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ON THE LOCAL DIFFEOMORPHISM AND SUBMANIFOLDS OF MATRICES WITH FIXED JORDAN BLOCK STRUCTURE

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Abstract. In this paper a local diffeomorphism, which allows to investigate different matrix submanifolds is introduced. With its help the submanifold of complex matrices with fixed structure of Jordan blocks is described. Then, explicit formulas of some special perturbations of Jordan structure are given.

Introduction

The submanifold of real symmetric matrices with fixed multiplicities of chosen eigenvalues for the first time was considered by V.I. Arnold [1]. He found a formula for codimension of this submanifold. The results of Arnold were generalized for the case of compact real self-adjoint operators in [2], where a special local diffeomorphism that takes Arnold's submanifold to a flat subspace was introduced. One of the authors (Ya. Dymarskii) developed the aforementioned results in [3], [4]. In the self-adjoint case, it was found that to track some (not all) of the eigenvalues is more complicated than to track all them. This is because we cannot use the full orbit of the matrix. In this paper we will study the subset of complex matrices with fixed structure of Jordan blocks associated with some (not all) chosen eigenvalues. The main purpose of this paper is to study the smooth structure of this subset. Similarly to the paper [2], we define a map (local diffeomorphism) Ψ in the space of matrices. This map locally parameterizes the submanifold of interest. The map has the following remarkable property: its domain is naturally decomposed into the direct sum of two subspaces which are invariant under linearization of the diffeomorphism. The first subspace "controls" the part of the spectrum we are interested in, while the second "controls" the corresponding invariant subspaces. This property allows to locally parameterize various subspaces of matrices generated by the given matrix such as its orbit, the bundle of matrices introduced in [5], and the submanifolds of matrices with fixed structure of Jordan blocks associated with some chosen eigenvalues. The domain decomposition is also based on the results of V. I. Arnold: the first subspace is the "versal family" of the matrix [5], the second is the tangent space of the orbit of the same matrix. The aforementioned invariance of the subspaces generates yet another remarkable property of the diffeomorphism: it locally maps the tangent space of the submanifold of interest on the submanifold itself.

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This paper is organized as follows. In the first section we introduce the basic notations and review some necessary facts from the theory of orbits and some families of matrices. In the second section the diffeomorphism Ψ is introduced and studied. The third section describes the set of matrices with a single eigenvalue that is associated with a fixed structure of Jordan blocks. In the fourth section we prove the main theorem about the submanifold of complex matrices with fixed structure of Jordan blocks associated with the chosen eigenvalue. In the fifth section we generalize last theorem to the case of several eigenvalues. Finally, in the last section we describe some one-parameter perturbations of a given matrix.

1. Notations and definitions

By $\mathbb{C}^{n\times n}$ we denote the space of n-by-n complex matrices with Hermitian inner product $(A,B) := Tr(A\cdot B^*)$, where B^* is the adjoint of B. Let J be a Jordan matrix with spectrum σ and $\lambda \in \sigma$. Then $J = J(\lambda)\oplus J(\sigma\setminus\lambda)$, where $J(\lambda)$ is the direct sum of Jordan blocks with the eigenvalue λ and $J(\sigma\setminus\lambda)$ is the direct sum of Jordan blocks with the other eigenvalues. Let m be the algebraic multiplicity of λ, s the geometric multiplicity of λ, and let $\overline{k} := \{k_1,...,k_s\}$ be the Segré characteristic at λ. We denote by $N_\varepsilon(J) \subset \mathbb{C}^{n\times n}$ an ε-neighborhood of J. We will study the set $N\left(J,\varepsilon,\lambda,\overline{k}\right)$ of matrices $A \in N_\varepsilon(J)$ such that A has a single eigenvalue λ' that is close to λ, and the corresponding Jordan blocks $J(A,\lambda')$ have the same structure as the blocks $J(\lambda)$.

Let P_λ be the Riesz projection operator that is associated with λ, and $P_{\sigma\setminus\lambda}$ the Riesz projection operator that is associated with $\sigma\setminus\lambda$. Space $\mathbb{C}^n$ can be represented in the form $\mathbb{C}^n = \mathbb{C}^m\oplus\mathbb{C}^{n-m}$, where summands are invariant subspaces; $\mathbb{C}^m = P_\lambda(\mathbb{C}^n)$ corresponds to the eigenvalue λ, $\mathbb{C}^{n-m} = P_{\sigma\setminus\lambda}(\mathbb{C}^n)$ corresponds to the rest of the spectrum σ. This decomposition generates the decomposition of the matrix space: an arbitrary matrix $B \in \mathbb{C}^{n\times n}$ has the block representation

$$B = \begin{bmatrix} B_{\mathbf{1,1}} & B_{\mathbf{1,2}} \\ B_{\mathbf{2,1}} & B_{\mathbf{2,2}} \end{bmatrix}, \tag{1}$$

where $B_{\mathbf{1,1}} \in M := \mathbb{C}^{m\times m}$, $B_{\mathbf{1,2}} \in \mathbb{C}^{m\times(n-m)}$, $B_{\mathbf{2,1}} \in \mathbb{C}^{(n-m)\times m}$, $B_{\mathbf{2,2}} \in \mathbb{C}^{(n-m)\times(n-m)}$. The structure of the Jordan blocks $J(\lambda)$ generates the block representation

$$B_{\mathbf{1,1}} = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1s} \\ B_{21} & B_{22} & \dots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s1} & B_{s2} & \dots & B_{ss} \end{bmatrix}. \tag{2}$$

It is well known [5] that the Lie group $GL(m,\mathbb{C})$ of all nonsingular $m\times m$ matrices acts on the linear manifold M by the formula

$$Ad_G : M \to M,\ Ad_G(J) := GJG^{-1} \quad (G \in GL(m,\mathbb{C})).$$

By $\mathscr{O}(A)$ we denote the orbit of an arbitrary fixed matrix $A \in M$ under the action of $GL(m,\mathbb{C})$. An orbit is a smooth submanifold that consists of all matrices similar to A.

