APPROXIMATE DOUBLE COMMUTANTS AND DISTANCE FORMULAS

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Abstract. We extend work of the first author concering relative double commutants and approximate double commutants of unital subalgebras of unital C*-algebras, including metric versions involving distance estimates. We prove metric results for AH subalgebras of von Neumann algebras or AF subalgebras of primitive C*-algebras. We prove other general results, including some for nonselfadjoint commutative subalgebras, using C*-algebraic versions of the Stone-Weierstrass and Bishop-Stone-Weierstrass theorems.

1. Introduction

We extend results in [14] on approximate double commutants of C*-subalgebras of C*-algebras. We also obtain some results for non-selfadjoint subalgebras. A key ingredient in the proof of the main result in [14] was S. Machado's version [18] of the Bishop-Stone-Weierstrass theorem [4]. In this paper we use Machado's vector version of his theorem [18], the factor state version of the Stone-Weierstrass theorem for C*-algebras of R. Longo [17], S. Popa [19], S. Teleman [22], and the first author's version of the Bishop-Stone-Weierstrass theorem for C*-algebras [13].

The classical double commutant theorem of von Neumann [25] is a key result in the theory of von Neumann algebras. The first author [11] proved an asymptotic version of von Neumann's result for unital C*-algebras, and later [12] proved a metric version with an analogue of Arveson's distance formula.

It was shown by R. Kadison [15] that inside a factor von Neumann algebra von Neumann's double commutant theorem fails, even for commutative subalgebras. However, the author [14] showed that, for commutative unital C*-subalgebras of a factor von Neumann algebra, the asymptotic version holds.

Suppose $\mathscr{S}$ is a subset of a ring $\mathscr{R}$. We define the *relative commutant* of $\mathscr{S}$ in $\mathscr{R}$, the *relative double commutant* of $\mathscr{S}$ in $\mathscr{R}$, respectively, by

$$(\mathscr{S},\mathscr{R})' = \{T \in \mathscr{R} : \forall S \in \mathscr{S}, TS = ST\},$$

$$(\mathscr{S},\mathscr{R})'' = \{T \in \mathscr{R} : \forall A \in (\mathscr{S},\mathscr{R})', TA = AT\}.$$

If $\mathscr{B}$ is a unital C*-algebra and $\mathscr{S} \subseteq \mathscr{B}$, we define the *relative approximate double commutant of* $\mathscr{S}$ *in* $\mathscr{B}$, denoted by $\mathrm{Appr}(\mathscr{S},\mathscr{B})''$ as the set of all $T \in \mathscr{B}$ such that

$$\|TA_\lambda - A_\lambda T\| \to 0$$

for every bounded net $\{A_\lambda\}$ in $\mathscr{B}$ for which

$$\|SA_\lambda - A_\lambda S\| \to 0$$

for every $S \in \mathscr{S}$. The approximate double commutant theorem [11] in $B(H)$ says that if $\mathscr{S} = \mathscr{S}^*$, then $\mathrm{Appr}(\mathscr{S})'' = C^*(\mathscr{S})$. Moreover, if we restrict the $\{A_\lambda\}$'s to be nets of unitaries or nets of projections that asymptotically commute with every element of $\mathscr{S}$, the resulting approximate double commutant is still $C^*(\mathscr{S})$.

It is clear that the *center* $\mathscr{Z}(\mathscr{B})$ of $\mathscr{B}$ is always contained in $\mathrm{Appr}(\mathscr{S},\mathscr{B})''$ and that $\mathrm{Appr}(\mathscr{S},\mathscr{B})''$ is a norm closed unital algebra. Thus $\mathrm{Appr}(\mathscr{S},\mathscr{B})''$ always contains the norm closed unital algebra generated by $\mathscr{S} \cup \mathscr{Z}(\mathscr{B})$. If $\mathscr{S} = \mathscr{S}^*$, then $\mathrm{Appr}(\mathscr{S},\mathscr{B})''$ is a C*-algebra and must contain $C^*(\mathscr{S} \cup \mathscr{Z}(\mathscr{B}))$. In [15] R. Kadison calls a subalgebra $\mathscr{A}$ of $\mathscr{B}$ *normal* if $\mathscr{A} = (\mathscr{A},\mathscr{B})''$. We say that $\mathscr{A}$ is *approximately normal* if $\mathscr{A} = Appr(\mathscr{A},\mathscr{B})''$.

We say that $\mathscr{A}$ is *metric-normal* in $\mathscr{B}$ if there is a constant $K < \infty$ such that, for every $T \in \mathscr{B}$,

$$dist(T,\mathscr{A}) \leqslant K \sup\{\|TU - UT\| : U \in (\mathscr{A},\mathscr{B})', U \text{ unitary}\}.$$

The smallest such K is the *constant of metric-normality* $K_n(\mathscr{A},\mathscr{B})$ of $\mathscr{A}$ in $\mathscr{B}$. We say that $\mathscr{A}$ is *approximately metric-normal* in $\mathscr{B}$ if there is a $K < \infty$ such that, for every $T \in \mathscr{B}$ there is a net $\{U_\lambda\}$ of unitaries in $\mathscr{B}$ such that, for every $A \in \mathscr{A}$, $\|AU_\lambda - U_\lambda A\| \to 0$, and such that

$$dist(T,\mathscr{A}) \leqslant K \lim_\lambda \|TU_\lambda - U_\lambda T\|.$$

The smallest such K is the *constant of approximate metric normality* $K_{an}(\mathscr{A},\mathscr{B})$.

Here is a summary of the results in this paper. In Section 2 we discuss a version of relative injectivity, summarize known results and prove a few new ones. We relate the forms of injectivity to the metric versions of normalilty and approximate normality. We also develop a number of useful basic results about the various versions of normality. We prove (Theorem 14) that if $\mathscr{A}$ is a unital AH C*-subalgebra $\mathscr{A}$ of a von Neumann algebra $\mathscr{B}$, then $C^*(\mathscr{A} \cup \mathscr{Z}(\mathscr{B}))$ is metric approximately normal in $\mathscr{B}$, and we prove (Theorem 15) that every unital AF C*-subalgebra of a primitive C*-algebra is metric approximately normal.

In Section 3, following ideas of Akemann and Pedersen [1], we prove (Theorem 17) that surjective unital $*$-homomormisms send the approximate double commutant of a set into the approximate double commutant of the image of the set. This result is a key ingredient to our results in Sections 4 and 5 where we prove general results (Theorem 21 and Theorem) that involve C*-algebraic versions of the Stone-Weierstrass or Bishop-Stone-Weierstrass theorems. We conclude in Section 6 with a list of open problems.

2. Metric results

If $\mathscr{A}$ is a unital C*-subalgebra of a C*-algebra $\mathscr{B}$, then $\mathscr{F}(\mathscr{A},\mathscr{B})$ is the convex hull of the maps $Ad_U : \mathscr{B} \to \mathscr{B}$ defined by $Ad_u(T) = U^*TU$, with a unitary

$U \in (\mathscr{A}, \mathscr{B})'$. We say that a unital C*-subalgebra $\mathscr{A}$ is *strongly injective* in a unital C*-algebra $\mathscr{B}$ if there is a conditional expectation $E : \mathscr{B} \to \mathscr{A}$, a faithful unital representation $\pi : \mathscr{B} \to B(H)$ for some Hilbert space H, and a net $\{\varphi_\lambda\}$ in $\mathscr{F}(\mathscr{A}, \mathscr{B})$ such that, for every $T \in \mathscr{B}$,

$$\pi(\varphi_\lambda(T)) \to \pi(E(T))$$

in the weak operator topology. It is clear that if $\mathscr{A}$ is strongly injective in $\mathscr{B}$, then $\mathscr{A}$ contains the center $\mathscr{Z}(\mathscr{B})$ of $\mathscr{B}$. If $\mathscr{A}$ and $\mathscr{B}$ are von Neumann algebras, we say that $\mathscr{A}$ is *weak* injective* in $\mathscr{B}$ if E and the net $\{\varphi_\lambda\}$ and be chosen so that, for every $T \in \mathscr{B}$,

$$\varphi_\lambda(T) \to E(T)$$

in the weak*-topology on $\mathscr{B}$, i.e., we can choose π to be the identity representation on $\mathscr{B}$.

PROPOSITION 1. *Suppose $\mathscr{B} \subseteq B(H)$ is a unital C*-algebra. Then*

1. *If $\pi : \mathscr{B} \to B(M)$ is a faithful unital $*$-homomorphism for some Hilbert space M, and $E : \mathscr{B} \to \mathscr{A}$ is a conditional expectation and there is a net $\{\psi_\lambda\}$ in $\mathscr{F}(\pi(\mathscr{A}), \pi(\mathscr{B})'')$ such that, for every $T \in \mathscr{B}$, $\{\psi_\lambda(\pi(T))\}$ converges in the weak operator topology to an element $\pi(E(T))$ of $\pi(\mathscr{A})$, then $\mathscr{A}$ is strongly injective in $\mathscr{B}$.*

2. *[10, Theorem C]If $\mathscr{B}$ is a von Neumann algebra, then $\mathscr{Z}(\mathscr{B})$ is weak* injective in $\mathscr{B}$.*

3. *[20] If $\mathscr{B}$ is a von Neumann algebra and $\mathscr{A}$ is a normal von Neumann subalgebra of $\mathscr{B}$ such that $(\mathscr{A}, \mathscr{B})'$ is hyperfinite (e.g., $\mathscr{A}$ is a masa in $\mathscr{B}$), then $\mathscr{A}$ is weak* injective in $\mathscr{B}$.*

4. *If $\mathscr{B}$ is a primitive C*-algebra, then $\mathscr{Z}(\mathscr{B}) = \mathbb{C}1$ is strongly injective in $\mathscr{B}$.*

5. *If $\mathscr{A} = \sum_{1 \leqslant j \leqslant m}^{\oplus} \mathscr{A}_j$ is a unital C*-subalgebra of $\mathscr{B}$ and, for $i = 1, \ldots, m$, $P_1 = 1 \oplus 0 \oplus \cdots \oplus 0$, $P_2 = 0 \oplus 1 \oplus \cdots \oplus 0, \ldots, P_m = 0 \oplus \cdots \oplus 0 \oplus 1$, then $\mathscr{A}$ is strongly injective (resp., normal, approximately normal) in $\mathscr{B}$ if $\mathscr{A}_i$ is strongly injective (resp., normal,approximately normal) in $P_i \mathscr{B} P_i$ for $1 \leqslant i \leqslant m$.*

6. *If $\mathscr{A}_i$ is strongly injective in $\mathscr{B}_i$ for $i = 1, 2$, then $\mathscr{A}_1 \otimes_{min} \mathscr{A}_2$ is strongly injective in $\mathscr{B}_1 \otimes_{min} \mathscr{B}_2$.*

7. *If $\mathscr{A}$ is strongly injective in $\mathscr{B}$ and $\mathscr{W}$ is any unital C*-algebra, then $\mathscr{W} \otimes_{min} \mathscr{A}$ is strongly injective in $\mathscr{W} \otimes_{min} \mathscr{B}$.*

8. *if $\mathscr{B} = \mathscr{M}_k(\mathscr{D}) = \mathscr{M}_k(\mathbb{C}) \otimes \mathscr{D}$ for $k \in \mathbb{N}$ and $\mathscr{A} = \mathscr{M}_k(\mathscr{E}) = \mathscr{M}_k(\mathbb{C}) \otimes \mathscr{E}$ and $\mathscr{E}$ is strongly injective (resp., normal) in $\mathscr{D}$, then $\mathscr{A}$ is strongly injective (resp., normal) in $\mathscr{B}$.*

Proof. (1). Suppose $U \in \pi(\mathscr{B})''$ is unitary. It follows that there is an $A = A^* \in \pi(\mathscr{B})''$ such that $U = e^{iA}$. It follows from the Kaplansky density theorem that there is a bounded net $\{A_m\}$ in $\mathscr{B}$ such that $\pi(A_m) \to A$ in the strong operator topology, and it follows, that if $U_m = e^{iA_m}$, then $\pi(U_m) \to U$ in the $*$-strong operator topology, and thus $\pi(Ad_{U_m}(B)) \to Ad_U(\pi(B))$ in the strong operator topology. It follows that the point-weak-operator closure of $\{\pi \circ \varphi : \varphi \in \mathscr{F}(\mathscr{A}, \mathscr{B})\}$ contains every $\psi_\lambda \circ \pi$, and thus contains $\pi \circ E$. It follows that $\mathscr{A}$ is strongly injective in $\mathscr{B}$.

