composition operators on weighted Hardy spaces are estimated.

1. Introduction

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) be the open unit disk in the complex plane \( \mathbb{C} \), \( \partial \mathbb{D} \) its boundary, \( H(\mathbb{D}) \) the space of all holomorphic functions on \( \mathbb{D} \), and \( H^\infty(\mathbb{D}) \) the space of all bounded analytic functions on \( \mathbb{D} \) with the norm \( \| f \|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \).

For \( a \in \mathbb{D} \), let \( \sigma_a \) be the involutive Möbius transformation of the unit disk, interchanging points \( a \) and 0, that is, \( \sigma_a(z) = (a - z)/(1 - \overline{a}z) \).

Let \( \omega \) be a positive continuous integrable function on \( [0, 1) \). If \( \omega(z) = \omega(|z|) \) for every \( z \in \mathbb{D} \), we call it a weight. We say that a weight \( \omega \) is almost standard if it is non-increasing and such that \( \omega(r)/(1 - r)^{1 + \gamma} \) is non-decreasing for some \( \gamma > 0 \). By \( H_\omega \) we denote the weighted Hardy space consisting of all \( f \in H(\mathbb{D}) \) such that

\[
\| f \|_{H_\omega}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty,
\]

where \( dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} r dr d\theta \) stands for the normalized area measure on \( \mathbb{D} \) (for this and some related spaces see, e.g. [1, 6]). By some calculation we see that a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) belongs to \( H_\omega \) if and only if

\[
\sum_{n=0}^{\infty} \omega_n |a_n|^2 < \infty,
\]

where \( \omega_0 = 1 \) and

\[
\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) dr, \quad n \in \mathbb{N}.
\]

The sequence \( (\omega_n)_{n \in \mathbb{N}_0} \) is called the weight sequence of the weighted Hardy space.


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Let $g \in H(D)$ and $\varphi$ be a holomorphic self-map of $D$. The next operator denoted by $J_{g, \varphi}$ was introduced by S. Li and S. Stević in [8]

$$J_{g, \varphi} f(z) = \int_0^z f'(\varphi(\zeta)) g(\zeta) \, d\zeta, \quad f \in H(D).$$  \hspace{1cm} (1)

It is called the generalized composition operator. The operator $J_{g, \varphi}$ is a generalization of the integral-type operator $J_g$, which is obtained for $\varphi(z) = z$.

When $g(z) = \varphi'(z)$, then $J_{g, \varphi}$ is reduced to the difference of a composition operator and a point evaluation operator, more precisely $J_{\varphi'} = C_{\varphi} - \delta_{\varphi(0)}$. Operator (1) is one of products of linear operators on $H(D)$, which have attracted some attention recently, mainly due to the fact that these kind of operators make a link between classical function theory and operator theory. For some results in the area see, e.g. [2]–[4], [6]–[35] and the references therein. Recall that

$$\beta(a, z) = \frac{1}{2} \log \frac{1 + |\sigma_a(z)|}{1 - |\sigma_a(z)|}$$

is the hyperbolic metric on $D$. Fix $r \in (0, 1)$ and consider the hyperbolic disk or the Bergman disk $D(a, r)$ of radius $r$ and hyperbolic center $a$. That is,

$$D(a, r) = \{ z \in D : \beta(a, z) < r \}, \quad a \in D.$$

It is well known that $D(a, r)$ is a Euclidean disk whose Euclidean center and Euclidean radius are given respectively by

$$\frac{(1 - s^2)a}{(1 - s^2|a|^2)} \quad \text{and} \quad \frac{(1 - |a|^2)s}{(1 - s^2|a|^2)},$$

where $s = \tanh r \in (0, 1)$.

In the following known lemmas (see e.g. [5] or [33]), we recall some useful properties of the hyperbolic disks.

**Lemma 1.** Let $r$ be a fixed positive number. Then for all $a$ and $z$ in $D$ satisfying $\beta(a, z) < r$, we have

$$A(D(a, r)) \asymp 1 - |a|^2 \asymp |1 - \overline{a}z| \asymp 1 - |z|^2,$$

where $A(D(a, r))$ denotes the area of $D(a, r)$.