In [5] V.I. Arnold introduced the special family of matrices (Arnold's family in the following) that differs from the original Jordan form in that some entries instead of being zero are holomorphic functions of parameters that vanish when the value of the parameters is zero. More specifically, a matrix from the Arnold's family is $A = J(\lambda) + B_{\mathbf{1,1}}$, where $B_{\mathbf{1,1}}$ has the form described in Figure 1: each oblique segment denotes a sequence of identical numbers, and the blank entries denote zero. Also in [5], it's proved that all matrices $J(\lambda) + B_{\mathbf{1,1}}$ (sufficiently close to $J(\lambda)$) can be simultaneously reduced by the transformation

$$J(\lambda) + B_{\mathbf{1,1}} \to C(B_{\mathbf{1,1}})^{-1}(J(\lambda) + B_{\mathbf{1,1}})C(B_{\mathbf{1,1}}) = J(\lambda) + B_{\mathbf{1,1}}, \tag{3}$$

where $C(B_{\mathbf{1,1}})$ is holomorphic at zero, $C(0) = I$, and $B_{\mathbf{1,1}}$ belongs to a submanifold, transversal to the centralizer and of complementary dimension (equal to the dimension of the orbit).

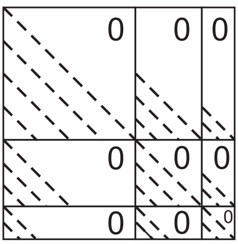

Figure 1: Matrix from $M_{\overline{k}}$

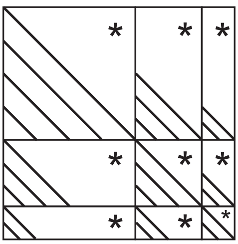

Figure 2: Matrix from tangent space

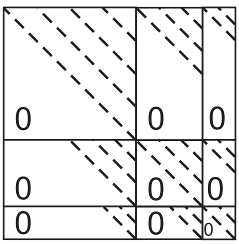

Figure 3: Solution of AX=XB

The set of all matrices in Fig. 1 forms a linear subspace $M_{\overline{k}} \subset M$ of dimension

$$\dim(M_{\overline{k}}) = k_1 + 3k_2 + \ldots + (2s-1)k_s. \tag{4}$$

This subspace is orthogonal and transversal to the tangent space $T_{J(\lambda)}\mathscr{O}(J(\lambda))$ to the orbit of $J(\lambda)$ at $J(\lambda)$. By

$$Fam := \left\{J(\lambda) + B_{\mathbf{1,1}} \mid B_{\mathbf{1,1}} \in M_{\overline{k}}\right\}$$

we denote the Arnold's family.

As has been shown in [5], the tangent vectors to the orbit of $J(\lambda)$ at $J(\lambda)$ are the matrices that can be represented in the form $[C, J(\lambda)] := C \cdot J(\lambda) - J(\lambda) \cdot C$. We claim that any matrix $S \in T_{J(\lambda)}\mathscr{O}(J(\lambda))$ has the form described in Figure 2: the sum of all entries in each line is equal to zero, all other entries $(*)$ are arbitrary. First, from the form of matrices S (see Figure 2), it is obvious that

$$\operatorname{codim}(T_{J(\lambda)}\mathscr{O}(J(\lambda))) = \dim(M_{\overline{k}}). \tag{5}$$

Second, matrices of aforementioned form are orthogonal to matrices from the subspace $M_{\overline{k}}$. We finally note that $M_{\overline{k}}$ is orthogonal and transversal to the tangent space $T_{J(\lambda)}\mathscr{O}(J(\lambda))$ to orbit of $J(\lambda)$ at $J(\lambda)$.

Thus, M is a direct sum of the orthogonal subspaces: $M = M_{\overline{k}} \oplus T_{J(\lambda)}\mathscr{O}(J(\lambda))$; an arbitrary matrix $B_{\mathbf{1,1}} \in M$ has the unique representation

$$B_{\mathbf{1,1}} = F(B_{\mathbf{1,1}}) + S(B_{\mathbf{1,1}}), \tag{6}$$

where

$$F : M \to M_{\overline{k}}, \;\; S : M \to T_{J(\lambda)}\mathscr{O}(J(\lambda))$$

are the orthogonal projections.

