Operators and Matrices
Volume 8, Number 2 (2014), 485–512
doi:10.7153/oam-08-25

CLASSICAL ADJOINT COMMUTING MAPPINGS ON ALTERNATE MATRICES AND SKEW–HERMITIAN MATRICES

WAI LEONG CHOOI AND WEI SHEAN NG

(*Communicated by C.-K. Li*)

Abstract. Let n be an even integer with $n \geqslant 4$. In this note we study classical adjoint commuting mappings ψ on the space of $n \times n$ alternate matrices, and on the space of $n \times n$ skew-Hermitian matrices with respect to a proper involution, satisfying one of the following conditions:

- $\psi(\operatorname{adj}(A+\alpha B)) = \operatorname{adj}(\psi(A)+\alpha\psi(B))$
- $\psi(\operatorname{adj}(A-B)) = \operatorname{adj}(\psi(A)-\psi(B))$ and ψ is surjective

for scalar α and matrices A,B in each respective matrix spaces. Here, $\operatorname{adj} A$ denotes the classical adjoint of a matrix A.

1. Introduction

Let $\mathbb{F}$ be a field and m,n be positive integers. By $\mathcal{M}_{m,n}(\mathbb{F})$ we denote the linear space of $m \times n$ matrices over $\mathbb{F}$. If $m = n$, we simply write $\mathcal{M}_n(\mathbb{F}) = \mathcal{M}_{n\times n}(\mathbb{F})$. Let $A \in \mathcal{M}_n(\mathbb{F})$. We say that A is an *alternate* matrix if $A^t = -A$ and the diagonal elements of A are all zero, or equivalently $u^tAu = 0$ for all $u \in \mathcal{M}_{n,1}(\mathbb{F})$, where A^t stands for the transpose of A. Suppose that $\mathbb{F}$ is a field which possesses an involution $^-$ of $\mathbb{F}$ (i.e., $^- : \mathbb{F} \to \mathbb{F}$ is an automorphism of $\mathbb{F}$ such that $\overline{\overline{a}} = a$ for all $a \in \mathbb{F}$). Then A is said to be *skew-Hermitian* (respectively, *Hermitian*) with respect to the involution $^-$ of $\mathbb{F}$ if $\overline{A}^t = -A$ (respectively, $\overline{A}^t = A$). Here, $\overline{A}$ is the matrix obtained from A by applying $^-$ entrywise. Let $\mathbb{F}^- := \{a \in \mathbb{F} : \overline{a} = a\}$ (respectively, $S\mathbb{F}^- := \{a \in \mathbb{F} : \overline{a} = -a\}$) denote the set of all symmetric elements (respectively, skew-symmetric elements) of $\mathbb{F}$ with respect to the involution $^-$ of $\mathbb{F}$. One can easily check that $\mathbb{F}^-$ forms a subfield of $\mathbb{F}$ and is called the *fixed field* with respect to the involution $^-$. Evidently, $\mathbb{F}^- = \mathbb{F}$ when the involution $^-$ is identity. Otherwise, the involution $^-$ is *proper*. Throughout, we shall use $\mathcal{K}_n(\mathbb{F})$, $\mathcal{SH}_n(\mathbb{F})$ and $\mathcal{H}_n(\mathbb{F})$ to designate the linear space of all $n \times n$ alternate matrices over $\mathbb{F}$, the $\mathbb{F}^-$-linear space of all $n \times n$ skew-Hermitian matrices over $\mathbb{F}$, and the $\mathbb{F}^-$-linear space of all $n \times n$ Hermitian matrices over $\mathbb{F}$, respectively.

The *classical adjoint*, sometimes called the *adjugate*, of a matrix $A \in \mathcal{M}_n(\mathbb{F})$, denoted by $\operatorname{adj} A$, is the $n \times n$ matrix whose (i,j)-th entry is the (j,i)-th cofactor of A. The notion of the classical adjoint is one of the important matrix functions on

Mathematics subject classification (2010): 15A03, 15A04, 15A86.
Keywords and phrases: Classical adjoint, alternate matrices, skew-Hermitian matrices, preserver problems, geometry of matrices.

© ELEMENT, Zagreb
Paper OaM-08-25

square matrices and has been employed to various studies of generalized invertibility of matrices, see [15]. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be matrix spaces such that $\operatorname{adj} A \in \mathscr{M}_i$ whenever $A \in \mathscr{M}_i$ for $i = 1,2$. A mapping $\psi : \mathscr{M}_1 \to \mathscr{M}_2$ is called *classical adjoint-commuting* if

$$\psi(\operatorname{adj} A) = \operatorname{adj} \psi(A) \quad \text{for every } A \in \mathscr{M}_1. \tag{1.1}$$

The study of classical adjoint commuting linear mappings was initiated by Sinkhorn in [17] over the complex field by using the classical result Frobenius [5] concerning determinant linear preservers. Later on, similar problems on various matrix spaces have been considered, see [18, 19, 20, 21], and [23, Chapter 10] and the references therein. Recently, inspired by the works of [3, 16], the present authors started the study of classical adjoint commuting mappings ψ on the space of square matrices over a field in [1], and on the space of Hermitian matrices over a field which possesses an involution, see [2], satisfying one of the following two conditions:

(A1) $\quad \psi(\operatorname{adj}(A + \alpha B)) = \operatorname{adj}(\psi(A) + \alpha\psi(B))$

(A2) $\quad \psi(\operatorname{adj}(A - B)) = \operatorname{adj}(\psi(A) - \psi(B))$

for scalar α and matrices A, B in each respective matrix space. One knows that ψ satisfies condition (**A1**) or (**A2**) implies that $\psi(0) = 0$, and so condition (1.1) holds true for ψ.

Note that when n is a positive even integer, by $\operatorname{adj}(-A) = (-1)^{n-1}\operatorname{adj} A$ for any $A \in \mathscr{M}_n(\mathbb{F})$, we see that if A is an alternate matrix (respectively, a skew-Hermitian matrix with respect to an involution $^-$ of $\mathbb{F}$), then $\operatorname{adj} A$ is alternate (respectively, skew-Hermitian) because $\operatorname{adj} A$ has zero diagonal entries and $(\operatorname{adj} A)^t = -\operatorname{adj} A$ (respectively, $\overline{(\operatorname{adj} A)}^t = -\operatorname{adj} A$).

Let n be an even integer with $n \geqslant 4$. In this present note, basically, by employing a similar idea and technique used in [1, 2], we continue to study classical adjoint commuting mappings ψ on the space of $n \times n$ alternate matrices, and on the space of $n \times n$ skew-Hermitian matrices with respect to a proper involution, satisfying either condition (**A1**) or condition (**A2**). Let $\mathbb{F}[x]$ denote the ring of polynomials in an indeterminate x over a field $\mathbb{F}$. More precisely, we prove the following results:

THEOREM 1. *Let n be an even integer such that $n \geqslant 4$. Let $\mathbb{F}$ be a field with at least $n+2$ elements such that $x^{n-1} - a \in \mathbb{F}[x]$ has a root for every $a \in \mathbb{F}$. Then $\psi : \mathscr{K}_n(\mathbb{F}) \to \mathscr{K}_n(\mathbb{F})$ is a mapping satisfying*

$$\psi(\operatorname{adj}(A + \alpha B)) = \operatorname{adj}(\psi(A) + \alpha\psi(B)) \tag{1.2}$$

for every $A, B \in \mathscr{K}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$ if and only if either $\psi(A) = 0$ for every invertible matrix $A \in \mathscr{K}_n(\mathbb{F})$ and $\operatorname{rank}(\psi(A) + \alpha\psi(B)) \leqslant n-2$ for every $A, B \in \mathscr{K}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$; or there exist an invertible matrix $P \in \mathscr{M}_n(\mathbb{F})$ with $P^tP = \mu I_n$, and nonzero scalars $\mu, \lambda \in \mathbb{F}$ with $(\lambda\mu)^{n-2} = 1$, such that either

$$\psi(A) = \lambda PAP^t \quad \textit{for every } A \in \mathscr{K}_n(\mathbb{F})$$

or when $n=4$,

$$\psi(A)=\lambda PA^{*}P^{t} \quad \text{for every } A\in\mathscr{K}_4(\mathbb{F}),$$

where

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^{*} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}. \tag{1.3}$$

THEOREM 2. *Let* m *and* n *be even integers such that* $m,n\geqslant 4$. *Let* $\mathbb{K}$ *be a field with at least three elements, and let* $\mathbb{F}$ *be a field with at least three elements such that* $x^{n-1}-a\in\mathbb{F}[x]$ *has a root for every* $a\in\mathbb{F}$. *Then* $\psi:\mathscr{K}_n(\mathbb{F})\to\mathscr{K}_m(\mathbb{K})$ *is a surjective mapping satisfying*

$$\psi(\operatorname{adj}(A-B))=\operatorname{adj}(\psi(A)-\psi(B)) \tag{1.4}$$

for every $A,B\in\mathscr{K}_n(\mathbb{F})$ *if and only if* $m=n$, $\mathbb{F}$ *and* $\mathbb{K}$ *are isomorphic, and there exist a field isomorphism* $\sigma:\mathbb{F}\to\mathbb{K}$, *an invertible matrix* $P\in\mathscr{M}_n(\mathbb{K})$ *with* $P^tP=\mu I_n$, *and nonzero scalars* $\mu,\lambda\in\mathbb{K}$ *with* $(\lambda\mu)^{n-2}=1$, *such that either*

$$\psi(A)=\lambda PA^{\sigma}P^{t} \quad \text{for every } A\in\mathscr{K}_n(\mathbb{F})$$

or when $n=4$,

$$\psi(A)=\lambda P(A^{*})^{\sigma}P^{t} \quad \text{for every } A\in\mathscr{K}_4(\mathbb{F}),$$

where

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^{*} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Here, A^{σ} *is the matrix obtained from* $A=(a_{ij})$ *by applying* σ *entrywise, i.e.,* $A^{\sigma}=(\sigma(a_{ij}))$.

THEOREM 3. *Let* m *and* n *be even integers such that* $m,n\geqslant 4$. *Let* $\mathbb{F}$ *be a field which possesses a proper involution* $^{-}$ *of* $\mathbb{F}$ *such that either* $|\mathbb{F}^{-}|=2$ *or* $|\mathbb{F}^{-}|>n+1$. *Then* $\psi:\mathscr{SH}_n(\mathbb{F})\to\mathscr{SH}_m(\mathbb{F})$ *is a mapping satisfying*

$$\psi(\operatorname{adj}(A+\alpha B))=\operatorname{adj}(\psi(A)+\alpha\psi(B))$$

for every $A,B\in\mathscr{SH}_n(\mathbb{F})$ *and* $\alpha\in\mathbb{F}^{-}$ *if and only if either* $\psi(A)=0$ *for every rank one matrix* $A\in\mathscr{SH}_n(\mathbb{F})$ *and* $\operatorname{rank}(\psi(A)+\alpha\psi(B))\leqslant m-2$ *for every* $A,B\in\mathscr{SH}_n(\mathbb{F})$ *and* $\alpha\in\mathbb{F}^{-}$; *or* $m=n$ *and*

$$\psi(A)=\lambda PA^{\sigma}\overline{P}^{t} \quad \text{for every } A\in\mathscr{SH}_n(\mathbb{F}),$$

where $\sigma:\mathbb{F}\to\mathbb{F}$ *is a field isomorphism satisfying* $\overline{\sigma(a)}=\sigma(\overline{a})$ *for all* $a\in\mathbb{F}$ *and* $\sigma(a)=a$ *for all* $a\in\mathbb{F}^{-}$, $P\in\mathscr{M}_n(\mathbb{F})$ *is invertible with* $\overline{P}^{t}P=\varsigma I_n$, *and* $\lambda,\varsigma\in\mathbb{F}^{-}$ *are scalars with* $(\lambda\varsigma)^{n-2}=1$.

THEOREM 4. *Let m and n be even integers such that $m,n \geqslant 4$. Let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess proper involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively, such that either $|\mathbb{K}^\wedge| = 2$, or $|\mathbb{F}^-|, |\mathbb{K}^\wedge| > 3$. Then $\psi : \mathscr{SH}_n(\mathbb{F}) \to \mathscr{SH}_m(\mathbb{K})$ is a surjective mapping satisfying*

$$\psi(\operatorname{adj}(A-B)) = \operatorname{adj}(\psi(A)-\psi(B))$$

for every $A,B \in \mathscr{SH}_n(\mathbb{F})$ if and only if $m=n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic, and

$$\psi(A) = \lambda P A^\sigma \widehat{P}^t \quad \textit{for every } A \in \mathscr{SH}_n(\mathbb{F}),$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a field isomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, $P \in \mathscr{M}_n(\mathbb{K})$ is invertible with $\widehat{P}^t P = \varsigma I_n$ and $\lambda, \varsigma \in \mathbb{K}^\wedge$ are scalars with $(\lambda\varsigma)^{n-2} = 1$.

Besides these results, we have also classified surjective classical adjoint commuting additive mappings on alternate matrices (in Corollary 1) and characterized classical adjoint commuting additive mappings on skew-Hermitian matrices (in Theorem 5). In Proposition 4, we address a general description of the structure of mappings $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{F})$ that satisfy

$$\varphi(\mu^{n-2}\operatorname{adj}(A+\alpha B)) = \mu^{m-2}\operatorname{adj}(\varphi(A)+\alpha\varphi(B))$$

for every $A,B \in \mathscr{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$, where μ is a fixed nonzero scalar in $\mathbb{F}^- \cup S\mathbb{F}^-$. This result serves as a tool in the proof of Theorem 3, and also it slightly improves a result and corrects a misprint in [2, Theorem 2.12].

Before starting our proofs, we give some examples of nonzero degenerate classical adjoint commuting mappings on alternate matrices sending invertible matrices to zero, and nonzero degenerate classical adjoint commuting mappings on skew-Herimitian matrices that map rank one matrices and invertible matrices to zero.

EXAMPLE 1. Let m and n be even integers such that $m,n \geqslant 4$.

(i) Let $\mathbb{F}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $f : \mathbb{F} \to \mathbb{F}$ be a nonzero function and let $\psi_1 : \mathscr{K}_n(\mathbb{F}) \to \mathscr{K}_m(\mathbb{F})$ be the mapping defined by

$$\psi_1(A) = \begin{cases} f(a_{12})(E_{12}-E_{21}) & \text{if } A=(a_{ij}) \text{ is of rank } k \text{ with } 2 \leqslant k \leqslant n-2 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let $\mathbb{F}$ be a field with $n-1$ elements. Let $g : \mathscr{K}_n(\mathbb{F}) \to \mathbb{F}$ and $h : \mathbb{F} \to \mathbb{F}$ be nonzero functions. Let $\psi_2 : \mathscr{K}_n(\mathbb{F}) \to \mathscr{K}_m(\mathbb{F})$ be the mapping defined by

$$\psi_2(A) = \begin{cases} \sum_{i=1}^{\frac{m}{2}-1} h(a_{12})(E_{2i-1,2i}-E_{2i,2i-1}) & \text{if } A=(a_{ij}) \text{ is of rank two} \\ g(A)(E_{12}-E_{21}) & \text{if } A \text{ is of rank } k,\ 2<k<n \\ 0 & \text{otherwise.} \end{cases}$$

Here, E_{ij} stands for the square matrix unit whose (i,j)-th entry is one and zero elsewhere. It is easily verified that each ψ_i satisfies conditions (**A1**) and (**A2**), and sends invertible matrices to zero.

EXAMPLE 2. Let m and n be even integers such that $m,n \geqslant 4$, and let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess proper involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively.