(4). This follows from (1) and (2).

(5) is obvious.

(6). Suppose, for $i \in \{1, 2\}$, $\pi_i : \mathscr{B}_i \to B(H_i)$ is a faithful representation, $E_i : \mathscr{B}_i \to \mathscr{A}_i$ is a conditional expectation and $\{\varphi_{\lambda, i}\}$ is a net in $\mathscr{F}(\mathscr{A}_i, \mathscr{B}_i)$ such that

$$\pi_i(\varphi_{\lambda, i}(T)) \to \pi_i(E_i(T))$$

in the weak operator topology. Then $E(T_1 \otimes T_2) = E_1(T_1) \otimes E_2(T_2)$ defines a conditional expectation $E : \mathscr{B}_1 \otimes_{\min} \mathscr{B}_2 \to \mathscr{A}_1 \otimes_{\min} \mathscr{A}_2$. Also $\pi(T_1 \otimes T_2) = \pi_1(T_1) \otimes \pi_2(T_2) \in B(H_1 \otimes H_2)$ defines a faithful representation of $\mathscr{B}_1 \otimes_{\min} \mathscr{B}_2$. Moreover, if $U_{k,i} \in (\mathscr{A}_i, \mathscr{B}_i)'$ is unitary for $i = 1, 2$ and $1 \leqslant k \leqslant m$ and if $0 \leqslant s_1, t_1, \ldots, s_m, t_m$ and $\sum_{k=1}^m s_k = \sum_{k=1}^m t_k = 1$, then $W_{k,r} = U_{k,1} \otimes U_{r,2} \in (\mathscr{A}_1 \otimes_{\min} \mathscr{A}_2, \mathscr{B}_1 \otimes_{\min} \mathscr{B}_2)'$ and

$$\sum_{k,r=1}^m s_k t_r W_{k,r}(T_1 \otimes T_2) W_{k,r}^* = \left(\sum_{k=1}^m s_k U_{k,1} T_1 U_{k,1}^* \right) \otimes \left(\sum_{r=1}^m t_r U_{r,2} T_2 U_{r,2}^* \right).$$

Thus

$$\varphi_\lambda(T_1 \otimes T_2) = \varphi_{\lambda,1}(T_1) \otimes \varphi_{\lambda,2}(T_2)$$

defines an element $\varphi_\lambda \in \mathscr{F}(\mathscr{A}_1 \otimes_{\min} \mathscr{A}_2, \mathscr{B}_1 \otimes_{\min} \mathscr{B}_2)$. Moreover,

$$\pi(\varphi_\lambda(T_1 \otimes T_2)) = \pi_1(\varphi_{\lambda,1}(T_1)) \otimes \pi_2(\varphi_{\lambda,2}(T_2)) \to \pi(E(T_1 \otimes T_2))$$

in the weak operator topology on $B(H_1 \otimes H_2)$. Hence $\mathscr{A}_1 \otimes_{\min} \mathscr{A}_2$ is strongly injective in $\mathscr{B}_1 \otimes_{\min} \mathscr{B}_2$.

(7) and (8) follow from (6). $\quad\square$

Suppose $\mathscr{A}$ is a unital C*-subalgebra of a unital C*-algebra $\mathscr{B}$. We define two seminorms on $\mathscr{B}$ as follows:

$$d_n(T, \mathscr{A}, \mathscr{B}) = \sup\{\|WT - TW\| : W \in (\mathscr{A}, \mathscr{B})', \|W\| \leqslant 1\},$$

and

$$d_{an}(T, \mathscr{A}, \mathscr{B}) = \sup_{\{W_\lambda\}} \limsup_\lambda \|W_\lambda T - TW_\lambda\|$$

taken over all nets $\{W_\lambda\}$ of contractions in $\mathscr{B}$ for which $\|AW_\lambda - W_\lambda A\| \to 0$ for every $A \in \mathscr{A}$.

The following lemma is obvious and the proof is omitted.

LEMMA 2. *Suppose $\mathscr{A}$ is a unital norm closed subalgebra of a unital C*-algebra $\mathscr{B}$ and $T \in \mathscr{B}$. Then*

1. $d_n(T,\mathscr{A},\mathscr{B})$ and $d_{an}(T,\mathscr{A},\mathscr{B})$ are seminorms on $\mathscr{B}$,

2. $d_n(T^*,\mathscr{A},\mathscr{B}) = d_n(T,\mathscr{A},\mathscr{B})$ and $d_{an}(T^*,\mathscr{A},\mathscr{B}) = d_{an}(T,\mathscr{A},\mathscr{B})$,

3. $d_n(T,\mathscr{A},\mathscr{B}) \leqslant d_{an}(T,\mathscr{A},\mathscr{B}) \leqslant 2 dist(T,\mathscr{A}) \leqslant 2\|T\|$

4. $d_n(T,\mathscr{A},\mathscr{B}) = 0$ if and only if $T \in (\mathscr{A},\mathscr{B})''$

5. $d_{an}(T,\mathscr{A},\mathscr{B}) = 0$ if and only if $T \in Appr(\mathscr{A},\mathscr{B})''$

We define $K_n(\mathscr{A},\mathscr{B})$ and $K_{an}(\mathscr{A},\mathscr{B})$ by

$$K_n(\mathscr{A},\mathscr{B}) = \sup\{dist(T,\mathscr{A}) : T \in \mathscr{B}, d_n(T,\mathscr{A},\mathscr{B}) \leqslant 1\},$$

$$K_{an}(\mathscr{A},\mathscr{B}) = \sup\{dist(T,\mathscr{A}) : T \in \mathscr{B}, d_{an}(T,\mathscr{A},\mathscr{B}) \leqslant 1\}.$$

Clearly $K_n(\mathscr{A},\mathscr{B})$ is the smallest $M \geqslant 0$ such that, for every $T \in \mathscr{B}$, we have $dist(T,\mathscr{A}) \leqslant M d_n(T,\mathscr{A},\mathscr{B})$ and $K_{an}(\mathscr{A},\mathscr{B})$ is the smallest $N \geqslant 0$ such that, for every $T \in \mathscr{B}$, we have $dist(T,\mathscr{A}) \leqslant N d_{an}(T,\mathscr{A},\mathscr{B})$. We say that $\mathscr{A}$ is *metric normal* in $\mathscr{A}$ if $K_n(\mathscr{A},\mathscr{B}) < \infty$ and $\mathscr{A}$ is *metric approximately normal* if $K_{an}(\mathscr{A},\mathscr{B}) < \infty$. It is also clear that $K_{an}(\mathscr{A},\mathscr{B}) \leqslant K_n(\mathscr{A},\mathscr{B})$, so metric normality implies metric approximate normality.

The following proposition shows the relationship between strong injectivity and metric normality.

PROPOSITION 3. *Suppose* $\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \cdots \subseteq \mathscr{A}_m$ *is are unital inclusions of C*-algebras and* $\mathscr{A}_k$ *is weakly injective in* $\mathscr{A}_{k+1}$ *for* $1 \leqslant k < m$. *Then*

$$K_n(\mathscr{A}_1,\mathscr{A}_m) \leqslant 1.$$

Proof. For each k, $2 \leqslant k \leqslant m$ choose a net $\{\varphi_{\lambda,k}\}$ in $\mathscr{F}(\mathscr{A}_{k-1},\mathscr{A}_k)$, a conditional expectation $E_k : \mathscr{A}_k \to \mathscr{A}_{k-1}$ and a faithful representation $\pi_k : \mathscr{A}_k \to B(H_k)$ such that

$$\pi_k(\varphi_{\lambda,k}(T)) \to \pi_k(E_k(T))$$

in the weak operator topology for every $T \in \mathscr{A}_k$. It is clear that

$$\mathscr{F}(\mathscr{A}_{k-1},\mathscr{A}_k) \subseteq \mathscr{F}(\mathscr{A}_1,\mathscr{A}_m)$$

for $2 \leqslant k \leqslant m$. Morover, if U is unitary and $U \in (\mathscr{A}_1,\mathscr{A}_m)'$, then

$$\|T - UTU^*\| = \|TU - UT\| \leqslant d_n(T,\mathscr{A}_1,\mathscr{A}_m)$$

for all $T \in \mathscr{A}_m$. Suppose $T \in \mathscr{A}_m$, and let B denote the closed ball in $\mathscr{A}_m$ centered at T with radius $d_n(T,\mathscr{A}_1,\mathscr{A}_m)$. Let $\mathscr{W}_m$ denote the set of all $A \in \mathscr{A}_m$ such that $\pi_m(A)$ is in the weak-operator closure of the convex hull of $\{\pi_m(UTU^*) : U \in (\mathscr{A}_1,\mathscr{A}_m)', U \text{ is unitary}\}$. Clearly $\mathscr{W}_m$ is convex and closed under conjugation by unitaries in $(\mathscr{A}_1,\mathscr{A}_m)'$, and, since π_m is an isometry, $\mathscr{W}_m \subseteq B$. It follows that $E_m(T) \in \mathscr{W}_m$.

Next we let $\mathscr{W}_{m-1}$ denote the set of all $A \in \mathscr{A}_m$ such that $\pi_{m-1}(A)$ is in the weak-operator closure of the convex hull of $\{\pi_{m-1}(UE_m(T)U^*) : U \in (\mathscr{A}_1, \mathscr{A}_m)', U \text{ is unitary}\}$. Clearly $\mathscr{W}_{m-1}$ is convex and closed under conjugation by unitaries in $(\mathscr{A}_1, \mathscr{A}_m)'$, and, since π_{m-1} is an isometry, $\mathscr{W}_{m-1} \subseteq B$, and it follows that $E_{m-1}(E_m(T)) \in \mathscr{W}_{m-1} \subseteq B$. Proceeding inductively, we see that

$$E_2(E_3(\cdots E_m(T))) \in B \cap \mathscr{A}_1,$$

from which it follows that

$$\mathrm{dist}(T, \mathscr{A}_1) \leqslant d_n(T, \mathscr{A}_1, \mathscr{A}_m).$$

Hence $K_n(\mathscr{A}_1, \mathscr{A}_m) \leqslant 1$.   □

The following corollaries follow from Proposition 1 and Proposition 3.

COROLLARY 4. *If $\mathscr{B}$ is a von Neumann algebra and $\mathscr{A}$ is a normal von Neumann subalgebra such that $(\mathscr{A}, \mathscr{B})'$ is hyperfinite, then $\mathscr{A}$ is metric normal in $\mathscr{B}$ and $K_n(\mathscr{M}_k(\mathscr{A}), \mathscr{M}_k(\mathscr{B})) \leqslant 1$ for every $k \in \mathbb{N}$.*

COROLLARY 5. *If $\mathscr{A}$ is a maximal abelian selfadjoint subalgebra of a von Neumann algebra $\mathscr{B}$, then $K_n(\mathscr{M}_k(\mathscr{A}), \mathscr{M}_k(\mathscr{B})) \leqslant 1$ for every $k \in \mathbb{N}$.*

COROLLARY 6. *If $\mathscr{A}$ is a maximal abelian selfadjoint subalgebra of a von Neumann algebra $\mathscr{B}$, and $\mathscr{W}$ is any von Neumann algebra, then $K_n(\mathscr{W} \otimes \mathscr{A}, \mathscr{W} \otimes \mathscr{B}) \leqslant 1$, where $\otimes$ denotes the spatial tensor product.*

COROLLARY 7. *If $\mathscr{B}$ is a hyperfinite von Neumann algebra, then every normal von Neumann subalgebra $\mathscr{A}$ of $\mathscr{B}$ is metric normal and*

$$K_n(\mathscr{A}, \mathscr{B}) \leqslant 1.$$

Without injectivity, this is the best analogue of Proposition 3.

