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SELF–ADJOINT BOUNDARY CONDITIONS AND INTERLACING OF EIGENVALUES FOR THE STURM–LIOUVILLE EQUATION ON GRAPHS

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Abstract. Applying the approach of Kostrykin and Schrader, [14], an explicit characterisation of self-adjoint boundary conditions at the nodes or vertices of a graph for the Sturm-Liouville equation is given. This is then proven to be equivalent to the conditions for self-adjointness of the corresponding system boundary value problem with separated boundary conditions. In addition, using an example, it is shown that the complete graph configuration is incorporated in the system boundary condition at the terminal end point. Making use of the separated system formulation, via matrix Prüfer angle techniques, an interlacing property of the eigenvalues for a self-adjoint Sturm-Liouville boundary value problem on a graph is ascertained.

1. Introduction

In this paper we consider the second order differential equation

$$ly := -\frac{d^2y}{dx^2} + q(x)y = \lambda y, \tag{1.1}$$

where q is real-valued and continuous, on the weighted graph G with boundary conditions at the nodes.

Differential operators on graphs often appear in mathematics, mechanics, physics, geophysics, chemistry and engineering, see [10, 11, 15, 16, 17, 18] and the bibliographies thereof. Second order boundary value problems on finite graphs also arise from quantum mechanical models of micro-electronic devices, [3, 12]. Multi-point boundary-value problems and periodic boundary-value problems can be considered as particular cases of boundary-value problems on graphs, [5]. In recent years interest in the spectral theory of Sturm-Liouville equations on graphs has grown considerably, see, for example [2, 20, 22], where properties of the spectrum are considered.

In our studies of differential operators on graphs we have on numerous occasions used the approach of rewriting the boundary value problem on a graph as a system boundary value problem with separated boundary conditions at 0 and 1, see [6, 7, 8]. In these papers we refer to the likes of Kostrykin and Schrader [14], Harmer [13] and

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Carlson [4] for the characterisation and details of self-adjoint boundary conditions on a graph. In addition, we impose certain constraints on the boundary condition coefficient matrices in the separated system formulation which ensure formal self-adjointness of the system boundary value problem. We however do not ever give the explicit link between the characterisation of self-adjoint boundary conditions on a graph, as given in [14], and those conditions on the boundary condition coefficient matrices which we assume for formal self-adjointness of the system boundary value problem. Thus, in this paper, we fill this gap and prove that the characterisation of self-adjoint boundary conditions on a graph as given by Kostrykin and Schrader in [14], using Hermitian symplectic geometry, is equivalent to the conditions on the boundary condition coefficient matrices used in the system formulation to guarantee formal self-adjointness.

It should be noted that Harmer, in [13], considers the special case of the Sturm-Liouville operator on a non-compact graph with a finite number of edges of infinite length connected at a single vertex i.e. a star shaped graph with infinite length rays. He uses the idea of Hermitian symplectic spaces to parametrise all self-adjoint boundary conditions at the single node in terms of a unitary matrix.

In the past it has been alluded to that once the boundary value problem on a graph is rewritten as a system boundary value problem with separated boundary conditions that the nature of the graph (or the explicit structure of the graph) is somehow ”lost”. This is most certainly not the case and to show this we will provide a natural, yet simple and physically meaningful example. This example will illustrate that in fact the entire graph structure is explicitly encapsulated in the system boundary conditions that occur at the end point 1. Consequently, using the formally self-adjoint system boundary value problem with separated boundary conditions together with the matrix Prüfer angle we prove a couple of interlacing results for the eigenvalues of the system boundary value problem and thus for the boundary value problem on the graph.

This paper is structured as follows: In Section 2, we recall how to rewrite a boundary value problem on a graph as a system boundary value problem with separated boundary conditions. In addition, we prove what conditions the boundary condition coefficient matrices must obey for the system problem to be formally self-adjoint.

We use the approach given by Kostrykin and Schrader, see [14], in Section 3, to give a characterisation of self-adjoint boundary conditions on a graph. Consequently, we show that this is equivalent to the conditions imposed on the boundary condition coefficient matrices in Section 2.

In Section 4, we show by means of an example, that the system boundary conditions are directly and explicitly formed from the boundary conditions on the graph, i.e. from the given graph structure. So the structure of the graph is entirely contained in the boundary condition matrices at the terminal end point. Thus by rewriting a graph boundary value problem as a separated system boundary value problem one does not lose the graph structure as it remains accessible (up to a class) in the terminal end boundary condition matrices.

Lastly, in Section 5, by considering the arguments of the characteristic roots of the matrix Prüfer angle for the system boundary value problem with Dirichlet boundary conditions, Neumann boundary conditions and general self-adjoint boundary conditions respectively we prove two interlacing results. The first result shows how the eigenvalues of the boundary value problem with Dirichlet boundary conditions and the eigenvalues of the boundary value problem with general self-adjoint boundary conditions interlace and using this we obtain an analogous result when the Dirichlet boundary conditions are replaced by Neumann boundary conditions.

2. Formal self-adjointness in the system setting

In this section we recap, from [6], the necessary details on how to rewrite a boundary value problem on a graph as a system boundary value problem with separated boundary conditions. The main result of this section provides constraints on the boundary condition coefficient matrices which ensure that the system boundary value problem is formally self-adjoint.

Let G denote a directed graph with a finite number of edges, say K, each of finite length and having the path-length metric. Each edge, e_i, of length say l_i can thus be considered as the interval $[0,l_i]$ where 0 is identified with the initial point of e_i and l_i with the terminal point.

The differential equation (1.1) on the graph G can now be considered as the system of equations

$$-\frac{d^2y_i}{dx^2}+q_i(x)y_i=\lambda y_i, \quad x\in[0,l_i], \quad i=1,\dots,K, \tag{2.1}$$

where q_i and y_i denote $q|_{e_i}$ and $y|_{e_i}$.