2. Local diffeomorphism

Decompositions (1) and (6) define the linear maps

$$\widehat{F},\, \widehat{S} :\; \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n},$$

$$\widehat{F}(B) := \begin{bmatrix} F(B_{\mathbf{1,1}}) & 0 \\ 0 & B_{\mathbf{2,2}} \end{bmatrix},\; \widehat{S}(B) := \begin{bmatrix} S(B_{\mathbf{1,1}}) & B_{\mathbf{1,2}} \\ B_{\mathbf{2,1}} & 0 \end{bmatrix}.$$

Consider now the map on a small neighborhood of zero $N(0)$:

$$\Psi :\; N(0) \subset \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n},$$

$$\Psi(B) := \left(I + \left(\widehat{S}(B)\right)^T\right)\left(J + \widehat{F}(B)\right)\left(I + \left(\widehat{S}(B)\right)^T\right)^{-1}. \tag{7}$$

THEOREM 1. *The following statements hold.*

1. $\Psi(0) = J$.
2. *Matrices* $J + \widehat{F}(B)$ *and* $\Psi(B)$ *are similar.*
3. *The map* Ψ *is holomorphic at zero.*
4. *The map* Ψ *is a diffeomorphism of a sufficiently small neighborhood* $N(0)$ *onto some neighborhood* $N_\varepsilon(J)$.

Proof. The first and the second statements follow from formula (7).

From formula (7) and linearity of the maps $\widehat{S}$ and $\widehat{F}$ it follows that Ψ is an analytic map.

To prove the fourth statement, we find the derivative $D\Psi(0)$ and prove that it is a linear isomorphism. Expanding $(I + (\widehat{S}(B))^T)^{-1}$ in Taylor series and separating the linear part, we obtain

$$\begin{gathered} \Psi(B) = (I + (\widehat{S}(B))^T)(J + \widehat{F}(B))(I + (\widehat{S}(B))^T)^{-1} = \\ = (I + (\widehat{S}(B))^T)(J + \widehat{F}(B))(I - (\widehat{S}(B))^T + ((\widehat{S}(B))^T)^2 - \ldots) = \\ = J + \widehat{F}(B) + (\widehat{S}(B))^T \cdot J - J \cdot (\widehat{S}(B))^T + o(B), \end{gathered}$$

where $o(B)$ is second-order infinitesimal. Therefore,

$$D\Psi(0)B = \widehat{F}(B) + \left[(\widehat{S}(B))^T, J\right] = \tag{8}$$

$$= \begin{bmatrix} F(B_{\mathbf{1,1}}) + \left[(S(B_{\mathbf{1,1}}))^T, J(\lambda)\right] & B_{\mathbf{2,1}}^T \cdot J(\sigma \setminus \lambda) - J(\lambda) \cdot B_{\mathbf{2,1}}^T \\ J(\sigma \setminus \lambda) \cdot B_{\mathbf{1,2}}^T - B_{\mathbf{1,2}}^T \cdot J(\lambda) & B_{\mathbf{2,2}} \end{bmatrix}.$$

Let us show that the map $D\Psi(0)$ is a bijection. To this end it is necessary and sufficient that the equation

$$D\Psi(0)B = 0 \tag{9}$$

has the unique trivial solution $B = 0$. Equation (9) is equivalent to the system of matrix equations:

$$\begin{cases} F(B_{\mathbf{1,1}}) + \left[(S(B_{\mathbf{1,1}}))^T, J(\lambda)\right] = 0 \in M, \\ B_{\mathbf{2,2}} = 0 & \in \mathbb{C}^{(n-m)\times(n-m)}, \\ B_{\mathbf{2,1}}^T \cdot J(\sigma \setminus \lambda) - J(\lambda) \cdot B_{\mathbf{2,1}}^T = 0 & \in \mathbb{C}^{m\times(n-m)}, \\ J(\sigma \setminus \lambda) \cdot B_{\mathbf{1,2}}^T - B_{\mathbf{1,2}}^T \cdot J(\lambda) = 0 & \in \mathbb{C}^{(n-m)\times m}. \end{cases} \tag{10}$$

Note that the equations are independent and each of them contains one of the blocks of decomposition (1) as an unknown. The last two equations are Sylvester equations. Since λ is not an eigenvalue of matrix $J(\sigma \setminus \lambda)$, then these equations have only trivial solutions [7]. It remains to show that the first equation has the unique trivial solution. Since

$$\begin{aligned} &F(B_{\mathbf{1,1}}) \in M_{\overline{k}}, \\ &\left[(S(B_{\mathbf{1,1}}))^T, J(\lambda)\right] \in T_{J(\lambda)}\mathscr{O}(J(\lambda)), \\ &M_{\overline{k}} \perp T_{J(\lambda)}\mathscr{O}(J(\lambda)) \end{aligned}$$

the first equation of system (10) is equivalent to the system

$$\begin{cases} F(B_{\mathbf{1,1}}) = 0 \in M_{\overline{k}}, \\ \left[(S(B_{\mathbf{1,1}}))^T, J(\lambda)\right] = 0 \in T_{J(\lambda)}\mathscr{O}(J(\lambda)). \end{cases} \tag{11}$$

From the first equation of system (11) and equality (6) we obtain that $B_{\mathbf{1,1}} = S(B_{\mathbf{1,1}})$. From Lemma in [5, §4], we have that $\left[(S(B_{\mathbf{1,1}}))^T, J(\lambda)\right] = 0$ if and only if $S(B_{\mathbf{1,1}})$ is perpendicular to the orbit of $J(\lambda)$. But, since $S(B_{\mathbf{1,1}}) \in T_{J(\lambda)}\mathscr{O}(J(\lambda))$, it must be $S(B_{\mathbf{1,1}}) = 0$. So the first equation of system (10) has only the trivial solution $B_{\mathbf{1,1}} = 0$. □

Note, that the map Ψ has the properties of exponential mapping at identity element of a group, i.e. takes tangent space to the submanifolds on submanifolds. Therefore, the following lemma is true.