(i) Let $\lambda, \lambda_1, \ldots, \lambda_{m-2} \in S\mathbb{K}^\wedge := \{a \in \mathbb{K} : \widehat{a} = -a\}$ be nonzero scalars. Let $\varphi_1 : \mathscr{SH}_n(\mathbb{F}) \to \mathscr{SH}_m(\mathbb{K})$ be the mapping defined by

$$\varphi_1(A) = \begin{cases} \lambda E_{11} & \text{if } A \text{ is of rank } k,\ 2 < k < n \\ \sum_{i=1}^{m-2} \lambda_i E_{ii} & \text{if } A \text{ is of rank two} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ be a field isomorphism such that $\sigma(\overline{a}) = \widehat{\sigma(a)}$ for all $a \in \mathbb{F}$. Let $\varphi_2 : \mathscr{SH}_n(\mathbb{F}) \to \mathscr{SH}_m(\mathbb{K})$ be the mapping defined by

$$\varphi_2(A) = \begin{cases} \sigma(a_{12})E_{12} + \sigma(a_{21})E_{21} & \text{if } A = (a_{ij}) \text{ is of rank } k \text{ with } 1 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

Each φ_i satisfies conditions (**A1**) and (**A2**) sending rank one matrices as well as invertible matrices to zero.

We remark that each nonzero degenerate classical adjoint commuting mapping provided in Examples 1 and 2 is neither injective nor surjective.

2. Alternate matrices

Let n be an integer such that $n \geqslant 2$ and $\mathbb{F}$ be a field. It is an elementary fact that each nonzero alternate matrix $A \in \mathscr{K}_n(\mathbb{F})$ is necessarily of even rank and can be expressed as

$$A = P(J_1 \oplus \cdots \oplus J_r \oplus 0_{n-2r})P^t \tag{2.5}$$

for some integer $1 \leqslant r \leqslant n/2$ and invertible matrix $P \in \mathscr{M}_n(\mathbb{F})$, where

$$J_1 = \cdots = J_r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathscr{M}_2(\mathbb{F}), \tag{2.6}$$

see for instance [10, p.g. 161] or [22, Proposition 1.34]. Denote $J_n := J_1 \oplus \cdots \oplus J_{n/2} \in \mathscr{K}_n(\mathbb{F})$. When n is even, J_n is invertible and $\operatorname{adj} J_n = -J_n$. If $A \in \mathscr{K}_n(\mathbb{F})$ is an alternate matrix with n even, then each (i,i)-th cofactor of A is zero. It follows that $\operatorname{adj} A$ has zero diagonal entries. Moreover, since $(\operatorname{adj} A)^t = (-1)^{n-1} \operatorname{adj} A = -\operatorname{adj} A$, we have $\operatorname{adj} A \in \mathscr{K}_n(\mathbb{F})$ and

$$\operatorname{rank} \operatorname{adj} A = \begin{cases} 0 & \text{if } \operatorname{rank} A \neq n, \\ n & \text{if } \operatorname{rank} A = n. \end{cases} \tag{2.7}$$

For the basic properties and preliminary results of classical adjoint matrices we refer the reader, for instance, to [23, Appendix D].

Let q be an integer such that $q \geqslant 2$. Let $\mathbb{F}$ be a field and $\mathbb{F}[x]$ be the ring of polynomials in an indeterminate x over $\mathbb{F}$. Evidently, if $\mathbb{F}$ is algebraically closed, then the following condition:

$$x^q - a \in \mathbb{F}[x] \text{ has a root in } \mathbb{F} \text{ for every } a \in \mathbb{F} \tag{2.8}$$

holds in $\mathbb{F}$. Besides algebraically closed fields, we see that

- if $\mathbb{F} = \mathbb{F}_p$ is a Galois field of p elements with $p = 2$ or $p^r = kq$ for some positive integers r and k, then, by the fact that $a^p = a$ for every $a \in \mathbb{F}_p$, condition (2.8) holds true in $\mathbb{F}_p$;
- if q is odd and $\mathbb{F}$ is the real field $\mathbb{R}$, then it follows from the intermediate value theorem that condition (2.8) holds in $\mathbb{R}$.

By this observation, we have the following result.

PROPOSITION 1. *Let n be an integer such that $n \geqslant 2$, and let $\mathbb{F}$ be a field. Then $\mathbb{F}$ satisfies condition (2.8) for $q = n-1$ if and only if for each invertible matrix $A \in \mathscr{M}_n(\mathbb{F})$, there exists an invertible matrix $B \in \mathscr{M}_n(\mathbb{F})$ such that $A = \operatorname{adj} B$.*

Proof. We prove the necessity part. Let $A \in \mathscr{M}_n(\mathbb{F})$ be invertible. Denote $\lambda := (\det A)^{n-2}$. Then $\lambda \neq 0$ and there is a nonzero scalar $\lambda_0 \in \mathbb{F}$ such that $\lambda_0^{n-1} = \lambda^{-1}$. Thus $A = \lambda^{-1}(\lambda A) = \operatorname{adj} B$, where $B = \lambda_0(\operatorname{adj} A) \in \mathscr{M}_n(\mathbb{F})$ is invertible. We are done.

We now consider the sufficiency part. Let $a \in \mathbb{F}$. We claim that there exists a scalar α_0 in $\mathbb{F}$ such that $\alpha_0^{n-1} - a = 0$. The result is trivial when $a = 0$. We consider $a \neq 0$. Then there exists an invertible matrix $B_0 \in \mathscr{M}_n(\mathbb{F})$ such that $\operatorname{adj} B_0 = aI_n$. Hence $(\det B_0)^{n-2} B_0 = \operatorname{adj}(\operatorname{adj} B_0) = \operatorname{adj}(aI_n) = a^{n-1} I_n$. So B_0 is diagonal. Let $B_0 = \alpha_0 I_n$ for some scalar $\alpha_0 \in \mathbb{F}$. Then $\alpha_0^{n-1} I_n = \operatorname{adj} B_0 = aI_n$ implies that $\alpha_0^{n-1} = a$. Consequently, $\mathbb{F}$ satisfies condition (2.8) for $q = n-1$. We are done. $\square$

Inspired by Proposition 1, we obtain the following lemma.

LEMMA 1. *Let n be a positive even integer and $\mathbb{F}$ be a field. Then $\mathbb{F}$ satisfies condition (2.8) for $q = n-1$ if and only if for each invertible matrix $A \in \mathscr{K}_n(\mathbb{F})$, there exists an invertible matrix $B \in \mathscr{K}_n(\mathbb{F})$ such that $A = \operatorname{adj} B$.*

Proof. Let $A \in \mathscr{K}_n(\mathbb{F})$ be invertible. By Proposition 1, there exists an invertible matrix $B \in \mathscr{M}_n(\mathbb{F})$ such that $A = \operatorname{adj} B$. Since $B = (\det B)^{-(n-2)} \operatorname{adj} A$, it follows that $B \in \mathscr{K}_n(\mathbb{F})$.

Conversely, let $a \in \mathbb{F}$. We claim that there exists $\alpha_0 \in \mathbb{F}$ such that $\alpha_0^{n-1} = a$. The result is clear when $a = 0$. Consider now $a \neq 0$. Then $aJ_n = \operatorname{adj} B_0$ for some invertible matrix $B_0 \in \mathscr{K}_n(\mathbb{F})$. Since $(\det B_0)^{n-2} B_0 = \operatorname{adj}(aJ_n) = -a^{n-1} J_n$, it follows that $B_0 = -\alpha_0 J_n$ for some scalar $\alpha_0 \in \mathbb{F}$. So $\alpha_0^{n-1} J_n = \operatorname{adj}(-\alpha_0 J_n) = aJ_n$. This yields $\alpha_0^{n-1} = a$, as desired. Then $\mathbb{F}$ satisfies condition (2.8) for $q = n-1$. This completes our proof. $\square$

In what follows, unless otherwise stated, we let m and n be even integers such that $m, n \geqslant 4$, and let $\mathbb{F}$ and $\mathbb{K}$ denote fields.

LEMMA 2. *Let $A, B \in \mathscr{K}_n(\mathbb{F})$. Then the following statements hold.*

(a) *If A is of rank r, then there exists a rank $n-r$ matrix $X_1 \in \mathscr{K}_n(\mathbb{F})$ such that* $\operatorname{rank}(A + X_1) = n$.

(b) *There exists a matrix* $X_2 \in \mathcal{K}_n(\mathbb{F})$ *such that* $\operatorname{rank}(A+X_2) = \operatorname{rank}(B+X_2) = n$.

(c) *There exists a nonzero matrix* $X_3 \in \mathcal{K}_n(\mathbb{F})$ *such that either* A *or* X_3 *is of rank* n *but not both with* $\operatorname{rank}(A+X_3) = n$.

(d) *If* $|\mathbb{F}| > n+1$ *and* $\operatorname{rank}(A+B) = n$, *then there exists a scalar* $\lambda \in \mathbb{F}$ *with* $\lambda \neq 1$ *such that* $\operatorname{rank}(A+\lambda B) = n$.

Proof. Recall that $J_1, \ldots, J_{n/2}$ denote the 2×2 alternate matrix defined in (2.6). Suppose that $A \in \mathcal{K}_n(\mathbb{F})$ is of rank r. It follows from (2.5) that $r \geqslant 0$ is necessarily even, and there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ such that

$$A = P(J_1 \oplus \cdots \oplus J_{r/2} \oplus 0_{n-r})P^t. \tag{2.9}$$

(a) In view of (2.9), we select $X_1 = P(0_r \oplus J_{r+1} \oplus \cdots \oplus J_{n/2})P^t \in \mathcal{K}_n(\mathbb{F})$. It is clear that X_1 is of rank $n-r$ and $A+X_1$ is of rank n, as required.

(b) Suppose that $A = B$. It follows from (a) that there exists a matrix $X_2 \in \mathcal{K}_n(\mathbb{F})$ such that $\operatorname{rank}(A+X_2) = n$. We consider $A \neq B$. Let $H := A - B \in \mathcal{K}_n(\mathbb{F})$ be of rank k with $0 < k \leqslant n$ even. By (2.5), there exists an invertible matrix $Q \in \mathcal{M}_n(\mathbb{F})$ such that $H = Q(J_1 \oplus \cdots \oplus J_{k/2} \oplus 0_{n-k})Q^t$. Let h be the odd integer such that $\frac{n}{2} - 1 \leqslant h \leqslant \frac{n}{2}$. We set

$$C = \begin{cases} QSQ^t & \text{if } k < \frac{n}{2}+1 \\ Q(S-T)Q^t & \text{if } k \geqslant \frac{n}{2}+1 \text{ and } h = \frac{n}{2}-1 \\ Q(U-V)Q^t & \text{if } k \geqslant \frac{n}{2}+1 \text{ and } h = \frac{n}{2} \end{cases}$$

where $S = (E_{1n} - E_{2,n-1}) + \cdots + (E_{n-1,2} - E_{n1}) \in \mathcal{K}_n(\mathbb{F})$, $T = J_1 \oplus \cdots \oplus J_{n/4} \oplus 0_{n/2} \in \mathcal{K}_n(\mathbb{F})$, $V = J_1 \oplus \cdots \oplus J_{(n+2)/4} \oplus 0_{(n-2)/2} \in \mathcal{K}_n(\mathbb{F})$, $Z_p = E_{1p} + E_{2,p-1} + \cdots + E_{p1} \in \mathcal{M}_p(\mathbb{F})$ with $p = (n-4)/2$, and

$$Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \in \mathcal{K}_4(\mathbb{F}) \text{ and } U = \begin{pmatrix} 0_{(n-4)/2} & 0 & Z_{(n-4)/2} \\ 0 & Z & 0 \\ -Z_{(n-4)/2} & 0 & 0_{(n-4)/2} \end{pmatrix} \in \mathcal{K}_n(\mathbb{F}).$$

It can be checked that $C \in \mathcal{K}_n(\mathbb{F})$ is of rank n and $\operatorname{rank}(H+C) = n$. Let $X_2 := C - B$. It is easy to see that $X_2 \in \mathcal{K}_n(\mathbb{F})$, and $A + X_2 = H + C$ and $B + X_2 = C$ are of rank n. We are done.

(c) If A is of rank n, then, by (2.9), we have $A = PJ_nP^t$. We select

$$X_3 := P(E_{1n} - E_{n1})P^t \in \mathcal{K}_n(\mathbb{F}).$$

It is clear that $\operatorname{rank} X_3 = 2 < n$ and $\operatorname{rank}(A+X_3) = n$, as required. We now consider $\operatorname{rank} A = r < n$. If $A = 0$, then we take $X_3 = J_n$. Suppose that $A \neq 0$. Let h be the odd integer such that $\frac{n}{2} - 1 \leqslant h \leqslant \frac{n}{2}$. In view of (2.9), we choose

$$X_3 = \begin{cases} PSP^t & \text{if } r < \frac{n}{2}+1 \\ P(S-T)P^t & \text{if } r \geqslant \frac{n}{2}+1 \text{ and } h = \frac{n}{2}-1 \\ P(U-V)P^t & \text{if } r \geqslant \frac{n}{2}+1 \text{ and } h = \frac{n}{2}, \end{cases}$$

where $S,T,U,V \in \mathscr{K}_n(\mathbb{F})$ are alternate matrices as defined in (b). Then $X_3 \in \mathscr{K}_n(\mathbb{F})$ is of rank n and $\operatorname{rank}(A+X_3)=n$. We are done.

(d) The result is clear when $B=0$. Consider now $B \neq 0$. For each $x \in \mathbb{F}$, we denote $p(x)=\det(A+xB)$. Then $p(x) \in \mathbb{F}[x]$ is a nonzero polynomial in x over $\mathbb{F}$. In view of (2.5), there exists an invertible matrix $N \in \mathscr{M}_n(\mathbb{F})$ such that $B=N(J_1 \oplus \cdots \oplus J_{s/2} \oplus 0_{n-s})N^t$ with $s \geqslant 1$ even. Then

$$p(x)=\zeta \det(G+x(J_1 \oplus \cdots \oplus J_{s/2} \oplus 0_{n-s})),$$

where $\zeta=\det(NN^t) \in \mathbb{F}$ is nonzero and $G=N^{-1}A(N^{-1})^t \in \mathscr{K}_n(\mathbb{F})$. Since $|\mathbb{F}| \geqslant n+2$ and $\deg p(x) \leqslant s \leqslant n$, it follows that there exists a scalar $\lambda_0 \in \mathbb{F}$ with $\lambda_0 \neq 1$ such that $p(\lambda_0) \neq 0$. Then $\operatorname{rank}(A+\lambda_0 B)=n$. We complete the proof. $\square$

LEMMA 3. *Let $\psi : \mathscr{K}_n(\mathbb{F}) \to \mathscr{K}_m(\mathbb{K})$ be a mapping satisfying condition (1.4). Let $A \in \mathscr{K}_n(\mathbb{F})$. Then the following statements hold.*

(a) *If $\mathbb{F}$ satisfies condition (2.8) for $q=n-1$, then A is invertible implies that $\psi(A)=0$ or $\psi(A)$ is invertible.*

(b) *If A is singular, then $\psi(A)$ is singular.*

(c) *ψ is injective if and only if* $\operatorname{rank}\psi(A)=m \Leftrightarrow \operatorname{rank} A=n$.

Proof. (a) If A is invertible, then there exists an invertible matrix $B \in \mathscr{K}_n(\mathbb{F})$ such that $A=\operatorname{adj} B$ by Lemma 1. Thus $\psi(A)=\operatorname{adj}\psi(B)$. If $\psi(B)$ is invertible, then $\psi(A)$ is invertible. If $\psi(B)$ is singular, then $\operatorname{rank}\psi(B) \leqslant m-2$, and so $\psi(A)=0$.

(b) If A is singular, then $\operatorname{rank} A \leqslant n-2$ and $\operatorname{adj} A=0$. So $\operatorname{adj}\psi(A)=\psi(\operatorname{adj} A)=\psi(0)=0$. Therefore, $\operatorname{rank}\psi(A) \leqslant m-2$, and thus $\psi(A)$ is singular.

