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# PRE–IMAGES OF BOUNDARY POINTS OF THE NUMERICAL RANGE

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*Abstract.* This paper considers matrices $A \in M_n(\mathbb{C})$ whose numerical range contains boundary points generated by multiple linearly independent vectors. Sharp bounds for the maximum number of such boundary points (excluding flat portions) are given for unitarily irreducible matrices of dimension $\leqslant 5$. An example is provided to show that there may be infinitely many for $n = 6$. For matrices unitarily similar to tridiagonal, however, a finite upper bound is found for all $n$. A somewhat unexpected byproduct of this is an explicit example of $A \in M_5(\mathbb{C})$ which is not tridiagonalizable via a unitary similarity.

## 1. Introduction

Let $\mathbb{C}^n$ be the standard $n$-dimensional vector space over the complex field $\mathbb{C}$, with the scalar product $\langle x,y\rangle = x^*y$ and the associated norm $\|x\| = \langle x,x\rangle^{1/2}$. The set of all $n$-by-$n$ matrices with elements in $\mathbb{C}$ will be denoted by $M_n(\mathbb{C})$. The *numerical range* (also called the *field of values*) of $A \in M_n(\mathbb{C})$ is by definition the set

$$F(A) = \{\langle Ax,x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}.$$

There is a vast number of publications on the numerical range and its properties; the proofs of all the basic properties used below can be found in standard references [9] or [10]. Note in particular that $F(A)$ is convex.

According to the definition, $F(A)$ is the range of the function $f_A(x) = \langle Ax,x\rangle$ when the domain is the unit sphere $\mathbb{C}S^n = \{x \in \mathbb{C}^n : \|x\| = 1\}$. The inverse $f_A^{-1}$ is a multivalued function. In fact, for any $z \in F(A)$ and $x \in f_A^{-1}(z)$, $e^{i\theta}x \in f_A^{-1}(A)$ for all $\theta \in [0,2\pi)$. Even after identifying vectors that are scalar multiples, the map $f_A^{-1}$ may still be multivalued. We say that a point $z \in F(A)$ is *multiply generated* if $f_A^{-1}(z)$ contains at least two linearly independent vectors, otherwise $z$ is *singularly generated*. The continuity properties of $f_A^{-1}$ were the subject of consideration in [5], and the notion of multiply generated points arose there as an auxiliary issue. This paper is devoted to its systematic treatment.

Note that all the points $z$ from the relative interior of $F(A)$ are multiply generated. Moreover, for such $z$ the set $f_A^{-1}(z)$ contains $n$ linearly independent vectors [3, Theorem 1] (for the $n = 2$ case, see also [24, Lemma 3(b)]). So, only points $z$ on the

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boundary $\partial F(A)$ of $F(A)$ are of interest. We classify them as follows: *flat portions* of the boundary are non-trivial line segments lying in $\partial F(A)$, *corner points* are $z \in \partial F(A)$ belonging to more than one supporting line of $F(A)$, and *round points* are all points of $\partial F(A)$ different from corner points and points in the relative interior of the flat portions. So, the set of extreme points of $F(A)$ consists of all corner points and round points of $\partial F(A)$.

It is not hard to show that the relative interiors of flat portions are always multiply generated. The corner points, being normal eigenvalues of $A$, are also easy to handle. This is done, along with some preliminary observations, in Section 2. The rest of the paper is thus devoted to round boundary points. In Section 3 it is shown how to reduce the treatment of arbitrary $A$ to that of its unitarily irreducible blocks. The case of 3-by-3 matrices is covered there in passing. The next two sections deal with 4-by-4 and 5-by-5 matrices, respectively. The bottom line here is that only finitely many multiply generated round boundary points can occur, and the sharp upper bound for their number is given, both for unitarily irreducible matrices and in general. A short Section 6 contains an example showing that the situation changes dramatically for matrices of bigger size, namely, that infinitely many (actually, all) round boundary points may be multiply generated, starting with $n = 6$. However, imposing an additional structure may keep the number of such points "under control" for matrices of arbitrary size. This is illustrated by tridiagonal matrices in Section 7. It is also shown there that certain 5-by-5 matrices constructed in Section 5 are not tridiagonalizable by a unitary similarity. Though the existence of such matrices has been known, we are not aware of specific examples in the literature. Finally, Section 8 provides an alternative point of view on the multiply generated boundary points, in terms of the so called critical curves of $A$.

