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ESSENTIAL NORM OF GENERALIZED COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

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Abstract. Upper and lower bounds for the essential norm of generalized composition operators on weighted Hardy spaces are estimated.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$, and $H^\infty(\mathbb{D})$ the space of all bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

For $a \in \mathbb{D}$, let σ_a be the involutive Möbius transformation of the unit disk, interchanging points a and 0, that is, $\sigma_a(z) = (a-z)/(1-\bar{a}z)$.

Let ω be a positive continuous integrable function on $[0,1)$. If $\omega(z) = \omega(|z|)$ for every $z \in \mathbb{D}$, we call it a *weight*. We say, that a weight ω is *almost standard* if it is non-increasing and such that $\omega(r)/(1-r)^{1+\gamma}$ is non-decreasing for some $\gamma > 0$. By H_ω we denote the weighted Hardy space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{H_\omega} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \omega(z)\, dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} r dr d\theta$ stands for the normalized area measure on $\mathbb{D}$ (for this and some related spaces see, e.g. [1, 6]). By some calculation we see that a function $f(z) = \sum_{n=0}^\infty a_n z^n$ belongs to H_ω if and only if

$$\sum_{n=0}^{\infty} \omega_n |a_n|^2 < \infty,$$

where $\omega_0 = 1$ and

$$\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r)\, dr, \quad n \in \mathbb{N}.$$

The sequence $(\omega_n)_{n\in\mathbb{N}_0}$ is called the *weight sequence* of the weighted Hardy space.

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Let $g \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. The next operator denoted by $J_{g,\varphi}$ was introduced by S. Li and S. Stević in [8]

$$J_{g,\varphi}f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)\,d\zeta, \quad f \in H(\mathbb{D}). \tag{1}$$

It is called the *generalized composition operator*. The operator $J_{g,\varphi}$ is a generalization of the integral-type operator J_g, which is obtained for $\varphi(z) = z$.

When $g(z) = \varphi'(z)$, then $J_{g,\varphi}$ is reduced to the difference of a composition operator and a point evaluation operator, more precisely $J_{\varphi',\varphi} = C_\varphi - \delta_{\varphi(0)}$. Operator (1) is one of products of linear operators on $H(\mathbb{D})$, which have attracted some attention recently, mainly due to the fact that these kind of operators make a link between classical function theory and operator theory. For some results in the area see, e.g. [2]–[4], [6]–[35] and the references therein. Recall that

$$\beta(a,z) = \frac{1}{2}\log\frac{1+|\sigma_a(z)|}{1-|\sigma_a(z)|}$$

is the hyperbolic metric on $\mathbb{D}$. Fix $r \in (0,1)$ and consider the hyperbolic disk or the Bergman disk $D(a,r)$ of radius r and hyperbolic center a. That is,

$$D(a,r) = \{z \in \mathbb{D} : \beta(a,z) < r\},\ a \in \mathbb{D}.$$

It is well known that $D(a,r)$ is a Euclidean disk whose Euclidean center and Euclidean radius are given respectively by

$$\frac{(1-s^2)a}{(1-s^2|a|^2)} \quad \text{and} \quad \frac{(1-|a|^2)s}{(1-s^2|a|^2)},$$

where $s = \tanh r \in (0,1)$.

In the following known lemmas (see e.g. [5] or [33]), we recall some useful properties of the hyperbolic disks.

LEMMA 1. *Let r be a fixed positive number. Then for all a and z in $\mathbb{D}$ satisfying $\beta(a,z) < r$, we have*

$$A(D(a,r)) \asymp 1-|a|^2 \asymp |1-\bar{a}z| \asymp 1-|z|^2, \tag{2}$$

where $A(D(a,r))$ denotes the area of $D(a,r)$.

LEMMA 2. *Let $r \in (0,1]$ be fixed. Then there exist a positive integer M and a sequence $\{a_j\}$ in $\mathbb{D}$ such that:*

(a) The disk $\mathbb{D}$ is covered by $\{D(a_j,r)\}_{j\in\mathbb{N}}$.