The boundary conditions at the node v are specified in terms of the values of y and y' at v on each of the incident edges. In particular if the edges which start at v are $e_i, i\in\Lambda_s(v)$ and the edges which end at v are $e_i, i\in\Lambda_e(v)$ then the boundary conditions at v can be expressed as

$$\sum_{j\in\Lambda_s(v)}\left[\alpha_{ij}y_j+\beta_{ij}y'_j\right](0)+\sum_{j\in\Lambda_e(v)}\left[\gamma_{ij}y_j+\delta_{ij}y'_j\right](l_j)=0, \quad i=1,\dots,N(v), \tag{2.2}$$

where $N(v)$ is the number of linearly independent boundary conditions at node v. For formally self-adjoint boundary conditions $N(v)=\sharp(\Lambda_s(v))+\sharp(\Lambda_e(v))$ and $\sum_v N(v)=2K$, see [4, 19] for more details.

Let $\alpha_{ij}=0=\beta_{ij}$ for $i=1,\dots,N(v)$ and $j\notin\Lambda_s(v)$ and similarly let $\gamma_{ij}=0=\delta_{ij}$ for $i=1,\dots,N(v)$ and $j\notin\Lambda_e(v)$. The boundary conditions (2.2) considered over all nodes v, after possible relabelling, may thus be written as

$$\sum_{j=1}^{K}\left[\alpha_{ij}y_j+\beta_{ij}y'_j\right](0)+\sum_{j=1}^{K}\left[\gamma_{ij}y_j+\delta_{ij}y'_j\right](l_j)=0, \quad i=1,\dots,2K, \tag{2.3}$$

where $2K$ is the total number of linearly independent boundary conditions.

We may define the Hilbert space $\mathscr{L}^2(G)$ as

$$\mathscr{L}^2(G)=\bigoplus_{i=1}^{K}\mathscr{L}^2(0,l_i),$$

with inner product

$$(f,g) = \sum_{i=1}^{K} \int_0^{l_i} f|_{e_i} \bar{g}|_{e_i}\, dt = \sum_{i=1}^{K} \int_0^{l_i} f_i \bar{g}_i\, dt = \sum_{i=1}^{K} (f_i, g_i)_{\mathscr{L}^2(0,l_i)}. \tag{2.4}$$

The boundary value problem (2.1), (2.3) on G can be formulated as an operator eigenvalue problem in $\mathscr{L}^2(G)$, [1, 4, 21], for the closed densely defined operator

$$Lf := -f'' + qf \tag{2.5}$$

with domain

$$D(L) = \{f \mid f, f' \in AC, Lf \in \mathscr{L}^2(G),\ f \text{ obeying } (2.3)\ \}. \tag{2.6}$$

In [6], it is shown that by letting $t = \frac{x}{l_i}$ and $\tilde{y}_i(t) = y_i(l_i t)$ the boundary value problem on G, (2.1), (2.3), can be reformulated as the following system boundary value problem on $[0,1]$:

$$-W\tilde{Y}'' + Q\tilde{Y} = \lambda \tilde{Y} \tag{2.7}$$

where $W = \operatorname{diag}\left[\frac{1}{l_1^2}, \ldots, \frac{1}{l_K^2}\right]$, $\tilde{Y} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_K \end{bmatrix}$ and $Q = \operatorname{diag}[Q_1, \ldots, Q_K]$. Here $Q_i(t) = q_i(l_i t)$.

The corresponding boundary conditions are given as

$$\tilde{A}\tilde{Y}(0) + \tilde{B}\tilde{Y}'(0) + \tilde{C}\tilde{Y}(1) + \tilde{D}\tilde{Y}'(1) = 0 \tag{2.8}$$

where $\tilde{A} = [\alpha_{ij}], \tilde{B} = \left[\frac{\beta_{ij}}{l_j}\right], \tilde{C} = [\gamma_{ij}]$ and $\tilde{D} = \left[\frac{\delta_{ij}}{l_j}\right]$.

Consequently, in [6], it is then also shown that by inserting a vertex, m_i, at the mid-point of each edge, e_i, and imposing the continuity boundary conditions

$$\begin{aligned} y(m_i^-) &= y(m_i^+), \\ y'(m_i^-) &= y'(m_i^+) \end{aligned}$$

for $i = 1, \ldots, K$ (it should be noted that these represent formally self-adjoint boundary conditions at the vertex m_i), the system (2.7), (2.8) may be rewritten as a system of twice the dimension with separated boundary conditions as given below:

$$-MY'' + PY = \lambda Y \tag{2.9}$$

with boundary conditions

$$A^*Y(0) - B^*Y'(0) = 0, \tag{2.10}$$

$$\Gamma^*Y(1) - \Delta^*Y'(1) = 0 \tag{2.11}$$

where $M = 4\begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix}$, $P = \begin{bmatrix} Q\left(\frac{t+1}{2}\right) & 0 \\ 0 & Q\left(\frac{1-t}{2}\right) \end{bmatrix}$, $A^* = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix}$, $-B^* = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ I & I \end{bmatrix}$, $\Gamma^* = [\tilde{C}, \tilde{A}]$ and $-\Delta^* = 2[\tilde{D}, -\tilde{B}]$.

Let

$$L_1Y := -MY'' + PY, \tag{2.12}$$

where

$$D(L_1) = \{Y \,|\, Y, Y' \in AC, L_1Y \in \mathscr{L}^2(0,1), Y \text{ obeys } (2.10), (2.11)\}. \tag{2.13}$$

In the following theorem we show what conditions are needed on the coefficients, A^*, B^*, Γ^* and Δ^*, in the boundary conditions (2.10) and (2.11) in order to establish that L_1 is formally self-adjoint with respect the the inner product

$$\langle U, V\rangle_M = \int_0^1 U^T M^{-\frac{1}{2}} \bar{V}\, dt. \tag{2.14}$$

THEOREM 2.1. *The system boundary value problem* (2.9)–(2.11), *where* $[\Gamma^*, -\Delta^*]$ *and* $[A^*, -B^*]$ *have maximal rank, is formally self-adjoint if and only if the following conditions hold:*

(1) $\Gamma^* M^{-\frac{1}{2}}\Delta = \Delta^* M^{-\frac{1}{2}}\Gamma$ *and* $A^* M^{-\frac{1}{2}} B = B^* M^{-\frac{1}{2}} A$*;*
(2) $\Gamma^*\Gamma + \Delta^*\Delta = I$ *and* $A^*A + B^*B = I$.

