Operators
and
Matrices
Volume 8, Number 3 (2014), 651–658
doi:10.7153/oam-08-35

ON CERTAIN FUNCTIONAL EQUATION RELATED TO A CLASS OF GENERALIZED INNER DERIVATIONS

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(*Communicated by F. Kittaneh*)

Abstract. The main purpose of this paper is to prove the following result. Let X be a real or complex Banach space, let $\mathscr{L}(X)$ be the algebra of all bounded linear operators on X and let $\mathscr{A}(X) \subseteq \mathscr{L}(X)$ be a standard operator algebra. Suppose there exists an additive mapping $T : \mathscr{A}(X) \to \mathscr{L}(X)$ satisfying the relation $T(A^n) = T(A)A^{n-1} - AT(A^{n-2})A + A^{n-1}T(A)$ for all $A \in \mathscr{A}(X)$ and some fixed integer $n > 2$. In this case T is of the form $T(A) = AB + BA$ for all $A \in \mathscr{A}(X)$ and some fixed $B \in \mathscr{L}(X)$.

Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0$, $x \in R$, implies $x = 0$. Recall that R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by $Q_s(R)$ the symmetric Martindale ring of quotients of a semiprime ring R (see [1, Chapter 2]). An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. A derivation $D : R \to R$ is inner in case D is of the form $D(x) = ax - xa$ for all $x \in R$ and some fixed $a \in R$. A mapping $T(x) = ax + xb$, where a and b are fixed elements of a ring, is sometimes called a generalized derivation. We follow Hvala [11] and call such mappings generalized inner derivations, as they present a generalization of the concept of inner derivation. In the theory of operator algebras, they are considered as an important class of so-called elementary operators - i.e., mappings of the form

$$x \longmapsto \sum_{i=1}^{n} a_i x b_i.$$

We refer the reader to [14] for a good account of the theory of elementary operators. A mapping $T(x) = ax + xa$, where a is a fixed element of a ring, will be called symmetric generalized inner derivation. Let X be a real or complex Banach space and let $\mathscr{L}(X)$ and $\mathscr{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathscr{L}(X)$, respectively. An algebra $\mathscr{A}(X) \subseteq \mathscr{L}(X)$ is said to be standard in case $\mathscr{F}(X) \subset \mathscr{A}(X)$. Any standard operator algebra is prime, which is a consequence of a Hahn-Banach theorem.

Motivated by the work of Brešar [2], Vukman, Kosi-Ulbl and Eremita [18] proved the following result (see [9] for a generalization).

Mathematics subject classification (2010): 16W10, 46K15, 39B05.
Keywords and phrases: Prime ring, semiprime ring, Banach space, standard operator algebra, derivation, generalized derivation.
This research has been supported by the Research Council of Slovenia.

Paper OaM-08-35

THEOREM 1. *Let R be a $2-$torsion free semiprime ring and let $T : R \to R$ be an additive mapping satisfying the relation*

$$T(xyx) = T(x)yx - xT(y)x + xyT(x) \tag{1}$$

for all pairs $x, y \in R$. In this case T is of the form $2T(x) = qx + xq$ for all $x \in R$ and some fixed $q \in Q_S(R)$.

Since any symmetric generalized inner derivation satisfies the functional equation (1), Theorem 1 characterizes symmetric generalized inner derivations among all additive mappings on $2-$torsion free semiprime rings. Fošner and Vukman [7] proved the result below (see [10] for a generalization).

THEOREM 2. *Let R be a $2-$torsion free prime ring and let $T : R \to R$ be an additive mapping satisfying the relation*

$$T(x^3) = T(x)x^2 - xT(x)x + x^2T(x) \tag{2}$$

for all $x \in R$. In this case T is of the form $4T(x) = qx + xq$ for all $x \in R$ and some fixed $q \in Q_S(R)$.

In the proof of Theorem 1 some ideas from [2] are used, while in the proof of Theorem 2 as the main tool Brešar-Beidar-Chebotar theory (the theory of functional identities) is used. We refer the reader to [3] for an introductory account on functional identities and to [4] for a full treatment of this theory. Let us point out that Theorem 1 and Theorem 2 were used in the solution of some functional equations arising from so-called bicircular projections (see [6, 7, 8, 10, 13, 16, 17, 19]).