(b) Every point in $\mathbb{D}$ belongs to at most M sets in $\{D(a_j,2r)\}_{j\in\mathbb{N}}$.

(c) If $j \neq m$, then $\beta(a_j,a_m) \geqslant \frac{r}{2}$.

In what follows, we make use of Carleson measure techniques, so we give a short introduction to Carleson windows and Carleson measures.

The *arcs* in the unit circle $\partial\mathbb{D}$ be sets of the form $I = \{z \in \partial\mathbb{D} : \theta_1 \leqslant \arg z < \theta_2\}$, where θ_1, $\theta_2 \in [0, 2\pi)$ and $\theta_1 < \theta_2$. Normalized length of an arc I will be denoted by $|I|$, that is,

$$|I| = \frac{1}{2\pi}\int_I |dz|.$$

Let I be an arc in $\partial\mathbb{D}$ and let $S(I)$ be the Carleson window defined by

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \leqslant |z| < 1, z/|z| \in I\}.$$

Let $0 < \alpha < \infty$. Recall that a positive Borel measure μ on $\mathbb{D}$ is an α-Carleson measure if

$$\|\mu\|_\alpha = \sup_{|I|>0} \frac{\mu(S(I))}{|I|^\alpha} < \infty.$$

A vanishing α-Carleson measure is one for which $\mu(S(I)) = o(|I|^\alpha)$ as $|I| \to 0$ uniformly in arcs $I \subset \partial\mathbb{D}$.

In this paper, we continued our work in [29], where we have established Carleson type Theorem for weighted Hardy spaces and characterized the boundedness of operator (1) on weighted Hardy spaces. The following results are proved in [29].

THEOREM 1. *Let ω be an almost standard weight, $r \in (0,1)$ fixed and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) The following quantity is bounded

$$C_1 := \sup_{a\in\mathbb{D}} \frac{\mu(D(a,r))}{\omega(a)(1-|a|^2)^2};$$

(ii) There is a constant $C_2 > 0$ such that, for every $f \in H_\omega$,

$$\int_{\mathbb{D}} |f'(z)|^2 d\mu(z) \leqslant C_2 \|f\|_{H_\omega}^2;$$

(iii) The following quantity is bounded

$$C_3 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}z|^{4+2\gamma}} d\mu(z).$$

Moreover, the following asymptotic relationships hold

$$C_1 \asymp C_2 \asymp C_3. \tag{3}$$

THEOREM 2. *Let ω be an almost standard weight, $g \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following statements are equivalent:*

(i) $J_{g,\varphi}$ *is bounded on* H_ω.

(ii) The pull-back measure $\mu_{g,\omega,\varphi} = \nu_{g,\omega} \circ \varphi^{-1}$ *of* $\nu_{g,\omega}$ *induced by* φ *is an* ω-*Carleson measure, where* $d\nu_{g,\omega}(z) = |g(z)|^2 \omega(z) dA(z)$.

(iii) $L := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}\varphi(z)|^{4+2\gamma}} |g(z)|^2 \omega(z) dA(z) < \infty.$

Moreover, if $J_{g,\varphi}$ *is bounded on* H_ω, *then*

$$\|J_{g,\varphi}\|^2 \asymp L.$$

The essential norm $\|T\|_e$ of a bounded linear operator T on a Banach space X is given by

$$\|T\|_e = \inf\{\|T+K\| : K \text{ is compact on } X\},$$

i.e., its distance in the operator norm from the space of compact operators on X. The essential norm provides a measure of non-compactness of T. Clearly T is compact if and only if $\|T\|_e = 0$. For some results in the area see, e.g. [4, 13, 15, 17, 21, 24, 28, 30] and the references therein.

Here we estimate the essential norm of the operator $J_{g,\varphi}$ on weighted Hardy space.