Proof. From the definition of A^* and B^* we have, automatically, that $A^* M^{-\frac{1}{2}} B = B^* M^{-\frac{1}{2}} A$ and $A^*A + B^*B = I$.

Let $U, V \in D(L_1)$ then, since P and $M^{-\frac{1}{2}}$ are diagonal with real entries, we obtain

$$\begin{aligned}
\langle L_1U, V\rangle_M - \langle U, L_1V\rangle_M &= \int_0^1 [(-U''^T M M^{-\frac{1}{2}} \bar{V} + U^T P M^{-\frac{1}{2}} \bar{V}) \\
&\quad - (-U^T M^{-\frac{1}{2}} M \bar{V}'' + U^T M^{-\frac{1}{2}} P \bar{V})]\, dt \\
&= \int_0^1 (-U''^T M^{\frac{1}{2}} \bar{V} + U^T M^{\frac{1}{2}} \bar{V}'')\, dt \\
&= [U^T M^{\frac{1}{2}} \bar{V}' - U'^T M^{\frac{1}{2}} \bar{V}]_0^1,
\end{aligned}$$

by integration by parts. Since $U, V \in D(L_1)$ they obey (2.10) and in addition $A^* M^{-\frac{1}{2}} B = B^* M^{-\frac{1}{2}} A$ giving that the above expression evaluated at 0 vanishes i.e. the Lagrange form $\mathscr{L}$ is given by

$$\mathscr{L}(U, V) := \langle L_1U, V\rangle_M - \langle U, L_1V\rangle_M = [U^T M^{\frac{1}{2}} \bar{V}' - U'^T M^{\frac{1}{2}} \bar{V}](1). \tag{2.15}$$

Assume that L_1 is formally self-adjoint with respect to the inner product (2.14) and that $[\Gamma^*, -\Delta^*]$ and $[A^*, -B^*]$ have rank $2K$. Then

$$[U^T M^{\frac{1}{2}} \bar{V}' - U'^T M^{\frac{1}{2}} \bar{V}](1) = 0.$$

This may be rewritten as

$$[U^T, U'^T] \begin{bmatrix} 0 & M^{\frac{1}{2}} \\ -M^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} \bar{V} \\ \bar{V}' \end{bmatrix} = 0.$$

Now, $[\Gamma^*, -\Delta^*]$ is of rank $2K$ and

$$\Gamma^* X - \Delta^* Y = 0 = \Gamma^* Z - \Delta^* W \;\Rightarrow\; [X^T, Y^T] \begin{bmatrix} 0 & M^{\frac{1}{2}} \\ -M^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} \bar{Z} \\ \bar{W} \end{bmatrix} = 0$$

$$\Leftrightarrow X^T M^{\frac{1}{2}} \bar{W} - Y^T M^{\frac{1}{2}} \bar{Z} = 0.$$

We may also write the null space of $[\Gamma^*, -\Delta^*]$ as $[D_1^T, D_2^T]^T p$, $p \in \mathbb{C}^{2K}$, for suitable $2K \times 2K$ matrices D_1 and D_2. In particular, for all p, $q \in \mathbb{C}^{2K}$, we require $\Gamma^* D_1 p - \Delta^* D_2 p = 0$ and $(D_1 q)^T M^{\frac{1}{2}} (\overline{D_2 p}) - (D_2 q)^T M^{\frac{1}{2}} (\overline{D_1 p}) = 0$.

I.e.

$$\begin{aligned} q^T D_1^T M^{\frac{1}{2}} \overline{D}_2 \overline{p} - q^T D_2^T M^{\frac{1}{2}} \overline{D}_1 \overline{p} = 0 &\Rightarrow D_1^T M^{\frac{1}{2}} \overline{D}_2 = D_2^T M^{\frac{1}{2}} \overline{D}_1 \\ &\Rightarrow D_1^* M^{\frac{1}{2}} D_2 = D_2^* M^{\frac{1}{2}} D_1. \end{aligned}$$

So

$$D_2^* M^{\frac{1}{2}} D_1 = D_1^* M^{\frac{1}{2}} D_2, \tag{2.16}$$

giving $D_2^* M^{\frac{1}{2}} D_1 p - D_1^* M^{\frac{1}{2}} D_2 p = 0$ for all $p \in \mathbb{C}^{2K}$. Hence $[D_2^* M^{\frac{1}{2}}, -D_1^* M^{\frac{1}{2}}]$ has the same null space as $[\Gamma^*, -\Delta^*]$ and the same rank. As we are only interested in the null space of $[\Gamma^*, -\Delta^*]$ we can without loss of generality let $\Gamma^* = D_2^* M^{\frac{1}{2}}$ and $\Delta^* = D_1^* M^{\frac{1}{2}}$. Therefore, by (2.16), we have that

$$\Gamma^* M^{-\frac{1}{2}} \Delta = \Delta^* M^{-\frac{1}{2}} \Gamma,$$

as required.