## 2. Preliminary observations

For every angle $\theta$, denote by $\ell_\theta$ the supporting line of $F(A)$ having slope $-\cot\theta$ and such that $e^{-i\theta}F(A)$ lies to the right of the vertical line $e^{-i\theta}\ell_\theta$. We will say that $\ell_\theta$ is an *exceptional supporting line* (and, respectively, $\theta$ is an *exceptional angle*) if $\ell_\theta$ contains at least one multiply generated point of $F(A)$.

We will be using the standard notation $\operatorname{Re} A = \frac{1}{2}(A + A^*)$ and $\operatorname{Im} A = \frac{1}{2i}(A - A^*)$ for any $A \in M_n(\mathbb{C})$. Let also

$$H(\theta) = \operatorname{Re}\left(e^{-i\theta}A\right), \quad K(\theta) = \operatorname{Im}\left(e^{-i\theta}A\right),$$

abbreviating $H(0)$ and $K(0)$ to $H$ and $K$, respectively.

Considering $A$ as a linear transformation on $\mathbb{C}^n$, for a subspace $L \subset \mathbb{C}^n$ we will denote by $A|L$ the compression of $A$ onto $L$.

THEOREM 2.1. *Given $A \in M_n(\mathbb{C})$, the angle $\theta$ is exceptional if and only if $H(\theta)$ has a multiple minimal eigenvalue.*

*Proof.* Indeed, $f_A(x) \in \ell_\theta$ if and only if $x$ is an eigenvector of $H(\theta)$ corresponding to its minimal eigenvalue $\lambda_0$. If the respective eigenspace $L$ is one-dimensional, then $F(A) \cap \ell_\theta$ is obviously a singleton $z$, and this $z$ is singularly generated.

Suppose now that $\dim L > 1$. Denote by $k_1$ (resp., $k_2$) the minimal (resp., maximal) eigenvalue of $K_0 = K(\theta)|L$. Then

$$F(A) \cap \ell_\theta = e^{i\theta}[\lambda_0 + ik_1, \lambda_0 + ik_2],$$

and for any $\xi \in [k_1, k_2]$,

$$f_A^{-1}(e^{i\theta}(\lambda_0 + i\xi)) = f_{K_0}^{-1}(\xi) = f_{K_0 - \xi I}^{-1}(0). \tag{2.1}$$

So, if $k_1 \neq k_2$ (that is, $K_0$ is different from a scalar multiple of the identity), then $e^{i\theta}[\lambda_0 + ik_1, \lambda_0 + ik_2]$ is a flat portion of $\partial F(A)$ lying on $\ell_\theta$. According to (2.1), for $\xi \in (k_1, k_2)$ the set $f_A^{-1}(e^{i\theta}(\lambda_0 + i\xi))$ consists of all the neutral vectors (having length 1) of the indefinite Hermitian matrix $K_0 - \xi I$. Thus, all the points from the relative interior of $F(A) \cap \ell_\theta$ are multiply generated. The endpoints of $F(A) \cap \ell_\theta$ may or may not be multiply generated, depending on whether $k_1$ and $k_2$ are multiple or simple eigenvalues of $K_0$. If, on the other hand, $k_1 = k_2 := k$ (that is, $K_0 = kI$), then $F(A) \cap \ell_\theta$ is a singleton $z = e^{i\theta}(\lambda_0 + ik)$, and $f_A^{-1}(z)$ is the whole unit sphere of $L$. Either way, some points of $F(A) \cap \ell_\theta$ are multiply generated. $\square$

As in [11], we say that a matrix $A$ is *generic* if the eigenvalues of $\mathrm{Re}(e^{-i\theta}A)$ as functions of $\theta$ do not cross. For generic $A$, the boundary of $F(A)$ is smooth and, according to Theorem 2.1, all its points are singularly generated. This is the case, in particular, for non-normal 2-by-2 matrices (which of course can also be proved directly, and was observed already a number of times). More generally, all points of $\partial F(A)$ are singularly generated if $A \in M_n(\mathbb{C})$ is *pure almost normal*, that is, unitarily irreducible and possesses $n-1$ orthogonal eigenvectors, see [17]. On the other hand, the following two observations can be easily extracted from the proof of Theorem 2.1.

COROLLARY 2.2. *Relative interiors of flat portions of $\partial F(A)$ consist entirely of multiply generated points.*

COROLLARY 2.3. *A corner point $z$ of $F(A)$ is multiply generated if and only if it is a multiple eigenvalue of $A$.*

Indeed, such $z$ is a normal eigenvalue of $A$, and thus an eigenvalue of $H(\theta)$ of the same multiplicity.

So, only round points of $\partial F(A)$ need to be investigated further. The following technical result will be useful in this regard.