LEMMA 8. *If $\mathscr{A} \subseteq \mathscr{D} \subseteq \mathscr{B}$ are unital C*-algebras, then*

$$K_n(\mathscr{A}, \mathscr{B}) \leqslant K_n(\mathscr{D}, \mathscr{B}) + K_n(\mathscr{A}, \mathscr{D})(2K_n(\mathscr{D}, \mathscr{B}) + 1),$$

and

$$K_{an}(\mathscr{A}, \mathscr{B}) \leqslant K_{an}(\mathscr{D}, \mathscr{B}) + K_{an}(\mathscr{A}, \mathscr{D})(2K_{an}(\mathscr{D}, \mathscr{B}) + 1).$$

Proof. We present the proof for K_n; the proof for K_{an} is similar. Suppose $T \in \mathscr{B}$ and $\varepsilon > 0$. Then

$$\mathrm{dist}(T, \mathscr{D}) < [K_n(\mathscr{D}, \mathscr{B}) + \varepsilon]d_n(T, \mathscr{D}, \mathscr{B}).$$

Hence there is a $D \in \mathscr{D}$ such that

$$\|T - D\| \leqslant [K_n(\mathscr{D}, \mathscr{B}) + \varepsilon]d_n(T, \mathscr{D}, \mathscr{B}).$$

Similarly, there is an $A \in \mathscr{A}$ such that

$$\|D - A\| \leqslant [K_n(\mathscr{A}, \mathscr{D}) + \varepsilon] d_n(D, \mathscr{A}, \mathscr{D}).$$

Hence

$$\begin{aligned} \|T - A\| &\leqslant \|T - D\| + \|D - A\| \\ &\leqslant [K_n(\mathscr{D}, \mathscr{B}) + \varepsilon] d_n(T, \mathscr{D}, \mathscr{B}) + [K_n(\mathscr{A}, \mathscr{D}) + \varepsilon] d_n(D, \mathscr{A}, \mathscr{D}). \end{aligned}$$

However,

$$d_n(T, \mathscr{D}, \mathscr{B}) \leqslant d_n(T, \mathscr{A}, \mathscr{B}),$$

and

$$\begin{aligned} d_n(D, \mathscr{A}, \mathscr{D}) &\leqslant d_n(D, \mathscr{A}, \mathscr{B}) \leqslant 2\|T - D\| + d_n(T, \mathscr{A}, \mathscr{B}) \\ &\leqslant 2[K_n(\mathscr{D}, \mathscr{B}) + \varepsilon] d_n(T, \mathscr{D}, \mathscr{B}) + d_n(T, \mathscr{A}, \mathscr{B}) \\ &\leqslant (2[K_n(\mathscr{D}, \mathscr{B}) + \varepsilon] + 1) d_n(T, \mathscr{A}, \mathscr{B}) \end{aligned}$$

Hence

$$\begin{aligned} dist(T, \mathscr{A}) &\leqslant \|T - A\| \\ &\leqslant [K_n(\mathscr{D}, \mathscr{B}) + \varepsilon] d_n(T, \mathscr{A}, \mathscr{B}) \\ &\quad + [K_n(\mathscr{A}, \mathscr{D}) + \varepsilon] (2[K_n(\mathscr{D}, \mathscr{B}) + \varepsilon] + 1) d_n(T, \mathscr{A}, \mathscr{B}). \end{aligned}$$

Letting $\varepsilon \to 0^+$, we see

$$dist(T, \mathscr{A}) \leqslant [K_n(\mathscr{D}, \mathscr{B}) + K_n(\mathscr{A}, \mathscr{D})(2K_n(\mathscr{D}, \mathscr{B}) + 1)] d_n(T, \mathscr{A}, \mathscr{B}).$$

It follows that

$$K_n(\mathscr{A}, \mathscr{B}) \leqslant K_n(\mathscr{D}, \mathscr{B}) + K_n(\mathscr{A}, \mathscr{D})(2K_n(\mathscr{D}, \mathscr{B}) + 1) \quad \square$$

We now consider the metric approximate normality for direct limits.

LEMMA 9. *Suppose $\mathscr{A}$ is a unital C*-subalgebra of a unital C*-algebra $\mathscr{B}$ and $\{\mathscr{A}_i : i \in I\}$ is an increasingly directed family of C*-subalgebras of $\mathscr{A}$. If $\mathscr{A}$ is the norm closure of $\cup_{i \in I} \mathscr{A}_i$, then*

$$K_{an}(\mathscr{A}, \mathscr{B}) \leqslant \liminf_i K_{an}(\mathscr{A}_i, \mathscr{B}).$$

Proof. Suppose $T \in \mathscr{B}$, $F \subseteq \mathscr{A}$ is finite, $\varepsilon > 0$, and let $\lambda = (F, \varepsilon)$. Then

$$\begin{aligned} dist(T, \mathscr{A}) &= \lim_i dist(T, \mathscr{A}_i) \\ &\leqslant \sup_i K_{an}(T, \mathscr{A}_i) \liminf_i d_n(T, \mathscr{A}_i, \mathscr{B}). \end{aligned}$$

We can choose i_0 sufficiently large so that there is a map $\alpha : F \to \mathscr{A}_{i_0}$ such that

$$\|A - \alpha(A)\| < \varepsilon/2$$

for every $A \in F$ and so that

$$\liminf_i d_{an}(T, \mathscr{A}_i, \mathscr{B}) \leqslant d_{an}(T, \mathscr{A}_{i_0}, \mathscr{B}) + \varepsilon.$$

We can choose a unitary U_λ in $\mathscr{B}$ so that

$$\|U_\lambda \alpha(A) - \alpha(A) U_\lambda\| < \varepsilon/3$$

for every $A \in F$ and so that

$$d_{an}(T, \mathscr{A}_{i_0}, \mathscr{B}) \leqslant \|U_\lambda T - TU_\lambda\| + \varepsilon.$$

It follows that

$$dist(T, \mathscr{A}) \leqslant [\|U_\lambda T - TU_\lambda\| + 2\varepsilon] \sup_i K_{an}(\mathscr{A}_i, \mathscr{B}).$$

and

$$\|U_\lambda A - AU_\lambda\| \leqslant \varepsilon.$$

If we let Λ be the set of all pairs $\lambda = (F, \varepsilon)$ directed by $(\subseteq, \geqslant)$, we see that $\{U_\lambda\}$ is a net such that

$$\|AU_\lambda - U_\lambda A\| \to 0$$

for every $A \in \mathscr{A}$ and such that

$$dist(T, \mathscr{A}) \leqslant \left[\lim_\lambda \|U_\lambda T - TU_\lambda\|\right] \sup_i K_{an}(\mathscr{A}_i, \mathscr{B}) \leqslant d_{an}(T, \mathscr{A}, \mathscr{B}) \sup_i K_{an}(\mathscr{A}_i, \mathscr{B}).$$

Hence $K_{an}(\mathscr{A}, \mathscr{B}) \leqslant \sup_i K_{an}(\mathscr{A}_i, \mathscr{B})$, and since the same holds for when we restrict to the set $\{\mathscr{A}_i : i \geqslant j\}$ for some j, we can replace $\sup_i K_{an}(\mathscr{A}_i, \mathscr{B})$ with $\liminf_i K_{an}(\mathscr{A}_i, \mathscr{B})$. □

We now want to extend a key result in [14]. Recall from [14] that a unital C*-algebra $\mathscr{B}$ is *centrally prime* if, whenever $0 \leqslant x, y \leqslant 1$ are in $\mathscr{B}$ and $x\mathscr{B}y = \{0\}$, then there is an $e \in \mathscr{Z}(\mathscr{B})$ such that $x \leqslant e \leqslant 1$ and $y \leqslant 1 - e$.

The following result is a generalization of [14, Theorem 1], in which $\mathscr{W} = \mathbb{C}$. That result required S. Machado's metric version [18] of the Bishop-Stone-Weierstrass theorem [4]. Here we require Machado's vector version of his result [18] (See [20] for an beautiful elementary proof.)

PROPOSITION 10. *Suppose $\mathscr{A} \subseteq \mathscr{D}$ are unital commutative C*-algebras, $\mathscr{W}$ is a unital C*-algebra, $\mathscr{B}$ is a centrally prime unital C*-algebra such that*

1. $\mathscr{A} \otimes \mathscr{W} \subseteq \mathscr{D} \otimes \mathscr{W} \subseteq \mathscr{B} \otimes_{min} \mathscr{W}$,

2. $\mathcal{L}(\mathcal{B}\otimes_{min}\mathcal{W}) \subseteq \mathcal{A}\otimes\mathcal{W}$.

   Then, for every $T \in \mathcal{D}\otimes\mathcal{W}$,

$$dist(T, \mathcal{A}\otimes\mathcal{W}) \leqslant d_{an}(T, \mathcal{A}\otimes\mathcal{W}, \mathcal{B}\otimes_{min}\mathcal{W}).$$

Proof. We can view $\mathcal{D} = C(X)$ for some compact Hausdorff space X and we can view $\mathcal{D}\otimes\mathcal{W}$ as $C(X,\mathcal{W})$, the C*-algebra of continuous functions from X to $\mathcal{W}$. We can write $T = f \in C(X,\mathcal{W})$. It follows that $\mathcal{A}\otimes\mathcal{W}$ is a C*-subalgebra of $C(X,\mathcal{W})$ that is an $\mathcal{A}$-module. It follows from Machado's theorem [18], that there is a closed $\mathcal{A}$-antisymmetric subset $E \subseteq X$ such that

$$dist(f, \mathcal{A}\otimes\mathcal{W}) = dist(f|_E, (\mathcal{A}\otimes\mathcal{W})|_E).$$

However, since $\mathcal{A} = \mathcal{A}^*$ and E is $\mathcal{A}$-antisymmetric, we see that every function in $\mathcal{A}$ is constant on E. Hence, if $u \in \mathcal{A}$ and $w \in \mathcal{W}$ we have, for every $x \in X$ that $(u\otimes w)(x) = u(x)w$. Hence, every function in $\mathcal{A}\otimes\mathcal{W}$ is constant on E. Thus

$$dist(f|_E, (\mathcal{A}\otimes\mathcal{W})|_E) = \inf\{\|f|_E - h\| : h : E \to \mathcal{W}, \ h \text{ is constant}\}$$
$$= \inf_{w\in\mathcal{W}} \sup_{x\in E} \|f(x) - w\|.$$

Since E is compact, we can choose $\alpha, \beta \in E$ such that

$$\|f(\alpha) - f(\beta)\| = \sup_{x,y\in E} \|f(x) - f(y)\|.$$

If we let $w = f(\beta)$, we see that

$$dist(f|_E, (\mathcal{A}\otimes\mathcal{W})|_E) \leqslant \sup_{x\in E} \|f(x) - f(\beta)\| = \|f(\alpha) - f(\beta)\|.$$

Hence,

$$dist(f, \mathcal{A}\otimes\mathcal{W}) \leqslant \|f(\beta) - f(\alpha)\|.$$

If $f(\alpha) = f(\beta)$, then $T = f \in \mathcal{A}\otimes\mathcal{W}$ and the desired inequality holds. Hence we can assume $\alpha \neq \beta$. Let Λ be the set of pairs (U,V), we U and V are disjoint open subsets of X such that $\alpha \in U$ and $\beta \in V$. Suppose $\lambda = (U,V) \in \Lambda$. We can define $g_\lambda, h_\lambda, r_\lambda, s_\lambda \in C(X)$ such that

1. $0 \leqslant g_\lambda, h_\lambda, r_\lambda, s_\lambda \leqslant 1$

2. $g_\lambda(\alpha) = h_\lambda(\alpha) = 1, \ g_\lambda h_\lambda = h_\lambda, \ g_\lambda|_{X\setminus U_\lambda} = 0$

3. $r_\lambda(\beta) = s_\lambda(\beta) = 1, \ r_\lambda s_\lambda = s_\lambda, \ r_\lambda|_{X\setminus V} = 0$.

   We then have, for every $F \in C(X,\mathcal{W})$

4. $h_\lambda F = Fh_\lambda$ and $\|h_\lambda F - (1\otimes F(\alpha))h_\lambda\| \to 0$

5. $s_\lambda F = F s_\lambda$ and $\|s_\lambda F - (1 \otimes F(\beta)) s_\lambda\| \to 0$.

Claim: $h_\lambda(\mathscr{B} \otimes 1) s_\lambda = (h_\lambda \mathscr{B} s_\lambda) \otimes 1 \neq \{0\}$. Since $\mathscr{B}$ is centrally prime, $h_\lambda \mathscr{B} s_\lambda = \{0\}$ implies that there is an $e \in \mathscr{Z}(\mathscr{B}) \subseteq \mathscr{A}$ such that $h_\lambda \leqslant e \leqslant 1$ and $s_\lambda \leqslant 1 - e \leqslant 1$. Thus $1 \leqslant e(\alpha)$ and $0 \leqslant e(\beta)$, which contradicts $e(\alpha) = e(\beta)$. This proves the claim.