We now show that $\Gamma^*\Gamma + \Delta^*\Delta = I$. Obviously $\Gamma^*\Gamma \geqslant 0$ and $\Delta^*\Delta \geqslant 0$ giving that $\Gamma^*\Gamma + \Delta^*\Delta \geqslant 0$. Since $\Gamma^*\Gamma + \Delta^*\Delta$ has rank 2K

$$\Gamma^*\Gamma + \Delta^*\Delta > 0$$

and is Hermitian symmetric. So there exists Υ and $\Phi > 0$ such that $\Upsilon^* = \Upsilon^{-1}$ and

$$\Gamma^*\Gamma + \Delta^*\Delta = \Upsilon^* \Phi \Upsilon$$

giving

$$\Upsilon \Gamma^* \Gamma \Upsilon^* + \Upsilon \Delta^* \Delta \Upsilon^* = \Phi.$$

Thus

$$\Phi^{-\frac{1}{2}} \Upsilon \Gamma^* \Gamma \Upsilon^* \Phi^{-\frac{1}{2}} + \Phi^{-\frac{1}{2}} \Upsilon \Delta^* \Delta \Upsilon^* \Phi^{-\frac{1}{2}} = I.$$

Let $\Psi = \Phi^{-\frac{1}{2}} \Upsilon \Gamma^*$ and $\Omega = \Phi^{-\frac{1}{2}} \Upsilon \Delta^*$. Then

$$\begin{aligned} [\Psi, -\Omega] \begin{bmatrix} Y(1) \\ Y'(1) \end{bmatrix} &= [\Phi^{-\frac{1}{2}} \Upsilon \Gamma^*, -\Phi^{-\frac{1}{2}} \Upsilon \Delta^*] \begin{bmatrix} Y(1) \\ Y'(1) \end{bmatrix} \\ &= \Phi^{-\frac{1}{2}} \Upsilon [\Gamma^* Y(1) - \Delta^* Y'(1)] \\ &= 0 \end{aligned}$$

if and only if $\Gamma^*Y(1)-\Delta^*Y'(1)=0$. Therefore $[\Psi, -\Omega]$ has the same null space as $[\Gamma^*,-\Delta^*]$ and the same rank. Again, since we are only interested in the null space of $[\Gamma^*,-\Delta^*]$ we can without loss of generality let $\Gamma^*=\Psi$ and $\Delta^*=\Omega$ and then we have $\Gamma^*\Gamma+\Delta^*\Delta=I$.

Note that $\Psi M^{-\frac{1}{2}}\Omega^*=\Phi^{-\frac{1}{2}}\Upsilon\Gamma^*M^{-\frac{1}{2}}\Delta\Upsilon^*(\Phi^{-\frac{1}{2}})^*=\Phi^{-\frac{1}{2}}\Upsilon\Delta^*M^{-\frac{1}{2}}\Gamma\Upsilon^*(\Phi^{-\frac{1}{2}})^*=\Omega M^{-\frac{1}{2}}\Psi^*$, so (1) is also preserved. This completes the proof in one direction and we now prove the converse.

Assume that (1) and (2) hold. We must show that L_1 is formally self-adjoint with respect to the inner product (2.14) and that $[\Gamma^*,-\Delta^*]$ and $[A^*,-B^*]$ have maximal rank. The Lagrange form $\mathscr{L}$, given in (2.15), is

$$\mathscr{L}(U,V)=[U^TM^{\frac{1}{2}}\bar{V}'-U'^TM^{\frac{1}{2}}\bar{V}](1)$$

for $U,V\in D(L_1)$. Since U,V obeys boundary condition (2.11) and condition (1) holds we have that $\mathscr{L}(U,V)=0$ i.e. L_1 is formally self-adjoint with respect to the inner product (2.14). In addition, since $\Gamma^*\Gamma+\Delta^*\Delta=I$ we have that $\Gamma^*\Gamma+\Delta^*\Delta$ has rank $2K$ and thus $[\Gamma^*,-\Delta^*]$ has maximal rank. Trivially, from the definition of A^* and B^*, $[A^*,-B^*]$ has maximal rank. □

It should be noted that $A^*,B^*,\Gamma^*,\Delta^*$ are not necessarily invertible. Hence, the system boundary value problem, (2.9)–(2.11), is more general than the boundary value problem studied in [23]. In [23] the author requires that at least one of the boundary condition matrices at each end point is invertible. In particular the boundary conditions considered in [23] are of the form

$$Y'(0)-hY(0)=0,\quad Y'(\pi)+HY(\pi)=0,$$

which is equivalent to $B^*=-\Delta^*=I$. Although this may appear to be a minor difference it in fact has significant consequences.

3. Self-adjoint boundary conditions on a graph

We now use the approach given by Kostrykin and Schrader, [14], in order to characterise self-adjoint boundary conditions on a compact graph. We then prove that this is equivalent to the conditions for formal self-adjointness given in Section 2.

Let L_0 be the minimal Sturm-Liouville operator such that

$$D(L_0)=\{f\,|\,f_i\in W^{2,2}(0,l_i), f_i^{(j)}(0)=f_i^{(j)}(l_j)=0, j=0,1, i=1,\dots,K\},\tag{3.1}$$

$$L_0f=\left(-\frac{d^2f_1}{dx^2}+q_1f_1,\dots,-\frac{d^2f_K}{dx^2}+q_Kf_K\right)\text{ for } f\in D(L_0),\tag{3.2}$$

where q_i is continuous and real valued for all $i=1,\dots,K$.

The defect indices of L_0 are then $(2K,2K)$.

Following the approach given in [14], we now wish to find all self-adjoint extensions of L_0. Let $D\subset\mathscr{L}^2(G)$ be defined as

$$D:=\{f\,|\,f_i\in W^{2,2}(0,l_i), i=1,\dots,K\}.$$

Set S to be the Lagrange form

$$S(f,g) = (Lf,g) - (f,Lg),$$

where L is as given in (2.5), (2.6).

Define $\Lambda : D \to \mathbb{C}^{4K}$ to be the surjective linear map which associates to each f the element $\Lambda(f)$ as

$$\Lambda(f) = \begin{pmatrix} f_1(0) \\ \vdots \\ f_K(0) \\ f_1(l_1) \\ \vdots \\ f_K(l_K) \\ f_1'(0) \\ \vdots \\ f_K'(0) \\ -f_1'(l_1) \\ \vdots \\ -f_K'(l_K) \end{pmatrix} := \begin{pmatrix} \underline{f} \\ \underline{f'} \end{pmatrix}.$$

Obviously $D(L_0)$ is the null space of the map Λ. Also, S vanishes identically on $D(L_0)$. Thus any self-adjoint extension of L_0 is given in terms of a maximal isotropic linear subspace on which the Lagrange form S vanishes identically. In order to find these maximal isotropic subspaces of D we use integration by parts as follows:

$$\begin{aligned} S(f,g) &= (Lf,g) - (f,Lg) \\ &= \sum_{i=1}^{K} [f_i \bar{g}_i' - f_i' \bar{g}_i]_0^{l_i} \\ &= (\Lambda(f), -J\Lambda(g))_{\mathbb{C}^{4K}} \end{aligned}$$

where $Lf_i = -f_i'' + q_i f_i$ for $f_i \in D(L)$, $J = \begin{bmatrix} 0 & I_{2K\times 2K} \\ -I_{2K\times 2K} & 0 \end{bmatrix}$ and $(\,,\,)_{\mathbb{C}^{4K}}$ is the scalar product on $\mathbb{C}^{4K}$.