LEMMA 1. *The map Ψ locally takes the tangent space $T_{J(\lambda)}\mathscr{O}(J(\lambda))$ on $\mathscr{O}(J(\lambda))$.*

Note that the restriction of the mapping (7) on M is a particular case of Arnold's transformation (3), and the definition of the matrix $S(B_{\mathbf{1,1}})$ is analogously to [2]. We will see the advantages of definition (7) in Theorems 2-4.

3. Case of a single eigenvalue

In this section $n=m$. By $V^m(Jor,\overline{k})\subset M$ we denote the subset of m-dimensional Jordan matrices $J(\lambda')$ with the same structure as the block $J(\lambda)$. Obviously, the subset $V^m(Jor,\overline{k})$ is the one-dimensional affine subspace of matrices $J(\lambda')=J(\lambda)+(\lambda'-\lambda)I$ $(\lambda'\in\mathbb{C})$. Consider the set $V^m(\overline{k})\subset M$ of all matrices, which have Jordan form from the space $V^m(Jor,\overline{k})$, i.e. $V^m(\overline{k})$ is a "bundle of matrices with the Jordan form $J(\lambda)$" in the terminology of [5]. By $N^m(\overline{k},\varepsilon)=V^m(\overline{k})\cap N_\varepsilon(J(\lambda))$ we denote a neighborhood of $J(\lambda)$ in the bundle $V^m(\overline{k})$. In Arnold's lemma [5, §5] proved that

$$\operatorname{codim}(N^m(\overline{k},\varepsilon))=\operatorname{codim}(V^m(\overline{k}))=\big(k_1+3k_2+5k_3+\ldots+(2s-1)k_s\big)-1. \tag{12}$$

Since $m=n$, then $N\left(J,\varepsilon,\lambda,\overline{k}\right)=N^m(\overline{k},\varepsilon)$, $J=J(\lambda)$, $B=B_{\mathbf{1,1}}$ and

$$\Psi(B)=\Big(I+\big(S(B)\big)^T\Big)\Big(J(\lambda)+F(B)\Big)\Big(I+\big(S(B)\big)^T\Big)^{-1}. \tag{13}$$

We will need the linear subspace

$$T^m:=\big\{(\lambda'-\lambda)I+S(B)\mid \lambda'\in\mathbb{C},\ S(B)\in T_{J(\lambda)}\mathscr{O}(J(\lambda))\big\}\subset M.$$

Using (4) and (5), we get

$$\operatorname{codim}(T^m)=\operatorname{codim}(T_{J(\lambda)}\mathscr{O}(J(\lambda)))-1=\big(k_1+3k_2+5k_3+\ldots+(2s-1)k_s\big)-1. \tag{14}$$

We give below a new local parametrization of the bundle $V^m(\overline{k})$.

THEOREM 2. *The next statements hold.*

1. *The subset $N^m(\overline{k},\varepsilon)$ is equal to*

$$N^m(\overline{k},\varepsilon)=\Psi(T^m\cap N(0)).$$

2. *The tangent space $T_{J(\lambda)}N^m(\overline{k},\varepsilon)$ is equal to*

$$T_{J(\lambda)}N^m(\overline{k},\varepsilon)=T^m.$$

3. *The map Ψ locally takes the tangent space $T_{J(\lambda)}N^m(\overline{k},\varepsilon)$ on $N^m(\overline{k},\varepsilon)$.*

Proof. First, map (13) takes a small matrix $B\in T^m\cap N(0)$ to the matrix

$$\Psi(B)=\Big(I+\big(S(B)\big)^T\Big)\Big(J(\lambda)+(\lambda'-\lambda)I\Big)\Big(I+\big(S(B)\big)^T\Big)^{-1}=$$

$$=\Big(I+\big(S(B)\big)^T\Big)\Big(J(\lambda')\Big)\Big(I+\big(S(B)\big)^T\Big)^{-1}.$$

Now, by statement 2 of Theorem 1, $\Psi(T^m\cap N(0))\subset N^m(\overline{k},\varepsilon)$. To conclude the proof of the first statement, it remains to note that Ψ ia a local diffeomorphism (statement 4 of Theorem 1) and $\operatorname{codim}(T^m)=\operatorname{codim}(N^m(\overline{k},\varepsilon))$ (see (12), (14)).

The derivative of Ψ (see (8)) takes the matrix B from the tangent space $T_0(T^m) = T^m$ to the matrix

$$D\Psi(0)B = (\lambda' - \lambda)I + \left[S(B)^T, J(\lambda)\right], \tag{15}$$

where $\left[S(B)^T, J(\lambda)\right] \in T_{J(\lambda)}\mathscr{O}(J(\lambda))$. Since Ψ is local diffeomorphism, it follows that the tangent space is

$$T_{J(\lambda)}N^m(\overline{k}, \varepsilon) := D\Psi(0)(T^m) = T^m. \tag{16}$$

Combining Theorem 1 and statements 1, 2, we obtain statement 3. □

4. Submanifold $N\left(J, \varepsilon, \lambda, \overline{k}\right)$

In this section $n > m$. We will need the linear subspace

$$T := \{B \mid B_{\mathbf{1,1}} \in T^m\} \subset \mathbb{C}^{n\times n}.$$

Now we state and prove the main result of the paper.