Throughout this paper constants are denoted by C and they are positive, but not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leqslant A \leqslant CB$.

2. Essential norm of $J_{g,\varphi}$ on H_w

To estimate the essential norm of operator $J_{g,\varphi}$, we define the next quantity

$$\|\mu\|_\omega = \sup_{0<|I|<1} \frac{\mu(S(I))}{\omega(1-|I|)|I|^2} < \infty. \tag{4}$$

The quantity $\|\mu\|_\omega$ in (4) and constants C_1, C_2 and C_3, in Theorem 1 are comparable. Indeed, let I be and arc in $\partial\mathbb{D}$ such that $0 < |I| < 1$ and $a = (1-|I|)e^{i\theta}$. Then $a \in \mathbb{D}$ and $|a| = 1 - |I|$. Thus

$$C_3 \geqslant \int_{\mathbb{D}} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}z|^{4+2\gamma}} d\mu(z) \geqslant \int_{S(I)} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}z|^{4+2\gamma}} d\mu(z).$$

By (2) and some standard geometric arguments, we can easily obtain that there is an absolute constant $C > 0$ such that

$$\frac{(1-|a|^2)^{2+2\gamma}}{|1-\bar{a}z|^{4+2\gamma}} \geqslant \frac{C}{|I|^2}, \quad z \in S(I).$$

Thus

$$C_3 \geqslant \frac{C}{\omega(1-|I|)|I|^2}\int_{S(I)} d\mu(z) = C\frac{\mu(S(I))}{\omega(1-|I|)|I|^2}.$$

Since I is an arbitrary arc, we have

$$\|\mu\|_\omega = \sup_{0<|I|<1}\frac{\mu(S(I))}{\omega(1-|I|)|I|^2} \leqslant CC_3. \tag{5}$$

Let $a\in\mathbb{D}$ be arbitrary. For a fixed $r\in(0,1)$ there is an arc I in $\partial\mathbb{D}$ such that $0<|I|<1$, $|I|\asymp 1-|a|$ and $D(a,r)\in S(I)$ [3]. Since ω is an almost standard weight we get

$$\frac{\mu(D(a,r))}{\omega(a)(1-|a|^2)^2} \leqslant C\sup_{0<|I|<1}\frac{\mu(S(I))}{\omega(1-|I|^2)|I|^2} = C||\mu||_\omega.$$

Taking supermum over $a\in\mathbb{D}$, we have

$$C_1 = \sup_{a\in\mathbb{D}}\frac{\mu(D(a,r))}{\omega(a)(1-|a|^2)^2} \leqslant C||\mu||_\omega. \tag{6}$$

Combining (3), (5) and (6), we have that $\|\mu\|_\omega$ and constants C_1, C_2 and C_3, in Theorem 1 are comparable. This settles the claim.

DEFINITION. A positive Borel measure μ on $\mathbb{D}$ is called an ω-Carleson measure if it satisfies either of the equivalent conditions in Theorem 1 or condition (4).

A positive Borel measure μ on $\mathbb{D}$ is called a vanishing ω-Carleson measure if it satisfies the following condition

$$\lim_{|I|\to 0}\frac{\mu(S(I))}{\omega(1-|I|)|I|^2} = 0 \ \left(\text{or equivalently, } \lim_{|a|\to 1}\frac{\mu(D(a,r))}{\omega(a)(1-|a|^2)^2} = 0\right).$$

For $g\in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$, define the next quantity

$$\Lambda_g^\varphi(a) := \int_{\mathbb{D}}\frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}\varphi(z)|^{4+2\gamma}}|g(z)|^2\omega(z)dA(z).$$

THEOREM 3. *Let ω be an almost standard weight, $g\in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Let $J_{g,\varphi}$ be bounded on H_ω. Then there is an absolute constant $C\geqslant 1$ such that*

$$\limsup_{|a|\to 1}\Lambda_g^\varphi(a) \leqslant \|J_{g,\varphi}\|_e^2 \leqslant C\limsup_{|a|\to 1}\Lambda_g^\varphi(a).$$

In order to prove Theorem 3, we need several lemmas. First, we quote an auxiliary result from [6].