(c) By (b), we have $\operatorname{rank}\psi(A)=m$ implies that $\operatorname{rank} A=n$. Let A be of rank n. By the injectivity of ψ, together with (a), we conclude that $\operatorname{rank}\psi(A)=m$. Conversely, we let $H,K \in \mathscr{K}_n(\mathbb{F})$ such that $\psi(H)=\psi(K)$. Let $\operatorname{rank}(H-K)=k$. By Lemma 2 (a), there exists a rank $n-k$ matrix $X \in \mathscr{K}_n(\mathbb{F})$ such that $H-K+X$ is of rank n. Then $\operatorname{adj}\psi(H-K+X)$ is of rank m. By (1.4), we see that $\operatorname{adj}\psi(X)=\operatorname{adj}\psi(K-(K-X))=\operatorname{adj}(\psi(K)-\psi(K-X))=\operatorname{adj}(\psi(H)-\psi(K-X))=\operatorname{adj}\psi(H-K+X)$. Thus $\operatorname{rank}\psi(X)=m$, and so $\operatorname{rank} X=n$. We thus have $k=0$, and hence $H=K$. Then ψ is injective. We are done. $\square$

LEMMA 4. *Let $\mathbb{F}$ be a field satisfying condition (2.8) for $q=n-1$. Let $\psi : \mathscr{K}_n(\mathbb{F}) \to \mathscr{K}_m(\mathbb{K})$ be a mapping satisfying condition (1.4). Let $P \in \mathscr{M}_n(\mathbb{F})$ be an invertible matrix and let $L_P : \mathscr{K}_n(\mathbb{F}) \to \mathscr{K}_m(\mathbb{K})$ be the mapping defined by*

$$L_P(A)=\psi(PAP^t) \quad \textit{for every } A \in \mathscr{K}_n(\mathbb{F}). \tag{2.10}$$

If $L_P(J_n)=0$, then $L_P(A)=0$ for every invertible matrix $A \in \mathscr{K}_n(\mathbb{F})$.

Proof. We first show that if $A,B\in\mathscr{K}_n(\mathbb{F})$ are invertible matrices such that $\operatorname{rank}(A-B)<n$, then

$$L_P(A)=0 \quad\Rightarrow\quad L_P(B)=0. \tag{2.11}$$

Since $\operatorname{rank}(A-B)<n$, it follows that $\operatorname{adj}(P(A-B)P^t)=0$, and so $\psi(\operatorname{adj}(P(A-B)P^t))=0$. It follows from (1.4) and (2.10) that $\operatorname{adj}(L_P(A)-L_P(B))=0$. Since $L_P(A)=0$, we have $\operatorname{adj} L_P(B)=0$, and so $\operatorname{rank}\psi(PBP^t)<m$. Hence $L_P(B)=\psi(PBP^t)=0$ by Lemma 3 (a). Denote

$$\mathscr{H}:=\{J\oplus X\mid X\in\mathscr{K}_{n-2}(\mathbb{F})\text{ and }\operatorname{rank}X=n-2\}\subseteq\mathscr{K}_n(\mathbb{F}).$$

Here, $J\in\mathscr{K}_2(\mathbb{F})$ is the 2×2 alternate matrix defined in (2.6). We now claim that

$$L_P(H)=0\quad\text{for every } H\in\mathscr{H}. \tag{2.12}$$

Let $H\in\mathscr{H}$. Then H is of rank n. Since $\operatorname{rank}(J_n-H)<n$, it follows from our assumption $L_P(J_n)=0$ and (2.11) that $L_P(H)=0$, as required.

Let $A\in\mathscr{K}_n(\mathbb{F})$ be an arbitrary invertible alternate matrix. Then A can be expressed as

$$A=\begin{pmatrix} aJ & B\\ -B^t & C\end{pmatrix}\in\mathscr{K}_n(\mathbb{F}) \tag{2.13}$$

where $a\in\mathbb{F}$, $B=(b_{ij})\in\mathscr{M}_{2,n-2}(\mathbb{F})$ and $C\in\mathscr{K}_{n-2}(\mathbb{F})$. We argue in the following two sub-cases:

Case I: $n=4$. Then we have $C=cJ$ for some scalar $c\in\mathbb{F}$. We first consider A is of form (2.13) with $b_{21}=b_{22}=0$. Since $\operatorname{rank}A=4$, it follows that $a,c\neq 0$. Let $H_1=J\oplus C\in\mathscr{H}$. Then $\operatorname{rank}(A-H_1)<4$. It follows from (2.11) and (2.12) that $L_P(A)=0$. Suppose now that A is an invertible alternate matrix of form (2.13) with $C\neq 0$. We select

$$H_2=\begin{pmatrix} \alpha J & \begin{pmatrix} b_{11} & b_{12}\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix} -b_{11} & 0\\ -b_{12} & 0\end{pmatrix} & C\end{pmatrix}\in\mathscr{K}_4(\mathbb{F}),$$

where

$$\alpha=\begin{cases} a \text{ if } a\neq 0,\\ 1 \text{ if } a=0.\end{cases}$$

In both cases, we see that each H_2 is invertible, $L_P(H_2)=0$ and $\operatorname{rank}(A-H_2)<4$. Then $L_P(A)=0$ by (2.11). Consider now A is of form (2.13) with $C=0$. Therefore B is invertible. If $a\neq 0$, then we choose

$$H_3=\begin{pmatrix} aJ & \begin{pmatrix} b_{11} & b_{12}\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix} -b_{11} & 0\\ -b_{12} & 0\end{pmatrix} & J\end{pmatrix}\in\mathscr{K}_4(\mathbb{F}).$$

Clearly, H_3 is invertible, $L_P(H_3)=0$ and $\operatorname{rank}(A-H_3)<4$. Then $L_P(A)=0$ by (2.11). If $a=0$, then we select

$$H_4=\begin{pmatrix} J & B \\ -B^t & 0 \end{pmatrix}\in\mathscr{K}_4(\mathbb{F}).$$

It is clear that H_4 is invertible, $L_P(H_4)=0$ and $\operatorname{rank}(A-H_4)<4$. Then $L_P(A)=0$ by (2.11). We are done.

Case II: $n\geqslant 6$. Let A be an invertible alternate matrix of form (2.13). If C is invertible, then we select $K_1=J\oplus C\in\mathscr{K}_n(\mathbb{F})$. Clearly, $K_1\in\mathscr{H}$ and $\operatorname{rank}(A-K_1)<n$, and so $L_P(A)=0$ by (2.11). We now consider C is singular. Since $\operatorname{rank}A=n$ and

$$\operatorname{rank}\begin{pmatrix} aJ & B \\ -B^t & 0 \end{pmatrix}\leqslant 4,$$

it follows that $\operatorname{rank}C=n-4$. By the fact of (2.5), there exists an invertible matrix $P\in\mathscr{M}_{n-2}(\mathbb{F})$ such that

$$C=P(J_1\oplus\cdots\oplus J_{(n-4)/2}\oplus 0_2)P^t, \tag{2.14}$$

where $J_i=J$ for $i=1,\ldots,(n-4)/2$. We argue in the following two cases:

Suppose that $n\geqslant 8$. We select $K_2=J\oplus P(J_1\oplus\cdots\oplus J_{(n-4)/2}\oplus J)P^t\in\mathscr{K}_{n-2}(\mathbb{F})$. It is clear that $K_2\in\mathscr{H}$, $L_P(K_2)=0$ by (2.12), and $\operatorname{rank}(A-K_2)<n$. It follows from (2.11) that $L_P(A)=0$, as desired.

Suppose that $n=6$. Let $\mathscr{N}$ denote the set of all 6×6 invertible alternate matrices of the form

$$N=\begin{pmatrix} xJ & X \\ -X^t & Y \end{pmatrix}\in\mathscr{K}_6(\mathbb{F})$$

for which $x\in\mathbb{F}$ is nonzero, $X=(x_{ij})\in\mathscr{M}_{2,4}(\mathbb{F})$ with $x_{2j}=0$ for $j=1,\ldots,4$, and $Y\in\mathscr{K}_4(\mathbb{F})$ is invertible. We claim that

$$L_P(N)=0 \quad \text{for every } N\in\mathscr{N}. \tag{2.15}$$

To see this, we take $K_3=J\oplus Y\in\mathscr{K}_6(\mathbb{F})$. Since $Y\in\mathscr{K}_4(\mathbb{F})$ is invertible, it follows that $K_3\in\mathscr{H}$, and so $L_P(K_3)=0$ by (2.12). Note that $\operatorname{rank}(N-K_3)<6$ yields $L_P(N)=0$ by (2.11). Let A be an invertible alternate matrix of form (2.13) with C singular. In view of (2.14), we have $C=P(J_1\oplus 0_2)P^t\in\mathscr{K}_4(\mathbb{F})$. We choose

$$K_4=\begin{pmatrix} J & \begin{pmatrix} b_{11} & \cdots & b_{14} \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} -b_{11} & 0 \\ \vdots & \vdots \\ -b_{14} & 0 \end{pmatrix} & P(J\oplus J)P^t \end{pmatrix}\in\mathscr{K}_6(\mathbb{F}).$$

Then $K_4 \in \mathcal{N}$. Since

$$\operatorname{rank}(A-K_4) = \operatorname{rank}\begin{pmatrix} (a-1)J & \begin{pmatrix} 0 & \cdots & 0 \\ b_{21} & \cdots & b_{24} \end{pmatrix} \\ \begin{pmatrix} 0 & -b_{21} \\ \vdots & \vdots \\ 0 & -b_{24} \end{pmatrix} & P(0_2 \oplus J)P^t \end{pmatrix} \leqslant 4,$$

it follows from (2.15) and (2.11) that $L_P(A)=0$. The proof is completed. □

LEMMA 5. *Let* $\mathbb{F}$ *be a field satisfying condition* (2.8) *for* $q=n-1$. *Let* $\psi:\mathcal{K}_n(\mathbb{F})\to\mathcal{K}_m(\mathbb{K})$ *be a mapping satisfying condition* (1.4)*. Then the following statements hold true.*

(a) $\psi(J_n)=0$ *if and only if* $\operatorname{rank}\psi(A)\leqslant m-2$ *for all* $A\in\mathcal{K}_n(\mathbb{F})$.

(b) $\psi(J_n)\neq 0$ *if and only if* ψ *injective.*

Proof. (a) Let $A\in\mathcal{K}_n(\mathbb{F})$. If A is singular, then $\psi(A)$ is singular by Lemma 3 (b). So, $\operatorname{rank}\psi(A)\leqslant m-2$, as desired. If A is invertible, then, since $\psi(J_n)=0$, in view of Lemma 4, by setting $P=I_n$, we have $\psi(A)=0$. Conversely, if $\operatorname{rank}\psi(A)\leqslant m-2$ for all $A\in\mathcal{K}_n(\mathbb{F})$, then $\operatorname{rank}\psi(J_n)\leqslant m-2$, and so $\psi(J_n)=0$ by Lemma 3 (a). We are done.

(b) Since $\psi(0)=0$, it follows from the injectivity of ψ that $\psi(J_n)\neq 0$. Conversely, suppose that $\psi(J_n)\neq 0$. We claim that $\operatorname{rank}A=n$ if and only if $\operatorname{rank}\psi(A)=m$. The sufficiency part follows from Lemma 3 (b). Let $A\in\mathcal{K}_n(\mathbb{F})$ be of rank n. By (2.5), there exists an invertible matrix $P\in\mathcal{M}_n(\mathbb{F})$ such that $A=PJ_nP^t$. We define the mapping $L_P:\mathcal{K}_n(\mathbb{F})\to\mathcal{K}_m(\mathbb{K})$ such as

$$L_P(X)=\psi(PXP^t)\quad\text{for all } X\in\mathcal{K}_n(\mathbb{F}).$$

Then $L_P(P^{-1}J_n(P^{-1})^t)=\psi(J_n)\neq 0$. Suppose that $\operatorname{rank}\psi(A)\neq m$. It follows from Lemma 3 (a) that $\psi(A)=0$, and so $L_P(J_n)=\psi(PJ_nP^t)=\psi(A)=0$. Then, by Lemma 4, we obtain $L_P(X)=0$ for every invertible matrix $X\in\mathcal{K}_n(\mathbb{F})$. In particular, we have $L_P(P^{-1}J_n(P^{-1})^t)=0$, a contradiction. Hence ψ is injective by Lemma 3 (c). □

Let k and n be even integers with $n\geqslant k\geqslant 4$, and let $\mathbb{F}$ be a field with at least three elements. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{K}_n(\mathbb{F})$. We define

$$\mathcal{S}^{\perp_k}:=\{A\in\mathcal{K}_n(\mathbb{F}):\operatorname{rank}(A-X)\leqslant k \text{ for all } X\in\mathcal{S}\}$$

and $\mathcal{S}^{\perp_k\perp_k}:=(\mathcal{S}^{\perp_k})^{\perp_k}$ if $\mathcal{S}^{\perp_k}$ is nonempty. Two alternate matrices $A,B\in\mathcal{K}_n(\mathbb{F})$ are said to be *adjacent* if $\operatorname{rank}(A-B)=2$. We recall the following result proved in [12, Lemmas 3.2 and 3.3].

LEMMA 6. *Let* k *and* m *be even integers with* $m\geqslant k\geqslant 4$*, and let* $\mathbb{F}$ *be a field with at least three elements. Let* $A,B\in\mathcal{K}_n(\mathbb{F})$ *be matrices such that* $\operatorname{rank}(A-B)\leqslant k$. *Then* A,B *is a pair of adjacent matrices if and only if* $\left|\{A,B\}^{\perp_k\perp_k}\right|\geqslant 3$.

A mapping $\varphi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ is said to *preserve adjacency in both directions* if $\operatorname{rank}(A-B)=2 \Leftrightarrow \operatorname{rank}(\varphi(A)-\varphi(B))=2$ for all $A,B\in\mathcal{K}_n(\mathbb{F})$. The following result is known, see the works of [6, 12, 7, 8].

PROPOSITION 2. *Let m and n be even integers with $m,n\geqslant 4$. Let $\mathbb{F}$ and $\mathbb{K}$ be fields with at least three elements. If $\varphi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ is a surjective mapping satisfying*

$$\operatorname{rank}(A-B)=n \quad\Leftrightarrow\quad \operatorname{rank}(\varphi(A)-\varphi(B))=m \tag{2.16}$$

for every $A,B\in\mathcal{K}_n(\mathbb{F})$, then φ is a bijective mapping preserving adjacency in both directions, $m=n$, and $\mathbb{F}$ and $\mathbb{K}$ are isomorphic.

Proof of Theorem 2. We note that $\operatorname{adj} A^* = (\operatorname{adj} A)^*$ for every $A\in\mathcal{K}_4(\mathbb{F})$ where $A^*\in\mathcal{K}_4(\mathbb{F})$ is the alternate matrix as defined in (1.3). The sufficiency part is clear.