**Lemma 2.** Let $r \in (0, 1]$ be fixed. Then there exist a positive integer $M$ and a sequence $\{a_j\}$ in $D$ such that:

(a) The disk $D$ is covered by $\{D(a_j, r)\}_{j \in \mathbb{N}}$.

(b) Every point in $D$ belongs to at most $M$ sets in $\{D(a_j, 2r)\}_{j \in \mathbb{N}}$.

(c) If $j \neq m$, then $\beta(a_j, a_m) \geq \frac{r}{2}$. 

In what follows, we make use of Carleson measure techniques, so we give a short introduction to Carleson windows and Carleson measures.

The arcs in the unit circle $\partial \mathbb{D}$ be sets of the form $I = \{z \in \partial \mathbb{D} : \theta_1 \leq \arg z < \theta_2\}$, where $\theta_1, \theta_2 \in [0, 2\pi)$ and $\theta_1 < \theta_2$. Normalized length of an arc $I$ will be denoted by $|I|$, that is,

$$|I| = \frac{1}{2\pi} \int_I |dz|.$$ 

Let $I$ be an arc in $\partial \mathbb{D}$ and let $S(I)$ be the Carleson window defined by

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \leq |z| < 1, z/|z| \in I\}.$$

Let $0 < \alpha < \infty$. Recall that a positive Borel measure $\mu$ on $\mathbb{D}$ is an $\alpha$-Carleson measure if

$$\|\mu\|_\alpha = \sup_{|I| > 0} \frac{\mu(S(I))}{|I|^\alpha} < \infty.$$

A vanishing $\alpha$-Carleson measure is one for which $\mu(S(I)) = o(|I|^\alpha)$ as $|I| \to 0$ uniformly in arcs $I \subset \partial \mathbb{D}$.

In this paper, we continued our work in [29], where we have established Carleson type Theorem for weighted Hardy spaces and characterized the boundedness of operator (1) on weighted Hardy spaces. The following results are proved in [29].

**Theorem 1.** Let $\omega$ be an almost standard weight, $r \in (0, 1)$ fixed and $\mu$ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:

(i) The following quantity is bounded

$$C_1 := \sup_{a \in \mathbb{D}} \frac{\mu(D(a, r))}{\omega(a)(1 - |a|^2)^2};$$

(ii) There is a constant $C_2 > 0$ such that, for every $f \in H_\omega$,

$$\int_\mathbb{D} |f'(z)|^2 d\mu(z) \leq C_2 \|f\|^2_{H_\omega};$$

(iii) The following quantity is bounded

$$C_3 := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^{2+2\gamma}}{\omega(a)|1 - \bar{a}z|^{4+2\gamma}} d\mu(z).$$

Moreover, the following asymptotic relationships hold

$$C_1 \asymp C_2 \asymp C_3.$$ (3)
THEOREM 2. Let \( \omega \) be an almost standard weight, \( g \in H(\mathbb{D}) \) and \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). Then the following statements are equivalent:

(i) \( J_{g,\varphi} \) is bounded on \( H_\omega \).

(ii) The pull-back measure \( \mu_{g,\omega,\varphi} = v_{g,\omega} \circ \varphi^{-1} \) of \( v_{g,\omega} \) induced by \( \varphi \) is an \( \omega \)-Carleson measure, where \( d v_{g,\omega}(z) = |g(z)|^2 \omega(z) dA(z) \).

(iii) \( L := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^{2+2\gamma}}{\omega(a)|1 - \overline{a}z|^{4+2\gamma}} |g(z)|^2 \omega(z) dA(z) < \infty \).

Moreover, if \( J_{g,\varphi} \) is bounded on \( H_\omega \), then

\[
\|J_{g,\varphi}\| \leq L.
\]

The essential norm \( \|T\|_e \) of a bounded linear operator \( T \) on a Banach space \( X \) is given by

\[
\|T\|_e = \inf \{ \|T + K\| : K \text{ is compact on } X \},
\]

i.e., its distance in the operator norm from the space of compact operators on \( X \). The essential norm provides a measure of non-compactness of \( T \). Clearly \( T \) is compact if and only if \( \|T\|_e = 0 \). For some results in the area see, e.g., [4, 13, 15, 17, 21, 24, 28, 30] and the references therein.