Consider the linear subspace $V(\mathscr{A},\mathscr{B})$ of all $\Lambda(f) \in \mathbb{C}^{4K}$ such that

$$\mathscr{A}\underline{f} + \mathscr{B}\underline{f'} = 0 \tag{3.3}$$

where $\mathscr{A}$, $\mathscr{B}$ are $2K \times 2K$ matrices.

To find all maximal isotropic subspaces in D with respect to S it suffices to find all maximal isotropic subspaces in $\mathbb{C}^{4K}$ with respect to S and take their pre-image under the map Λ. Since J is non-degenerate such spaces all have complex dimension equal to $2K$.

The following theorem by Kostrykin and Schrader, [14], gives the conditions on $\mathscr{A}$ and $\mathscr{B}$ which define self-adjointness of the boundary value problem on a graph:

THEOREM 3.1. *If $[\mathscr{A},\mathscr{B}]$ has maximal rank then $V(\mathscr{A},\mathscr{B})$ is maximal isotropic if and only if $\mathscr{A}\mathscr{B}^*$ is self-adjoint.*

Proof. If the $2K \times 4K$ matrix $[\mathscr{A}, \mathscr{B}]$ has maximal rank equal to $2K$ then obviously V has dimension equal to $2K$. Also, the image of $\mathbb{C}^{4K}$ under the map $[\mathscr{A}, \mathscr{B}]$ is then all of $\mathbb{C}^{2K}$ since we have the general result that for any linear map T from $\mathbb{C}^{4K}$ into $\mathbb{C}^{2K}$

$$\dim \ker(T) + \dim \operatorname{range}(T) = 4K.$$

Noting that $J^2 = -I$ and $J^* = -J$ we have that a linear subspace V of $\mathbb{C}^{4K}$ is maximal isotropic if and only if $V^\perp = -JV$ and $V^\perp$ is maximal isotropic, where $V^\perp$ denotes the orthogonal complement with respect to $(\,,\,)_{\mathbb{C}^{4K}}$ of V.

We now rewrite (3.3) as $(F^{(i)}, \Lambda(f))_{\mathbb{C}^{4K}} = 0$ for $1 \leqslant i \leqslant 2K$, where $F^{(i)}$ is the ith column vector of the $4K \times 2K$ matrix $[\overline{\mathscr{A}}, \overline{\mathscr{B}}]^T = \begin{bmatrix} \mathscr{A}^* \\ \mathscr{B}^* \end{bmatrix}$. Clearly the $F^{(i)}$'s are linearly independent, so from above $V(\mathscr{A}, \mathscr{B})$ is maximal isotropic if and only if the space spanned by $F^{(i)}$ is maximal isotropic. This condition is tantamount to the condition that

$$[\mathscr{A}, \mathscr{B}] J \begin{bmatrix} \mathscr{A}^* \\ \mathscr{B}^* \end{bmatrix} = 0$$

which means $\mathscr{A}\mathscr{B}^*$ must be self-adjoint. $\square$

It should be noted that Kostrykin and Schrader, in [14], also characterise self-adjoint boundary conditions for non-compact graphs with both trivial compact part and non-trivial compact part.

The next theorem proves that the conditions for formal self-adjointness given in Theorem 2.1 are equivalent to those given in Theorem 3.1.

THEOREM 3.2. *The following are equivalent:*
(I) $\Gamma^* M^{-\frac{1}{2}} \Delta = \Delta^* M^{-\frac{1}{2}} \Gamma$ *and* $\Gamma^*\Gamma + \Delta^*\Delta = I$*;*
(II) $[\mathscr{A}, \mathscr{B}]$ *has maximal rank and* $\mathscr{A}\mathscr{B}^* = \mathscr{B}\mathscr{A}^*$.

Proof. From (3.3), (2.3) and (2.8) we obtain that

$$\mathscr{A} = [\alpha_{ij}, \gamma_{ij}] = [\tilde{A}, \tilde{C}] \text{ and } \mathscr{B} = [\beta_{ij}, -\delta_{ij}] = 2[\tilde{B}, -\tilde{D}] M^{-\frac{1}{2}},$$

where M is the weight matrix given in (2.9). Thus $\mathscr{A}\mathscr{B}^*$ self-adjoint implies that

$$[\tilde{A}, \tilde{C}] \begin{bmatrix} W^{-\frac{1}{2}} & 0 \\ 0 & W^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \tilde{B}^* \\ -\tilde{D}^* \end{bmatrix} = [\tilde{B}, -\tilde{D}] \begin{bmatrix} W^{-\frac{1}{2}} & 0 \\ 0 & W^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \tilde{A}^* \\ \tilde{C}^* \end{bmatrix}.$$

Hence we have the condition

$$\tilde{A} W^{-\frac{1}{2}} \tilde{B}^* - \tilde{C} W^{-\frac{1}{2}} \tilde{D}^* = \tilde{B} W^{-\frac{1}{2}} \tilde{A}^* - \tilde{D} W^{-\frac{1}{2}} \tilde{C}^*. \tag{3.4}$$

Now consider $\Delta^* M^{-\frac{1}{2}} \Gamma = \Gamma^* M^{-\frac{1}{2}} \Delta$. From the definition of Γ^* and Δ^* we get

$$[-\tilde{D}, \tilde{B}] \begin{bmatrix} W^{-\frac{1}{2}} & 0 \\ 0 & W^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \tilde{C}^* \\ \tilde{A}^* \end{bmatrix} = [\tilde{C}, \tilde{A}] \begin{bmatrix} W^{-\frac{1}{2}} & 0 \\ 0 & W^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} -\tilde{D}^* \\ \tilde{B}^* \end{bmatrix},$$

which when multiplying out gives exactly (3.4).