We now consider the necessity part. Suppose that $\psi(J_n)=0$. By Lemma 5 (a), we have $\psi(A)$ is singular for all $A\in\mathcal{K}_n(\mathbb{F})$. This contradicts to the surjectivity of ψ. Then $\psi(J_n)\neq 0$, and so ψ is injective by Lemma 5 (b). Let $A,B\in\mathcal{K}_n(\mathbb{F})$. Then, in view of Lemma 3 (c) and by condition (1.4), we have

$$\begin{aligned}\operatorname{rank}(A-B)=n \quad&\Leftrightarrow\quad \operatorname{rank}\, \psi(\operatorname{adj}(A-B))=m\\ &\Leftrightarrow\quad \operatorname{rank}\operatorname{adj}(\psi(A)-\psi(B))=m\\ &\Leftrightarrow\quad \operatorname{rank}(\psi(A)-\psi(B))=m.\end{aligned}$$

It follows from Proposition 2 that ψ is a bijective mapping preserving adjacency in both directions, $m=n$, and $\mathbb{F}$ and $\mathbb{K}$ are isomorphic. By the fundamental theorem of the geometry of alternate matrices, see [13] or [22, Theorem 4.4], together with $\psi(0)=0$, we see that there exist a field isomorphism $\sigma:\mathbb{F}\to\mathbb{K}$, an invertible matrix $P\in\mathcal{M}_n(\mathbb{K})$ and a nonzero scalar $\lambda\in\mathbb{K}$ such that either

$$\psi(A)=\lambda P A^{\sigma} P^t \quad \text{for every } A\in\mathcal{K}_n(\mathbb{F}) \tag{2.17}$$

or when $n=4$, we also have

$$\psi(A)=\lambda P (A^*)^{\sigma} P^t \quad \text{for every } A\in\mathcal{K}_4(\mathbb{F}). \tag{2.18}$$

We next claim that $P^tP=\mu I_n$ for some nonzero scalar $\mu\in\mathbb{F}$ such that $(\lambda\mu)^{n-2}=1$. Since $\operatorname{adj}(A^*)=(\operatorname{adj}A)^*$ for every $A\in\mathcal{K}_4(\mathbb{F})$, we consider only the first case (2.17) as the second case (2.18) can be verified similarly. By (2.17), we obtain

$$\lambda P \operatorname{adj}(A^{\sigma}-B^{\sigma})P^t=\psi(\operatorname{adj}(A-B))=\operatorname{adj}\psi(A-B)=\lambda^{n-1}\operatorname{adj}P^t\operatorname{adj}(A^{\sigma}-B^{\sigma})\operatorname{adj}P$$

for all $A,B\in\mathcal{K}_n(\mathbb{F})$. This implies that $\lambda^{n-2}(\det Q)Q^{-1}\operatorname{adj}(A^{\sigma}-B^{\sigma})Q^{-1}=\operatorname{adj}(A^{\sigma}-B^{\sigma})$ for every $A,B\in\mathcal{K}_n(\mathbb{F})$, where $Q=P^tP$ is invertible with $Q^t=Q$. In particular, we have $\lambda^{n-2}(\det Q)Q^{-1}XQ^{-1}=X$ for every invertible $X\in\mathcal{K}_n(\mathbb{F})$. Let $1\leqslant i\neq j\leqslant n$. Since $J_n+\lambda(E_{ij}-E_{ji})$ is invertible, it can be verified that $\lambda^{n-2}(\det Q)Q^{-1}(E_{ij}-E_{ji})Q^{-1}=E_{ij}-E_{ji}$ for every $1\leqslant i\neq j\leqslant n$. Consequently, we obtain

$$Q(E_{ij}-E_{ji})=\lambda^{n-2}(E_{ij}-E_{ji})\operatorname{adj}Q \quad \text{for every } 1\leqslant i\neq j\leqslant n. \tag{2.19}$$

Let $Q = (q_{ij})$. Since $Q^t = Q$, it follows from (2.19) that $q_{ij} = 0$ for every $1 \leqslant i \neq j \leqslant n$ and $q_{ii}q_{jj} - q_{ij}^2 = \lambda^{n-2}(\det Q)$ for every $1 \leqslant i \neq j \leqslant n$. Thus $P^tP = Q = \mu I_n$ for some nonzero scalar $\mu \in \mathbb{F}$ such that $\mu^2 = \lambda^{n-2}(\det Q)$. Since $\det Q = \mu^n$, we obtain $(\lambda\mu)^{n-2} = 1$. This completes our proof. $\square$

As an immediate consequence of Theorem 2, we have

COROLLARY 1. *Let m and n be even integers with $m,n \geqslant 4$. Let $\mathbb{K}$ be a field with at least three elements, and let $\mathbb{F}$ be a field with at least three elements such that $x^{n-1} - a \in \mathbb{F}[x]$ has a root for every $a \in \mathbb{F}$. Then $\varphi : \mathscr{K}_n(\mathbb{F}) \to \mathscr{K}_m(\mathbb{K})$ is a surjective classical adjoint commuting additive mapping if and only if $m = n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic, and there exist a field isomorphism $\sigma : \mathbb{F} \to \mathbb{K}$, an invertible matrix $P \in \mathscr{M}_n(\mathbb{K})$ with $P^tP = \mu I_n$, and nonzero scalars $\mu, \lambda \in \mathbb{K}$ with $(\lambda\mu)^{n-2} = 1$, such that either*

$$\psi(A) = \lambda P A^{\sigma} P^t \quad for\ every\ A \in \mathscr{K}_n(\mathbb{F})$$

or when $n = 4$,

$$\psi(A) = \lambda P (A^*)^{\sigma} P^t \quad for\ every\ A \in \mathscr{K}_n(\mathbb{F}),$$

where

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

We now proceed to prove Theorem 1.

LEMMA 7. *Let m and n be even integers such that $m,n \geqslant 4$, and let $\mathbb{F}$ be a field with $|\mathbb{F}| = 2$ or $|\mathbb{F}| > n+1$ satisfying condition (2.8) for $q = n-1$. If $\psi : \mathscr{K}_n(\mathbb{F}) \to \mathscr{K}_m(\mathbb{F})$ is a mapping satisfying condition (1.2) with $\psi(J_n) \neq 0$, then ψ is linear.*

Proof. If ψ satisfies condition (1.2), then it satisfies condition (1.4), and so ψ is injective by Lemma 5 (b). In view of Lemma 3 (c), together with condition (1.2), we have

$$\operatorname{rank} \psi(A + \alpha B) = m \Leftrightarrow \operatorname{rank}(A + \alpha B) = n \Leftrightarrow \operatorname{rank}(\psi(A) + \alpha\psi(B)) = m \quad (2.20)$$

for all $A, B \in \mathscr{K}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$. Let $H, K \in \mathscr{K}_n(\mathbb{F})$ and $\lambda \in \mathbb{F}$ such that $\operatorname{rank}(H + \lambda K) = n$. By using the fact of (2.20), we obtain $\psi(H+\lambda K) \operatorname{adj} \psi(H+\lambda K) = \det\psi(H+\lambda K) I_m$, and also $(\psi(H) + \lambda\psi(K)) \operatorname{adj}(\psi(H) + \lambda\psi(K)) = \det(\psi(H) + \lambda\psi(K)) I_m$. Further, since $\operatorname{adj} \psi(H+\lambda K) = \operatorname{adj}(\psi(H) + \lambda\psi(K))$, it follows that

$$\psi(H + \lambda K) = \frac{\det\psi(H+\lambda K)}{\det(\psi(H) + \lambda\psi(K))}(\psi(H) + \lambda\psi(K)). \quad (2.21)$$

By a similar argument as in (2.21), we have

$$\psi(H + \lambda K) = \frac{\det\psi(H+\lambda K)}{\det(\psi(H) + \psi(\lambda K))}(\psi(H) + \psi(\lambda K)). \quad (2.22)$$

If we choose $H=0$ and λK is of rank n, then, by the same argument as in (2.21), we obtain

$$\psi(\lambda K)=\frac{\det\psi(\lambda K)}{\det\lambda\,\psi(K)}\lambda\,\psi(K). \tag{2.23}$$

We first claim that

$$\psi(\alpha A)=\alpha\psi(A) \tag{2.24}$$

for every invertible matrix $A\in\mathscr{K}_n(\mathbb{F})$ and $\alpha\in\mathbb{F}$. The result holds when $\alpha=0$. Consider $\alpha\neq 0$. By Lemma 2 (c), there is a nonzero singular matrix $C_1\in\mathscr{K}_n(\mathbb{F})$ such that rank $(C_1+\alpha A)=n$. By the facts of (2.21) and (2.22), we have

$$\lambda_1\psi(\alpha A)-\lambda_2\alpha\psi(A)=(\lambda_2-\lambda_1)\psi(C_1), \tag{2.25}$$

where $\lambda_1=\det(\psi(C_1)+\alpha\psi(A))$ and $\lambda_2=\det(\psi(C_1)+\psi(\alpha A))$ are nonzero scalars in $\mathbb{F}$. Suppose $\lambda_1\neq\lambda_2$. Since A is invertible, it follows from (2.23) that $\psi(\alpha A)$ and $\psi(A)$ are linearly dependent. So $\psi(\alpha A)=\beta\psi(A)$ for some nonzero scalar $\beta\in\mathbb{F}$. Substituting into (2.25), we obtain

$$(\lambda_1\beta-\lambda_2\alpha)\psi(A)=(\lambda_2-\lambda_1)\psi(C_1).$$

By the injectivity of ψ, $\psi(A)$ and $\psi(C_1)$ are nonzero, and so rank $\psi(A)=$ rank $\psi(C_1)$. This leads to a contradiction since rank $\psi(A)=m$ but rank $\psi(C_1)<m$ by Lemma 3 (b). Hence $\lambda_1=\lambda_2$, and thus the desired conclusion follows immediately from (2.25).

We next claim that if $H,K\in\mathscr{K}_n(\mathbb{F})$ such that $H+K$ is invertible, then

$$H,K \text{ are linearly independent } \Rightarrow\ \psi(H),\psi(K) \text{ are linearly independent.} \tag{2.26}$$

Suppose to the contrary that $\psi(H)$ and $\psi(K)$ are linearly dependent. By the injectivity of ψ, we have $\psi(H)$ and $\psi(K)$ are distinct nonzero matrices. Then there exists a nonzero scalar $\gamma\in\mathbb{F}$ such that $\psi(K)=\gamma\psi(H)$. Since rank $(H+K)=n$, it follows from (2.20) that rank $((1+\gamma)\psi(H))=$ rank $(\psi(H)+\psi(K))=m$. Thus rank $\psi(H)=m$, and so rank $H=n$ by Lemma 3 (b). It follows from (2.24) that $\psi(K)=\gamma\psi(H)=\psi(\gamma H)$. By the injectivity of ψ, we obtain $K=\gamma H$. This contradicts to the assumption that H and K are linearly independent.

We show that

$$\psi(H+K)=\psi(H)+\psi(K) \tag{2.27}$$

for all alternate matrices $H,K\in\mathscr{K}_n(\mathbb{F})$ such that $H+K$ and K are invertible and H is singular. The result holds when $H=0$. Consider $H\neq 0$. By the fact of (2.21), we have

$$\frac{\psi(H+K)}{\det\psi(H+K)}=\frac{\psi(H)+\psi(K)}{\det(\psi(H)+\psi(K))}.$$

The result clearly holds when $|\mathbb{F}|=2$. We now consider $\mathbb{F}$ is a field with at least $n+2$ elements satisfying condition (2.8) for $q=n-1$. By Lemma 2 (d), there is a nonzero scalar $\lambda_0\in\mathbb{F}$ such that $H+(1+\lambda_0)K$ is invertible. Again, by the fact of (2.21), we get

$$\frac{\psi(H+K)+\psi(\lambda_0K)}{\det(\psi(H+K)+\psi(\lambda_0K))}=\frac{\psi(H+K+\lambda_0K)}{\det\psi(H+K+\lambda_0K)}=\frac{\psi(H)+\psi((1+\lambda_0)K)}{\det(\psi(H)+\psi((1+\lambda_0)K))}.$$

Since K is invertible, it follows from (2.24) that $\psi((1+\lambda_0)K)=(1+\lambda_0)\psi(K)=\psi(K)+\psi(\lambda_0 K)$. Then we have

$$\alpha_2\psi(H+K)-\alpha_1(\psi(H)+\psi(K))=(\alpha_1-\alpha_2)\psi(\lambda_0 K). \tag{2.28}$$

where $\alpha_1=\det(\psi(H+K)+\psi(\lambda_0 K))$ and $\alpha_2=\det(\psi(H)+\psi((1+\lambda_0)K))$ are nonzero scalars in $\mathbb{F}$. Since $\psi(H+K)$ and $\psi(H)+\psi(K)$ are invertible linearly dependent matrices, we have $\psi(H)+\psi(K)=\alpha_0\psi(H+K)$ for some nonzero scalar $\alpha_0\in\mathbb{F}$. It follows from (2.28) that

$$(\alpha_2-\alpha_1\alpha_0)\psi(H+K)=(\alpha_1-\alpha_2)\psi(\lambda_0 K). \tag{2.29}$$

H and K are linearly independent implies that $H+K$ and $\lambda_0 K$ are linearly independent. So $\psi(H+K)$ and $\psi(\lambda_0 K)$ are linearly independent by (2.26). By (2.29), we have $\alpha_1=\alpha_2$, and the desired result follows immediately from (2.28).

We now claim that

$$\psi(\alpha A)=\alpha\psi(A)\quad\text{for every } A\in\mathscr{K}_n(\mathbb{F}) \text{ and } \alpha\in\mathbb{F}. \tag{2.30}$$

The result holds when $\alpha=0$, $A=0$ or A is invertible. Consider now $\alpha\neq 0$ and A is a nonzero singular alternate matrix. By Lemma 2 (c), we can find an invertible matrix $C_2\in\mathscr{K}_n(\mathbb{F})$ such that $\operatorname{rank}(\alpha A+C_2)=n$. In view of (2.24) and (2.27), we see that $\psi(\alpha A)+\psi(C_2)=\psi(\alpha A+C_2)=\psi(\alpha(A+\alpha^{-1}C_2))=\alpha\psi(A+\alpha^{-1}C_2)=\alpha(\psi(A)+\alpha^{-1}\psi(C_2))=\alpha\psi(A)+\psi(C_2)$. Then we have $\psi(\alpha A)=\alpha\psi(A)$, and so the homogeneity of ψ is shown.

We finally show that ψ is additive. Let $A,B\in\mathscr{K}_n(\mathbb{F})$. If either $A=0$ or $B=0$, then the result holds. Suppose that A and B are nonzero. We first consider $A+B$ is invertible. Again, by (2.21), we have

$$\frac{\psi(A+B)}{\det\psi(A+B)}=\frac{\psi(A)+\psi(B)}{\det(\psi(A)+\psi(B))}. \tag{2.31}$$

The result holds true when $|\mathbb{F}|=2$. Consider now $\mathbb{F}$ is a field with at least $n+2$ elements satisfying condition (2.8) for $q=n-1$. If A and B are linearly dependent, then $A=\mu_0 B$ for some nonzero scalar $\mu_0\in\mathbb{F}$. By the homogeneity of ψ, we have $\psi(A+B)=\psi((\mu_0+1)B)=(\mu_0+1)\psi(B)=\mu_0\psi(B)+\psi(B)=\psi(A)+\psi(B)$, as desired. If A and B are linearly independent. By Lemma 2 (d), there exists a nonzero scalar $\mu_1\in\mathbb{F}$ such that $A+(1+\mu_1)B$ is invertible. By (2.21) and the homogeneity of ψ, we obtain

$$\frac{\psi(A+B)+\psi(\mu_1 B)}{\det(\psi(A+B)+\psi(\mu_1 B))}=\frac{\psi(A)+\psi(B)+\psi(\mu_1 B)}{\det(\psi(A)+\psi(B)+\psi(\mu_1 B))},$$

and so, together with (2.31), we have

$$(a_1-a_2a_3)\psi(A+B)=(a_2-a_1)\psi(\mu_1 B)$$

where $a_1=\det(\psi(A)+\psi(B)+\psi(\mu_1 B))$, $a_2=\det(\psi(A+B)+\psi(\mu_1 B))$ and $a_3=\frac{\det(\psi(A)+\psi(B))}{\det\psi(A+B)}$ are nonzero scalars. On the other hand, since A and B are linearly independent, it follows that $\psi(A+B)$ and $\psi(\mu_1 B)$ are linearly independent. Therefore, we conclude that $a_1 = a_2$, and so $a_3 = 1$. The desired result follows immediately from (2.31). Next, we consider $A+B$ is singular. By Lemma 2 (b), there exists an alternate matrix $C_3 \in \mathscr{K}_n(\mathbb{F})$ such that $\operatorname{rank}(A+C_3) = \operatorname{rank}(A+B+C_3) = n$. Then we have $\psi(A+C_3) = \psi(A)+\psi(C_3)$, and also

$$\psi(A+B)+\psi(C_3) = \psi(A+B+C_3) = \psi(A+C_3)+\psi(B) = \psi(A)+\psi(C_3)+\psi(B).$$

Hence $\psi(A+B) = \psi(A)+\psi(B)$, as required. Consequently, together with (2.30), ψ is a linear mapping. The proof is complete. $\square$

Proof of Theorem 1. The sufficiency part is clear. We now prove the necessity part. Evidently, ψ satisfies condition (1.4). We argue in the following two sub-cases:

Case I: $\psi(J_n) = 0$. In view of Lemma 5 (a), we have $\operatorname{rank}\psi(A) \leqslant n-2$ for all $A \in \mathscr{K}_n(\mathbb{F})$, and thus

$$\operatorname{rank}\psi(A+\alpha B) \leqslant n-2 \quad \text{for all } A,B \in \mathscr{K}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}.$$

Let $H \in \mathscr{K}_n(\mathbb{F})$ be an invertible matrix. By Lemma 1, there exists an invertible matrix $K \in \mathscr{K}_n(\mathbb{F})$ such that $H = \operatorname{adj} K$. So $\psi(H) = \psi(\operatorname{adj} K) = \operatorname{adj}\psi(K) = 0$ as $\operatorname{rank}\psi(K) \leqslant n-2$. Hence $\psi(A) = 0$ for every invertible matrix $A \in \mathscr{K}_n(\mathbb{F})$.