Here we estimate the essential norm of the operator \( J_{g,\varphi} \) on weighted Hardy space.

Throughout this paper constants are denoted by \( C \) and they are positive, but not necessarily the same at each occurrence. The notation \( A \asymp B \) means that there is a positive constant \( C \) such that \( B/C \leq A \leq CB \).

2. Essential norm of \( J_{g,\varphi} \) on \( H_w \)

To estimate the essential norm of operator \( J_{g,\varphi} \), we define the next quantity

\[
\|\mu\|_\omega = \sup_{0 < |I| < 1} \frac{\mu(S(I))}{\omega(1 - |I||I|)} < \infty.
\]

The quantity \( \|\mu\|_\omega \) in (4) and constants \( C_1, C_2 \) and \( C_3 \), in Theorem 1 are comparable. Indeed, let \( I \) be arid arc in \( \partial \mathbb{D} \) such that \( 0 < |I| < 1 \) and \( a = (1 - |I|) e^{i\theta} \). Then \( a \in \mathbb{D} \) and \( |a| = 1 - |I| \). Thus

\[
C_3 \geq \int_{\mathbb{D}} \frac{(1 - |a|^2)^{2+2\gamma}}{\omega(a)|1 - \overline{a}z|^{4+2\gamma}} d\mu(z) \geq \int_{S(I)} \frac{(1 - |a|^2)^{2+2\gamma}}{\omega(a)|1 - \overline{a}z|^{4+2\gamma}} d\mu(z).
\]

By (2) and some standard geometric arguments, we can easily obtain that there is an absolute constant \( C > 0 \) such that

\[
\frac{(1 - |a|^2)^{2+2\gamma}}{|1 - \overline{a}z|^{4+2\gamma}} \geq \frac{C}{|I|^2}, \quad z \in S(I).
\]
Thus
\[ C_3 \geq \frac{C}{\left| \omega(1 - |I|)|I|^2 \right|} \int_{S(I)} d\mu(z) = C \frac{\mu(S(I))}{\omega(1 - |I|)|I|^2}. \]

Since \( I \) is an arbitrary arc, we have
\[ \| \mu \|_\omega = \sup_{0 < |I| < 1} \frac{\mu(S(I))}{\omega(1 - |I|)|I|^2} \leq CC_3. \] (5)

Let \( a \in \mathbb{D} \) be arbitrary. For a fixed \( r \in (0, 1) \) there is an arc \( I \) in \( \partial \mathbb{D} \) such that \( 0 < |I| < 1, |I| > 1 - |a| \) and \( D(a, r) \in S(I) \) [3]. Since \( \omega \) is an almost standard weight we get
\[ \frac{\mu(D(a, r))}{\omega(a)(1 - |a|^2)^2} \leq C \sup_{0 < |I| < 1} \frac{\mu(S(I))}{\omega(1 - |I|^2)|I|^2} = C\| \mu \|_\omega. \]

Taking supermum over \( a \in \mathbb{D} \), we have
\[ C_1 = \sup_{a \in \mathbb{D}} \frac{\mu(D(a, r))}{\omega(a)(1 - |a|^2)^2} \leq C\| \mu \|_\omega. \] (6)

Combining (3), (5) and (6), we have that \( \| \mu \|_\omega \) and constants \( C_1, C_2 \) and \( C_3 \), in Theorem 1 are comparable. This settles the claim.

**Definition.** A positive Borel measure \( \mu \) on \( \mathbb{D} \) is called an \( \omega \)-Carleson measure if it satisfies either of the equivalent conditions in Theorem 1 or condition (4).

A positive Borel measure \( \mu \) on \( \mathbb{D} \) is called a vanishing \( \omega \)-Carleson measure if it satisfies the following condition
\[ \lim_{|I| \to 0} \frac{\mu(S(I))}{\omega(1 - |I|)|I|^2} = 0 \quad \text{or equivalently,} \quad \lim_{|a| \to 1} \frac{\mu(D(a, r))}{\omega(a)(1 - |a|^2)^2} = 0. \]

For \( g \in H(\mathbb{D}) \) and \( \varphi \) a holomorphic self-map of \( \mathbb{D} \), define the next quantity
\[ \Lambda_g^\varphi(a) := \int_\mathbb{D} \frac{(1 - |a|^2)^{2+2\gamma}}{\omega(a)|1 - \bar{a}\varphi(z)|^{4+2\gamma}} |g(z)|^2 \omega(z)dA(z). \]