For each $\lambda \in \Lambda$ we can choose $Q_\lambda \in h_\lambda \mathscr{B} s_\lambda \otimes 1$ with $\|Q_\lambda\| = 1$. We then have, for every $F \in C(X, \mathscr{W})$

$$\|[FQ_\lambda - Q_\lambda F] - [(1 \otimes F(\alpha))Q_\lambda - Q_\lambda(1 \otimes F(\beta))]\| \to 0,$$

so

$$\|\|FQ_\lambda - Q_\lambda F\| - \|(1 \otimes F(\alpha))Q_\lambda - Q_\lambda(1 \otimes F(\beta))\|\| \to 0.$$

However,

$$(1 \otimes F(\alpha))Q_\lambda = Q_\lambda \otimes F(\alpha), \text{ and } Q_\lambda(1 \otimes F(\beta)) = Q_\lambda \otimes F(\beta).$$

Hence, $\|FQ_\lambda - Q_\lambda F\| \to 0$ for every $F \in \mathscr{A} \otimes \mathscr{W}$ and

$$\lim_\lambda \|fQ_\lambda - Q_\lambda f\| = \lim_\lambda \|Q_\lambda \otimes (f(\beta) - f(\alpha))\| = \|f(\beta) - f(\alpha)\|$$
$$\geqslant dist(f, \mathscr{A} \otimes \mathscr{W}). \quad \square$$

COROLLARY 11. *Suppose $\mathscr{A}$ is a commutative unital C*-subalgebra of a centrally prime unital C*-algebra $\mathscr{B}$ such that $\mathscr{Z}(\mathscr{B}) \subseteq \mathscr{A}$ and $\mathscr{W}$ is any unital C*-algebra. Then $\mathscr{A} \otimes \mathscr{W}$ is approximately normal in $\mathscr{B} \otimes_{min} \mathscr{W}$.*

Proof. Suppose $\mathscr{D} \subseteq \mathscr{B}$ is a masa in $\mathscr{B}$ that contains $\mathscr{A}$. It follows that $(\mathscr{D} \otimes \mathscr{W}, \mathscr{B} \otimes_{min} \mathscr{W})' = \mathscr{D} \otimes \mathscr{Z}(\mathscr{W})$, and $(\mathscr{D} \otimes \mathscr{W}, \mathscr{B} \otimes_{min} \mathscr{W})'' = \mathscr{D} \otimes \mathscr{W}$. Hence $\mathscr{D} \otimes \mathscr{W}$ is normal in $\mathscr{B} \otimes_{min} \mathscr{W}$. Hence, if $T \in Appr(\mathscr{A} \otimes \mathscr{W}, \mathscr{B} \otimes_{min} \mathscr{W})''$, then $T \in \mathscr{D} \otimes \mathscr{W}$, and it follows from Proposition 10 that $T \in \mathscr{A} \otimes \mathscr{W}$. $\quad \square$

COROLLARY 12. *Suppose $\mathscr{A}$ is a commutative unital C*-subalgebra of a von Neumann algebra $\mathscr{B}$ and $\mathscr{Z}(\mathscr{B}) \subseteq \mathscr{A}$, and $\mathscr{W}$ is any unital C*-algebra. Then $\mathscr{A} \otimes_{min} \mathscr{W}$ is metric approximately normal in $\mathscr{B} \otimes_{min} \mathscr{W}$ and*

$$K_{an}(\mathscr{A} \otimes_{min} \mathscr{W}, \mathscr{B} \otimes_{min} \mathscr{W}) \leqslant 4.$$

Proof. Let $\mathscr{D}$ be a masa in $\mathscr{B}$ that contains $\mathscr{A}$. It follows from Proposition 1 that $\mathscr{D}$ is weak*-injective in $\mathscr{B}$, and that $\mathscr{D} \otimes \mathscr{W}$ is strongly injective in $\mathscr{B} \otimes_{min} \mathscr{W}$. Suppose $T \in \mathscr{B} \otimes_{min} \mathscr{W}$. We can assume that $\mathscr{B} \subseteq \mathscr{B}(H)$ is a von Neumann algebra and $\mathscr{W} \subseteq B(M)$ and $\mathscr{B} \otimes_{min} \mathscr{W}$ is the spatial tensor product of $\mathscr{B}$ and $\mathscr{W}$ in $B(H \otimes M)$. Then there is a net $\{\varphi_\lambda\}$ in $\mathscr{F}(\mathscr{D}, \mathscr{B})$ such that $E(S) = w^*\text{-}\lim_\lambda \varphi_\lambda(S)$ is a conditional expectation from $\mathscr{B}$ to $\mathscr{D}$. Then $E \otimes 1 : \mathscr{B} \otimes_{min} \mathscr{W} \to \mathscr{D} \otimes \mathscr{W}$ defined, for every R in $B \otimes W$, by

$$(E \otimes 1)(R) = w^*\text{-}\lim_\lambda (\varphi_\lambda \otimes 1)(R)$$

is a conditional expectation and each

$$\varphi_\lambda \otimes 1 \in \mathscr{F}(\mathscr{D} \otimes \mathscr{W}, \mathscr{B} \otimes_{\min} \mathscr{W}) \subseteq \mathscr{F}(\mathscr{A} \otimes \mathscr{W}, \mathscr{B} \otimes_{\min} \mathscr{W}).$$

Hence $T_1 = (E \otimes 1)(T) \in B$, where B is the closed ball in $\mathscr{B} \otimes_{\min} \mathscr{W}$ centered at T with radius $d_n(T, \mathscr{A} \otimes \mathscr{W}, \mathscr{B} \otimes_{\min} \mathscr{W})$. However, Theorem 10 implies that

$$
\begin{aligned}
dist(T_1, \mathscr{A} \otimes \mathscr{W}) &\leqslant d_{an}(T_1, \mathscr{A} \otimes \mathscr{W}, \mathscr{B} \otimes_{\min} \mathscr{W}) \\
&\leqslant d_{an}(T, \mathscr{A} \otimes \mathscr{W}, \mathscr{B} \otimes_{\min} \mathscr{W}) + 2\|T - T_1\| \\
&\leqslant d_{an}(T, \mathscr{A} \otimes \mathscr{W}, \mathscr{B} \otimes_{\min} \mathscr{W}) + 2d_n(T, \mathscr{A} \otimes \mathscr{W}, \mathscr{B} \otimes_{\min} \mathscr{W}) \\
&\leqslant 3d_{an}(T, \mathscr{A} \otimes \mathscr{W}, \mathscr{B} \otimes_{\min} \mathscr{W}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
dist(T, \mathscr{A} \otimes \mathscr{W}) &\leqslant dist(T_1, \mathscr{A} \otimes \mathscr{W}) + \|T - T_1\| \\
&\leqslant 4d_{an}(T, \mathscr{A} \otimes \mathscr{W}, \mathscr{B} \otimes_{\min} \mathscr{W}). \quad \square
\end{aligned}
$$

THEOREM 13. *If $\mathscr{B}$ is a unital centrally prime C*-algebra and $\mathscr{Z}(\mathscr{B}) \subseteq \mathscr{A}$ is a unital C*-subalgebra that is isomorphic to a finite direct sum of tensor products of algebras of the form $\mathscr{D} \otimes \mathscr{M}_k(\mathbb{C})$, with $\mathscr{D}$ commutative, then $\mathscr{A}$ is approximately normal in $\mathscr{B}$. Moreover, if $\mathscr{B}$ is a von Neumann algebra, then $K_{an}(\mathscr{A}, \mathscr{B}) \leqslant 4$.*

Proof. Write $\mathscr{A} = \mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_n$ where each $\mathscr{A}_k$ is isomorphic to $\mathscr{D}_k \otimes \mathscr{M}_{s_k}(\mathbb{C})$ for some s_k in $\mathbb{N}$, and let $P_1 = 1 \oplus 0 \oplus \cdots \oplus 0, P_2 = 0 \oplus 1 \oplus \cdots \oplus 0, \ldots, P_n = 0 \oplus \cdots \oplus 0 \oplus 1$. It follows from Proposition 1 that $\sum_{j=1}^n P_j \mathscr{B} P_j$ is strongly injective in $\mathscr{B}$. Since $\mathscr{D}_k \otimes \mathscr{M}_{s_k}(\mathbb{C}) \subseteq P_k \mathscr{B} P_k$ we can write

$$P_k \mathscr{B} P_k = \mathscr{B}_k \otimes \mathscr{M}_{s_k}(\mathbb{C})$$

with $\mathscr{D}_k \subseteq \mathscr{B}_k$. Since $\mathscr{B}$ is centrally prime, so is each $P_k \mathscr{B} P_k$, and thus so does each $\mathscr{B}_k$. Since $\mathscr{Z}(\mathscr{B}) \subseteq \mathscr{D}$, we know that

$$\mathscr{Z}(\mathscr{B}_k \otimes \mathscr{M}_{s_k}(\mathbb{C})) = \mathscr{Z}(\mathscr{B}_k) \otimes 1 \subseteq \mathscr{D}_k \otimes \mathscr{M}_{s_k}(\mathbb{C}),$$

which implies $\mathscr{Z}(\mathscr{B}_k) \subseteq \mathscr{D}_k$ for $1 \leqslant k \leqslant n$. Since, by [14], $\mathscr{D}_k$ is normal in $\mathscr{B}_k$, we know that $P_k \mathscr{A} P_k = \mathscr{D}_k \otimes \mathscr{M}_{s_k}(\mathbb{C})$ is normal in $\mathscr{B}_k \otimes \mathscr{M}_{s_k}(\mathbb{C}) = P_k \mathscr{B} P_k$ for $1 \leqslant k \leqslant n$. Hence by Proposition 1, $\mathscr{A}$ is normal in $\mathscr{B}$. If $\mathscr{B}$ is a von Neumann algebra and if, for each k, $\mathscr{E}_k$ is a masa in $\mathscr{B}_k$ containing $\mathscr{D}_k$ for $1 \leqslant k \leqslant n$, then

$$\sum_{1 \leqslant k \leqslant n}^{\oplus} \mathscr{E}_k \otimes \mathscr{M}_{s_k}(\mathbb{C})$$

is weak* injective in $\mathscr{B}$. It follows from Theorem 10 that, for every $S = S_1 \oplus \cdots \oplus S_n \in \sum_{1 \leqslant k \leqslant n}^{\oplus} \mathscr{E}_k \otimes \mathscr{M}_{s_k}(\mathbb{C})$

$$
\begin{aligned}
dist(S, \mathscr{A}) &\leqslant \max_{1 \leqslant k \leqslant n} dist(S_k, \mathscr{A}_k \otimes \mathscr{M}_{s_k}(\mathbb{C})) \\
&\leqslant \max_{1 \leqslant k \leqslant n} d_{an}(S_k, \mathscr{A}_k \otimes \mathscr{M}_{s_k}(\mathbb{C}), \mathscr{B}_k \otimes \mathscr{M}_{s_k}(\mathbb{C})) \\
&\leqslant d_{an}(S, \mathscr{A}, \mathscr{B}).
\end{aligned}
$$

If $T \in \mathscr{B}$, it follows that there is an $S \in \sum_{1 \leqslant k \leqslant n}^{\oplus} \mathscr{E}_k \otimes \mathscr{M}_{s_k}(\mathbb{C})$ such that

$$\|T - S\| \leqslant d_{an}(T, \mathscr{A}, \mathscr{B}).$$

It follows that

$$
\begin{aligned}
dist\,(T, \mathscr{A}) &\leqslant dist\,(S, \mathscr{A}) + \|S - T\| \\
&\leqslant d_{an}(S - T, \mathscr{A}, \mathscr{B}) + 2d_{an}(T, \mathscr{A}, \mathscr{B}) \\
&\leqslant 2\|S - T\| + 2d_{an}(T, \mathscr{A}, \mathscr{B}) \leqslant 4d_{an}(T, \mathscr{A}, \mathscr{B}). \quad \square
\end{aligned}
$$

THEOREM 14. *If $\mathscr{A}$ is a unital AH C*-subalgebra of a von Neumann algebra $\mathscr{B}$, then*

$$K_{an}(\mathscr{A}, \mathscr{B}) \leqslant 4.$$

Proof. This follows from Theorem 13 and Lemma 9. $\square$

THEOREM 15. *If $\mathscr{B}$ is a primitive unital C*-algebra and $\mathscr{A}$ is a unital AF C*-subalgebra of $\mathscr{B}$, then*

$$K_{an}(\mathscr{A}, \mathscr{B}) \leqslant 1.$$

Proof. Suppose $\mathscr{A} = \mathscr{M}_{s_1}(\mathbb{C}) \oplus \cdots \oplus \mathscr{M}_{s_k}(\mathbb{C})$ and let $P_1 = 1 \oplus 0 \oplus \cdots \oplus 0$, $P_2 = 0 \oplus 1 \oplus \cdots \oplus 0, \ldots, P_k = 0 \oplus \cdots \oplus 0 \oplus 1$. Then $\sum_{1 \leqslant k \leqslant k} P_i \mathscr{B} P_i$ is strongly injective in $\mathscr{B}$ and we can write $P_i \mathscr{B} P_i = \mathscr{M}_{s_i}(\mathscr{B}_i)$ for $1 \leqslant i \leqslant k$. Since $\mathscr{B}$ is primitive, it follows that each $\mathscr{B}_i$ is primitive, and thus $\mathbb{C}$ is strongly injective in $\mathscr{B}_i$ for $1 \leqslant i \leqslant k$. Hence by Proposition 1, $\mathscr{M}_{s_i}(\mathbb{C})$ is strongly injective in $\mathscr{M}_{s_i}(\mathscr{B}_i)$ for $1 \leqslant i \leqslant k$. Whence, by Proposition 1, $\mathscr{A}$ is strongly injective in $\sum_{1 \leqslant i \leqslant k} P_i \mathscr{B} P_i$. Hence, by Proposition 3, $K_n(\mathscr{A}, \mathscr{B}) \leqslant 1$. The general case easily follows from Lemma 9. $\square$

The reason we can get better metric results (AH instead of AF) for von Neumann algebras than primitive C*-algebras is that we know that every masa is strongly injective, or that $K_{an}(\mathscr{A}, \mathscr{B}) < \infty$ when $\mathscr{A}$ is a masa in a von Neumann algebra $\mathscr{B}$.