LEMMA 3. *Let ω be an almost standard weight. Then*

$$\int_{\mathbb{D}}\frac{\omega(z)}{|1-\bar{a}z|^{4+2\gamma}}dA(z) \asymp \frac{\omega(a)}{(1-|a|^2)^{2+2\gamma}}.$$

Moreover, if

$$f_a(z) = \frac{1}{\sqrt{\omega(a)}} \frac{(1-|a|^2)^{1+\gamma}}{(1-\bar{a}z)^{1+\gamma}}, \tag{7}$$

then $\|f_a\|_{H_\omega} \asymp 1$.

LEMMA 4. *Let* $0 < r < 1$, $\mathbb{D}(0,r) = \{z \in \mathbb{D} : |z| < r\}$ *and* μ *be a finite positive Borel measures on* $\mathbb{D}$. *Set*

$$M_r^*(\mu) = M_r^* = \sup_{|a| \geqslant r} \int_{\mathbb{D}} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}z|^{4+2\gamma}} d\mu(z).$$

Then, if μ *is an* ω*-Carleson measure for the weighted Hardy space* H_ω, *so is* $\widetilde{\mu}_r = \mu|_{\mathbb{D}\setminus\mathbb{D}(0,r)}$. *Moreover,*

$$\|\widetilde{\mu}_r\|_\omega \leqslant N M_r^*,$$

where N *is a positive constant.*

Proof. Let

$$M_r = \sup_{0<|I|\leqslant 1-r} \frac{\mu(S(I))}{\omega(1-|I|)|I|^2}.$$

Let $I \subset \partial\mathbb{D}$ be a non-degenerate arc. Then $|I| = \gamma(1-r)$ for some $\gamma \in (0, 1/(1-r)]$. If $0 < \gamma \leqslant 1$, then $S(I) \subset \mathbb{D} \setminus \mathbb{D}(0,r)$, and so

$$\widetilde{\mu}_r(S(I)) = \mu(S(I)) \leqslant M_r \omega(1-|I|)|I|^2.$$

If $\gamma > 1$. Then $1 < ([\gamma]+1)/\gamma \leqslant 2$. Let $m = [\gamma]+1$. Then I can be covered by m arcs $I_1, I_2, \ldots, I_m$, such that $|I_k| = 1-r$, $k = 1, 2, \ldots, m$. We have

$$\begin{aligned}
\widetilde{\mu}_r(S(I)) &= \mu(S(I) \cap (\mathbb{D} \setminus \mathbb{D}(0,r))) \leqslant \sum_{k=1}^m \mu(S(I_k)) \\
&\leqslant M_r \sum_{k=1}^m \omega(1-|I_k|)|I_k|^2 = M_r m \omega(1-|I_1|)|I_1|^2 \\
&\leqslant \frac{4M_r}{m} \omega(1-|I_1|)|I|^2 = \frac{4M_r}{m} \omega\Big(1 - \frac{|I|}{\gamma}\Big)|I|^2 \leqslant 4M_r \omega(1-|I|)|I|^2,
\end{aligned}$$

where in the last inequality we have used the monotonicity of $\omega(r)$. This implies that $\|\widetilde{\mu}_r\|_\omega \leqslant 4M_r$, which means that $\tilde{\mu}_r$ is an ω-Carleson measure.