THEOREM 3. *The following statements hold.*

1. *The subset $N\left(J, \varepsilon, \lambda, \overline{k}\right) \subset \mathbb{C}^{n\times n}$ is equal to*

$$N\left(J, \varepsilon, \lambda, \overline{k}\right) = \Psi(T \cap N(0)).$$

2. *The subset $N\left(J, \varepsilon, \lambda, \overline{k}\right) \subset \mathbb{C}^{n\times n}$ is an analytic submanifold with complex codimension in $\mathbb{C}^{n\times n}$:*

$$\operatorname{codim} N\left(J, \varepsilon, \lambda, \overline{k}\right) = \left(k_1 + 3k_2 + 5k_3 + \ldots + (2s-1)k_s\right) - 1.$$

3. *The tangent space $T_J N\left(J, \varepsilon, \lambda, \overline{k}\right)$ is equal to*

$$T_J N\left(J, \varepsilon, \lambda, \overline{k}\right) = T.$$

4. *The map Ψ locally takes the tangent space $T_J N\left(J, \varepsilon, \lambda, \overline{k}\right)$ on $N\left(J, \varepsilon, \lambda, \overline{k}\right)$.*

Proof. By definition (7), the map Ψ takes the small matrix B to the matrix

$$\Psi(B) = \left(I + \left(\widehat{S}(B)\right)^T\right)\left(J + \widehat{F}(B)\right)\left(I + \left(\widehat{S}(B)\right)^T\right)^{-1} =$$

$$= \left(I + \begin{bmatrix} S(B_{\mathbf{1,1}})^T & B_{\mathbf{2,1}}^T \\ B_{\mathbf{1,2}}^T & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} J(\lambda) + F(B_{\mathbf{1,1}}) & 0 \\ 0 & J(\sigma \setminus \lambda) + B_{\mathbf{2,2}} \end{bmatrix} \cdot \left(I + \begin{bmatrix} S(B_{\mathbf{1,1}})^T & B_{\mathbf{2,1}}^T \\ B_{\mathbf{1,2}}^T & 0 \end{bmatrix}\right)^{-1}. \tag{17}$$

Now, the matrix $J+\widehat{F}(B)$ has block diagonal form and the spectra of diagonal blocks $J(\lambda)+F(B_{\mathbf{1,1}})$ and $J(\sigma\setminus\lambda)+B_{\mathbf{2,2}}$ do not intersect. If we combine this with statement 2 of Theorem 1 and statement 1 of Theorem 2, we get

$$\Psi(B)\in N\left(J,\varepsilon,\lambda,\overline{k}\right)\ \Leftrightarrow\ F(B_{\mathbf{1,1}})+S(B_{\mathbf{1,1}})\in T^m\cap N(0).$$

By definition of T,

$$F(B_{\mathbf{1,1}})+S(B_{\mathbf{1,1}})\in T^m\cap N(0)\ \Leftrightarrow\ B\in T\cap N(0). \tag{18}$$

Now, the second statement follows from (18) and formula (14).

Using (8), we get that the tangent space is

$$T_JN\left(J,\varepsilon,\lambda,\overline{k}\right)=D\Psi(0)(T)=$$

$$\left\{\left[\begin{array}{c|c}(\lambda'-\lambda)I+[(S(B_{\mathbf{1,1}}))^T,J(\lambda)] & B_{\mathbf{2,1}}^T\cdot J(\sigma\setminus\lambda)-J(\lambda)\cdot B_{\mathbf{2,1}}^T\\ \hline J(\sigma\setminus\lambda)\cdot B_{\mathbf{1,2}}^T-B_{\mathbf{1,2}}^T\cdot J(\lambda) & B_{\mathbf{2,2}}\end{array}\right],\ B\in T\right\}.$$

Now, if we recall equalities (15) and (16) for $(\lambda'-\lambda)I+[(S(B_{\mathbf{1,1}}))^T,J(\lambda)]$, we get $T_JN\left(J,\varepsilon,\lambda,\overline{k}\right)=\{B\mid B_{\mathbf{1,1}}\in T^m\}=T$.

Combining statements 1 and 3, we obtain the last statement of the Theorem. □

Let us now consider the special cases of Theorem 3. If the eigenvalue λ has a single m-dimensional Jordan block (i.e. $\overline{k}=\{m\}$), then the subset $N(J,\varepsilon,\lambda,\{m\})\subset M$ is an analytic submanifold of complex codimension $m-1$ [8]. If the eigenvalue λ has m one-dimensional Jordan's blocks (i.e. $\overline{k}=\{1,1,\dots,1\}$), then the subset $N(J,\varepsilon,\lambda,\{1,1,...1\})\subset M$ is an analytic submanifold of complex codimension m^2-1. The tangent space $T_JN(J,\varepsilon,\lambda,\{1,1,...1\})$ consists of the matrices $[B_{\mathbf{i,j}}]_{i,j=1}^2$ such that $B_{\mathbf{1,1}}$ is an m-dimensional scalar matrix (diagonal matrix with all its main diagonal entries equal) and the rest of the blocks are arbitrary.