Suppose I is an infinite set and $\{\mathscr{B}_i : i \in I\}$ is a family of unital C*-algebras and, for each $i \in I$, $\mathscr{A}_i$ is a unital C*-subalgebra of $\mathscr{B}_i$. Suppose α is a nontrivial ultrafilter on I and $\pi : \prod_{i \in I} \mathscr{B}_i \to \prod_{i \in I} \mathscr{B}_i / \sum_{i \in I}^{\oplus} \mathscr{B}_i$ and $\rho : \prod_{i \in I} \mathscr{B}_i \to \prod_{i \in I}^{\alpha} \mathscr{B}_i$ are the quotient maps, where $\prod_{i \in I}^{\alpha} \mathscr{B}_i$ is the C*-ultraproduct of the $\mathscr{B}_i$'s with respect to the ultrafilter α. Let $\mathscr{A} = \prod_{i \in I} \mathscr{A}_i$.

PROPOSITION 16. *The following are true.*

1. $K_{an}\left(\pi(\mathscr{A}), \prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) \leqslant \sup_{i \in I} K_{an}(\mathscr{A}_i, \mathscr{B}_i)$

2. $K_n \left(\pi(\mathscr{A}), \prod_{i \in I} \mathscr{B}_i / \Sigma_{i \in I} \mathscr{B}_i \right) \leqslant \sup_{i \in I} K_n(\mathscr{A}_i, \mathscr{B}_i).$

3. $K_{an} \left(\rho(\mathscr{A}), \prod^{\alpha} \mathscr{B}_i \right) \leqslant \lim_{i \to \alpha} K_{an}(\mathscr{A}_i, \mathscr{B}_i)$

4. $K_n \left(\rho(\mathscr{A}), \prod^{\alpha} \mathscr{B}_i \right) \leqslant \lim_{i \to \alpha} K_n(\mathscr{A}_i, \mathscr{B}_i)$

5. *If each $\mathscr{A}_i$ is a masa in a von Neumann algebra $\mathscr{B}_i$, then*

$$K_n \left(\pi(\mathscr{A}), \prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) \leqslant 1 \text{ and } K_n \left(\rho(\mathscr{A}), \prod^{\alpha} \mathscr{B}_i \right) \leqslant 1.$$

6. *If each $\mathscr{B}_i$ is primitive or a von Neumann algebra, then*

$$\mathscr{L} \left(\prod^{\alpha} \mathscr{B}_i \right) = \prod^{\alpha} \mathscr{L}(\mathscr{B}_i) \text{ and}$$

$$\mathscr{L} \left(\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) = \pi \left(\prod_{i \in I} \mathscr{L}(\mathscr{B}_i) \right)$$

7. *If, for each $i \in I$, $\mathscr{B}_i = B(H_i)$ for some Hilbert space H_i, then*

$$K_{an} \left(\pi(\mathscr{A}), \prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) \leqslant 29 \text{ and } K_{an} \left(\rho(\mathscr{A}), \prod^{\alpha} \mathscr{B}_i \right) \leqslant 29.$$

8. *If each $\mathscr{B}_i$ is a von Neumann algebra and $\mathscr{D}$ is a unital commutative C*-subalgebra of $\prod_{i \in I} B_i$, then*

$$K_{an} \left(C^* \left(\pi(\mathscr{D}) \cup \mathscr{L} \left(\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) \right), \prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) \leqslant 4 \text{ and}$$

$$K_{an} \left(C^* \left(\rho(\mathscr{D}) \cup \mathscr{L} \left(\prod^{\alpha} \mathscr{B}_i \right) \right), \prod^{\alpha} \mathscr{B}_i \right) \leqslant 4,$$

9. *If $\mathscr{D} \subseteq \prod_{i \in I} \mathscr{B}_i$ is norm separable and $I = \mathbb{N}$, then*

$$K_n \left(C^* \left(\pi(\mathscr{D}) \cup \mathscr{L} \left(\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) \right), \prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right)$$

$$= K_{an} \left(C^* \left(\pi(\mathscr{D}) \cup \mathscr{L} \left(\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) \right), \prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) \text{ and}$$

$$K_n\left(C^*\left(\rho\left(\mathscr{D}\right)\cup\mathscr{L}\left(\prod^\alpha \mathscr{B}_i\right)\right),\prod^\alpha\mathscr{B}_i\right)$$

$$=K_{an}\left(C^*\left(\rho\left(\mathscr{D}\right)\cup\mathscr{L}\left(\prod^\alpha \mathscr{B}_i\right)\right),\prod^\alpha\mathscr{B}_i\right).$$

Proof. (1) Let $\Delta = \sup_{i\in I}K_{an}\left(\mathscr{A}_i,\mathscr{B}_i\right)$. If $\Delta = \infty$, there is nothing to prove, so we can assume that $0 < \Delta < \infty$. Suppose $T = \pi(\{T_i\}) \in \prod_{i\in I}\mathscr{B}_i/\Sigma_{i\in I}\mathscr{B}_i$ and, for $1 \leqslant k \leqslant m$, $A_k = \{A_{k,i}\} \in \mathscr{A}$ and suppose $\varepsilon > 0$. Then, for each $i \in I$,

$$dist\left(T_i,\mathscr{A}_i\right) \leqslant K_{an}\left(\mathscr{A}_i,\mathscr{B}_i\right)d_{an}\left(T_i,\mathscr{A}_i,\mathscr{B}_i\right) \leqslant \Delta d_{an}\left(T_i,\mathscr{A}_i,\mathscr{B}_i\right).$$

Hence there is a unitary $U_i \in \mathscr{B}_i$ such that

$$\left\|U_iA_{k,i} - A_{k,i}U_i\right\| < \varepsilon$$

for $1 \leqslant k \leqslant m$ and such that

$$dist\left(T_i,\mathscr{A}_i\right) < \Delta\left(\left\|U_iT_i - T_iU_i\right\| + \varepsilon\right).$$

Hence, for each $i \in I$ there is a $C_i \in \mathscr{A}_i$ such that

$$\left\|T_i - C_i\right\| < \Delta\left(\left\|U_iT_i - T_iU_i\right\| + \varepsilon\right)$$

Then $U = \{U_i\} \in \prod_{i\in I}\mathscr{B}_i$ is a unitary and $C = \{C_i\} \in \mathscr{A}$. Moreover

$$dist\left(T,\pi\left(\mathscr{A}\right)\right) \leqslant \left\|T - \pi\left(C\right)\right\| = \limsup_{i\to\infty}\left\|T_i - C_i\right\|$$

$$\leqslant \limsup_{i\to\infty}\Delta\left(\left\|U_iT_i - T_iU_i\right\| + \varepsilon\right) = \Delta\left[\left\|\pi\left(U\right)T - T\pi\left(U\right)\right\| + \varepsilon\right],$$

and, for $1 \leqslant k \leqslant m$,

$$\left\|\pi\left(U\right)A_k - A_k\pi\left(U\right)\right\| = \limsup_{i\to\infty}\left\|U_iA_{k,i} - A_{k,i}U_i\right\| \leqslant \varepsilon.$$

If we let Λ be the set of all pairs $\lambda = (\mathscr{F},\varepsilon)$ with $\varepsilon > 0$ and $\mathscr{F} = \{A_1,\ldots,A_m\}$ a finite subset of $\pi\left(\mathscr{A}\right)$ and we let $V_\lambda = \pi\left(U\right)$ constructed above, then $\{V_\lambda\}$ is a net of unitary elements of $\prod_{i\in I}\mathscr{B}_i/\Sigma_{i\in I}\mathscr{B}_i$ such that, for every $A \in \pi\left(\mathscr{A}\right)$

$$\left\|V_\lambda A - AV_\lambda\right\| \to 0,$$

and such that

$$dist\left(T,\pi\left(\mathscr{A}\right)\right) \leqslant \Delta\limsup\left\|TV_\lambda - V_\lambda T\right\| \leqslant \Delta d_{an}\left(T,\pi\left(\mathscr{A}\right),\prod_{i\in I}\mathscr{B}_i/\sum_{i\in I}\mathscr{B}_i\right).$$

Hence $K_{an}\left(\pi\left(\mathscr{A}\right),\prod_{i\in I}\mathscr{B}_i/\Sigma_{i\in I}\mathscr{B}_i\right)\leqslant\Delta$.

(2). Now we let $\Delta=\sup_{i\in I}K_n\left(\mathscr{A}_i,\mathscr{B}_i\right)$. Suppose $T=\pi\left(\{T_i\}\right)\in\prod_{i\in I}\mathscr{B}_i/\Sigma_{i\in I}\mathscr{B}_i$ and $\varepsilon>0$. As in the proof of (1), for each $i\in I$, we can choose a unitary $U_i\in\left(\mathscr{A}_i,\mathscr{B}_i\right)'$ and a $C_i\in\mathscr{A}_i$ such that

$$\|C_i-T_i\|\leqslant\Delta\left[\|U_iT_i-T_iU_i\|+\varepsilon\right].$$

Hence $U=\pi\left(\{U_i\}\right)$ is a unitary in $\left(\pi\left(\mathscr{A}\right),\prod_{i\in I}\mathscr{B}_i/\Sigma_{i\in I}\mathscr{B}_i\right)'$ and

$$dist\left(T,\pi\left(\mathscr{A}\right)\right)\leqslant\limsup_{i\to\infty}\|T_i-C_i\|\leqslant\Delta\|UT-TU\|$$

$$\leqslant\Delta d_n\left(T,\pi\left(\mathscr{A}\right),\prod_{i\in I}\mathscr{B}_i/\sum_{i\in I}\mathscr{B}_i\right).$$

(3) and (4). The proofs are almost the same as those of (1) and (2).

(5). This follows from (2) and (4).

(6). This follows from (2) and (4) and the fact that $\mathscr{L}\left(\mathscr{B}\right)$ is weakly injective when $\mathscr{B}$ is primitive or a von Neumann algebra, which implies $K_n\left(\mathscr{L}\left(\mathscr{B}\right),\mathscr{B}\right)=1$.

(7). This follows from (1) and (3) and the fact from [12] that $K_{an}\left(\mathscr{C},B\left(H\right)\right)\leqslant 29$ for every Hilbert space H and every unital C*-subalgebra $\mathscr{C}\subseteq B\left(H\right)$.