To complete the proof, it is enough to prove that $M_r \leqslant N M_r^*$ for some $N > 0$. Take $|I| \leqslant 1-r$. Let $a = (1-|I|)e^{i\theta}$. Then $|a| = 1-|I| \geqslant r$. By using the standard geometric arguments it is easy to see that there is a positive constant C such that

$$\frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}z|^{4+2\gamma}} \geqslant \frac{C}{\omega(1-|I|)|I|^2},$$

when $z \in S(I)$ and $e^{i\theta}$ is the mid point of I. Hence

$$\begin{aligned}\frac{\mu(S(I))}{\omega(1-|I|)|I|^2} &\leqslant \frac{1}{C}\int_{S(I)} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}z|^{4+2\gamma}}d\mu(z)\\ &\leqslant \frac{1}{C}\int_{\mathbb{D}} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}z|^{4+2\gamma}}d\mu(z) \leqslant \frac{M_r^*}{C}. \end{aligned} \tag{8}$$

From this and by taking the supremum over all I with $0<|I|\leqslant 1-r$, we get $M_r \leqslant M_r^*/C$, as desired. □

Let R_n be the orthogonal projection of H_ω onto $z^n H_\omega$ and $Q_n = I - R_n$, that is, for $f=\sum_{k=0}^{\infty} a_k z^k$ in H_ω, let

$$(R_n f)(z) = \sum_{k=n}^{\infty} a_k z^k \quad \text{and} \quad (Q_n f)(z) = \sum_{k=0}^{n-1} a_k z^k.$$

We recall the following lemma, ([3, Proposition 3.15]).

LEMMA 5. *Let H_w be a weighted Hardy space. Then for each $r \in (0,1)$ and $f \in H_w$*

1. $|(R_n f)(z)| \leqslant \|f\|_{H_w}\Big(\sum_{k=n}^{\infty} \frac{r^{2k}}{w_k}\Big)^{1/2}$ *for* $|z|\leqslant r$

2. $|(R_n f)'(z)| \leqslant \|f\|_{H_w}\Big(\sum_{k=n}^{\infty} k^2\frac{r^{2(k-1)}}{w_k}\Big)^{1/2}$ *for* $|z|\leqslant r$,

where $w_k = \|z^k\|^2_{H_\omega}$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

LEMMA 6. *Let H_w be a weighted Hardy space and φ be a holomorphic self-map of $\mathbb{D}$. Then*

$$\|J_{g,\varphi}\|_e \leqslant \liminf_{n\to\infty} \|J_{g,\varphi}R_n\|. \tag{9}$$

Proof. Since $R_n + Q_n = I$ and Q_n is compact on H_ω, we have that for each $n \in \mathbb{N}$

$$\|J_{g,\varphi}\|_e = \|J_{g,\varphi}R_n + J_{g,\varphi}Q_n\|_e \leqslant \|J_{g,\varphi}R_n\|_e \leqslant \|J_{g,\varphi}R_n\|,$$

from which inequality (9) follows. □

Now we are in a position to estimate the essential norm of $J_{g,\varphi}: H_\omega \to H_\omega$, that is, we are in a position to prove Theorem 3.

Proof of Theorem 3. Upper bound. By Lemma 6, we have

$$\|J_{g,\varphi}\|_e^2 \leqslant \liminf_{n\to\infty} \|J_{g,\varphi}R_n\|_e^2 = \liminf_{n\to\infty} \sup_{\|f\|_{H_\omega}\leqslant 1} \|(J_{g,\varphi}R_n)f\|^2_{H_\omega}.$$

Thus

$$\begin{aligned}\|(J_{g,\varphi}R_n)f\|_{H_\omega}^2 &= \int_{\mathbb{D}} |(R_nf)'(\varphi(z))|^2|g(z)|^2\omega(z)dA(z)\\ &= \int_{\mathbb{D}} |(R_nf)'(z)|^2 d\mu_{g,\omega,\varphi}(z)\\ &= \left(\int_{\mathbb{D}\setminus\mathbb{D}(0,r)} + \int_{\mathbb{D}(0,r)}\right)|(R_nf)'(z)|^2 d\mu_{g,\omega,\varphi}(z)\\ &= I_1(n)+I_2(n).\end{aligned}$$