5. Case of several eigenvalues

By $\Lambda=\{\lambda_1,\dots,\lambda_p\}\subset\sigma$ we denote the subset of fixed eigenvalues. Then $J=J(\Lambda)\oplus J(\sigma\setminus\Lambda)=J(\lambda_1)\oplus\ldots\oplus J(\lambda_p)\oplus J(\sigma\setminus\Lambda)$, where $J(\lambda_i)$ $(i=1,...,p)$ is a direct sum of Jordan blocks with the eigenvalue λ_i and $J(\sigma\setminus\Lambda)$ is a direct sum of Jordan blocks with other eigenvalues. Let m_i be the algebraic multiplicity of λ_i, s_i the geometric multiplicity of λ_i, and let $\overline{k}(i):=(k_1(i),k_2(i),\dots k_{s_i}(i))$ be the Segré characteristic at λ_i. By $K=(\overline{k}(1),\overline{k}(2),...,\overline{k}(p))$ we denote the ordered list of these vectors. We will study the set $N(J,\varepsilon,\Lambda,K)\subset\mathbb{C}^{n\times n}$ of matrices $A\in N_\varepsilon(J)$ such that A has a unique eigenvalue λ_i', which is close to λ_i, and the corresponding Jordan block $J(A,\lambda_i')$ has the same structure as the block $J(\lambda_i)$, for $i=1,\dots,p$.

We use notations similar to the notations from Sections 1 and 3. Invariant spaces corresponding to the selected eigenvalues and the complement subspace generate the block representation of an arbitrary matrix:

$$B = \left[\begin{array}{ccccc} B_{\mathbf{1,1}} & & & & \\ & B_{\mathbf{2,2}} & & \overline{B} & \\ & & \ddots & & \\ & & & B_{\mathbf{p,p}} & \\ & \underline{B} & & & B_{\mathbf{p+1,p+1}} \end{array}\right], \tag{19}$$

where

$$B_{\mathbf{i,i}} = \left[\begin{array}{c|c|c|c} B_{\mathbf{i},11} & B_{\mathbf{i},12} & \dots & B_{\mathbf{i},1s_i} \\ \hline B_{\mathbf{i},21} & B_{\mathbf{i},22} & \dots & B_{\mathbf{i},2s_i} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline B_{\mathbf{i},s_i1} & B_{\mathbf{i},s_i2} & \dots & B_{\mathbf{i},s_is_i} \end{array}\right],$$

$$\overline{B} = \left[\begin{array}{cccccc} 0 & B_{\mathbf{1,2}} & \dots & & B_{\mathbf{1,p}} & \\ & 0 & B_{\mathbf{2,3}} & \dots & & \\ & & \ddots & \ddots & \vdots & \\ & & & 0 & B_{\mathbf{p,p+1}} & \\ & 0 & & & 0 & \end{array}\right], \quad \underline{B} = \left[\begin{array}{cccccc} 0 & & & & & \\ B_{\mathbf{2,1}} & 0 & & & 0 & \\ \vdots & B_{\mathbf{2,3}} & \ddots & & & \\ & \vdots & \ddots & 0 & & \\ B_{\mathbf{p,1}} & \dots & & B_{\mathbf{p+1,p}} & 0 & \end{array}\right].$$

Analogously to the maps $\widehat{F}(B)$ and $\widehat{S}(B)$, we define

$$\widehat{F}_{\mathbf{p}}(B) := \left[\begin{array}{ccccc} F(B_{\mathbf{1,1}}) & & & & \\ & F(B_{\mathbf{2,2}}) & & 0 & \\ & & \ddots & & \\ & & & F(B_{\mathbf{p,p}}) & \\ & 0 & & & B_{\mathbf{p+1,p+1}} \end{array}\right],$$

$$\widehat{S}_{\mathbf{p}}(B) := \left[\begin{array}{ccccc} S(B_{\mathbf{1,1}}) & & & & \\ & S(B_{\mathbf{2,2}}) & & \overline{B} & \\ & & \ddots & & \\ & & & S(B_{\mathbf{p,p}}) & \\ & \underline{B} & & & 0 \end{array}\right].$$

Analogously to (7), define the map

$$\Psi_{\mathbf{p}}(B) := (I + (\widehat{S}_{\mathbf{p}}(B))^T)(J + \widehat{F}_{\mathbf{p}}(B))(I + (\widehat{S}_{\mathbf{p}}(B))^T)^{-1}.$$

It is easily verified that statements of Lemma 1 hold for the map $\Psi_{\mathbf{p}}$.