**Theorem 3.** Let \( \omega \) be an almost standard weight, \( g \in H(\mathbb{D}) \) and \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). Let \( J_{g,\varphi} \) be bounded on \( H_\omega \). Then there is an absolute constant \( C \geq 1 \) such that
\[ \limsup_{|a| \to 1} \Lambda_g^\varphi(a) \leq \| J_{g,\varphi} \|_e^2 \leq C \limsup_{|a| \to 1} \Lambda_g^\varphi(a). \]

In order to prove Theorem 3, we need several lemmas. First, we quote an auxiliary result from [6].

**Lemma 3.** Let \( \omega \) be an almost standard weight. Then
\[ \int_\mathbb{D} \frac{\omega(z)}{|1 - \bar{a}z|^{4+2\gamma}}dA(z) \leq \frac{\omega(a)}{(1 - |a|^2)^{2+2\gamma}}. \]
Moreover, if
\[ f_a(z) = \frac{1}{\sqrt{\omega(a)}} \frac{(1 - |a|^2)^{1+\gamma}}{(1 - \overline{a}z)^{1+\gamma}}, \quad (7) \]
then \( \|f_a\|_{H\omega} \leq 1. \)

Lemma 4. Let \( 0 < r < 1, \mathbb{D}(0, r) = \{ z \in \mathbb{D} : |z| < r \} \) and \( \mu \) be a finite positive Borel measures on \( \mathbb{D} \). Set
\[ M_r^\ast(\mu) = M_r^\ast = \sup_{|a| \geq r} \int_{\mathbb{D}} \frac{(1 - |a|^2)^{2+2\gamma}}{\omega(a)|1 - \overline{a}z|^{4+2\gamma}} d\mu(z). \]
Then, if \( \mu \) is an \( \omega \)-Carleson measure for the weighted Hardy space \( H\omega \), so is \( \tilde{\mu}_r = \mu|_{\mathbb{D} \setminus \mathbb{D}(0, r)} \). Moreover,
\[ \|\tilde{\mu}_r\|_\omega \leq NM_r^\ast, \]
where \( N \) is a positive constant.

Proof. Let
\[ M_r = \sup_{0 < |I| \leq 1 - r} \frac{\mu(S(I))}{\omega(1 - |I|)|I|^2}. \]
Let \( I \subset \partial \mathbb{D} \) be a non-degenerate arc. Then \( |I| = \gamma(1 - r) \) for some \( \gamma \in (0, 1/(1 - r)] \). If \( 0 < \gamma \leq 1 \), then \( S(I) \subset \mathbb{D} \setminus \mathbb{D}(0, r) \), and so
\[ \tilde{\mu}_r(S(I)) = \mu(S(I)) \leq M_r \omega(1 - |I|)|I|^2. \]
If \( \gamma > 1 \). Then \( 1 < ([\gamma] + 1)/\gamma \leq 2 \). Let \( m = [\gamma] + 1 \). Then \( I \) can be covered by \( m \) arcs \( I_1, I_2, \ldots, I_m \), such that \( |I_k| = 1 - r, \ k = 1, 2, \ldots, m \). We have
\[ \tilde{\mu}_r(S(I)) = \mu(S(I) \cap (\mathbb{D} \setminus \mathbb{D}(0, r))) \leq \sum_{k=1}^m \mu(S(I_k)) \]
\[ \leq M_r \sum_{k=1}^m \omega(1 - |I_k|)|I_k|^2 = M_r m \omega(1 - |I_1|)|I_1|^2 \]
\[ \leq \frac{4M_r}{m} \omega(1 - |I_1|)|I_1|^2 \leq \frac{4M_r}{m} \omega \left( \frac{1 - |I|}{\gamma} \right)|I|^2 \leq 4M_r \omega(1 - |I|)|I|^2, \]
where in the last inequality we have used the monotonicity of \( \omega(r) \). This implies that \( \|\tilde{\mu}_r\|_\omega \leq 4M_r \), which means that \( \tilde{\mu}_r \) is an \( \omega \)-Carleson measure.