Note that $[\mathscr{A},\mathscr{B}]$ has maximal rank if and only if $[\tilde{A},\tilde{C},\tilde{B}W^{-\frac{1}{2}},-\tilde{D}W^{-\frac{1}{2}}]$ has maximal rank if and only if $[\tilde{C},\tilde{A},-\tilde{D}W^{-\frac{1}{2}},\tilde{B}W^{-\frac{1}{2}}]$ has maximal rank if and only if $[\Gamma^*,\Delta^* M^{-\frac{1}{2}}]$ has maximal rank if and only if $[\Gamma^*,-\Delta^*]$ has maximal rank and this implies that $\Gamma^*\Gamma+\Delta^*\Delta$ may, without loss of generality, be taken to be equal to the identity, as shown in Theorem 2.1. $\square$

REMARK. In Theorem 3.2 we only consider the system boundary conditions at the end point 1 since the end point 0 represents the artificial node inserted at the mid-point of each edge of the graph where we imposed continuity boundary conditions which are undoubtedly self-adjoint. In addition, as is shown in the next section, the complete graph structure as well as the original graph boundary conditions are totally incorporated in the system boundary condition at the end point 1.

4. Example

We now illustrate by means of an easy, yet applicable example, that the configuration of the graph is completely encapsulated in the boundary condition matrices at the terminal end point. In particular, given a self-adjoint graph boundary value problem we may construct the terminal end boundary condition matrices such that the system boundary value problem is self-adjoint and the boundary conditions are co-normal. In addition, Jordan reduction of the terminal end boundary condition matrices enables one to find graphs with the minimal number of loops on which the boundary value problem can be posed.

Consider the graph boundary value problem given by equation

$$-y_i''(x)+q_i(x)y_i(x)=\lambda y_i(x), \quad \text{on } [0,l_i], \tag{4.1}$$

where $i=1,2$ and $l_1=l_2=1$, with boundary conditions

$$y_2(0)=0, \tag{4.2}$$

$$y_1(0)=y_1(1), \tag{4.3}$$

$$y_1(0)=y_2(1) \tag{4.4}$$

and

$$y_1'(0)=y_1'(1)+y_2'(1). \tag{4.5}$$

We observe that equations (4.1) are equivalent to the system

$$-\tilde{Y}''+Q\tilde{Y}=\lambda\tilde{Y}, \tag{4.6}$$

on $[0,1]$, where $Q=\operatorname{diag}[q_1,q_2]$ and $\tilde{Y}=[\tilde{y}_1,\tilde{y}_2]^T$.

Now the boundary conditions, (4.2)–(4.5), on the graph may be written as

$$\sum_{j=1}^{2}\left[\alpha_{ij}y_j(0)+\beta_{ij}y'_j(0)\right]+\sum_{j=1}^{2}\left[\gamma_{ij}y_j(l_j)+\delta_{ij}y'_j(l_j)\right]=0, \quad i=1,\dots,4, \tag{4.7}$$

where $\alpha_{12} = \alpha_{21} = \alpha_{31} = \beta_{41} = 1$, $\gamma_{21} = \gamma_{32} = \delta_{41} = \delta_{42} = -1$ and all the others are equal to zero.

These boundary conditions transform to the system boundary conditions

$$\tilde{A}\tilde{Y}(0) + \tilde{B}\tilde{Y}'(0) + \tilde{C}\tilde{Y}(1) + \tilde{D}\tilde{Y}'(1) = 0, \tag{4.8}$$

where

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}.$$

Now, by inserting a node with continuity boundary conditions at the middle of each edge, the formally self-adjoint boundary value problem, (4.6), (4.8), is equivalent to a formally self-adjoint boundary value problem of dimension 4 with separated boundary conditions, i.e., is equivalent to a system of the form

$$-MY'' + PY = \lambda Y, \tag{4.9}$$

with boundary conditions

$$A^*Y(0) - B^*Y'(0) = 0, \tag{4.10}$$
$$\Gamma^*Y(1) - \Delta^*Y'(1) = 0, \tag{4.11}$$

where P is a diagonal potential matrix,

$$M = 4\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} := 4I_4, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad A^* = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B^* = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}, \quad \Gamma^* = [\tilde{C}, \tilde{A}] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\Delta^* = 2[-\tilde{D}, \tilde{B}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix}.$$

In addition, we require that $\Gamma^*\Gamma + \Delta^*\Delta = I = A^*A = B^*B$, $\Gamma^*M^{-\frac{1}{2}}\Delta = \Delta^*M^{-\frac{1}{2}}\Gamma$ and $A^*M^{-\frac{1}{2}}B = B^*M^{-\frac{1}{2}}A$. It is easy to verify that $I = A^*A = B^*B$ and $A^*M^{-\frac{1}{2}}B = B^*M^{-\frac{1}{2}}A$. In order for $\Gamma^*\Gamma + \Delta^*\Delta = I$ we rewrite Γ and Δ appropriately ensuring that the null space remains the same. Using Gram-Schmidt gives

$$\Gamma^* = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

By routine calculation one can show that these represent exactly the same boundary conditions as before and that $\Gamma^*\Gamma + \Delta^*\Delta = I$ and $\Gamma^* M^{-\frac{1}{2}}\Delta = \Delta^* M^{-\frac{1}{2}}\Gamma$.