Since $\mu_{g,\omega,\varphi}$ is an ω-Carleson measure for the weighted Hardy space H_ω, so by Lemma 5, we have that

$$I_2(n) \leqslant \sup_{|z|\leqslant r}|(R_nf)'(z)|^2\int_{\mathbb{D}(0,r)} d\mu_{g,\omega,\varphi}(z) \leqslant C\|f\|_{H_\omega}^2\left(\sum_{k=n}^{\infty}k^2\frac{r^{2(k-1)}}{\omega_k}\right) \to 0,$$

as $n\to\infty$. Thus for a fixed r we have $\sup_{\|f\|_{H_\omega}\leqslant 1} I_2(n)\to 0$, as $n\to\infty$.

On the other hand, if we denote by $\mu_{g,\omega,\varphi_r} = \mu_{g,\omega,\varphi}|_{\mathbb{D}\setminus\mathbb{D}(0,r)}$, then by Theorem 1 *(ii)* and Lemma 4, we have

$$\begin{aligned}I_1(n) &= \int_{\mathbb{D}} |(R_nf)'(z)|^2 d\mu_{g,\omega,\varphi_r}(z)\\ &\leqslant C\|\mu_{g,\omega,\varphi_r}\|_\omega \int_{\mathbb{D}} |(R_nf)'(z)|^2\omega(z)dA(z)\\ &\leqslant CNM_r^*(\mu_{g,\omega,\varphi})\|f\|_{H_\omega}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\|J_{g,\varphi}R_n\|_e^2 &\leqslant CN\lim_{r\to 1}M_r^*(\mu_{g,\omega,\varphi})\\ &= CN\limsup_{|a|\to 1}\int_{\mathbb{D}}\frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}\varphi(z)|^{4+2\gamma}}|g(z)|^2\omega(z)dA(z),\end{aligned}$$

which gives the desired upper bound.

Lower bound. Consider the function f_a defined as in Lemma 3. Then $\|f_a\|_{H_\omega}\asymp 1$ and $f_a\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a|\to 1$. Fix a compact operator K on H_ω. Then $\|Kf_a\|_{H_\omega}\to 0$ as $|a|\to 1$ (see [3] for the original idea). Therefore,

$$\begin{aligned}\|J_{g,\varphi}+K\| &\geqslant \limsup_{|a|\to 1}\|(J_{g,\varphi}+K)f_a\|_{H_\omega}\\ &\geqslant \limsup_{|a|\to 1}\left(\|J_{g,\varphi}f_a\|_{H_\omega}-\|Kf_a\|_{H_\omega}\right)\\ &= \limsup_{|a|\to 1}\|J_{g,\varphi}f_a\|_{H_\omega}.\end{aligned}$$

Thus

$$\|J_{g,\varphi}\|_e^2 = \inf_K\|J_{g,\varphi}+K\|^2 \geqslant \limsup_{|a|\to 1}\int_{\mathbb{D}}\frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}\varphi(z)|^{4+2\gamma}}|g(z)|^2\omega(z)dA(z). \qquad \square$$

Before we formulate and prove the next corollary, for a Borel measure μ, we define the following Dirichlet-type space

$$\mathfrak{D}_\mu(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|^2_{\mathfrak{D}_\mu} := \int_{\mathbb{D}} |f'(z)|^2 d\mu(z) < \infty \right\}.$$

COROLLARY 2. *Let $g \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following statements are equivalent:*

(i) *$J_{g,\varphi}$ is compact on H_ω.*

(ii) *The inclusion $i : H_\omega \to \mathfrak{D}_{\mu_{g,\omega,\varphi}}$ is compact.*

(iii) *The pull-back measure $\mu_{g,\omega,\varphi} = \nu_{g,\omega} \circ \varphi^{-1}$ of $\nu_{g,\omega}$ induced by φ is a vanishing ω-Carleson measure.*

(iv) $\displaystyle \lim_{|a|\to 1} \int_{\mathbb{D}} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\bar{a}\varphi(z)|^{4+2\gamma}} |g(z)|^2 \omega(z) dA(z) = 0.$

Proof. By definition *(i)* is equivalent to *(ii)*. Theorem 3 implies that *(i)* is equivalent to *(iv)*. Applying (8) with $\mu = \mu_{g,\omega,\varphi}$ we get that *(iv)* implies *(iii)*.