The substitution $y = x^{n-2}$ in (1) gives

$$T(x^n) = T(x)x^{n-1} - xT(x^{n-2})x + x^{n-1}T(x). \tag{3}$$

It is our aim in this paper to prove the following result, which is related to the functional equation (3).

THEOREM 3. *Let X be a real or complex Banach space and let $\mathscr{A}(X)$ be a standard operator algebra on X. Suppose there exists an additive mapping $T : \mathscr{A}(X) \to \mathscr{L}(X)$ satisfying the relation*

$$T(A^n) = T(A)A^{n-1} - AT(A^{n-2})A + A^{n-1}T(A)$$

for all $A \in \mathscr{A}(X)$ and some fixed integer $n > 2$. In this case T is of the form $T(A) = AB + BA$ for all $A \in \mathscr{A}(X)$ and some fixed $B \in \mathscr{L}(X)$.

Let us point out that in Theorem 3 we obtain as a result the continuity of T under purely algebraic assumptions concerning T, which means that Theorem 2 might be of some interest from the automatic continuity point of view. For results concerning automatic continuity we refer to [5] and [15]. In the proof of Theorem 3 we use Theorem 2 and the fact that for any standard operator algebra $\mathscr{A}(X)$ we have $Q_S(\mathscr{A}(X)) = \mathscr{L}(X)$.

Proof. We have the relation

$$T(A^n) = T(A)A^{n-1} - AT(A^{n-2})A + A^{n-1}T(A). \tag{4}$$

Let us first restrict our attention to $\mathscr{F}(X)$.

Let A be from $\mathscr{F}(X)$ and let $P \in \mathscr{F}(X)$ be a projection with $AP = PA = A$. Putting $A+P$ for A in the above relation, we obtain

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}T(A^{n-i}P^i) = T(A+P)&\left(\sum_{i=0}^{n-1}\binom{n-1}{i}A^{n-1-i}P^i\right)\\ &-\sum_{i=0}^{n-2}\binom{n-2}{i}(A+P)T(A^{n-2-i}P^i)(A+P)\\ &+\left(\sum_{i=0}^{n-1}\binom{n-1}{i}A^{n-1-i}P^i\right)T(A+P).\end{aligned} \tag{5}$$

Using (4) and rearranging the relation (5) in sense of collecting together terms involving equal number of factors of P, we obtain

$$\sum_{i=1}^{n-1} f_i(A,P) = 0, \tag{6}$$

where $f_i(A,P)$ stands for the expression of terms involving i factors of P.

Replacing A by $A+2P$, $A+3P$, $\ldots$, $A+(n-1)P$ in turn in the relation (4) and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(A,P)$, $i = 1,2,\ldots,n-1$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \ldots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution. In particular,

$$\begin{aligned} f_{n-1}(A,P) = &\binom{n}{n-1}T(A) - \binom{n-1}{n-1}T(A)P - \binom{n-1}{n-2}T(P)A \\ &+\binom{n-2}{n-2}AT(P)P + \binom{n-2}{n-2}PT(P)A + \binom{n-2}{n-3}PT(A)P \\ &-\binom{n-1}{n-1}PT(A) - \binom{n-1}{n-2}AT(P). \end{aligned}$$

The above relation reduces to

$$\begin{aligned} nT(A) = &\, T(A)P + (n-1)T(P)A - AT(P)P - PT(P)A \\ &-(n-2)PT(A)P + PT(A) + (n-1)AT(P). \end{aligned} \tag{7}$$

Multiplying the above relation from both sides by P, we obtain

$$2PT(A)P = PT(P)A + AT(P)P, \tag{8}$$

which reduces (7) to

$$2nT(A) = 2T(A)P + 2PT(A) + 2(n-1)(T(P)A + AT(P)) - n(PT(P)A + AT(P)P). \tag{9}$$