We will use the subspace

$$T^{\mathbf{p}} := \{B \mid B_{\mathbf{i,i}} \in T^{m_i}, \quad i = 1,...,p\} \subset \mathbb{C}^{n\times n},$$

where

$$T^{m_i} := \left\{(\lambda_i' - \lambda_i)I + S_{\mathbf{i,i}} \mid \lambda_i' \in \mathbb{C},\ S_{\mathbf{i,i}} \in T_{J(\lambda_i)}\mathscr{O}(J(\lambda_i))\right\}$$

THEOREM 4. *The following statements hold.*

1. *The subset $N(J,\varepsilon,\Lambda,K)$ is defined as follows*

$$N(J,\varepsilon,\Lambda,K) = \Psi_{\boldsymbol{p}}\left(T^{\boldsymbol{p}} \cap N(0)\right).$$

2. *The subset $N(J,\varepsilon,\Lambda,K) \subset \mathbb{C}^{n\times n}$ is an analytic submanifold of complex codimension*

$$\operatorname{codim} N(J,\varepsilon,\Lambda,K) = \\ = \sum_{i=1}^{p}\left(k_1(i) + 3k_2(i) + 5k_3(i) + \ldots + (2s_i - 1)k_{s_i}(i)\right) - p,$$

3. *The tangent space $T_J N(J,\varepsilon,\Lambda,K)$ is equal to*

$$T_J N(J,\varepsilon,\Lambda,K) = T^{\boldsymbol{p}}.$$

4. *The map $\Psi_{\boldsymbol{p}}$ locally takes the tangent space $T_J N(J,\varepsilon,\Lambda,K)$ on $N(J,\varepsilon,\Lambda,K)$.*

The proof of Theorem 4 is completely analogous to the proof of Theorem 3.

The special case is $\Lambda = \sigma$. In this case, the map $\Psi_{\mathbf{p}}$ locally takes the tangent space $T_J Orb$ on $Orb(J)$. This case is similar to the case of a single eigenvalue (see statement 4 of Theorem 2).

REMARK 1. In [5, §4] Arnold claims that there exists other form of versal families $J(\lambda) + B_{\mathbf{1,1}}$ in which the number of non-zero entries in $B_{\mathbf{1,1}}$ is minimal (and equal to the number of parameters). To do this it is sufficient to take for the family of matrices $B_{\mathbf{1,1}}$ matrices for which on each oblique segment in Figure 1 one entry is an independent parameter, and all the other entries are zero. The non-zero element can be chosen on each oblique segment at an arbitrary place. Each of these families generates a diffeomorphism likewise Ψ. Note that Lemma 1 and statement 4 of Theorems 2, 3 and 4 are valid exactly because of the special family $J(\lambda) + M_{\overline{k}}$.

6. Perturbation theory

One-parameter perturbations (families), which keep (or change) the Jordan structure of the initial matrix are often used in applications. Using the local diffeomorphism Ψ, we can write down some of these perturbations.

Let E_m be the block matrix of the form (1), where blocks $B_{\mathbf{1,2}}, B_{\mathbf{2,1}}, B_{\mathbf{2,2}}$ equal to zero and block $B_{\mathbf{1,1}}$ is the $m \times m$ identity matrix. Denote by T' the linear space of block matrices of the form (1), where $B_{\mathbf{1,1}} \in T_{J(\lambda)}\mathscr{O}(J(\lambda))$ and the other blocks are arbitrary. An arbitrary matrix $B \in T$ has unique representation in the form

$$B = \gamma \cdot E_m + B', \quad \text{where } \gamma \in \mathbb{C}, \ \ B' \in T'. \tag{20}$$

Let $S^{2m-1} \subset \mathbb{C}^m$ be the sphere (of real dimension $2m-1$) of normalized right eigenvectors of J corresponding to the eigenvalue λ. Let $u \in S^{2m-1}$ and $N(u) \subset S^{2m-1}$ be a small neighborhood of u.

Now we describe the one-parameter families of matrices (under small parameter $t \in \mathbb{C}$), which keep the Jordan structure.

THEOREM 5. *Suppose*

$$\begin{array}{ll} \gamma(t) \in \mathbb{C}, & \gamma(0) = 0, \\ B'(t) \in T' \subset \mathbb{C}^{n\times n}, & B'(0) = 0 \end{array}$$

are arbitrary small analytic functions. The following assertions hold.

1. *The matrix function*

$$A(t) = \Psi(\gamma(t) \cdot E_m + B'(t)), \quad A(0) = J$$

is analytic and $A(t)$ has eigenvalue $\lambda(t) = \lambda + \gamma(t)$ with the same Jordan structure as $J(\lambda)$.

2. *Conversely, if an analytic function $A(t) \in \mathbb{C}^{n\times n}$ $(A(0) = J)$ preserves the Jordan structure of the eigenvalue λ, then there exists a unique pair of analytic functions $\gamma(t)$ and $B(t)$ for which $A(t) \equiv \Psi(\gamma(t) \cdot E_m + B'(t))$.*

3. *If $w(t) \in N(u) \subset S^{2m-1}$, $w(0) = u$ is a continuous function then $u(t) = \Big(I + \big(\widehat{S}(B'(t))\big)^T\Big) w(t)$ is the family of corresponding normalized right eigenvectors of $A(t)$.*

Proof. Let us prove the first statement. We first note that $\gamma(t) \cdot E_m + B'(t) \in T$. If we combine this with the first statement of Theorem 3, we get that $A(t)$ is an analytic function and $A(t) \in N\left(J, \varepsilon, \lambda, \overline{k}\right)$. Substituting $\gamma(t) \cdot I$ for $F(B_{\mathbf{1,1}})$ in (17), we get that $A(t)$ has eigenvalue $\lambda(t) = \lambda + \gamma(t)$ with the same Jordan structure as $J(\lambda)$.