Case II: $\psi(J_n) \neq 0$. Then ψ is an injective linear mapping by Lemmas 5 (b) and 7, and hence ψ is surjective. By Corollary 1, together with the homogeneity of ψ, we prove the desired result. $\square$

3. Skew-Hermitian matrices

Throughout this section, unless otherwise stated, we let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess proper involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively. We recall that $\mathbb{F}^- = \{a \in \mathbb{F} : \overline{a} = a\}$ and $S\mathbb{F}^- = \{a \in \mathbb{F} : \overline{a} = -a\}$ (respectively, $\mathbb{K}^\wedge = \{a \in \mathbb{K} : \widehat{a} = a\}$ and $S\mathbb{K}^\wedge = \{a \in \mathbb{K} : \widehat{a} = -a\}$). Since $^-$ is proper, there exists an element $i \in \mathbb{F}$, with $\overline{i} = -i$ when $\operatorname{char}\mathbb{F} \neq 2$, and $\overline{i} = 1+i$ when $\operatorname{char}\mathbb{F} = 2$, such that $\mathbb{F} = \mathbb{F}^- \oplus i\mathbb{F}^-$ (see [14, p.g. 601]), and also $1 \in S\mathbb{F}^-$ when $\operatorname{char}\mathbb{F} = 2$, and $1 \in \mathbb{F}^-$. It follows that $S\mathbb{F}^- \neq \{0\}$ and $\mathbb{F}^- \neq \{0\}$. Note that if n is a positive even integer, then $\mu^n \in \mathbb{F}^-$ and $\eta^n \in \mathbb{K}^\wedge$ for every elements $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$.

We start with the following basic result.

LEMMA 8. *Let m and n be even integers with $m,n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars and $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{K})$ be a map satisfying*

$$\varphi(\mu^{n-2}\operatorname{adj}(H-K)) = \eta^{m-2}\operatorname{adj}(\varphi(H)-\varphi(K)) \ \textit{for every } H,K \in \mathscr{H}_n(\mathbb{F}). \tag{3.32}$$

Let $A,B \in \mathscr{H}_n(\mathbb{F})$. Then the following statements hold.

(a) $\varphi(\mu^{n-2}\operatorname{adj} A) = \eta^{m-2}\operatorname{adj}\varphi(A)$.

(b) $\operatorname{adj} \varphi(A-B) = \operatorname{adj}(\varphi(A)-\varphi(B))$.

(c) $\operatorname{rank} \varphi(A) \leqslant 1$ *if* $\operatorname{rank} A = 1$.

(d) $\operatorname{rank} \varphi(A) \leqslant m-1$ *if* $\operatorname{rank} A = n-1$.

(e) $\operatorname{rank} \varphi(A) \leqslant m-2$ *if* $\operatorname{rank} A \leqslant n-2$.

(f) φ *is injective if and only if* $\operatorname{rank} \varphi(A) = m \Leftrightarrow \operatorname{rank} A = n$.

Proof. (a) It is clear that $\varphi(0) = 0$. So $\varphi(\mu^{n-2}\operatorname{adj} A) = \varphi(\mu^{n-2}\operatorname{adj}(A-0)) = \eta^{m-2}\operatorname{adj}(\varphi(A)-\varphi(0)) = \eta^{m-2}\operatorname{adj}\varphi(A)$.

(b) By (a) and (3.32), we see that $\eta^{m-2}\operatorname{adj}\varphi(A-B) = \varphi(\mu^{n-2}\operatorname{adj}(A-B)) = \eta^{m-2}\operatorname{adj}(\varphi(A)-\varphi(B))$, and the result follows.

(c) If A is of rank one, then there is a rank $n-1$ matrix $B \in \mathscr{H}_n(\mathbb{F})$ such that $\operatorname{adj} B = \frac{1}{\mu^{n-2}}A$ by [2, Lemma 2.2]. Then $\varphi(A) = \varphi(\mu^{n-2}\operatorname{adj} B) = \eta^{m-2}\operatorname{adj}\varphi(B)$. Since $\eta^{m-2}\operatorname{adj}\varphi(A) = \varphi(\mu^{n-2}\operatorname{adj} A) = \varphi(0) = 0$, we obtain $\operatorname{rank}\varphi(A) < m$, and so $\operatorname{rank}\varphi(B) < m$. Hence $\operatorname{rank}\varphi(A) = \operatorname{rank}(\eta^{m-2}\operatorname{adj}\varphi(B)) \leqslant 1$, as required.

(d) If A is of rank $n-1$, then $\operatorname{rank}\varphi(\mu^{n-2}\operatorname{adj} A) \leqslant 1$ by (c). So $\operatorname{adj}\varphi(\mu^{n-2}\operatorname{adj} A) = 0$. On the other hand, $\operatorname{adj}\varphi(\mu^{n-2}\operatorname{adj} A) = \operatorname{adj}(\eta^{m-2}\operatorname{adj}\varphi(A)) = (\eta^{m-2})^{m-1}\operatorname{adj}(\operatorname{adj}\varphi(A))$. Thus $\operatorname{adj}(\operatorname{adj}\varphi(A)) = 0$ implies that $\operatorname{rank}\varphi(A) \leqslant m-1$.

(e) If $\operatorname{rank} A \leqslant n-2$, then $\eta^{m-2}\operatorname{adj}\varphi(A) = \varphi(\mu^{n-2}\operatorname{adj} A) = \varphi(0) = 0$. Therefore $\operatorname{rank}\varphi(A) \leqslant m-2$.

(f) Since $\psi(0) = 0$, by the injectivity of φ, we have $\operatorname{Ker}\varphi = \{0\}$. By (d) and (e), we see that $\operatorname{rank}\varphi(A) = m$ implies that $\operatorname{rank} A = n$. We now consider A is of rank n. Suppose that $\operatorname{rank}\varphi(A) < m$. Then $\eta^{-(m-2)}\varphi(\mu^{n-2}\operatorname{adj}(\mu^{n-2}\operatorname{adj} A)) = (\eta^{m-2})^{m-1}\operatorname{adj}(\operatorname{adj}\varphi(A)) = 0$, which implies that $\mu^{n-2}\operatorname{adj}(\mu^{n-2}\operatorname{adj} A) = 0$ because $\operatorname{Ker}\varphi = \{0\}$. This contradicts to the assumption that $\operatorname{rank} A = n$. So $\operatorname{rank}\varphi(A) = m$.

Conversely, suppose that $\varphi(H) = \varphi(K)$ for some $H, K \in \mathscr{H}_n(\mathbb{F})$. We let $\operatorname{rank}(H-K) = k$. It follows from [2, Lemma 2.4 (a)] that we can choose a rank $n-k$ matrix $Y \in \mathscr{H}_n(\mathbb{F})$ such that $\operatorname{rank}(H-K+Y) = n$. Then $\operatorname{rank}\varphi(H-K+Y) = m$. By (b), we see that

$$\operatorname{adj}\varphi(Y) = \operatorname{adj}(\varphi(K)-\varphi(K-Y)) = \operatorname{adj}(\varphi(H)-\varphi(K-Y)) = \operatorname{adj}\varphi(H-K+Y)$$

is of rank m. Thus $\operatorname{rank}\varphi(Y) = m$ implies that $\operatorname{rank} Y = n$, and so $k = 0$. Hence $H = K$ and φ is injective, as desired. $\square$

LEMMA 9. *Let m and n be even integers with $m, n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars and $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{K})$ be a mapping satisfying condition (3.32). Suppose that $P \in \mathscr{M}_n(\mathbb{F})$ is invertible, and that $T_P : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{K})$ is the mapping defined by*

$$T_P(A) = \varphi(PA\overline{P}^t) \quad for\ every\ A \in \mathscr{H}_n(\mathbb{F}).$$

Then the following statements hold.

(i) *If* $\operatorname{rank} T_P(I_n) \neq m$, *then* $\operatorname{rank} T_P(A) \leqslant m-2$ *for all* $A \in \mathscr{H}_n(\mathbb{F})$ *and* $T_P(A) = 0$ *for all rank one matrices* $A \in \mathscr{H}_n(\mathbb{F})$.

(ii) *If* $\operatorname{rank} T_P(I_n) = m$, *then* $\operatorname{rank} T_P(aE_{ii}) = 1$ *for all integers* $1 \leqslant i \leqslant n$ *and nonzero scalars* $a \in \mathbb{F}^-$.

Proof. Throughout this proof, we denote $\theta := \mu^{n(n-2)} \det(P\overline{P})^{n-2}$, $\vartheta := \mu^{n-2}\theta^{n-1}$ and $U := \operatorname{adj} P$. It is clear that $\theta, \vartheta \in \mathbb{F}^-$ are nonzero scalars and $\operatorname{rank} U = n$. Certainly, by the definition of T_P, we see that Lemma 8 (c), (d) and (e) hold true for T_P, and

$$\operatorname{adj}\, T_P(A-B) = \operatorname{adj}\,(T_P(A) - T_P(B)) \quad \textit{for every}\ A,B \in \mathscr{H}_n(\mathbb{F}). \tag{3.33}$$

(a) We note that $\mu^{n-2}\operatorname{adj}\,(\mu^{n-2}\operatorname{adj}\,(P\overline{P}^t)) = \theta P\overline{P}^t$. It follows that $T_P(\theta I_n) = \varphi(\theta P\overline{P}^t) = \varphi(\mu^{n-2}\operatorname{adj}\,(\mu^{n-2}\operatorname{adj}\,(P\overline{P}^t)) = \eta^{m-2}\operatorname{adj}\,(\eta^{m-2}\operatorname{adj}\, T_P(I_n))$. Since $\operatorname{rank} T_P(I_n) < m$, we get

$$T_P(\theta I_n) = 0. \tag{3.34}$$

Also, $\varphi(\vartheta\overline{U}^t U) = \varphi(\mu^{n-2}\operatorname{adj}\,(\theta P\overline{P}^t)) = \eta^{m-2}\operatorname{adj}\, T_P(\theta I_n)$. It follows from (3.34) that

$$\varphi(\vartheta\overline{U}^t U) = 0. \tag{3.35}$$

We next claim that

$$\varphi(\overline{U}^t \vartheta E_{ii} U) = 0 \quad \text{for } i = 1, \ldots, n. \tag{3.36}$$

Let $1 \leqslant i \leqslant n$. By using the fact that $\theta^{n-1}E_{ii} = \operatorname{adj}\,(\theta(I_n - E_{ii}))$ as well as (3.33), (3.34) and Lemma 8 (a), we get $\varphi(\overline{U}^t \vartheta E_{ii}U) = \varphi(\overline{U}^t(\mu^{n-2}\theta^{n-1})E_{ii}U) = \varphi(\mu^{n-2}\operatorname{adj}\,(P\theta(I_n - E_{ii})\overline{P}^t)) = \eta^{m-2}\operatorname{adj}\, T_P(\theta I_n - \theta E_{ii}) = \eta^{m-2}\operatorname{adj}\,(T_P(\theta I_n) - T_P(\theta E_{ii})) = \eta^{m-2}\operatorname{adj}\,(-T_P(\theta E_{ii})) = 0$ because $\operatorname{rank} T_P(\theta E_{ii}) \leqslant 1$ by Lemma 8 (c). We next show, for each $1 \leqslant i \leqslant n$, that

$$T_P(\alpha E_{ii}) = 0 \quad \text{for every } \alpha \in \mathbb{F}^-. \tag{3.37}$$

The result clearly holds for $\alpha = 0$. Suppose that $\alpha \neq 0$. Let $\beta = \mu^{(n-2)(n-1)}\alpha \in \mathbb{F}^-$. By the fact of $\operatorname{adj}\,(\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\beta E_{jj}) = \theta^{-1}\beta E_{ii}$ with $i \neq j$, $\operatorname{adj} U = (\det P)^{n-2}P$, (3.35) and (3.36), we have

$$\begin{aligned}
T_P(\alpha E_{ii}) &= \varphi((\mu^{-1})^{(n-2)(n-1)}\theta P(\theta^{-1}\beta)E_{ii}\overline{P}^t) \\
&= \varphi((\mu^{-1})^{(n-2)(n-1)}\mu^{(n-2)n}\det(P\overline{P})^{n-2}P(\theta^{-1}\beta)E_{ii}\overline{P}^t) \\
&= \varphi(\mu^{n-2}(\det P)^{n-2}P(\theta^{-1}\beta E_{ii})(\det\overline{P})^{n-2}\overline{P}^t) \\
&= \varphi(\mu^{n-2}(\operatorname{adj} U)\operatorname{adj}\,(\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\beta E_{jj})(\operatorname{adj}\overline{U}^t)) \\
&= \varphi(\mu^{n-2}\operatorname{adj}\,(\overline{U}^t(\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\beta E_{jj})U)) \\
&= \eta^{m-2}\operatorname{adj}\,(\varphi(\overline{U}^t(\vartheta I_n - \vartheta E_{ii} + \theta^{-1}\vartheta^{2-n}\beta E_{jj})U) - \varphi(\overline{U}^t\vartheta E_{jj}U)) \\
&= \eta^{m-2}\operatorname{adj}\, \varphi(\overline{U}^t(\vartheta I_n + \theta^{-1}\vartheta^{2-n}\beta E_{jj})U - \overline{U}^t\vartheta E_{ii}U) \\
&= \eta^{m-2}\operatorname{adj}\,(\varphi(\overline{U}^t(\vartheta I_n + \theta^{-1}\vartheta^{2-n}\beta E_{jj})U) - \varphi(\overline{U}^t\vartheta E_{ii}U)) \\
&= \eta^{m-2}\operatorname{adj}\, \varphi(\vartheta\overline{U}^t U + \overline{U}^t(\theta^{-1}\vartheta^{2-n}\beta E_{jj})U) \\
&= \eta^{m-2}\operatorname{adj}\,(\varphi(\vartheta\overline{U}^t U) - \varphi(-\overline{U}^t(\theta^{-1}\vartheta^{2-n}\beta E_{jj})U)) \\
&= \eta^{m-2}\operatorname{adj}\,(-\varphi(-\overline{U}^t(\theta^{-1}\vartheta^{2-n}\beta E_{jj})U)) = 0
\end{aligned}$$

since rank $\varphi(-\overline{U}^t(\theta^{-1}\vartheta^{2-n}\beta)E_{jj}U) \leqslant 1$. By the fact of (3.33) and (3.37), we have

$$\operatorname{adj} T_P(A+\alpha_1 E_{11}+\cdots+\alpha_n E_{nn}) = \operatorname{adj} T_P(A) \tag{3.38}$$

for every matrix $A \in \mathscr{H}_n(\mathbb{F})$ and scalars $\alpha_1,\ldots,\alpha_n \in \mathbb{F}^-$. We next claim, for each $1 \leqslant i \leqslant n$, that

$$\varphi(\overline{U}^t(\alpha E_{ii})U) = 0 \quad \text{for every } \alpha \in \mathbb{F}^-. \tag{3.39}$$

Since $\operatorname{adj}(I_n - E_{ii} - E_{jj} + \gamma E_{jj}) = \gamma E_{ii}$ with $i \neq j$ and $\gamma = (\mu^{-1})^{n-2}\alpha \in \mathbb{F}^-$, together with (3.33) and (3.38), we have

$$\begin{aligned}
\varphi(\overline{U}^t(\alpha E_{ii})U) &= \varphi(\mu^{n-2}(\operatorname{adj}\overline{P}^t)(\operatorname{adj}(I_n - E_{ii} - E_{jj} + \gamma E_{jj}))\operatorname{adj} P) \\
&= \eta^{m-2}\operatorname{adj} T_P(I_n - E_{ii} - E_{jj} + \gamma E_{jj}) \\
&= \eta^{m-2}\operatorname{adj} T_P(\gamma E_{jj}) = 0.
\end{aligned}$$

It follows from Lemma 8 (b) and (3.39) that

$$\operatorname{adj} \varphi(A+\overline{U}^t(\alpha_1 E_{11}+\cdots+\alpha_n E_{nn})U) = \operatorname{adj} \varphi(A) \tag{3.40}$$

for every matrix $A \in \mathscr{H}_n(\mathbb{F})$ and scalars $\alpha_1,\ldots,\alpha_n \in \mathbb{F}^-$. Let $1 \leqslant i,j,k \leqslant n$ be distinct integers. Denote $X_{ijk} := I_n - E_{ii} - E_{jj} - 2E_{kk}$. Let $a \in \mathbb{F}^-$ be a nonzero scalar. Then $\overline{a}a \in \mathbb{F}^-$ and $\operatorname{adj}(aE_{ij}+\overline{a}E_{ji}+X_{ijk}) = aE_{ij}+\overline{a}E_{ji}+\overline{a}aX_{ijk}$. By Lemma 8 (a) and (3.38), we obtain

$$\begin{aligned}
\varphi(\mu^{n-2}\overline{U}^t(aE_{ij}+\overline{a}E_{ji}+\overline{a}aX_{ijk})U) &= \varphi(\mu^{n-2}\operatorname{adj}(P(aE_{ij}+\overline{a}E_{ji}+X_{ijk})\overline{P}^t)) \\
&= \eta^{m-2}\operatorname{adj} T_P(aE_{ij}+\overline{a}E_{ji}+X_{ijk}) \\
&= \eta^{m-2}\operatorname{adj} T_P(aE_{ij}+\overline{a}E_{ji}) = 0
\end{aligned}$$

since rank $T_P(aE_{ij}+\overline{a}E_{ji}) \leqslant m-2$. Consequently, we have

$$\varphi(\mu^{n-2}\overline{U}^t(aE_{ij}+\overline{a}E_{ji}+\overline{a}aX_{ijk})U) = 0 \tag{3.41}$$

for every distinct integers $1 \leqslant i,j,k \leqslant n$ and scalar $a \in \mathbb{F}^-$.