To complete the proof, it is enough to prove that \( M_r \leq NM_r^\ast \) for some \( N > 0 \). Take \( |I| \leq 1 - r \). Let \( a = (1 - |I|)e^{i\theta} \). Then \( |a| = 1 - |I| \geq r \). By using the standard geometric arguments it is easy to see that there is a positive constant \( C \) such that
\[ \frac{(1 - |a|^2)^{2+2\gamma}}{\omega(a)|1 - \overline{a}z|^{4+2\gamma}} \geq \frac{C}{\omega(1 - |I|)|I|^2}. \]
when \( z \in S(I) \) and \( e^{i\theta} \) is the mid point of \( I \). Hence
\[
\frac{\mu(S(I))}{\omega(1-|I|)|I|^2} \leq \frac{1}{C} \int_{S(I)} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\overline{a}z|^{4+2\gamma}} d\mu(z)
\]
\[
\leq \frac{1}{C} \int_{D} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\overline{a}z|^{4+2\gamma}} d\mu(z) \leq \frac{M_r^*}{C}.
\]
(8)

From this and by taking the supremum over all \( I \) with \( 0 < |I| \leq 1-r \), we get
\[
Mr_r \leq \frac{M_r^*}{C},
\]
as desired. □

Let \( R_n \) be the orthogonal projection of \( H_\omega \) onto \( z^n H_\omega \) and \( Q_n = I - R_n \), that is, for \( f = \sum_{k=0}^{\infty} a_k z^k \) in \( H_\omega \), let
\[
(R_nf)(z) = \sum_{k=n}^{\infty} a_k z^k \quad \text{and} \quad (Q_nf)(z) = \sum_{k=0}^{n-1} a_k z^k.
\]

We recall the following lemma, ([3, Proposition 3.15]).

**Lemma 5.** Let \( H_\omega \) be a weighted Hardy space. Then for each \( r \in (0,1) \) and \( f \in H_\omega \)

1. \( |(R_nf)(z)| \leq \|f\|_{H_\omega} \left( \sum_{k=n}^{\infty} \frac{r^{2k}}{w_k} \right)^{1/2} \) for \( |z| \leq r \)

2. \( |(R_nf)'(z)| \leq \|f\|_{H_\omega} \left( \sum_{k=n}^{\infty} k^2 \frac{r^{2(k-1)}}{w_k} \right)^{1/2} \) for \( |z| \leq r \),

where \( w_k = \|z^k\|_{H_\omega}^2, \ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

**Lemma 6.** Let \( H_\omega \) be a weighted Hardy space and \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). Then
\[
\|J_{g,\varphi}\|_e \leq \liminf_{n \to \infty} \|J_{g,\varphi}R_n\|.
\]

**Proof.** Since \( R_n + Q_n = I \) and \( Q_n \) is compact on \( H_\omega \), we have that for each \( n \in \mathbb{N} \)
\[
\|J_{g,\varphi}\|_e = \|J_{g,\varphi}R_n + J_{g,\varphi}Q_n\|_e \leq \|J_{g,\varphi}R_n\|_e \leq \|J_{g,\varphi}R_n\|,
\]
from which inequality (9) follows. □

Now we are in a position to estimate the essential norm of \( J_{g,\varphi} : H_\omega \to H_\omega \), that is, we are in a position to prove Theorem 3.