Left multiplication of the above relation by P gives, using the relation (8), the relation

$$2PT(A) = 2AT(P) + PT(P)A - AT(P)P \tag{10}$$

and similary we obtain

$$2T(A)P = 2T(P)A + AT(P)P - PT(P)A. \tag{11}$$

Applying (10) and (11) in (9), we obtain

$$2T(A) = A(2T(P) - T(P)P) + (2T(P) - PT(P))A. \tag{12}$$

Multiplying the relation (8) from both sides by A, we obtain

$$2AT(A)A = A^2T(P)A + AT(P)A^2. \tag{13}$$

Applying the relations (12) and (13), we obtain

$$\begin{aligned} &2T(A)A^2 + 2A^2T(A) \\ &= (A^2T(P)A + AT(P)A^2) + (2T(P) - PT(P))A^3 + A^3(2T(P) - T(P)P) \\ &= 2AT(A)A + 2T(A^3). \end{aligned}$$

We therefore have

$$T(A^3) = T(A)A^2 - AT(A)A + A^2T(A). \tag{14}$$

From the relation (12) one can conclude that T maps $\mathscr{F}(X)$ into itself. Therefore we have an additive mapping $T : \mathscr{F}(X) \to \mathscr{F}(X)$ satisfying the relation (14) for all $A \in \mathscr{F}(X)$. Since $\mathscr{F}(X)$ is prime, one can apply Theorem 2, which implies that T is of the form $4T(A) = AC + CA$ for all $A \in \mathscr{F}(X)$ and some $C \in Q_S(\mathscr{F}(X))$. Since $Q_S(\mathscr{F}(X)) = \mathscr{L}(X)$ (this is a direct consequence of [1, Theorem 4.3.8] and [12, p.78, Example 5]), it follows that $C \in \mathscr{L}(X)$. Therefore, one can conclude that T is of the form

$$T(A) = AB + BA, \tag{15}$$

for all $A \in \mathscr{F}(X)$ and some $B \in \mathscr{L}(X)$. It remains to prove that the relation (15) holds on $\mathscr{A}(X)$ as well. Let us introduce $T_1 : \mathscr{A}(X) \to \mathscr{L}(X)$ by $T_1(A) = AB + BA$ and consider $T_0 = T - T_1$. Obviously, the mapping T_0 is additive and satisfies the relation (4). Besides, T_0 vanishes on $\mathscr{F}(X)$. It is our aim to prove that T_0 vanishes on $\mathscr{A}(X)$ as well. Let $A \in \mathscr{A}(X)$, let P be a one-dimensional projection and $S = A + PAP - (AP + PA)$. It is clear that $T_0(S) = T_0(A)$ and $SP = PS = 0$. We have the relation

$$T_0(A^n) = T_0(A)A^{n-1} - AT_0(A^{n-2})A + A^{n-1}T_0(A)$$

for all $A \in \mathscr{A}(X)$. Applying the above relation, we obtain

$$
\begin{aligned}
&T_0(S)S^{n-1} - ST_0(S^{n-2})S + S^{n-1}T_0(S) = T_0(S^n) = T_0(S^n + P) = T_0((S+P)^n) \\
&= T_0(S+P)(S+P)^{n-1} - (S+P)T_0((S+P)^{n-2})(S+P) + (S+P)^{n-1}T_0(S+P) \\
&= T_0(S)(S^{n-1}+P) - (S+P)T_0(S^{n-2})(S+P) + (S^{n-1}+P)T_0(S) \\
&= T_0(S)S^{n-1} + T_0(S)P - ST_0(S^{n-2})S - ST_0(S^{n-2})P - PT_0(S^{n-2})S \\
&\quad -PT_0(S^{n-2})P + S^{n-1}T_0(S) + PT_0(S).
\end{aligned}
$$

From the above relation it follows that

$$T_0(S)P - ST_0(S^{n-2})P - PT_0(S^{n-2})S - PT_0(S^{n-2})P + PT_0(S) = 0$$

and since $T_0(S) = T_0(A)$, we can write

$$T_0(A)P - ST_0(A^{n-2})P - PT_0(A^{n-2})S - PT_0(A^{n-2})P + PT_0(A) = 0. \tag{16}$$