Let's prove the second statement. Since Ψ is local diffeomorphism, the superposition of mappings

$$f(t) := \Psi^{-1}(A(t))$$

is an analytic function. Since $A(t) \in N\left(J, \varepsilon, \lambda, \overline{k}\right)$, it follows that $f(t) \in T$. Using (20), we get $f(t) = \gamma(t) \cdot E_m + B'(t)$. The functions $\gamma(t)$ and $B'(t)$ are analytic because representation (20) is unique.

The third statement follows from first statement and definition (7) of the map Ψ. □

Now we study some special cases of perturbation of the Jordan structure. First, consider the case when the Jordan structure fall into direct sum of one-dimensional

Jordan blocks. Let $D_m(t) \in \mathbb{C}^{n\times n}$ be a matrix function of the form (1), where blocks $B_{\mathbf{1,2}}(t), B_{\mathbf{2,1}}(t), B_{\mathbf{2,2}}(t)$ are equal to zero and

$$B_{\mathbf{1,1}}(t) = \begin{pmatrix} \gamma_1(t) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \gamma_m(t) \end{pmatrix}.$$

THEOREM 6.

Suppose

$$\begin{array}{l} \gamma_i(t) \in \mathbb{C},\ \gamma_i(0) = 0 \ \ (i = 1,\dots,m), \\ \gamma_i(t) \neq \gamma_j(t)\, \textit{for}\ \ i \neq j\ \ \textit{and}\, t \neq 0, \\ B'(t) \in T' \subset \mathbb{C}^{n\times n},\ B'(0) = 0 \end{array}$$

are arbitrary small analytic functions. Then the matrix function

$$A(t) = \Psi\left(D_m(t) + B'(t)\right)$$

is analytic and $A(t)$ *has* m *different simple eigenvalues* $\lambda_i = \lambda + \gamma_i(t)$ $(i = 1,\dots,m)$ *that are close to* λ.

Proof. The proof is analogous to the proof of the first statement of Theorem 5. The only difference among them is the structure of Jordan blocks: substituting $D_m(t)$ by $F(B_{\mathbf{1,1}})$ in (17), we get that matrix $J(\lambda)+D_m(t)$ has m different simple eigenvalues $\lambda_i = \lambda + \gamma_i(t)$, $i = 1,\dots,m$. $\square$

Consider now the case when the Jordan structure transfigures into single Jordan block. Let $U_m(t) \in \mathbb{C}^{n\times n}$ be the matrix function of the form (1), where blocks $B_{\mathbf{1,2}}(t)$, $B_{\mathbf{2,1}}(t)$, $B_{\mathbf{2,2}}(t)$ are equal to zero and and block $B_{\mathbf{1,1}}(t) := U_{\mathbf{1,1}}(t)$ has the form described in Figure 4: entries on positions k_1, k_1+1; k_1+k_2, k_1+k_2+1;...; $\sum_{i=1}^{s-1} k_i$, $\sum_{i=1}^{s-1} k_i + 1$ are correspondingly equal to $\gamma_1(t), \gamma_2(t), \dots, \gamma_{s-1}(t)$ and all other entries equal to zero.

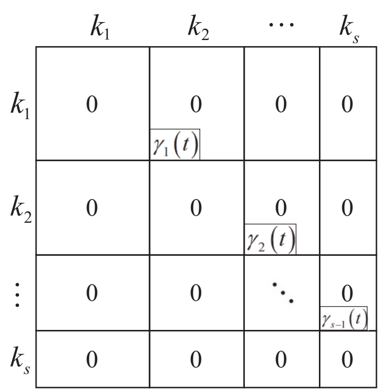

Figure 4: Matrix $U_{\mathbf{1,1}}(t)$

THEOREM 7. *Suppose*

$$\gamma_i(t) \in \mathbb{C},\ \gamma_i(0) = 0\ (i = 0, \dots, s-1),$$
$$B'(t) \in T' \subset \mathbb{C}^{n \times n},\ B'(0) = 0$$

are arbitrary small analytic functions. Then the matrix function

$$A(t) = \Psi\left(\gamma_0(t) \cdot E_m + U_m(t) + B'(t)\right)$$

is analytic and $A(t)$ *has unique eigenvalue* $\lambda' := \lambda + \gamma_0(t)$ *(that is close to* λ *) with a single* m*-dimensional Jordan block.*

Proof. The matrix $A(t)$ is similar to the matrix $J + \gamma_0(t) \cdot E_m + U_m(t)$. By definition of $U_{\mathbf{1,1}}(t)$ (see Figure 4), the matrix $J(\lambda) + \gamma_0(t) \cdot I + U_{\mathbf{1,1}}(t)$ has eigenvalue $\lambda + \gamma_0(t)$ of algebraic multiplicity m. Since

$$\operatorname{rank}\Big(\big(J(\lambda) + \gamma_0(t) \cdot I + U_{\mathbf{1,1}}(t)\big) - (\lambda + \gamma_0(t)) \cdot I\Big) = m - 1,$$

it follows that geometric multiplicity of $\lambda + \gamma_0(t)$ is equal to one [7]. So, matrix $J + \gamma_0(t) \cdot E_m + U_m(t)$ has unique eigenvalue $\lambda' := \lambda + \gamma_0(t) \approx \lambda$ with a single m-dimensional Jordan block. $\square$

Theorems 5-7 can be restated for the cases when we track some or all of the eigenvalues.

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