**Proof of Theorem 3. Upper bound.** By Lemma 6, we have
\[
\|J_{g,\varphi}\|_e^2 \leq \liminf_{n \to \infty} \|J_{g,\varphi}R_n\|_e^2 = \liminf_{n \to \infty} \sup_{\|f\|_{H_\omega} \leq 1} \|(J_{g,\varphi}R_n)f\|_{H_\omega}^2.
\]
Thus
\[
\| (J_{g, \varphi} R_n f) \|_{H_{\omega}}^2 = \int_{\mathbb{D}} |(R_n f)'(\varphi(z))|^2 |g(z)|^2 \omega(z) dA(z)
\]
\[
= \int_{\mathbb{D}} |(R_n f)'(z)|^2 d\mu_{g, \omega, \varphi}(z)
\]
\[
= \left( \int_{\mathbb{D}\setminus \mathbb{D}(0, r)} + \int_{\mathbb{D}(0, r)} \right) |(R_n f)'(z)|^2 d\mu_{g, \omega, \varphi}(z)
\]
\[
= I_1(n) + I_2(n).
\]
Since \( \mu_{g, \omega, \varphi} \) is an \( \omega \)-Carleson measure for the weighted Hardy space \( H_{\omega} \), so by Lemma 5, we have that
\[
I_2(n) \leq \sup_{|z| \leq r} |(R_n f)'(z)|^2 \int_{\mathbb{D}(0, r)} d\mu_{g, \omega, \varphi}(z) \leq C \| f \|_{H_{\omega}}^2 \left( \sum_{k=n}^{\infty} k^2 \frac{2^{(k-1)}}{\omega_k} \right) \to 0,
\]
as \( n \to \infty \). Thus for a fixed \( r \) we have \( \sup_{\| f \|_{H_{\omega}} \leq 1} I_2(n) \to 0, \) as \( n \to \infty \).

On the other hand, if we denote by \( \mu_{g, \omega, \varphi_r} = \mu_{g, \omega, \varphi} \big|_{\mathbb{D} \setminus \mathbb{D}(0, r)} \), then by Theorem 1 (ii) and Lemma 4, we have
\[
I_1(n) = \int_{\mathbb{D}} |(R_n f)'(z)|^2 d\mu_{g, \omega, \varphi_r}(z)
\]
\[
\leq C \| \mu_{g, \omega, \varphi_r} \|_{\omega} \int_{\mathbb{D}} |(R_n f)'(z)|^2 \omega(z) dA(z)
\]
\[
\leq CNM_r^* (\mu_{g, \omega, \varphi}) \| f \|_{H_{\omega}}^2.
\]
Therefore,
\[
\lim_{n \to \infty} \| J_{g, \varphi} R_n \|_c^2 \leq CN \lim_{r \to 1} M_r^* (\mu_{g, \omega, \varphi})
\]
\[
= CN \limsup_{|a| \to 1} \int_{\mathbb{D}} \frac{(1 - |a|^2)^{1+2\gamma}}{\omega(a) |1 - \overline{a} \varphi(z)|^{4+2\gamma}} |g(z)|^2 \omega(z) dA(z),
\]
which gives the desired upper bound.

**Lower bound.** Consider the function \( f_a \) defined as in Lemma 3. Then \( \| f_a \|_{H_{\omega}} \cong 1 \) and \( f_a \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( |a| \to 1 \). Fix a compact operator \( K \) on \( H_{\omega} \). Then \( \| K f_a \|_{H_{\omega}} \to 0 \) as \( |a| \to 1 \) (see [3] for the original idea). Therefore,
\[
\| J_{g, \varphi} + K \| \geq \limsup_{|a| \to 1} \| (J_{g, \varphi} + K) f_a \|_{H_{\omega}}
\]
\[
\geq \limsup_{|a| \to 1} \left( \| J_{g, \varphi} f_a \|_{H_{\omega}} - \| K f_a \|_{H_{\omega}} \right)
\]
\[
= \limsup_{|a| \to 1} \| J_{g, \varphi} f_a \|_{H_{\omega}}.
\]
Thus
\[
\| J_{g, \varphi} \|_c^2 = \inf_{K} \| J_{g, \varphi} + K \|^2 \geq \limsup_{|a| \to 1} \int_{\mathbb{D}} \frac{(1 - |a|^2)^{1+2\gamma}}{\omega(a) |1 - \overline{a} \varphi(z)|^{4+2\gamma}} |g(z)|^2 \omega(z) dA(z). \quad \square
\]
Before we formulate and prove the next corollary, for a Borel measure $\mu$, we define the following Dirichlet-type space

$$\mathcal{D}_\mu(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{D}_\mu}^2 := \int_\mathbb{D} |f'(z)|^2 d\mu(z) < \infty \right\}.$$ 

**COROLLARY 2.** Let $g \in H(\mathbb{D})$ and $\phi$ be a holomorphic self-map of $\mathbb{D}$. Then the following statements are equivalent:

(i) $J_{g,\phi}$ is compact on $H_\omega$.