(8) We can find, for each $i\in I$, a masa $\mathscr{A}_i$ in $\mathscr{B}_i$ so that $\mathscr{A}=\prod_{i\in I}\mathscr{A}_i$ contains $\mathscr{D}$. We know from (5) that

$$K_n\left(\pi\left(\mathscr{A}\right),\prod_{i\in I}\mathscr{B}_i/\sum_{i\in I}\mathscr{B}_i\right)\leqslant 1 \text{ and } K_n\left(\rho\left(\mathscr{A}\right),\overset{\alpha}{\prod}\mathscr{B}_i\right)\leqslant 1.$$

Suppose $T=\pi(\{A_i\})\in\pi(\mathscr{A})$ with $A=\{A_i\}\in\mathscr{A}$ and suppose $\varepsilon>0$. We know from (5) that $\mathscr{L}\left(\prod_{i\in I}\mathscr{B}_i/\Sigma_{i\in I}\mathscr{B}_i\right)=\pi\left(\prod_{i\in I}\mathscr{L}\left(\mathscr{B}_i\right)\right)$, so $\pi\left(C^*\left(\mathscr{D}\cup\prod_{i\in I}\mathscr{L}\left(\mathscr{B}_i\right)\cup\Sigma^\oplus\mathscr{A}_i\right)\right)$

$=C^*\left(\pi\left(\mathscr{D}\right)\cup\mathscr{L}\left(\prod_{i\in I}\mathscr{B}_i/\Sigma_{i\in I}\mathscr{B}_i\right)\right)$ and thus $\mathscr{E}=_{\text{def}}C^*\left(\mathscr{D}\cup\prod_{i\in I}\mathscr{L}\left(\mathscr{B}_i\right)\cup\Sigma^\oplus\mathscr{A}_i\right)$

$\subseteq\mathscr{A}$. It is clear that $dist\left(T,C^*\left(\pi\left(\mathscr{D}\right)\cup\mathscr{L}\left(\prod_{i\in I}\mathscr{B}_i/\Sigma_{i\in I}\mathscr{B}_i\right)\right)\right)$ is the same as

$dist\left(A,\mathscr{E}\right)$. However, it follows from Proposition 10 (with $\mathscr{W}=\mathbb{C}$) that there is a net $\{U_\lambda\}$ of unitary elements of $\prod_{i\in I}\mathscr{B}_i$ such that

$$\|U_\lambda S-SU_\lambda\|\to 0$$

for every $S\in\mathscr{E}$ and such that

$$dist\left(A,\mathscr{E}\right)\leqslant\lim_\lambda\|U_\lambda A-AU_\lambda\|.$$

If $J \subseteq I$ and $S = \{S_i\} \in \prod_{i \in I} \mathscr{B}_i$, we define $P_J S = \{S_i'\} \in \prod_{i \in I} \mathscr{B}_i$, where

$$S_i' = \begin{cases} S_i & \text{if } i \in J \\ 0 & \text{if } i \notin J \end{cases}.$$

Since $A \in \mathscr{A}$, it follows, for every finite subset $J \subseteq I$, that $P_J A \in \sum_{i \in I} \mathscr{A}_i \subseteq \mathscr{E}$. Hence, for every finite $J \subseteq I$, we have

$$\lim \|U_\lambda A - A U_\lambda\| = \lim \|U_\lambda P_{I \setminus J} A - P_{I \setminus J} A U_\lambda\|.$$

Suppose $F \subseteq \mathscr{D}$ is finite and $\varepsilon > 0$. We write $U_\lambda = \{U_\lambda(i)\}$ and, for each $D \in F$, we write $D = \{D_i\}$. It follows that the set $I_{(F,\varepsilon)}$ of $i \in I$ for which there is a unitary $W_i \in \mathscr{B}_i$ with

$$\max_{D \in F} \|W_i D_i - D_i W_i\| < \varepsilon \text{ and}$$

$$dist(A, \mathscr{E}) \leqslant \|W_i A_i - A_i W_i\| + \varepsilon$$

must be infinite. Hence we can choose a unitary $W_{(F,\varepsilon)} = \{W_{(F,\varepsilon)}(i)\}$ so that

$$W_{(F,\varepsilon)}(i) = \begin{cases} W_i & \text{if } i \in I_{(F,\varepsilon)} \\ 1 & \text{otherwise} \end{cases}.$$

It follows that

$$\max_{D \in F} \|D W_{(F,\varepsilon)} - W_{(F,\varepsilon)} D\| < \varepsilon$$

and

$$dist(A, \mathscr{E}) \leqslant \|\pi(W_{(F,\varepsilon)} A - A W_{(F,\varepsilon)})\| = \|\pi(W_{(F,\varepsilon)}) T - T \pi(W_{(F,\varepsilon)})\|.$$

It follows that $\{\pi(W_{(F,\varepsilon)})\}$ is a net of unitary elements of $\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i$ such that

$$\|\pi(W_{(F,\varepsilon)}) S - S \pi(W_{(F,\varepsilon)})\| \to 0$$

for every $S \in \pi(\mathscr{D})$ and such that

$$dist\left(T, C^*\left(\pi(\mathscr{D}) \cup \mathscr{L}\left(\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i\right)\right)\right)$$

$$= dist(A, \mathscr{E}) \leqslant \limsup_{(F,\varepsilon)} \|\pi(W_{(F,\varepsilon)}) T - T \pi(W_{(F,\varepsilon)})\|$$

$$\leqslant d_{an}\left(T, C^*\left(\pi(\mathscr{D}) \cup \mathscr{L}\left(\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i\right)\right), \prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i\right).$$

The fact that $K_n(\mathscr{A}) \leqslant 1$ (by part (5)) implies, reasoning as in the proof of Lemma 8, we see that

$$K_{an}\left(C^*\left(\pi(\mathscr{D}) \cup \mathscr{L}\left(\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i\right)\right)\right) \leqslant 4.$$

The argument for ultraproducts is the same except for considering finite subsets $J \subseteq I$ we consider subsets J not in the ultrafilter α, which shows that $I_{(F,\varepsilon)} \in \alpha$.

(9). This follows using arguments in the proof of [14, Theorem 4].   □

3. Representations

In [1] C. Akemann and G. Pedersen showed that central sequences from a quotient $\mathscr{B}/\mathscr{J}$ can be lifted to a central sequence in $\mathscr{B}$. The ideas in their proof can be used here. Recall from [1] and [2] that if $\mathscr{J}$ is a closed ideal in a unital C*-algebra $\mathscr{B}$ there is a *quasicentral approximate unit*, i.e., a net $\{e_\lambda\}_{\lambda \in \Lambda}$ in $\mathscr{J}$ such that

1. $0 \leqslant e_\lambda \leqslant 1$ for every $\lambda \in \Lambda$,

2. $\|(1 - e_\lambda)x\| + \|x(1 - e_\lambda)\| \to 0$ for every $x \in \mathscr{J}$,

3. $\|be_\lambda - e_\lambda b\| \to 0$ for every $b \in \mathscr{B}$.

It is well-known [1] that if $\pi : \mathscr{B} \to \mathscr{B}/\mathscr{J}$ is the quotient homomorphism, then $\|(1 - e_\lambda)b\| \to \|\pi(b)\|$ for every $b \in \mathscr{B}$.

THEOREM 17. *Suppose $\mathscr{B}$ and $\mathscr{E}$ are unital C*-algebras and $\pi : \mathscr{B} \to \mathscr{E}$ is a unital surjective $*$-homomorphism. If $\mathscr{S} \subseteq \mathscr{B}$, then*

$$\pi\left(Appr(\mathscr{S}, \mathscr{B})''\right) \subseteq Appr(\pi(S), \mathscr{E})''.$$

Proof. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a quasicentral approximate unit for $\ker \pi$. Then, for every $x, y \in \mathscr{B}$,
$$\|(1 - e_\lambda)x\| \to \|\pi(x)\|,$$
and
$$\|[(1 - e_\lambda)x]y - y[(1 - e_\lambda)x]\| \to \|\pi(x)\pi(y) - \pi(y)\pi(x)\|.$$
The second follows from the first statement and
$$\|y(1 - e_\lambda) - (1 - e_\lambda)y\| \to 0.$$
Suppose $x \in \mathscr{B}$ and $\pi(x) \notin Appr(\pi(S), \mathscr{E})''$. Then there is an $\varepsilon > 0$ such that for every finite subset $\mathscr{F}$ of $\mathscr{S}$ and every $\eta > 0$ there is a $y \in \mathscr{B}$ such that
$$\|\pi(y)\| < 1$$
$$\|\pi(y)\pi(w) - \pi(w)\pi(y)\| < \eta$$
for every $w \in \mathscr{S}$ and
$$\|\pi(y)\pi(x) - \pi(x)\pi(y)\| > \varepsilon.$$
It follows from the above remarks that there is a $\lambda \in \Lambda$ such that if $y_{(\mathscr{F}, \eta)} = (1 - e_\lambda)y$, then
$$\|y_{(\mathscr{F}, \eta)}\| < 1,$$
$$\|y_{(\mathscr{F}, \eta)}w - wy_{(\mathscr{F}, \eta)}\| < \eta$$
for every $w \in \mathscr{S}$, and
$$\|y_{(\mathscr{F}, \eta)}x - xy_{(\mathscr{F}, \eta)}\| > \varepsilon.$$

Then $\{y_{(\mathscr{F},\eta)}\}$ is a bounded net such that $\|y_{(\mathscr{F},\eta)}w - wy_{(\mathscr{F},\eta)}\| \to 0$ for every $w \in \mathscr{S}$ and such that $\|y_{(\mathscr{F},\eta)}x - xy_{(\mathscr{F},\eta)}\| \nrightarrow 0$. Hence $x \notin Appr(\mathscr{S},\mathscr{B})''$. $\square$

It is easy to show that a direct product of unital centrally prime C*-algebras is centrally prime. The following result shows that the same is not true for subdirect products. This gives a way to construct examples of commutative unital C*-subalgebras of a C*-algebra $\mathscr{B}$ for which $Appr(\mathscr{A},\mathscr{B})''$ is much larger than $C^*(\mathscr{A} \cup \mathscr{Z}(\mathscr{B}))$. Note that in the following lemma the algebra $\mathscr{A}$ is not assumed to be selfadjoint.

LEMMA 18. *Suppose $\mathscr{B}_1$, $\mathscr{B}_2,\ldots,\mathscr{B}_n$ are unital C*-algebras and $\mathscr{B} \subseteq \mathscr{B}_1 \oplus \mathscr{B}_2 \oplus \cdots \oplus \mathscr{B}_n$ is a unital C*-algebra such that the coordinate projection $\pi_j : \mathscr{B} \to \mathscr{B}_j$ is surjective for $j = 1,2,\ldots,n$. Then, for every unital norm closed subalgebra $\mathscr{A}$ of $\mathscr{B}$, we have*

$$Appr(\mathscr{A},\mathscr{B})'' = \left[Appr(\pi_1(\mathscr{A}),\mathscr{B}_1)'' \oplus \cdots \oplus Appr(\pi_n(\mathscr{A}),\mathscr{B}_n)''\right] \cap \mathscr{B}.$$

Proof. It follows from Theorem 17 and the surjectivity of π_j that

$$\pi_j\left(Appr(\mathscr{A},\mathscr{B})''\right) \subseteq Appr(\pi_j(\mathscr{A}),\mathscr{B}_j)''$$

for $j = 1,2,\ldots,n$. Hence

$$Appr(\mathscr{A},\mathscr{B})'' \subseteq \left[Appr(\pi_1(\mathscr{A}),\mathscr{B}_1)'' \oplus \cdots \oplus Appr(\pi_n(\mathscr{A}),\mathscr{B}_n)''\right] \cap \mathscr{B}.$$

Next suppose $b_j \in Appr(\pi_j(\mathscr{A}),\mathscr{B}_j)''$ for $j = 1,2,\ldots,n$ and $b = b_1 \oplus b_2 \oplus \cdots \oplus b_n \in \mathscr{B}$. Suppose $\{x_\lambda = x_{\lambda,1} \oplus x_{\lambda,2} \oplus \cdots \oplus x_{\lambda,n}\}$ is a bounded net in $\mathscr{B}$ such that, for every $a = \pi_1(a) \oplus \pi_2(a) \oplus \cdots \oplus \pi_n(a) \in \mathscr{A}$,

$$\|ax_\lambda - x_\lambda a\| \to 0.$$

Then

$$\left\|\pi_j(a)x_{\lambda,j} - x_{\lambda,j}\pi_j(a)\right\| \to 0 \text{ for } 1 \leqslant j \leqslant n.$$