We now show that T_P sends all rank one matrices into zero. Let $H \in \mathscr{H}_n(\mathbb{F})$ be a rank one matrix. By [2, Lemma 2.2], there exists a rank $n-1$ matrix $R = (r_{ij}) \in \mathscr{H}_n(\mathbb{F})$ such that $\theta^{-1}H = \operatorname{adj} R$. By Lemma 8 (a), we have

$$\begin{aligned}
T_P(H) &= \varphi(\theta P(\theta^{-1}H)\overline{P}^t) \\
&= \varphi(\mu^{(n-2)n}\det(P\overline{P})^{n-2}P(\operatorname{adj} R)\overline{P}^t) \\
&= \varphi(\mu^{(n-2)n}(\operatorname{adj} U)(\operatorname{adj} R)(\operatorname{adj}\overline{U}^t)) \\
&= \varphi(\mu^{n-2}\operatorname{adj}(\mu^{n-2}\overline{U}^t RU)) \\
&= \eta^{m-2}\operatorname{adj} \varphi(\mu^{n-2}\overline{U}^t RU).
\end{aligned}$$

By using (3.40) and following by Lemma 8 (b) and (3.41), we see that

$$\begin{aligned}
\operatorname{adj}\,\varphi(\mu^{n-2}\overline{U}^{t}RU) &= \operatorname{adj}\,\varphi\left(\sum_{1\leqslant i<j\leqslant n}\mu^{n-2}\overline{U}^{t}(r_{ji}E_{ji}+\overline{r_{ji}}E_{ij})U+\sum_{i=1}^{n}\overline{U}^{t}(\mu^{n-2}r_{ii}E_{ii})U\right)\\
&= \operatorname{adj}\,\varphi\left(\sum_{1\leqslant i<j\leqslant n}\mu^{n-2}\overline{U}^{t}(r_{ji}E_{ji}+\overline{r_{ji}}E_{ij})U\right)\\
&= \operatorname{adj}\,\varphi\left(\sum_{1\leqslant i<j\leqslant n}\mu^{n-2}\overline{U}^{t}(r_{ji}E_{ji}+\overline{r_{ji}}E_{ij})U+\mu^{n-2}\overline{U}^{t}(\overline{r_{21}}r_{21}X_{12k})U\right)\\
&= \operatorname{adj}\,\varphi\left(\sum_{\substack{1\leqslant i<j\leqslant n,\\ i\neq 1\text{ and } j\neq 2}}\mu^{n-2}\overline{U}^{t}(r_{ji}E_{ji}+\overline{r_{ji}}E_{ij})U\right).
\end{aligned}$$

Continuing in this way, we obtain

$$\operatorname{adj}\,\varphi(\mu^{n-2}\overline{U}^{t}RU)=\operatorname{adj}\,\varphi(\mu^{n-2}\overline{U}^{t}(r_{n,n-1}E_{n,n-1}+\overline{r_{n,n-1}}E_{n-1,n})U)=0$$

since $\operatorname{rank}\varphi(\mu^{n-2}\overline{U}^{t}(r_{n,n-1}E_{n,n-1}+\overline{r_{n,n-1}}E_{n-1,n})U)\leqslant m-2$. Consequently, we conclude that $T_P(H)=0$ for every rank one matrix $H\in\mathscr{H}_n(\mathbb{F})$.

We now prove that $\operatorname{adj}\,T_P(A)=0$ for all $A\in\mathscr{H}_n(\mathbb{F})$. The result obviously holds if $A=0$. Suppose that $\operatorname{rank}A=k$ with $1\leqslant k\leqslant n$. In view of [2, Lemma 2.3], there exist rank one matrices $A_1,\ldots,A_h\in\mathscr{H}_n(\mathbb{F})$, with $k\leqslant h\leqslant k+1$, such that $A=A_1+\cdots+A_h$. By (3.33), we have $\operatorname{adj}\,T_P(A)=\operatorname{adj}\,(T_P(A_1+\cdots+A_{h-1})-T_P(-A_h))=\operatorname{adj}T_P(A_1+\cdots+A_{h-1})=\cdots=\operatorname{adj}\,T_P(A_1)=0$. So $\operatorname{rank}T_P(A)\leqslant m-2$ for every $A\in\mathscr{H}_n(\mathbb{F})$, as desired.

(b) Since $\varphi(\mu^{n-2}\overline{U}^{t}U)=\varphi(\mu^{n-2}\operatorname{adj}\,(P\overline{P}^{t}))=\eta^{m-2}\operatorname{adj}\,T_P(I_n)$, it follows that $\varphi(\mu^{n-2}\overline{U}^{t}U)$ is of rank m. Suppose that $T_P(a_0E_{i_0i_0})=0$ for some integer $1\leqslant i_0\leqslant n$ and some nonzero scalar $a_0\in\mathbb{F}^-$. Let s,t be two distinct integers such that $1\leqslant s,t\leqslant n$ with $s,t\neq i_0$. Since

$$\operatorname{adj}\,(I_n-E_{ss}-(1+a_0)E_{i_0i_0}-(1-a_0^{-1})E_{tt})=-E_{ss},$$

it follows from (3.33) and Lemma 8 (e) that

$$\begin{aligned}
\varphi(\mu^{n-2}\overline{U}^{t}(-E_{ss})U) &= \varphi(\mu^{n-2}\operatorname{adj}\,(P(I_n-E_{ss}-(1+a_0)E_{i_0i_0}-(1-a_0^{-1})E_{tt})\overline{P}^{t}))\\
&= \eta^{m-2}\operatorname{adj}\,T_P(I_n-E_{ss}-(1+a_0)E_{i_0i_0}-(1-a_0^{-1})E_{tt})\\
&= \eta^{m-2}\operatorname{adj}\,(T_P(I_n-E_{ss}-E_{i_0i_0}-(1-a_0^{-1})E_{tt})-T_P(a_0E_{i_0i_0}))\\
&= \eta^{m-2}\operatorname{adj}\,T_P(I_n-E_{ss}-E_{i_0i_0}-(1-a_0^{-1})E_{tt})=0
\end{aligned}$$

because $\operatorname{rank}\,(I_n-E_{ss}-E_{i_0i_0}-(1-a_0^{-1})E_{tt})=n-2$. By Lemma 8 (b) and (d), we have

$$\begin{aligned}
\operatorname{adj}\,\varphi(\mu^{n-2}\overline{U}^{t}U) &= \operatorname{adj}\,\varphi(\mu^{n-2}\overline{U}^{t}(I_n-E_{ss}+E_{ss})U)\\
&= \operatorname{adj}\,(\varphi(\mu^{n-2}\overline{U}^{t}(I_n-E_{ss})U)-\varphi(\mu^{n-2}\overline{U}^{t}(-E_{ss})U)\\
&= \operatorname{adj}\,\varphi(\mu^{n-2}\overline{U}^{t}(I_n-E_{ss})U),
\end{aligned}$$

which implies that $\operatorname{rank}\varphi(\mu^{n-2}\overline{U}^t U) \neq m$, a contradiction. Thus $T_P(aE_{ii}) \neq 0$ for every nonzero $a \in \mathbb{F}^-$. By Lemma 8 (c), $\operatorname{rank} T_P(aE_{ii}) = 1$ for every integer $1 \leqslant i \leqslant n$ and nonzero scalar $a \in \mathbb{F}^-$. The proof is complete. $\square$

LEMMA 10. *Let m and n be even integers with $m,n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars, and let $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{K})$ be a mapping satisfying condition (3.32). If $\operatorname{rank}\varphi(I_n) = m$, then φ is injective and*

$$\operatorname{rank}(H-K) = n \quad \Leftrightarrow \quad \operatorname{rank}(\varphi(H)-\varphi(K)) = m$$

for every $H,K \in \mathscr{H}_n(\mathbb{F})$.

Proof. Let $A \in \mathscr{H}_n(\mathbb{F})$ be of rank one. It follows from [2, Lemma 2.1] that there exist an invertible matrix $P \in \mathscr{M}_n(\mathbb{F})$ and a nonzero scalar $\alpha \in \mathbb{F}^-$ such that $A = P(\alpha E_{11})\overline{P}^t$. We define the mapping $T_P : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{F})$ such as

$$T_P(H) = \varphi(PH\overline{P}^t) \quad \text{for every } H \in \mathscr{H}_n(\mathbb{F}).$$

Then $T_P(P^{-1}\overline{P^{-1}}^t)$ is of rank m. Suppose that $\operatorname{rank} T_P(I_n) \neq m$. Then, by Lemma 9 (a), we have $\operatorname{rank} T_P(H) \leqslant m-2$ for every matrix $H \in \mathscr{H}_n(\mathbb{F})$, which contradicts to the fact that $\operatorname{rank} T_P(P^{-1}\overline{P^{-1}}^t) = m$. So $\operatorname{rank} T_P(I_n) = m$, and thus $\operatorname{rank} T_P(aE_{ii}) = 1$ for all integers $1 \leqslant i \leqslant n$ and nonzero scalars $a \in \mathbb{F}^-$ by Lemma 9 (b). Therefore, $\operatorname{rank}\varphi(A) = \operatorname{rank} T_P(\alpha E_{ii}) = 1$. Hence φ preserves rank one matrices.

Let $X,Y \in \mathscr{H}_n(\mathbb{F})$ with $\varphi(X) = \varphi(Y)$. Suppose that $X - Y \neq 0$. By [2, Lemma 2.4 (d)], there is a matrix $Z \in \mathscr{H}_n(\mathbb{F})$ with $\operatorname{rank} Z \leqslant n-2$ such that $\operatorname{rank}(X-Y+Z) = n-1$. Then $\operatorname{adj}(X-Y+Z) = 1$, and so $\operatorname{rank}\varphi(\mu^{n-2}\operatorname{adj}(X-Y+Z)) = 1$. On the other hand, $\varphi(\mu^{n-2}\operatorname{adj}(X-Y+Z)) = \eta^{m-2}\operatorname{adj}\varphi(X+Z-Y) = \eta^{m-2}\operatorname{adj}(\varphi(X+Z) - \varphi(Y)) = \eta^{m-2}\operatorname{adj}(\varphi(X+Z)-\varphi(X)) = \eta^{m-2}\operatorname{adj}\varphi(Z) = 0$, a contradiction. Hence $X = Y$, and so φ is injective.

Let $H,K \in \mathscr{H}_n(\mathbb{F})$. By the injectivity of φ, it follows from Lemmas 8 (a), (b) and (f) that

$$\begin{aligned}\operatorname{rank}(H-K) = n &\Leftrightarrow \operatorname{rank}\ \psi(\mu^{n-2}\operatorname{adj}(H-K)) = m\\ &\Leftrightarrow \operatorname{rank}\ \eta^{m-2}\operatorname{adj}(\psi(H-K)) = m\\ &\Leftrightarrow \operatorname{rank}(\psi(H)-\psi(K)) = m.\end{aligned}$$

The proof is complete. $\square$

PROPOSITION 3. *Let m and n be even integers with $m,n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be any fixed nonzero scalars. Then $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{K})$ is an additive mapping satisfying*

$$\varphi(\mu^{n-2}\operatorname{adj} H) = \eta^{m-2}\operatorname{adj}\varphi(H) \ \textit{for every } H \in \mathscr{H}_n(\mathbb{F})$$

if and only if either $\varphi = 0$, *or* $m = n$ *and*

$$\varphi(A) = \lambda P A^{\sigma} \widehat{P}^{t} \quad \textit{for every } A \in \mathscr{H}_n(\mathbb{F}),$$

where $\sigma : (\mathbb{F}, ^{-}) \to (\mathbb{K}, ^{\wedge})$ *is a nonzero field homomorphism satisfying* $\widehat{\sigma(a)} = \sigma(\overline{a})$ *for all* $a \in \mathbb{F}$, $P \in \mathscr{M}_n(\mathbb{K})$ *is invertible with* $\widehat{P}^t P = \varsigma I_n$, *and* $\lambda, \varsigma \in \mathbb{K}^{\wedge}$ *are scalars with* $(\lambda \varsigma \eta \sigma(\mu)^{-1})^{n-2} = 1$.

Proof. The sufficiency part is clear. We now consider the necessity part. By the additivity of φ, we see that φ satisfies (3.32). We argue in the following two sub-cases:

Case I: $\operatorname{rank} \varphi(I_n) \neq m$. In view of Lemma 9 (a), by considering $P = I_n$, we have $\varphi(A) = 0$ for every rank one matrix $A \in \mathscr{H}_n(\mathbb{F})$. By the additivity of φ, we show that $\varphi = 0$, as desired.