Multiplying the above relation from both sides by P, we obtain

$$2PT_0(A)P - PT_0(A^{n-2})P = 0. \tag{17}$$

Putting $2A$ for A in the above relation, we obtain

$$4PT_0(A)P - 2^{n-2}PT_0(A^{n-2})P = 0. \tag{18}$$

In case $n = 3$, the relation (17) gives

$$PT_0(A)P = 0. \tag{19}$$

In case $n > 3$, the relations (17) and (18) give (19). Considering the above relation in the relation (17), we obtain

$$PT_0(A^{n-2})P = 0,$$

which reduces the relation (16) to

$$T_0(A)P - ST_0(A^{n-2})P - PT_0(A^{n-2})S + PT_0(A) = 0. \tag{20}$$

Putting $2A$ for A (in this case S becomes $2S$) in the above relation, we obtain

$$2T_0(A)P - 2^{n-1}ST_0(A^{n-2})P - 2^{n-1}PT_0(A^{n-2})S + 2PT_0(A) = 0,$$

which together with the relation (20) implies that

$$T_0(A)P + PT_0(A) = 0.$$

Right multiplication of the above relation by P gives

$$T_0(A)P + PT_0(A)P = 0$$

and the relation (19) reduces the above relation to

$$T_0(A)P = 0.$$

Since P is an arbitrary one-dimensional projection, if follows from the above relation that $T_0(A) = 0$ for any $A \in \mathscr{A}(X)$, which completes the proof of the theorem. □

We proceed with the following purely algebraic conjecture.

CONJECTURE 4. Let $n > 2$ be a fixed integer and let R be a semiprime ring with suitable torsion restrictions. Suppose there exists an additive mapping $T : R \to R$ satisfying the relation

$$T(x^n) = T(x)x^{n-1} - xT(x^{n-2})x + x^{n-1}T(x)$$

for all $x \in R$. In this case T is of the form $2T(x) = qx + xq$ for all $x \in R$ and some fixed $q \in Q_S(R)$.

We conclude the article with the result below, which proves the above conjecture in case a ring has the identity element.

THEOREM 5. *Let $n > 2$ be a fixed integer and let R be a $n!-$torsion free semiprime ring with the identity element. Suppose there exists an additive mapping $T : R \to R$ satisfying the relation*

$$T(x^n) = T(x)x^{n-1} - xT(x^{n-2})x + x^{n-1}T(x)$$

for all $x \in R$. In this case T is of the form $2T(x) = ax + xa$ for all $x \in R$ and some fixed $a \in R$.

Proof. We have the relation

$$T(x^n) = T(x)x^{n-1} - xT(x^{n-2})x + x^{n-1}T(x) \tag{21}$$

Let y be any element of the center $Z(R)$. Putting $x + y$ in the above relation, we obtain

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} T(x^{n-i}y^i) = T(x+y)&\left(\sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}y^i\right) \\ &- \sum_{i=0}^{n-2} \binom{n-2}{i} (x+y)T(x^{n-2-i}y^i)(x+y) \\ &+ \left(\sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}y^i\right) T(x+y). \end{aligned} \tag{22}$$

Using (21) and rearranging the relation (22) in sense of collecting together terms involving equal number of factors of y, we obtain

$$\sum_{i=1}^{n-1} f_i(x,y) = 0,$$

where $f_i(x,y)$ stands for the expression of terms involving i factors of y. Replacing x by $x+2y$, $x+3y$, ..., $x+(n-1)y$ in turn in the relation (21) and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(x,y)$, $i=1,2,\dots,n-1$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution. In particular, putting the identity element e for y, we obtain

$$\begin{aligned} f_{n-1}(x,e) = {} & \tbinom{n}{n-1}T(x) - \tbinom{n-1}{n-1}T(x) - \tbinom{n-1}{n-2}ax \\ & + \tbinom{n-2}{n-2}xa + \tbinom{n-2}{n-2}ax + \tbinom{n-2}{n-3}T(x) \\ & - \tbinom{n-1}{n-1}T(x) - \tbinom{n-1}{n-2}xa, \end{aligned}$$

where a denotes $T(e)$. The above relation reduces to

$$2(n-2)T(x) = (n-2)ax + (n-2)xa.$$

Since R is $n!$-torsion free, it follows from the above relation that

$$2T(x) = ax + xa$$

for all $x \in R$. The proof of the theorem is now complete. □

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(Received January 26, 2012)

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