In addition, it can be shown that the boundary conditions (4.11) are co-normal in the sense of [8, Appendix A]. To do this we must show that Δ and Γ may be written as

$$\Delta = \left[\frac{\underline{w}_1}{\sqrt{1+|\mu_1|^2}}, \ldots, \frac{\underline{w}_n}{\sqrt{1+|\mu_n|^2}}, 0, \ldots, 0 \right] \tag{4.12}$$

and

$$\Gamma = \left[\frac{\mu_1 \underline{w}_1}{\sqrt{1+|\mu_1|^2}}, \ldots, \frac{\mu_n \underline{w}_n}{\sqrt{1+|\mu_n|^2}}, \underline{w}_{n+1}, \ldots, \underline{w}_4 \right], \tag{4.13}$$

where $\underline{w}_1, \ldots, \underline{w}_4$ form an orthonormal basis for $\mathbb{C}^4$ and $\mu_1, \ldots, \mu_n$ are real numbers. We may, without altering the boundary conditions at all, write

$$\Gamma^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Delta^* = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

i.e.

$$\Gamma = \begin{bmatrix} 0 & -\sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Setting

$$\underline{w}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \ \underline{w}_2 = \begin{pmatrix} -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{pmatrix}, \ \underline{w}_3 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{w}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

we have that

$$\Delta = [\underline{w}_1, 0, 0, 0] \quad \text{and} \quad \Gamma = [0, \underline{w}_2, \underline{w}_3, \underline{w}_4].$$

So Γ and Δ are in the form (4.13) and (4.12) where $n = 1$, $\mu_1 = 0$ and $\underline{w}_1, \ldots, \underline{w}_4$ form an orthonormal basis for $\mathbb{C}^4$.

Thus the system boundary conditions (4.11) are directly and explicitly formed from the boundary conditions on the graph (4.2)–(4.5), i.e. from the given graph structure.

Although three different forms for Γ^* and Δ^* are found in the example above, the minimal graph on which the boundary conditions given by Γ^* and Δ^* can be posed is, in all three cases, (i.e. the graph with a minimal number of interactions at the nodes), a lasso type graph. Note that, one can give Γ^* and Δ^* which express equivalent boundary conditions to those above but which, from their form, appear to belong to a figure of eight graph (the maximal graph). That there is redundancy becomes apparent after Jordan reduction of the augmented matrix $[\Gamma^* : \Delta^*]$.

5. Interlacing of eigenvalues

To obtain interlacing results for the eigenvalues of the system we consider the arguments of the characteristic roots of the matrix Prüfer angle for Dirichlet boundary conditions, Neumann boundary conditions and general self-adjoint boundary conditions as given in (2.10) and (2.11).

In order to define the matrix Prüfer angle for the system boundary value problem we need the following theorem:

THEOREM 5.1. [9, Thm C] *Let* $\{Y(x,\lambda), Z(x,\lambda)\}$ *be the solution pair of*

$$Y' = Z, \quad Z' = -G(x,\lambda)Y$$

where $G(x,\lambda) = M^{-1}(\lambda I - P)$*, i.e the second order operator rewritten as a first order system, satisfying the the conditions*

$$Y(0,\lambda) \equiv B, \quad Z(0,\lambda) \equiv A.$$

Then there exists a continuous, symmetric matrix $H(x,\lambda)$ *and a nonsingular, continuously differentiable (in* x*) matrix* $T(x,\lambda)$ *such that*

$$Y(x,\lambda) = S^*(x,\lambda)T(x,\lambda), \quad Z(x,\lambda) = C^*(x,\lambda)T(x,\lambda)$$

for each λ*, where* $\{S(x,\lambda), C(x,\lambda)\}$ *is the solution of*

$$S' = H(x,\lambda)C, \quad C' = -H(x,\lambda)S, \tag{5.1}$$

$$S(0,\lambda) = B^*, \quad C(0,\lambda) = A^*. \tag{5.2}$$

Moreover , $T(x,\lambda)$ *is the solution of*

$$T' = [SC^* - CGS^*]T, \quad T(0,\lambda) = I$$

and

$$H(x,\lambda) = CC^* + SGS^*. \tag{5.3}$$

In [6] it was shown that for the system (2.9)–(2.11) the matrix Prüfer angle is given by

$$F(x,\lambda) = (V - iU)^{-1}(V + iU), \tag{5.4}$$

where

$$U(x,\lambda) = S(x,\lambda)\Gamma - C(x,\lambda)\Delta, \quad V(x,\lambda) = C(x,\lambda)\Gamma + S(x,\lambda)\Delta \tag{5.5}$$

and S, C are as given in Theorem 5.1.

If we have purely Dirichlet boundary conditions i.e. $\Delta = 0$ and $\Gamma = I$, then we have that the matrix Prüfer angle is given as

$$F^D(x,\lambda) = (C - iS)^{-1}(C + iS) = E(x,\lambda) \tag{5.6}$$

where C and S are as above since we have not altered A and B.

Let $f_j(x,\lambda)$, $j = 1\dots 2K$, denote the characteristic roots of $F(x,\lambda)$ and let $\beta_j(x,\lambda) = \arg f_j(x,\lambda)$ for each j, with the assumption that $\beta_j(x,\lambda)$ is a continuous function and $\beta_j(0,\lambda) \in [0,2\pi)$.

Similarly let $\phi_j(x,\lambda)$, $j = 1\dots 2K$, denote the characteristic roots of $E(x,\lambda)$ and let $\omega_j(x,\lambda) = \arg\phi_j(x,\lambda)$ for each j, where $\phi_j(x,\lambda)$ is a continuous function.

Then, from [6], we have that both the $\beta_j(x,\lambda)$'s and the $\omega_j(x,\lambda)$'s are monotonically increasing in λ and we have the equation

$$\sum_{j=1}^{2K}[\omega_j(1,\lambda) - \omega_j(0,\lambda)] = \sum_{j=1}^{2K}[\beta_j(1,\lambda) - \beta_j(0,\lambda)]. \tag{5.7}$$

Now since the boundary conditions are independent of λ we have that $\sum_{j=1}^{2K}\omega_j(0,\lambda)$ and $\sum_{j=1}^{2K}\beta_j(0,\lambda)$ are constants independent of λ.

THEOREM 5.2. *Let λ_i denote the eigenvalues of the system (2.9)–(2.11). Let λ_i^D denote the eigenvalues for the boundary value problem (2.9), (2.10) and*

$$Y(1) = 0. \tag{5.8}$$

Then there is at least one λ_i in the interval $[\lambda_n^D, \lambda_{n+4K-1}^D]$ and at least two λ_i's in the interval $[\lambda_n^D, \lambda_{n+4K}^D]$.