Hence, for $1 \leqslant j \leqslant n$, $\{x_{\lambda,j}\}$ is a bounded net in $\mathscr{B}_j$ such that, for every $c \in \pi(\mathscr{A}_j)$

$$\left\|x_{\lambda,j}c - cx_{\lambda,j}\right\| \to 0.$$

Hence

$$\|x_\lambda b - bx_\lambda\| = \left\|\left(b_1 x_{\lambda,1} - x_{\lambda,1}b_1\right) \oplus \cdots \oplus \left(b_n x_{\lambda,n} - x_{\lambda,n}b_n\right)\right\| \to 0.$$

Hence $b \in Appr(\mathscr{A},\mathscr{B})''$. $\square$

COROLLARY 19. *Suppose $\mathscr{B}_1$, $\mathscr{B}_2,\ldots,\mathscr{B}_n$ are unital centrally prime C*-algebras and $\mathscr{B} \subseteq \mathscr{B}_1 \oplus \cdots \oplus \mathscr{B}_n$ is a unital C*-algebra such that the coordinate projection $\pi_j : \mathscr{B} \to \mathscr{B}_j$ is surjective for $j = 1,2,\ldots,n$. Then, for every unital commutative C*-subalgebra $\mathscr{A}$ of $\mathscr{B}$, we have*

$$Appr(\mathscr{A},\mathscr{B})'' = \left[C^*(\pi_1(\mathscr{A}) \cup \mathscr{Z}(\mathscr{B}_1)) \oplus \cdots \oplus C^*(\pi_n(\mathscr{A}) \cup \mathscr{Z}(\mathscr{B}_n))\right] \cap \mathscr{B}$$

EXAMPLE 20. *Let S be the unilateral shift operator on ℓ^2, and let $\mathscr{B} = C^*(S^* \oplus S)$. It follows that $\mathscr{K}(\ell^2) \oplus \mathscr{K}(\ell^2) \subseteq \mathscr{B} \neq C^*(\mathscr{S}^*) \oplus C^*(S)$ and $\mathscr{Z}(\mathscr{B}) = \mathbb{C}1 \subseteq \mathscr{A}$. If $0 \neq A = A^* \in \mathscr{K}(\ell^2)$ and $\mathscr{A} = C^*(A \oplus A)$, then $\mathscr{A}$ a unital commutative C^*-subalgebra of $\mathscr{B}$, $\mathscr{Z}(\mathscr{B}) \subseteq \mathscr{A}$, but*

$$Appr(\mathscr{A}, \mathscr{B})'' = C^*(A) \oplus C^*(A),$$

which is much larger than $\mathscr{A}$.

4. C*-algebraic Stone-Weierstrass and continuous fields

Here is our main result in this section. The proof is based on the factor state version of the Stone-Weierstrass theorem of Longo [17], Popa [19], (and Teleman [22]).

THEOREM 21. *Suppose $\mathscr{B}$ is a unital separable C^*-algebra and $\mathscr{A}$ is a unital C^*-subalgebra of $\mathscr{B}$ with $\mathscr{Z}(\mathscr{B}) \subseteq \mathscr{A}$. Suppose also $\{\mathscr{J}_i : i \in I\}$ is a family of closed two-sided ideals of $\mathscr{B}$ such that*

1. *If $i \neq j$ are in I, then*

$$(\mathscr{A} \cap \mathscr{J}_i) + (\mathscr{A} \cap \mathscr{J}_j) = \mathscr{A}$$

2. *$\mathscr{A}/(\mathscr{A} \cap \mathscr{J}_i)$ is approximately normal in $\mathscr{B}/\mathscr{J}_i$ for each $i \in I$.*

3. *If $\mathscr{J}$ is a primitive ideal in $\mathscr{B}$, then there is an $i \in I$ such that $\mathscr{J} \subseteq \mathscr{J}_i$.*

Then $\mathscr{A}$ is approximately normal in $\mathscr{B}$.

Proof. Assume via contradiction that $T \in Appr(\mathscr{A}, \mathscr{B})''$ and $T \notin \mathscr{A}$. It follows from the factor state Stone-Weierstrass theorem [17], [19], that there are factor states $\alpha \neq \beta$ on $C^*(\mathscr{A} \cup \{T\})$ such that $\alpha(A) = \beta(A)$ for every $A \in \mathscr{A}$. We can choose $S \in C^*(\mathscr{A} \cup \{T\})$ so that $\alpha(S) \neq \beta(S)$. Since $Appr(\mathscr{A}, \mathscr{B})''$ is a C*-algebra containing $\mathscr{A} \cup \{T\}$, we see that $S \in Appr(\mathscr{A}, \mathscr{B})$. It follows from Longo's extension theorem [17] that we can extend α and β to factor states on $\mathscr{B}$. Let $(\pi_\alpha, H_\alpha, e_\alpha)$ and $(\pi_\beta, H_\beta, e_\beta)$ be the GNS representations for α and β, respectively. Since α and β are factor states, $\pi_\alpha(\mathscr{B})''$ and $\pi_\beta(\mathscr{B})''$ are factor von Neumann algebras; whence $\ker \pi_\alpha$ and $\ker \pi_\beta$ are prime ideals, which by [8] are primitive. Hence there are $i, j \in I$ such that $\mathscr{J}_i \subseteq \ker \pi_\alpha$ and $\mathscr{J}_j \subseteq \ker \pi_\beta$.

Case 1. $i = j$. Define $\rho_i : \mathscr{B} \to \mathscr{B}/\mathscr{J}_i$ to be the quotient homomorphism. It follows that $\rho_i(Appr(\mathscr{A}, \mathscr{B})'') \subseteq Appr(\rho_i(\mathscr{A}), \rho_i(\mathscr{B})) = \rho_i(\mathscr{A})$ since $\rho_i(\mathscr{A}) = \mathscr{A}/(\mathscr{A} \cap \mathscr{J}_i)$ is approximately normal in $\rho_i(\mathscr{B})$. It follows that $\rho_i(S) \in \rho_i(\mathscr{A})$, so there is an $A \in \mathscr{A}$ such that $S - A \in \ker \rho_i = \mathscr{J}_i$. But $\mathscr{J}_i \subseteq \ker \pi_\alpha$ and $\mathscr{J}_i = \mathscr{J}_j \subseteq \ker \pi_\beta$. Hence $\pi_\alpha(S) = \pi_\alpha(A)$ and $\pi_\beta(S) = \pi_\beta(A)$, which implies $\alpha(S) = \alpha(A) = \beta(A) = \beta(S)$, a contradiction. Hence this case is impossible.

Case 2. $i \neq j$. It follows from assumption (2) that $(\rho_i \oplus \rho_j)(\mathscr{A}) = \rho_i(\mathscr{A}) \oplus \rho_j(\mathscr{A})$. It follows that $(\rho_i \oplus \rho_j)(\mathscr{B}) = \rho_i(\mathscr{B}) \oplus \rho_j(\mathscr{B})$, and we know from Theorem 17 that

$$(\rho_i \oplus \rho_j)(S) \in (\rho_i \oplus \rho_j)\left(Appr(\mathscr{A},\mathscr{B})''\right)$$
$$\subseteq Appr(\rho_i(\mathscr{A}) \oplus \rho_j(\mathscr{A}), \rho_i(\mathscr{B}) \oplus \rho_j(\mathscr{B}))$$
$$= Appr(\rho_i(\mathscr{A}), \rho_i(\mathscr{B}))'' \oplus Appr(\rho_j(\mathscr{A}), \rho_j(\mathscr{B}))''$$
$$= \rho_i(\mathscr{A}) \oplus \rho_j(\mathscr{A}) = (\rho_i \oplus \rho_j)(\mathscr{A}).$$

Hence there is an $A \in \mathscr{A}$ such that

$$S - A \in \ker \rho_i \cap \ker \rho_j \subseteq \ker \pi_\alpha \cap \ker \pi_\beta.$$

Hence,

$$\alpha(S) = \alpha(A) = \beta(A) = \beta(S),$$

which is also a contradiction.

Since Cases 1 and 2 are both impossible, our assumption that $\mathscr{A}$ is not approximately normal must be false. This completes the proof. $\square$

COROLLARY 22. *If in Theorem 21 we replace condition (3) with any one of*

1. $\mathscr{A}$ *is commutative,* $\mathscr{Z}(\mathscr{B}/\mathscr{J}_i) \subseteq \mathscr{A}/(\mathscr{A} \cap \mathscr{J}_i)$ *and* $\mathscr{B}/\mathscr{J}_i$ *is centrally prime for every* $i \in I$,

2. $\mathscr{A} = \mathscr{C}^*(\mathscr{A}_0 \cup \mathscr{Z}(\mathscr{B}))$ *where* $\mathscr{A}_0$ *is an AF algebra and each* $\mathscr{J}_i$ *is a primitive ideal,*

3. $\mathscr{A} = \mathscr{C}^*(\mathscr{A}_0 \cup \mathscr{Z}(\mathscr{B}))$ *where* $\mathscr{A}_0$ *is an AH algebra and each* $\mathscr{B}/\mathscr{J}_i$ *is a von Neumann algebra,*

4. $\mathscr{Z}(\mathscr{B}) \subseteq \mathscr{A}$ *and each* $\mathscr{B}/\mathscr{J}_i$ *is finite-dimensional*

then $\mathscr{A}$ *is approximately normal in* $\mathscr{B}$.

COROLLARY 23. *Suppose* $\mathscr{D}$ *is a separable unital commutative C*-algebra and* $\mathscr{W}$ *is a unital C*-algebra, and* $\mathscr{A}_0$ *is a C*-subalgebra of* $\mathscr{B} = \mathscr{D} \otimes \mathscr{W}$. *If any one of the following holds,*

1. $\mathscr{A}_0$ *is commutative and* $\mathscr{W}$ *is centrally prime,*

2. $\mathscr{A}_0$ *is AF and* $\mathscr{W}$ *is primitive,*

3. $\mathscr{A}_0$ *is AH and* $\mathscr{W}$ *is a von Neumann algebra,*

then

$$Appr(\mathscr{A}_0, \mathscr{B})'' = C^*(\mathscr{A}_0 \cup \mathscr{Z}(\mathscr{B})).$$

5. C*-algebraic Bishop-Stone-Weierstrass and nonselfadjoint subalgebras

In this section we prove a modest result that applies to commutative nonselfadjoint subalgebras. The proof relies on the first author's version of the Bishop-Stone-Weierstrass theorem for C*-algebras [13]. Suppose $\mathscr{A}$ is a unital closed (not necessarily selfadjoint) subalgebra of a unital C*-algebra $\mathscr{B}$. A set $\mathscr{E}$ of states on $\mathscr{B}$ is called $\mathscr{A}$-*antisymmetric* if whenever $a \in \mathscr{A}$ and $a|_{\mathscr{E}}$ is real (i.e., $\varphi(a) \in \mathbb{R}$ for all φ in $\mathscr{E}$), we have $a|_{\mathscr{E}}$ is constant. Here is the first author's Bishop-Stone-Weierstrass theorem for C*-algebras [13].

THEOREM 24. [13] *Suppose $\mathscr{A}$ is a separable commutative unital closed subalgebra of a unital C*-algebra $\mathscr{B}$ and $b \in \mathscr{B}$ and suppose for every $\mathscr{A}$-antisymmetric set of pure states on $\mathscr{B}$ there is an $a \in \mathscr{A}$ such that $b|_E = a|_E$. Then $b \in \mathscr{A}$.*

THEOREM 25. *Suppose $\mathscr{B}$ is a unital separable C*-algebra $\mathscr{A}$ is a unital commutative norm-closed subalgebra of $\mathscr{B}$ with $\mathscr{L}(\mathscr{B}) \subseteq \mathscr{A}$. Suppose also $\{\mathscr{J}_i : i \in I\}$ is a family of closed two-sided ideals of $\mathscr{B}$ such that*

1. *If $i \neq j$ are in I, then*

$$(\mathscr{L}(\mathscr{B}) \cap \mathscr{J}_i) + (\mathscr{L}(\mathscr{B}) \cap \mathscr{J}_j) = \mathscr{L}(\mathscr{B})$$

2. *$\mathscr{A}/(\mathscr{A} \cap \mathscr{J}_i)$ is approximately normal in $\mathscr{B}/\mathscr{J}_i$ for each $i \in I$.*

3. *If $\mathscr{J}$ is a primitive ideal in $\mathscr{B}$, then there is an $i \in I$ such that $\mathscr{J}_i \subseteq \mathscr{J}$.*

Then $\mathscr{A}$ is approximately normal in $\mathscr{B}$.