(ii) The inclusion $i : H_\omega \to \mathcal{D}_{\mu_{g,\omega,\phi}}$ is compact.

(iii) The pull-back measure $\mu_{g,\omega,\phi} = v_{g,\omega} \circ \phi^{-1}$ of $v_{g,\omega}$ induced by $\phi$ is a vanishing $\omega$-Carleson measure.

(iv) $\lim_{|a| \to 1} \int_\mathbb{D} \frac{(1 - |a|^2)^2 + 2\gamma}{|\omega(a)| - \overline{\omega(z)}|z|^2 + 2\gamma} |g(z)|^2 |\omega(z)| dA(z) = 0$.

**Proof.** By definition (i) is equivalent to (ii). Theorem 3 implies that (i) is equivalent to (iv). Applying (8) with $\mu = \mu_{g,\omega,\phi}$ we get that (iv) implies (iii).

(iii) $\Rightarrow$ (ii) Assume $(f_n)_{n \in \mathbb{N}}$ is a bounded sequence in $H_\omega$, say by $L$, such that $f_n \to 0$ on compacta of $\mathbb{D}$ as $n \to \infty$. For an $\varepsilon > 0$ we choose $\rho \in (0,1)$ such that

$$\sup_{|a| > \rho} \frac{\mu_{g,\omega,\phi}(D(a,r))}{\omega(a)(1 - |a|^2)^2} < \varepsilon.$$

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence as in Lemma 2, that is, $(z_n)_{n \in \mathbb{N}}$ is a sequence with a positive separation constant such that $\bigcup_{n=1}^{\infty} D(z_n, r) = \mathbb{D}$ and that every point in $\mathbb{D}$ belongs to at most $M$ sets in the family $\{D(z_n, 2r)\}_{n \in \mathbb{N}}$.

For each $\rho \in (0,1)$ we have

$$\int_\mathbb{D} |f'_n(z)|^2 d\mu_{g,\omega,\phi}(z) = \left( \int_{D(0,\rho)} + \int_{\mathbb{D}\setminus D(0,\rho)} \right) |f'_n(z)|^2 d\mu_{g,\omega,\phi}(z) = J_1(n) + J_2(n).$$

Clearly, for each $\rho \in (0,1)$, we have

$$\lim_{n \to \infty} J_1(n) = 0. \quad (10)$$

Since there are $\rho_1 \in (0,1)$ and $k \in \mathbb{N}$, such that $\bigcup_{n \geq k} D(z_n, r) \subset \mathbb{D}\setminus D(0,\rho_1)$, we have

$$\int_{\mathbb{D}\setminus D(0,\rho_1)} |f'_n(z)|^2 d\mu_{g,\omega,\phi}(z) \leq \sum_{n=k}^{\infty} \int_{D(z_n, r)} |f'_n(z)|^2 d\mu_{g,\omega,\phi}(z) \leq \sum_{n=k}^{\infty} \mu_{g,\omega,\phi}(D(z_n, r)) \sup_{w \in D(z_n, r)} |f'_n(w)|^2 \leq C \sum_{n=k}^{\infty} \omega(z_n)(1 - |z_n|^2)^2 \int_{D(z_n, 3r)} |f'_n(z)|^2 |\omega(z)| dA(z) \leq C \sum_{n=k}^{\infty} \mu_{g,\omega,\phi}(D(z_n, r)).$$
\[ \leq C e \sum_{n=k}^{\infty} \int_{D(z_n,3r)} |f_n'(z)|^2 \omega(z) dA(z) \]
\[ \leq C M e \int_D |f_n'(z)|^2 \omega(z) dA(z) = C M L^2 \varepsilon. \quad (11) \]

From (11) we have that
\[ \limsup_{n \to \infty} \int_{D \setminus B(0,\rho_1)} |f_n'(z)|^2 d\mu_{g,\omega,\varphi}(z) \leq C M L^2 \varepsilon. \quad (12) \]

Since \( \varepsilon > 0 \) is arbitrary from (10) with \( \rho = \rho_1 \) and (12) we get \( \lim_{n \to \infty} \|f_n\|_{D_{g,\omega,\varphi}} = 0 \), from which the compactness of the inclusion \( i : H_\omega \to D_{g,\omega,\varphi} \) follows. \( \square \)

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