Proof. For the case of purely Dirichlet boundary conditions i.e. for the boundary value problem (2.9), (2.10) and (5.8) we have that the eigencondition which needs to be satisfied in order to get an eigenvalue of the system is given by

$$\omega_j(1,\lambda) = 0(mod 2\pi), \tag{5.9}$$

for some $j \in \{1,\dots,2K\}$, whereas for the general boundary conditions i.e. for the boundary value problem (2.9)–(2.11) we have the eigencondition

$$\beta_j(1,\lambda) = 0(mod 2\pi) \tag{5.10}$$

for some $j \in \{1, \dots, 2K\}$.

In order to guarantee that at least one of the $\beta_j(1,\lambda)$'s, $j \in \{1, \dots, 2K\}$, has increased by 2π i.e. to guarantee that we have an eigenvalue of the general system, we require $\sum_{j=1}^{2K} \omega_j(1,\lambda)$ to increase by at least $4K\pi$. This is since, from equation (5.7), we would then have that $\sum_{j=1}^{2K} \beta_j(1,\lambda)$ must increase by at least $4K\pi$ and if none of the $\beta_j(1,\lambda)$'s increase by at least 2π then we have the situation where $\sum_{j=1}^{2K} \beta_j(1,\lambda)$ increases by $(2\pi - \varepsilon)2K = 4K\pi - 2\varepsilon K$ which is less that $4K\pi$ and gives us a contradiction.

From equation (5.9) we have that as λ increases from λ_n^D to λ_{n+2K}^D we are ensured that $\sum_{j=1}^{2K} \omega_j(1,\lambda)$ increases by at least 2π since (5.9) has been solved $2K+1$ times. However, if λ increases from λ_n^D to λ_{n+2K+1}^D then $\sum_{j=1}^{2K} \omega_j(1,\lambda)$ increases by at least 4π. Proceeding in this manner, as λ increases from λ_n^D to λ_{n+N}^D, $\sum_{j=1}^{2K} \omega_j(1,\lambda)$ increases by at least $2\pi(N-(2K-1))$. So, in order for $\sum_{j=1}^{2K} \omega_j(1,\lambda)$ to increase by at least $4K\pi$ we need $2\pi(N-2K+1) \geqslant 4K\pi$, giving that $n \geqslant 4K-1$. Hence in the interval $[\lambda_n^D, \lambda_{n+4K-1}^D]$ we have at least one eigenvalue of (2.9)–(2.11).

Now consider the interval $[\lambda_n^D, \lambda_{n+4K}^D]$. Then by the above calculations $\sum_{j=1}^{2K} \omega_j(1,\lambda)$ increases by at least $2\pi(2K+1)$ as λ increases from λ_n^D to λ_{n+4K}^D. Hence $\sum_{j=1}^{2K} \beta_j(1,\lambda)$ increases by at least $2\pi(2K+1)$ as λ increases from λ_n^D to λ_{n+4K}^D. Then at least one β_j has increased by at least 2π, say β_1. If more than one β_j has increased by 2π, the result is proved. So assume that only one β_j has increased by at least 2π. Then β_j, $j = 2, \dots, 2K$, each increase by at most $2\pi - \varepsilon$ for some $0 < \varepsilon < 2\pi$. Thus the increase in $\sum_{j=1}^{2K} \beta_j(1,\lambda)$ for λ increasing from λ_n^D to λ_{n+4K}^D is at most $(2K-1)(2\pi - \varepsilon)$, making the increase in β_1 at least $2\pi(2K+1) - (2K-1)(2\pi - \varepsilon) = 4\pi(2K-1)\varepsilon > 4\pi$. Thus there are at least two eigenvalues of (2.9)–(2.11) in $[\lambda_n^D, \lambda_{n+4K}^D]$. $\square$

Next we consider the case where we have purely Neumann boundary conditions, i.e. $\Delta = I$ and $\Gamma = 0$ in (2.11). The matrix Prüfer angle is then given by

$$F^N(x,\lambda) = (S - iC)^{-1}(S + iC) \tag{5.11}$$

where C and S are as defined in Theorem 5.1.

Let $f_j^N(x,\lambda)$, $j = 1 \dots 2K$, denote the characteristic roots of $F^N(x,\lambda)$ and let $\beta_j^N(x,\lambda) = \arg f_j^N(x,\lambda)$ for each j, with the assumption that $\beta_j^N(x,\lambda)$ is a continuous function.

THEOREM 5.3. *Let λ_i denote the eigenvalues of the system (2.9)–(2.11). Let λ_i^N denote the eigenvalues for the boundary value problem (2.9), (2.10) and*

$$Y'(1) = 0. \tag{5.12}$$

Then there is at least one λ_i in the interval $[\lambda_n^N, \lambda_{n+4K-1}^N]$ and at least two λ_i's in the interval $[\lambda_n^N, \lambda_{n+4K}^N]$.

Proof. We begin by relating $F^N(x,\lambda)$ and $F^D(x,\lambda)$.

$$F^N(x,\lambda) = (S - iC)^{-1}(S + iC) = (-C - iS)^{-1}(C - iS) = -(F^D(x,\lambda))^{-1}.$$

Thus

$$e^{i\beta_j^N} = -e^{-i\omega_j} = e^{i(\pi-\omega_j)},$$

which implies that

$$\beta_j^N = (2\pi k_j - \pi) - \omega_j$$

where k_j is a constant. The interlacing result now follows directly from the analysis used in Theorem 5.2 since adding an additional constant term into equation (5.7) does not change any of the reasoning. Hence we obtain that in the interval $[\lambda_n^N, \lambda_{n+4K-1}^N]$ we have at least one eigenvalue of (2.9)–(2.11) and in the interval $[\lambda_n^N, \lambda_{n+4K}^N]$ we have at least two eigenvalues of (2.9)–(2.11). $\square$

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