Proof. Suppose E is an $\mathscr{A}$-antisymmetric set of pure states on $\mathscr{B}$. Since $\mathscr{L}(\mathscr{B})$ $= \mathscr{L}(\mathscr{B})^* \subseteq \mathscr{A}$, it follows that each element of $\mathscr{L}(\mathscr{B})$ is constant on E. Suppose, for $k = 1, 2$, that $\alpha_k \in E$ with GNS representation π_k and, by (3), choose $i_k \in I$ so that $\mathscr{J}_{i_k} \subseteq \ker \pi_k$. If $i_1 \neq i_2$, it follows from that there is an $x \in \mathscr{L}(\mathscr{B})$ such that $x - 1 \in \mathscr{J}_{i_1}$ and $x \in \mathscr{J}_{i_2}$, which implies $\pi_1(x) = 1$ and $\pi_2(x) = 0$, contradicting $\alpha_1(x) = \alpha_2(x)$. Hence there is an $i \in I$ such that, for every $\alpha \in E$ with GNS representation π, we have $\mathscr{J}_i \subseteq \ker \pi$. Let $\rho : \mathscr{B} \to \mathscr{B}/\mathscr{J}_i$ be the quotient map. We know from Theorem 17 that $\rho(T) \in Appr(\rho(\mathscr{A}), \rho(\mathscr{B}))''$. However, it follows from (2) that $Appr(\rho(\mathscr{A}), \rho(\mathscr{B}))'' = \rho(\mathscr{A})$. Hence there is an $A \in \mathscr{A}$ such that $T - A \in \ker \rho = \mathscr{J}_i$. Hence, for every $\alpha \in E$, $\alpha(T) = \alpha(A)$. It follows from Theorem 24 that $T \in \mathscr{A}$. $\square$

One example of an algebra $\mathscr{B}$ with a family of ideals satisfying (1) and (3) in Theorem 25 is by letting $\mathscr{B} = C(X) \otimes \mathscr{W} = \mathscr{C}(X, \mathscr{W})$ for some unital C*-algebra $\mathscr{W}$ and some compact Hausdorff space X, and, for each $i \in X$, letting $\mathscr{J}_i = \{f \in C(X, \mathscr{W}) : f(i) = 0\}$. The trick is guaranteeing condition (2).

In [23] T. Rolf Turner proved that if T is an algebraic operator on a Hilbert space H, then $(\{T\}, B(H))'' = \{p(T) : p \in \mathbb{C}[z]\}$. This leads to the first statement in the following lemma.

LEMMA 26. *Suppose $n \in \mathbb{N}$. Then*

1. *If $T \in M_n(\mathbb{C})$, then the algebra of polynomials in T is normal.*

2. *If $n \geqslant 2$, the following are equivalent:*

    (a) *$n \in \{2,3\}$.*
    (b) *Every unital commutative subalgebra of $\mathcal{M}_n(\mathbb{C})$ is normal.*

Proof. (1). This follows from Turner's result [14].

(2). $(a) \Longrightarrow (b)$. First suppose $n = 2$. It follows from Wedderburn's theorem that any commutative algebra $\mathscr{A} \subseteq \mathcal{M}_2(\mathbb{C})$ is upper triangular with respect to some basis for $\mathbb{C}^2$; whence $\dim \mathscr{A}$ is at most 2. This means that there is a $T \in \mathcal{M}_2(\mathbb{C})$ such that $\mathscr{A}$ is the set of polynomials in T; whence, by (1) above, $\mathscr{A}$ is normal..

Next suppose $n = 3$ and $\mathscr{A}$ is a commutative unital subalgebra of $\mathcal{M}_3(\mathbb{C})$. If $\mathscr{A}$ contains a nontrivial idempotent, then $\mathscr{A}$ is the direct sum of a subalgebra of $\mathcal{M}_2(\mathbb{C})$ and $\mathcal{M}_1(\mathbb{C})$, and the desired conclusion follows from the case $n = 2$. If $\mathscr{A}$ contains no nontrivial idempotents, then every element of $\mathscr{A}$ is the sum of a nilpotent and a scalar multiple of the identity. Since the algebra generated by a 3×3 nilpotent of order 3 is maximal abelian, the desired conclusion follows from (1) above whenever $\mathscr{A}$ contains a nilpotent of order 3. If the subalgebra $\mathscr{N}$ of nilpotents in $\mathscr{A}$ has dimension 1 then the desired conclusion follows from (1). Since $\mathscr{N}$ is commutative and is unitarily equivalent to a subalgebra of the strictly upper-triangular 3×3 matrices, we conclude that $\dim \mathscr{N} = 2$. Moreover, every nonzero element of $\mathscr{N}$ is a nilpotent of order 2, and therefore has rank 1. A linear space of rank-one operators must have all have the form $e \otimes x$ with e fixed or with x fixed and $\langle e, x \rangle = 0$ (see, for example, [14, Lemma 4.2]). Here

$$(e \otimes x)(h) = \langle h, x \rangle e.$$

Hence $\mathscr{N}$ is unitarily equivalent to

$$\mathscr{N}_1 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

or

$$\mathscr{N}_2 = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\},$$

and it is easily shown that $\mathscr{N}_j' = \mathscr{N}_j$ for $j = 1, 2$. Hence $\mathscr{N}$ is normal. $\square$

The algebra $\mathscr{A}$ of 4×4 matrices of the form $\begin{pmatrix} \alpha, I_2 & A \\ 0 & \alpha I_2 \end{pmatrix}$, where $\alpha \in \mathbb{C}$ and $A \in \mathcal{M}_2(\mathbb{C})$ and $trace(A) = 0$, is commutative and not normal, since $(\mathscr{A}, \mathcal{M}_4(\mathbb{C}))''$ is the set of 4×4 matrices of the same form without the restriction $trace(A) = 0$.

The following result is an immediate consequence of Theorem 25 and Lemma 26.

THEOREM 27. *Suppose K is a compact metric space and $\mathscr{B} = C(K) \otimes \mathscr{M}_n(\mathbb{C})$.*
Then

1. *If $T \in \mathscr{B}$, then the norm closed algebra $\mathscr{A}$ generated by $\{T\} \cup \mathscr{Z}(\mathscr{B})$ is approximately normal, i.e.,*

$$Appr(\{T\}, \mathscr{B})'' = \mathscr{A}.$$

2. *If $n = 2$ or $n = 3$, then every unital commutative closed subalgebra $\mathscr{A}$ of $\mathscr{B}$ that contains $\mathscr{Z}(\mathscr{B})$ is approximately normal, i.e., if $\mathscr{S} \subseteq \mathscr{B}$ is a commuting family, then $Appr(\mathscr{S}, \mathscr{B})''$ is the norm closed algebra generated by $\mathscr{S} \cup \mathscr{Z}(\mathscr{B})$.*

6. Questions and comments

We conclude with a list of questions and comments.

1. If $\mathscr{B}$ is any unital C*-algebra, it is clear that $\mathscr{Z}(\mathscr{B})$ is normal. When is $\mathscr{Z}(\mathscr{B})$ metric normal or metric approximately normal? It is clear that for $T \in \mathscr{B}$, the inner derivation δ_T on $\mathscr{B}$ defined by $\delta_T(S) = TS - ST$ extends to a weak*-continuous operator on $\mathscr{B}^{\#\#}$, and since the closed unit ball of $\mathscr{B}$ is weak*-dense in the closed unit ball of $\mathscr{B}^{\#\#}$, it follows that

$$\|\delta_T\| = \|\delta_T|_{\mathscr{B}^{\#\#}}\| = 2dist\left(T, \mathscr{Z}\left(\mathscr{B}^{\#\#}\right)\right).$$

On the other hand $\|\delta_T\|$ is clearly equal to $d_n(T, \mathscr{Z}(B), \mathscr{B})$. Hence, for every $T \in \mathscr{B}$,

$$dist(T, \mathscr{Z}(\mathscr{B})) \leqslant 2K_n(\mathscr{Z}(\mathscr{B}), \mathscr{B}) dist\left(T, \mathscr{Z}\left(\mathscr{B}^{\#\#}\right)\right).$$

The same argument applies if we replace $\mathscr{B}^{\#\#}$ with $\pi(\mathscr{B})''$, where $\pi : \mathscr{B} \to B(H)$ is a faithful representation. This makes it easy to see that if $\mathscr{B}$ is primitive, there is a faithful irreducible representation π, so

$$d_n(T, \mathscr{Z}(B), \mathscr{B}) = \left\|\delta_{\pi(T)}|_{\pi(\mathscr{B})''}\right\| = 2dist\left(\pi(T), \mathscr{Z}\left(\pi(\mathscr{B})''\right)\right)$$
$$= 2dist\left(\pi(T), \mathbb{C}1\right) = 2dist\left(T, \mathscr{Z}(\mathscr{B})\right),$$

which implies $K_n(\mathscr{Z}(\mathscr{B}), \mathscr{B}) = 1/2$. It is not hard to show that $\mathscr{Z}(\mathscr{B})$ is metric normal when $\mathscr{B}$ has a finite separating family of irreducible representations. However, it is also true for $\mathscr{M}_2(C(X))$ when X is compact Hausdorff space.

2. For which unital C*-algebras is every masa metric normal or metric approximately normal? In these algebras we know that every commutative unital C*-algebra containing the center is metric approximately normal. Morover, if, for a centrally prime algebra $\mathscr{B}$ there is an upper bound for the $\mathscr{K}_{an}(\mathscr{A}, P\mathscr{B}P)$ for all masas $\mathscr{A} \subseteq P\mathscr{B}P$ with P a projection in $\mathscr{B}$, then it follows that every *AH* C*-subalgebra of $\mathscr{B}$ containing $\mathscr{Z}(\mathscr{B})$ is metric approximately normal.

3. It was shown in Proposition 16 that, if each B_i is a von Neumann algebra, then any commutative C*-subalgebra $\mathscr{A}$ containing the center of $\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i$ that lifts to a commutative C*-subalgebra of $\prod_{i \in I} \mathscr{B}_i$, is metric approximately normal. What about those commutative C*-algebras $\mathscr{A}$ that do not lift? We see that the general problem almost reduces to masas that do not lift.

Interesting special cases are when $\mathscr{A}$ is the C*-algebra generated by a single normal element or two unitary elements or three selfadjoint elements and $I = \mathbb{N}$. It was shownn by H. Lin [16] that when each $\mathscr{B}_i$ is finite-dimensional, then every normal element in $\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i$ lifts to a normal element in $\prod_{i \in I} \mathscr{B}_i$. P. Friis and M. Rørdam [9] gave a simple proof of Lin's result when each $\mathscr{B}_i$ is a finite von Neumann algebra. If I is infinite and $\mathscr{B}_i$ is an infinite von Neumann algebra for infinitely many $i \in I$, then there is a normal element T in $\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i$ that does not lift to a normal element of $\prod_{i \in I} \mathscr{B}_i$. Indeed, if S is a nonunitary isometry and

$$T_n = \left[S^n (S^*)^n + \sum_{k=1}^{n} \frac{k}{n} S^K (1 - SS^*) (S^*)^k \right] S,$$

then $\|T_n T_n^* - T_n^* T_n\| \leqslant 2/n$ and the distance from T_n to the normal operators is 1. Is $C^* \left(\{T\} \cup \mathscr{Z} \left(\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i \right) \right)$ normal or approximately normal? What is a masa in $\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i$ that contains T? There is a similar example (see [7]) when $I = \mathbb{N}$ and $\mathscr{B}_n = \mathscr{M}_n(\mathbb{C})$ for each n. There is a commuting family $\{T_1, T_2, T_3\}$ of selfadjoint operators in $\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i$ that does not lift to commuting selfadjoints in $\prod_{i \in I} \mathscr{B}_i$. There is also [24] a commuting pair U, V of unitaries $\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i$ that do not lift to commuting unitaries in $\prod_{i \in I} \mathscr{B}_i$. Are the associated C*-algebras genereated by these families and the center normal or approximately normal? What are the masas in $\prod_{i \in I} \mathscr{B}_i / \sum_{i \in I} \mathscr{B}_i$ in this case?

4. Let $\mathbb{F}_3$ denote the free group with 3 generators u, v, w. Is $C^* (u, v)$ approximately normal in $C^* (\mathbb{F}_3)$? In $C_r^* (\mathbb{F}_3)$? In $\mathscr{L}_{\mathbb{F}_3}$?

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