Case II: $\operatorname{rank} \varphi(I_n) = m$. By Lemma 10, φ is injective, and so φ preserves rank one matrices by Lemma 8 (c). Note that $m = \operatorname{rank}(\varphi(E_{11}) + \cdots + \varphi(E_{nn})) \leqslant \sum_{i=1}^{n} \operatorname{rank} \varphi(E_{ii}) = n$. Suppose that $n > m$. Then $\operatorname{rank}(\varphi(E_{11}) + \cdots + \varphi(E_{nn})) < n$. It follows from [4, Theorem 2.1] that there exist integers $1 \leqslant s_1 < \cdots < s_p \leqslant n$, with $m \leqslant p < n$, such that $\operatorname{rank} \varphi(E_{s_1 s_1} + \cdots + E_{s_p s_p}) = m$. Then $m = \operatorname{rank}(\eta^{m-2} \operatorname{adj} \varphi(E_{s_1 s_1} + \cdots + E_{s_p s_p})) = \operatorname{rank} \varphi(\mu^{n-2} \operatorname{adj}(E_{s_1 s_1} + \cdots + E_{s_p s_p})) \leqslant 1$, a contradiction. Hence $m = n$. By [14, Main Theorem, p.g. 603] and [11, Theorem 2.1 and Remark 2.4], we have

$$\varphi(A) = \lambda Q A^{\sigma} \widehat{Q}^{t} \quad \text{for every } A \in \mathscr{H}_n(\mathbb{F}),$$

where $\sigma : (\mathbb{F}, ^{-}) \to (\mathbb{K}, ^{\wedge})$ is a nonzero field homomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$, $Q \in \mathscr{M}_n(\mathbb{K})$ is an invertible matrix and $\lambda \in \mathbb{K}^{\wedge}$ is a nonzero scalar. We now claim that there exists a nonzero scalar $\varsigma \in \mathbb{K}^{\wedge}$ such that

$$Q\widehat{Q}^{t} = \varsigma I_n \quad \text{and} \quad (\eta \lambda \varsigma \sigma(\mu)^{-1})^{n-2} = 1. \tag{3.42}$$

In view of Lemma 8 (a), we see that $\eta^{n-2} \operatorname{adj} \varphi(I_n) = \varphi(\mu^{n-2} I_n)$. Then $\eta^{n-2} \lambda^{n-1} \operatorname{adj}(Q\widehat{Q}^t) = \lambda \sigma(\mu)^{n-2} Q\widehat{Q}^t$, and so $Q\widehat{Q}^t = (\lambda \eta \sigma(\mu)^{-1})^{n-2} (\operatorname{adj} \widehat{Q}^t)(\operatorname{adj} Q)$. Let $\xi := (\lambda \eta \sigma(\mu)^{-1})^{n-2} \in \mathbb{K}^{\wedge}$. Then

$$(\widehat{Q}^t Q)^2 = \widehat{Q}^t (Q\widehat{Q}^t) Q = \xi \widehat{Q}^t (\operatorname{adj} \widehat{Q}^t)(\operatorname{adj} Q) Q = \xi \det(\widehat{Q}^t Q) I_n. \tag{3.43}$$

Let $1 \leqslant i < j \leqslant n$. Since $\operatorname{adj}(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = -(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})$, it follows from Lemma 8 (a) that

$$\eta^{n-2} \operatorname{adj} \varphi(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = -\varphi(\mu^{n-2}(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})).$$

Then $\eta^{n-2} \operatorname{adj}(\lambda Q (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) \widehat{Q}^t) = -\lambda Q (\sigma(\mu)^{n-2} (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})) \widehat{Q}^t$, and by (3.43), we have

$$\begin{aligned} \widehat{Q}^t Q (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) \widehat{Q}^t Q &= \xi \det(\widehat{Q}^t Q)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) \\ &= (\widehat{Q}^t Q)^2 (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}). \end{aligned}$$

Pre-multiplying by $(\widehat{Q}^t Q)^{-1}$ gives $\widehat{Q}^t Q(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t Q$ for every $1 \leqslant i < j \leqslant n$. Then $\widehat{Q}^t Q = \varsigma I_n$ for some nonzero scalar $\varsigma \in \mathbb{K}^\wedge$, and so $Q\widehat{Q}^t = \varsigma I_n$. Further, since $\eta^{n-2}\mathrm{adj}\,(\lambda \varsigma I_n) = \eta^{n-2}\mathrm{adj}\,(\lambda Q\widehat{Q}^t) = \eta^{n-2}\mathrm{adj}\,\psi(I_n) = \psi(\mu^{n-2}I_n) = \lambda\,\sigma(\mu)^{n-2}\varsigma I_n$, it follows that $(\eta\lambda\varsigma\,\sigma(\mu)^{-1})^{n-2} = 1$. Claim (3.42) is shown. We complete the proof. $\square$

Proposition 3 gives a slight extension of Theorem 2.10 in [2].

Let m and n be even integers with $m,n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ be a fixed but arbitrarily chosen nonzero scalar, and let $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{F})$ be a mapping satisfying

$$\varphi(\mu^{n-2}\mathrm{adj}\,(H + \alpha K)) = \mu^{m-2}\mathrm{adj}\,(\varphi(H) + \alpha\varphi(K)) \tag{3.44}$$

for every $H,K \in \mathscr{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. Then φ satisfies condition (3.32) for $(\mathbb{K},^\wedge) = (\mathbb{F},^-)$ and $\eta = \mu$. Thus Lemmas 8, 9 and 10 hold true for φ. In particular, by an argument analogous to the proof of Lemma 8 (b), we have

$$\mathrm{adj}\;\varphi(H + \alpha K) = \mathrm{adj}\,(\varphi(H) + \alpha\varphi(K))$$

for every $H,K \in \mathscr{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. Further, if $\mathrm{rank}\,\varphi(I_n) = m$, then, by Lemma 10, we see that φ is injective and, in view of Lemma 8 (f) and by a similar argument as in the last paragraph of the proof of Lemma 10, we have

$$\mathrm{rank}\;\varphi(H + \alpha K) = m \;\Leftrightarrow\; \mathrm{rank}\,(H + \alpha K) = n \;\Leftrightarrow\; \mathrm{rank}\,(\varphi(H) + \alpha\varphi(K)) = m$$

for every $H,K \in \mathscr{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. Therefore, by following the lines of the analogous proof in Lemma 7 applied on Hermitian matrices or [2, Lemma 2.9], it can be shown that φ is additive and $\varphi(\alpha A) = \alpha\varphi(A)$ for every matrix $A \in \mathscr{H}_n(\mathbb{F})$ and scalar $\alpha \in \mathbb{F}^-$. We formulate this observation as a lemma:

LEMMA 11. *Let m and n be even integers with $m,n \geqslant 4$. Let $\mathbb{F}$ be a field which possesses a proper involution $^-$ of $\mathbb{F}$ such that either $|\mathbb{F}^-| = 2$ or $|\mathbb{F}^-| > n+1$. Let $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{F})$ be a mapping satisfying condition (3.44). If $\mathrm{rank}\,\varphi(I_n) = m$, then φ is additive and $\varphi(aA) = a\varphi(A)$ for every matrix $A \in \mathscr{H}_n(\mathbb{F})$ and scalar $a \in \mathbb{F}^-$.*

PROPOSITION 4. *Let m and n be even integers with $m,n \geqslant 4$, and $\mathbb{F}$ be a field which possesses a proper involution $^-$ of $\mathbb{F}$ such that either $|\mathbb{F}^-| = 2$ or $|\mathbb{F}^-| > n+1$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ be a fixed but arbitrarily chosen nonzero scalar. Then $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{F})$ is a mapping satisfying*

$$\varphi(\mu^{n-2}\mathrm{adj}\,(H + \alpha K)) = \mu^{m-2}\mathrm{adj}\,(\varphi(H) + \alpha\varphi(K))$$

for every $H,K \in \mathscr{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$ if and only if $\varphi(A) = 0$ for every rank one matrix $A \in \mathscr{H}_n(\mathbb{F})$ and $\mathrm{rank}\,(\varphi(A) + \alpha\varphi(B)) \leqslant m-2$ for every $A,B \in \mathscr{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$; or $m = n$ and

$$\varphi(A) = \lambda P A^\sigma \overline{P}^t \quad \textit{for every } A \in \mathscr{H}_n(\mathbb{F}),$$

where $\sigma : \mathbb{F} \to \mathbb{F}$ is a field isomorphism satisfying $\overline{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$ and $\sigma(a) = a$ for all $a \in \mathbb{F}^-$, $P \in \mathscr{M}_n(\mathbb{F})$ is invertible satisfying $\widehat{P}^t P = \varsigma I_n$, and $\lambda, \varsigma \in \mathbb{F}^-$ are scalars satisfying $(\lambda\varsigma\mu\sigma(\mu)^{-1})^{n-2} = 1$.

Proof. The sufficiency part is clear. To prove the necessity part, we first see that if $\operatorname{rank}\varphi(I_n) \neq m$, then it follows from Lemma 9 (a), by considering $P = I_n$, that $\varphi(A) = 0$ for every rank one matrix $A \in \mathcal{H}_n(\mathbb{F})$, and $\operatorname{rank}\varphi(A) \leqslant m-2$ for every $A \in \mathcal{H}_n(\mathbb{F})$.

We next consider $\operatorname{rank}\varphi(I_n) = m$. By Lemma 11 and Proposition 3, we conclude that $m = n$ and $\varphi(A) = \lambda Q A^{\sigma} \overline{Q}^t$ for every $A \in \mathcal{H}_n(\mathbb{F})$, where $\sigma : \mathbb{F} \to \mathbb{F}$ is a nonzero field homomorphism satisfying $\overline{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, $Q \in \mathcal{M}_n(\mathbb{K})$ is invertible with $\widehat{Q}^t Q = \varsigma I_n$, and $\lambda, \varsigma \in \mathbb{F}^-$ are scalars with $(\lambda\varsigma\mu\sigma(\mu)^{-1})^{n-2} = 1$. It follows from $\varphi(aI_n) = a\varphi(I_n)$ for every $a \in \mathbb{F}^-$, and hence $\sigma(a) = a$ for every $a \in \mathbb{F}^-$. Furthermore, since ${}^-$ is proper, there exists a scalar $i \in \mathbb{F}$ with $\overline{i} = -i$ when $\operatorname{char}\mathbb{F} \neq 2$, and $\overline{i} = 1+i$ when $\operatorname{char}\mathbb{F} = 2$, such that $\mathbb{F} = \mathbb{F}^- \oplus i\mathbb{F}^-$. It is easily verified that $\overline{\sigma(i)} = -\sigma(i)$ when $\operatorname{char}\mathbb{F} \neq 2$, and $\overline{\sigma(i)} = 1 + \sigma(i)$ when $\operatorname{char}\mathbb{F} = 2$. We thus have $\mathbb{F} = \mathbb{F}^- \oplus \sigma(i)\mathbb{F}^-$. Let $\alpha \in \mathbb{F}$. Then there exist scalars $\beta_1, \beta_2 \in \mathbb{F}^-$ such that $\alpha = \beta_1 + \sigma(i)\beta_2$. Let $\gamma = \beta_1 + i\beta_2 \in \mathbb{F}$. We see that $\sigma(\gamma) = \alpha$. Hence σ is surjective, and so it is an isomorphism. The proof is complete. $\square$

We remark that Proposition 4 gives a slight improvement, as well as a correction for a misprint, of Theorem 2.12 in [2]. When $\mathbb{F}$ is the complex field $\mathbb{C}$, we have the field isomorphism σ on $\mathbb{C}$ is either the identity or the complex conjugate of $\mathbb{C}$.

Let $\mu \in S\mathbb{F}^-$ be a nonzero scalar. Then $\mu^{-1} \in S\mathbb{F}^-$. We note that if $A \in \mathcal{SH}_n(\mathbb{F})$, then $(\overline{\mu A})^t = \overline{\mu}\overline{A}^t = -\mu(-A) = \mu A$. This implies that $\mu A \in \mathcal{H}_n(\mathbb{F})$. Conversely, if $\mu A \in \mathcal{H}_n(\mathbb{F})$, then $\mu A = (\overline{\mu A})^t = \overline{\mu}\overline{A}^t = (-\mu)\overline{A}^t = -\mu\overline{A}^t$. Thus $\overline{A}^t = -A$, and so $A \in \mathcal{SH}_n(\mathbb{F})$. We thus obtain

$$A \in \mathcal{SH}_n(\mathbb{F}) \quad \Leftrightarrow \quad \mu A \in \mathcal{H}_n(\mathbb{F}) \tag{3.45}$$

for any fixed nonzero scalar $\mu \in S\mathbb{F}^-$. Likewise, we also have

$$A \in \mathcal{H}_n(\mathbb{F}) \quad \Leftrightarrow \quad \mu A \in \mathcal{SH}_n(\mathbb{F}) \tag{3.46}$$

for any fixed nonzero scalar $\mu \in S\mathbb{F}^-$. Then (3.45) and (3.46) lead to

$$\mathcal{SH}_n(\mathbb{F}) = \mu\mathcal{H}_n(\mathbb{F}) := \{\mu A : A \in \mathcal{H}_n(\mathbb{F})\} \tag{3.47}$$

$$\mathcal{H}_n(\mathbb{F}) = \mu\mathcal{SH}_n(\mathbb{F}) := \{\mu A : A \in \mathcal{SH}_n(\mathbb{F})\} \tag{3.48}$$

for any fixed nonzero scalar $\mu \in S\mathbb{F}^-$.

LEMMA 12. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mu \in S\mathbb{F}^-$ and $\eta \in S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars. Let $\psi : \mathcal{SH}_n(\mathbb{F}) \to \mathcal{SH}_m(\mathbb{K})$ be a mapping. If $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ is the mapping defined by*

$$\varphi(H) = \eta^{-1}\psi(\mu H) \quad \textit{for every } H \in \mathcal{H}_n(\mathbb{F}),$$

then the following statements hold:

(a) $\psi(\operatorname{adj}(A-B)) = \operatorname{adj}(\psi(A) - \psi(B))$ *for every* $A, B \in \mathcal{SH}_n(\mathbb{F})$ *if and only if* $\varphi(\mu^{n-2}\operatorname{adj}(H-K)) = \eta^{m-2}\operatorname{adj}(\varphi(H) - \varphi(K))$ *for every* $H, K \in \mathcal{H}_n(\mathbb{F})$.

(b) *If* $(\mathbb{K},^\wedge) = (\mathbb{F},^-)$ *and* $\eta = \mu$, *then* $\psi(\mathrm{adj}\,(A+\alpha B)) = \mathrm{adj}\,(\psi(A)+\alpha\psi(B))$ *for every* $A,B \in \mathscr{SH}_n(\mathbb{F})$ *and* $\alpha \in \mathbb{F}^-$ *if and only if* $\varphi(\mu^{n-2}\mathrm{adj}\,(H+\alpha K)) = \mu^{m-2}\mathrm{adj}\,(\varphi(H)+\alpha\varphi(K))$ *for every* $H,K \in \mathscr{H}_n(\mathbb{F})$ *and* $\alpha \in \mathbb{F}^-$.

Proof. It suffices to prove the lemma only for statement (a) as statement (b) can be shown similarly. Let $H,K \in \mathscr{H}_n(\mathbb{F})$. By the definition of φ and (3.47), we see that $\eta^{m-2}\mathrm{adj}\,(\varphi(H)-\varphi(K)) = \eta^{m-2}\mathrm{adj}\,(\eta^{-1}\psi(\mu H)-\eta^{-1}\psi(\mu K)) = \eta^{-1}\mathrm{adj}\,(\psi(\mu H)-\psi(\mu K)) = \eta^{-1}\psi(\mathrm{adj}\,(\mu(H-K))) = \eta^{-1}\psi(\mu^{n-1}\mathrm{adj}\,(H-K)) = \varphi(\mu^{n-2}\mathrm{adj}\,(H-K))$, as required.

Conversely, consider $A,B \in \mathscr{SH}_n(\mathbb{F})$. By the definition of φ and (3.48), we see that $\mathrm{adj}\,(\psi(A)-\psi(B)) = \eta^{m-1}\mathrm{adj}\,(\eta^{-1}\psi(\mu(\mu^{-1}A))-\eta^{-1}\psi(\mu(\mu^{-1}B))) = \eta^{m-1}$ $\mathrm{adj}\,(\varphi(\mu^{-1}A)-\varphi(\mu^{-1}B)) = \eta(\eta^{m-2}\mathrm{adj}\,(\varphi(\mu^{-1}A)-\varphi(\mu^{-1}B))) = \eta\varphi(\mu^{n-2}\mathrm{adj}\,\mu^{-1}$ $(A-B)) = \psi(\mu(\mu^{-1}\mathrm{adj}\,(A-B))) = \psi(\mathrm{adj}\,(A-B))$. We are done. □

We are now ready to prove our main theorems of this section.

THEOREM 5. *Let m and n be even integers with $m,n \geqslant 4$. Let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess proper involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively. Then $\psi : \mathscr{SH}_n(\mathbb{F}) \to \mathscr{SH}_m(\mathbb{K})$ is a classical adjoint commuting additive mapping if and only if either $\psi = 0$, or $m = n$ and*

$$\psi(A) = \lambda P A^\sigma \widehat{P}^t \quad \textit{for every } A \in \mathscr{SH}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F},^-) \to (\mathbb{K},^\wedge)$ is a nonzero field homomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, $P \in \mathscr{M}_n(\mathbb{K})$ is invertible with $\widehat{P}^t P = \varsigma I_n$, and $\lambda,\varsigma \in \mathbb{K}^\wedge$ are scalars with $(\lambda\varsigma)^{n-2} = 1$.