(iii)⇒(ii) Assume $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in H_ω, say by L, such that $f_n \to 0$ on compacta of $\mathbb{D}$ as $n \to \infty$. For an $\varepsilon > 0$ we choose $\rho \in (0,1)$ such that

$$\sup_{|a|>\rho} \frac{\mu_{g,\omega,\varphi}(D(a,r))}{\omega(a)(1-|a|^2)^2} < \varepsilon.$$

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence as in Lemma 2, that is, $(z_n)_{n\in\mathbb{N}}$ is a sequence with a positive separation constant such that $\cup_{n=1}^{\infty} D(z_n,r) = \mathbb{D}$ and that every point in $\mathbb{D}$ belongs to at most M sets in the family $\{D(z_n,2r)\}_{n\in\mathbb{N}}$.

For each $\rho \in (0,1)$ we have

$$\int_{\mathbb{D}} |f_n'(z)|^2 d\mu_{g,\omega,\varphi}(z) = \left(\int_{\mathbb{D}(0,\rho)} + \int_{\mathbb{D}\setminus\mathbb{D}(0,\rho)} \right) |f_n'(z)|^2 d\mu_{g,\omega,\varphi}(z) = J_1(n) + J_2(n).$$

Clearly, for each $\rho \in (0,1)$, we have

$$\lim_{n\to\infty} J_1(n) = 0. \tag{10}$$

Since there are $\rho_1 \in (0,1)$ and $k \in \mathbb{N}$, such that $\cup_{n\geqslant k} D(z_n,r) \subseteq \mathbb{D} \setminus \mathbb{D}(0,\rho_1)$, we have

$$\begin{aligned}
\int_{\mathbb{D}\setminus\mathbb{D}(0,\rho_1)} |f_n'(z)|^2 d\mu_{g,\omega,\varphi}(z) &\leqslant \sum_{n=k}^{\infty} \int_{D(z_n,r)} |f_n'(z)|^2 d\mu_{g,\omega,\varphi}(z) \\
&\leqslant \sum_{n=k}^{\infty} \mu_{g,\omega,\varphi}(D(z_n,r)) \sup_{w\in D(z_n,r)} |f_n'(w)|^2 \\
&\leqslant C \sum_{n=k}^{\infty} \frac{\mu_{g,\omega,\varphi}(D(z_n,r))}{\omega(z_n)(1-|z_n|^2)^2} \int_{D(z_n,3r)} |f_n'(z)|^2 \omega(z) dA(z)
\end{aligned}$$

$$\leqslant C\varepsilon \sum_{n=k}^{\infty} \int_{D(z_n,3r)} |f_n'(z)|^2 \omega(z) dA(z)$$
$$\leqslant CM\varepsilon \int_{\mathbb{D}} |f_n'(z)|^2 \omega(z) dA(z) = CML^2\varepsilon. \tag{11}$$

From (11) we have that

$$\limsup_{n\to\infty} \int_{\mathbb{D}\setminus\mathbb{D}(0,\rho_1)} |f_n'(z)|^2 d\mu_{g,\omega,\varphi}(z) \leqslant CML^2\varepsilon. \tag{12}$$

Since $\varepsilon > 0$ is arbitrary from (10) with $\rho = \rho_1$ and (12) we get $\lim_{n\to\infty} \|f_n\|_{\mathfrak{D}_{\mu_{g,\omega,\varphi}}} = 0$, from which the compactness of the inclusion $i : H_\omega \to \mathfrak{D}_{\mu_{g,\omega,\varphi}}$ follows. $\square$

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