Proof. The sufficiency part is clear. We now consider the necessity part. By the additivity of ψ, we have $\psi(\mathrm{adj}\,(A-B)) = \mathrm{adj}\,(\psi(A)-\psi(B))$ for every $A,B \in \mathscr{SH}_n(\mathbb{F})$. Let $\mu \in S\mathbb{F}^-$ and $\eta \in S\mathbb{K}^\wedge$ be two fixed nonzero scalars. In view of (3.47), we define the mapping $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{K})$ such as

$$\varphi(H) = \eta^{-1}\psi(\mu H) \quad \text{for every } H \in \mathscr{H}_n(\mathbb{F}). \tag{3.49}$$

By Lemma 12 (a) and $\psi(0) = 0$, we have $\varphi(\mu^{n-2}\mathrm{adj}\,H) = \eta^{m-2}\mathrm{adj}\,\varphi(H)$ for every $H \in \mathscr{H}_n(\mathbb{F})$. We now claim that φ is additive. Let $H,K \in \mathscr{H}_n(\mathbb{F})$. Then $\varphi(H+K) = \eta^{-1}\psi(\mu(H+K)) = \eta^{-1}(\psi(\mu H)+\psi(\mu K)) = \varphi(H)+\varphi(K)$. By Proposition 3, together with (3.49), we have either $\varphi = 0$, or $m = n$ and there exist a nonzero field homomorphism $\sigma : (\mathbb{F},^-) \to (\mathbb{K},^\wedge)$ with $\widehat{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, an invertible matrix $P \in \mathscr{M}_n(\mathbb{K})$ with $\widehat{P}^t P = \varsigma I_n$, and scalars $\alpha,\varsigma \in \mathbb{K}^\wedge$ with $(\eta\alpha\varsigma\,\sigma(\mu)^{-1})^{n-2} = 1$, such that $\varphi(H) = \alpha P H^\sigma \widehat{P}^t$ for all $H \in \mathscr{H}_n(\mathbb{F})$. By (3.49), we have

$$\psi(\mu H) = \eta\alpha P H^\sigma \widehat{P}^t = (\eta\alpha\,\sigma(\mu)^{-1})P(\mu H)^\sigma \widehat{P}^t \quad \text{for every } H \in \mathscr{H}_n(\mathbb{F}).$$

Let $\lambda := \eta\alpha\,\sigma(\mu)^{-1}$. Then $\lambda \in \mathbb{K}^\wedge$ since $\eta, \sigma(\mu)^{-1} \in S\mathbb{K}^\wedge$ and $\alpha \in \mathbb{K}^\wedge$. It follows from (3.47) that

$$\psi(A) = \lambda P A^\sigma \widehat{P}^t \quad \text{for every } A \in \mathscr{SH}_n(\mathbb{F})$$

with $\widehat{P}^t P = \varsigma I_n$ and $(\lambda\varsigma)^{n-2} = 1$. We are done. $\square$

Proof of Theorem 3. The sufficiency part is clear. To prove the necessity part, we let $\mu \in S\mathbb{F}^-$ be a fixed nonzero scalar and $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{F})$ be the mapping defined by

$$\varphi(H) = \mu^{-1}\psi(\mu H) \quad \text{for every } H \in \mathscr{H}_n(\mathbb{F}). \tag{3.50}$$

By the assumption of ψ and Lemma 12 (b), we see that φ satisfies (3.44). By Proposition 4, we have either

(a) $\varphi(H) = 0$ for every rank one matrix $H \in \mathscr{H}_n(\mathbb{F})$, and $\operatorname{rank}\varphi(H) \leqslant m-2$ for every $H \in \mathscr{H}_n(\mathbb{F})$; or

(b) $m = n$ and $\varphi(A) = \alpha P A^\sigma \overline{P}^t$ for every $A \in \mathscr{H}_n(\mathbb{F})$, where $\sigma : \mathbb{F} \to \mathbb{F}$ is a field isomorphism satisfying $\overline{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$ and $\sigma(a) = a$ for all $a \in \mathbb{F}^-$, $P \in \mathscr{M}_n(\mathbb{F})$ is invertible with $\widehat{P}^t P = \varsigma I_n$, and $\alpha, \varsigma \in \mathbb{F}^-$ are scalars with $(\alpha\varsigma\mu\sigma(\mu)^{-1})^{n-2} = 1$.

If Case **(a)** holds, then $\psi(A) = \psi(\mu(\mu^{-1}A)) = \mu\varphi(\mu^{-1}A) = 0$ for all rank one matrices $A \in \mathscr{SH}_n(\mathbb{F})$. Let $A \in \mathscr{SH}_n(\mathbb{F})$ be any matrix. Then we have $\operatorname{rank}\psi(A) = \operatorname{rank}\psi(\mu(\mu^{-1}A)) = \operatorname{rank}\varphi(\mu^{-1}A) \leqslant m-2$ by (3.50). We are done.

If Case **(b)** holds, then, by (3.50) and (3.47), we have $\psi(A) = \psi(\mu(\mu^{-1}A)) = \mu\varphi(\mu^{-1}A) = \lambda P A^\sigma \widehat{P}^t$ for every $A \in \mathscr{SH}_n(\mathbb{F})$, where $\widehat{P}^t P = \varsigma I_n$, and $\lambda = \mu\alpha\sigma(\mu)^{-1}$, $\varsigma \in \mathbb{F}^-$ with $(\lambda\varsigma)^{n-2} = 1$. This completes our proof. $\square$

Proof of Theorem 4. The sufficiency part is clear. We now consider the necessity part. Let $\mu \in S\mathbb{F}^-$ and $\eta \in S\mathbb{K}^\wedge$ be any fixed nonzero scalars and $\varphi : \mathscr{H}_n(\mathbb{F}) \to \mathscr{H}_m(\mathbb{K})$ be the mapping defined by

$$\varphi(H) = \eta^{-1}\psi(\mu H) \quad \text{for every } H \in \mathscr{H}_n(\mathbb{F}). \tag{3.51}$$

By Lemma 12 (a), we see that φ satisfies (3.32). We claim that φ is surjective. Let $Y \in \mathscr{SH}_m(\mathbb{K})$. Then $\eta Y \in \mathscr{H}_m(\mathbb{K})$ by (3.45). By the surjectivity of ψ, there is a matrix $X \in \mathscr{SH}_n(\mathbb{F})$ such that $\psi(X) = \eta Y$. Then $\mu^{-1}X \in \mathscr{H}_n(\mathbb{F})$ and $\varphi(\mu^{-1}X) = \eta^{-1}\psi(X) = Y$, as desired. Suppose that $\operatorname{rank}\varphi(I_n) \neq m$. It follows from Lemma 9 (a), by considering $P = I_n$, that $\operatorname{rank}\varphi(H) \leqslant m-2$ for all $H \in \mathscr{H}_n(\mathbb{F})$. This contradicts to the surjectivity of φ, and so $\operatorname{rank}\varphi(I_n) = m$. In view of Lemma 10, we see that φ is a bijection satisfying

$$\operatorname{rank}(H-K) = n \quad \Leftrightarrow \quad \operatorname{rank}(\varphi(H) - \varphi(K)) = m$$

for every $H, K \in \mathscr{H}_n(\mathbb{F})$. We now show that $m = n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic and

$$\varphi(A) = \alpha P A^\sigma \widehat{P}^t \quad \text{for every } A \in \mathscr{H}_n(\mathbb{F}), \tag{3.52}$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a field isomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, $P \in \mathscr{M}_n(\mathbb{K})$ is invertible with $\widehat{P}^t P = \varsigma I_n$, and $\alpha, \varsigma \in \mathbb{K}^\wedge$ are nonzero scalars with $(\alpha\varsigma\eta\sigma(\mu)^{-1})^{n-2} = 1$. We divide our proof into the following two cases.

Case I: $|\mathbb{K}^{\wedge}| = 2$. Thus $-1 = 1$ since $0, 1, -1 \in \mathbb{K}$. Then $\operatorname{rank}(H-K) = n$ if and only if $\operatorname{rank}(\varphi(H)+\varphi(K)) = m$ for $H, K \in \mathscr{H}_n(\mathbb{F})$. We claim that φ is additive. Let $A, B \in \mathscr{H}_n(\mathbb{F})$. If $\operatorname{rank}(A+B) = n$, then, together with Lemma 8 (f), we have $\operatorname{rank}\varphi(A+B) = \operatorname{rank}(A-(-B)) = \operatorname{rank}(\varphi(A)+\varphi(-B)) = m$. By Lemma 8 (b) and a similar argument as in the proof of (2.21), we obtain

$$\frac{\varphi(A+B)}{\det\varphi(A+B)} = \frac{\varphi(A)+\varphi(-B)}{\det(\varphi(A)+\varphi(-B))}.$$

Since $\det\varphi(A+B) = 1 = \det(\varphi(A)+\varphi(-B))$, it follows that $\varphi(A+B) = \varphi(A) + \varphi(-B)$ for every $A, B \in \mathscr{H}_n(\mathbb{F})$ with $\operatorname{rank}(A+B) = n$. By the injectivity of ψ, we see that $\varphi(-I_n) = \varphi(0-I_n) = \varphi(0)+\varphi(I_n) = \varphi(I_n)$ implies that $I_n = -I_n$. Then $\mathbb{F}$ is of characteristic 2, and so the claim holds. We now consider $\operatorname{rank}(A+B) < n$. By [2, Lemma 2.4 (b)], there exists $C \in \mathscr{H}_n(\mathbb{F})$ such that $\operatorname{rank}(A+C) = \operatorname{rank}(A+B+C) = n$. Then $\varphi(A+C) = \varphi(A)+\varphi(C)$ and $\varphi(A+B)+\varphi(C) = \varphi(A+B+C) = \varphi(A+C)+\varphi(B) = \varphi(A)+\varphi(C)+\varphi(B)$. Hence $\varphi(A+B) = \varphi(A)+\varphi(B)$, as required. By Proposition 3 and the bijectivity of φ, Claim (3.52) is proved.

Case II: $|\mathbb{F}^{-}|, |\mathbb{K}^{\wedge}| > 3$. Since $\varphi(0) = 0$, by combining [9, Theorem 3.6] and the fundamental theorem of the geometry of Hermitian matrices [22, Theorem 6.4], we have $m = n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic, and

$$\varphi(A) = \alpha P A^{\sigma} \widehat{P}^{t} \quad \text{for every } A \in \mathscr{H}_n(\mathbb{F}),$$

where $\sigma : (\mathbb{F},^{-}) \to (\mathbb{K},^{\wedge})$ is a field isomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$, $P \in \mathscr{M}_n(\mathbb{K})$ is invertible, and $\alpha \in \mathbb{K}^{\wedge}$ is nonzero. By an argument analogous to Claim (3.42), we see that $P\widehat{P}^{t} = \varsigma I_n$ for some nonzero scalar $\varsigma \in \mathbb{K}^{\wedge}$ and $(\alpha\varsigma\eta\sigma(\mu)^{-1})^{n-2} = 1$. So Claim (3.52) is proved.

In view of (3.51) and (3.52), we obtain $\psi(\mu H) = \eta\varphi(H) = \lambda P(\mu H)^{\sigma}\widehat{P}^{t}$ for every $H \in \mathscr{H}_n(\mathbb{F})$, where $\lambda := \alpha\eta\sigma(\mu)^{-1} \in \mathbb{K}^{\wedge}$, $P\widehat{P}^{t} = \varsigma I_n$ and $(\lambda\varsigma)^{n-2} = 1$. Then $\psi(A) = \lambda P A^{\sigma}\widehat{P}^{t}$ for every $A \in \mathscr{SH}_n(\mathbb{F})$ by (3.47). We are done. □

Acknowledgement.

The authors wish to express their gratitude to the referee for helpful comments and suggesting a simple proof in the last paragraph of the proof of Theorem 2. They gratefully acknowledge the valuable advices of Professor Lim Ming Huat during the preparation of the manuscript. The first author's research was partially supported by the University of Malaya UMRG Research Grants RG156-11AFR.

REFERENCES

[1] W. L. CHOOI AND W. S. NG, *On classical adjoint-commuting mappings between matrix algebras*, Linear Algebra Appl. **432** (2010), 2589–2599.

[2] W. L. CHOOI AND W. S. NG, *Classical adjoint-commuting mappings on Hermitian and symmetric matrices*, Linear Algebra Appl. **435** (2011), 202–223.

[3] G. DOLINAR AND P. ŠEMRL, *Determinant preserving maps on matrix algebras*, Linear Algebra Appl. **348** (2002), 189–192.

[4] A. FOŠNER AND P. ŠEMRL, *Additive maps on matrix algebras preserving invertibility or singularity*, Acta Mathematica Sinica, English Series, **21**, 4 (2005), 681–684.

[5] G. FROBENIUS, *Über die Darstellung der endlichen Gruppen durch Lineare Substitutionen*, Sitzungsber. Deustch. Akad. Wiss. Berlin, 1897, 944–1015.

[6] L. P. HUANG, *Diameter preserving surjections on alternate matrices*, Acta Math. Sinica, English Series, **25**, 9 (2009), 1517–1528.

[7] L. P. HUANG, *Good distance graphs and the geometry of matrices*, Linear Algebra Appl. **433** (2010), 221–232.

[8] W. L. HUANG, *Bounded distance preserving surjections in the geometry of matrices*, Linear Algebra Appl. **433** (2010), 1973–1987.

[9] W. L. HUANG AND H. HAVLICEK, *Diameter preserving surjections in the geometry of matrices*, Linear Algebra Appl. **429** (2008), 376–386.

[10] N. JACOBSON, *Lectures in abstract algebra*, Linear algebra, D. Van Nostrand II, New York, 1953.

[11] B. KUZMA AND M. OREL, *Additive rank-one nonincreasing additive maps on Hermitian matrices over the field* $GL(2^2)$, Electronic Journal of Linear Algebra **18** (2009), 482–499.

[12] M. H. LIM AND J. J. H. TAN, *Preservers of pairs of bivectors with bounded distance*, Linear Algebra Appl. **430** (2009), 564–573.

[13] M. LIU, *Geometry of alternate matrices*, Acta Math. Sinica **16** (1996), 104–135. (English Translation: Chinese Math. **8** (1966), 108–143)

[14] M. OREL AND B. KUZMA , *Additive maps on Hermitian matrices*, Linear and Multilinear Algebra **55**, 6 (2007), 599–617.

[15] D. W. ROBINSON, *The classical adjoint*, Linear Algebra Appl. **411** (2005), 254–276.

[16] P. ŠEMRL, *Hua's fundamental theorems of the geometry of matrices and related results*, Linear Algebra Appl. **361** (2003), 161–179.

[17] R. SINKHORN, *Linear adjugate preservers on the complex matrices*, Linear and Multilinear Algebra **12** (1982), 215–222.

[18] X. M. TANG, *Linear operators preserving adjoint matrix between matrix spaces*, Linear Algebra Appl. **372** (2003), 287–293.

[19] X. M. TANG, *Additive rank-1 preservers between Hermitian matrix spaces and applications*, Linear Algebra Appl. **395** (2005), 333–342.

[20] X. M. TANG AND X. ZHANG, *Additive adjoint preservers between matrix spaces*, Linear and Multilinear Algebra **54**, 4 (2006), 285–300.

[21] X. M. TANG, X. ZHANG AND C. G. CAO, *Additive adjugate preservers on the matrices over fields*, Northeast. Math. J. **15** (1999), 246–252.

[22] Z. X. WAN, *Geometry of matrices*, World Scientific, Singapore, 1996.

[23] X. ZHANG, X. M. TANG, AND C. G. CAO, *Preserver problems on spaces of matrices*, Science Press, Beijing, 2007.

(Received September 12, 2012)

Wai Leong Chooi
Institute of Mathematical Sciences
University of Malaya
Kuala Lumpur, Malaysia
e-mail: wlchooi@um.edu.my

Wei Shean Ng
Department of Mathematical and Actuarial Sciences
University Tunku Abdul Rahman
Kuala Lumpur, Malaysia
e-mail: ngws@utar.edu.my