PROPOSITION 2.4. *If $z$ is a multiply generated round point of $\partial F(A)$, then there is a 2-dimensional subspace onto which the compression of $A$ equals $zI$.*

*Proof.* In the notation of Theorem 2.1 and its proof, the restriction (and thus the compression $H_0$) of $H(\theta)$ onto $L$ is a scalar multiple of the identity. If $F(A) \cap \ell_\theta$ is a singleton, then the compression $K_0$ of $K(\theta)$ is a scalar multiple of the identity as well, and thus so is $A|L$. This is not true any more if $F(A)\cap\ell_\theta$ contains an interval. However, in the latter case $z$ is an endpoint of this interval, and it corresponds to a multiple eigenvalue of $K_0$. Any 2-dimensional subspace of the respective eigenspace of $K_0$ has the desired property. $\square$

In different terms, Proposition 2.4 means that multiply generated round points of $\partial F(A)$ belong to the *rank-2 numerical range* of $A$, see e.g. [4].

We end this section with the following simple observation which will be repeatedly used below. Let us say, following [14], that two matrices $A,B$ are *affine equivalent* if the pair $\{\operatorname{Re} B, \operatorname{Im} B\}$ can be obtained from $\{\operatorname{Re} A, \operatorname{Im} A\}$ by an invertible affine transformation:

$$\operatorname{Re} B = c_{11}\operatorname{Re} A + c_{12}\operatorname{Im} A + c_1 I, \quad \operatorname{Im} B = c_{21}\operatorname{Re} A + c_{22}\operatorname{Im} A + c_2 I, \tag{2.2}$$

where $[c_{ij}]_{i,j=1}^2$ is an invertible real matrix, and $c_1, c_2$ are real as well. Under condition (2.2), the numerical range $W(B)$ is obtained from $W(A)$ via the same affine transformation. In particular, parallel supporting lines of $W(A)$ are mapped to also parallel supporting lines of $W(B)$, while the angle between the intersecting supporting lines can be changed arbitrarily. In particular, two intersecting supporting lines of $W(A)$ can be mapped onto orthogonal supporting lines of $B$. Moreover, affine equivalent matrices are unitarily reducible (or irreducible) only simultaneously.

## 3. Reduction via unitary similarity

Recall now that the numerical range is invariant under unitary similarities, that is, $F(A) = F(B)$ whenever $A \cong B$. (Here and below $\cong$ stands for unitary similarity, which is an equivalence relation on $M_n(\mathbb{C})$.) Moreover, for block diagonal matrices

$$A = A_1 \oplus \cdots \oplus A_m, \tag{3.1}$$

$F(A)$ is the convex hull of $F(A_1), \ldots, F(A_m)$. The latter is therefore true whenever $A$ is *unitarily reducible*, that is, unitarily similar to a block diagonal matrix with at least two blocks. Our next result concerns round points of $\partial F(A)$ for unitarily reducible $A$.

THEOREM 3.1. *Let $A \cong A_1 \oplus \cdots \oplus A_m$ and let $z$ be a round point of $\partial F(A)$. Then $z$ is multiply generated as a point of $F(A)$ if and only if it either* (a) *belongs to $F(A_i)$ for more than one $i$, or* (b) *is multiply generated as a point of $F(A_i)$ for some $i$.*

*Proof.* Without loss of generality we may suppose that (3.1) holds. Representing $x \in \mathbb{C}S^n$ as $x = x_1 \oplus \cdots \oplus x_m$ in accordance with (3.1), we see that

$$f_A(x) = \sum_{i=1}^m \langle A_i x_i, x_i\rangle = \sum_{i=1}^m t_i \zeta_i, \tag{3.2}$$

where $\zeta_i \in F(A_i) \subset F(A)$ and $t_i = \|x_i\|^2$, so that $\sum_{i=1}^m t_i = 1$, $t_i \geqslant 0$ $(i = 1, \ldots, m)$.

Since $z$ is an extreme point of $F(A)$, for (3.2) to hold it is necessary and sufficient that for each $i=1,\dots,m$ either $x_i=0$ or $\zeta_i=z$. In other words,

$$f_A^{-1}(z)=\left\{\sum \alpha_i x_i\colon \sum |\alpha_i|^2=1,\ x_i\in f_{A_i}^{-1}(z)\right\}, \tag{3.3}$$

with the summation over $i$'s satisfying $z\in F(A_i)$ and $x_i$ from the domain of $A_i$ identified with its imbedding into $\mathbb{C}^n$. So, $f_A^{-1}(z)$ contains linearly independent vectors, unless the right hand side of (3.3) reduces to one summand and the respective $f_{A_i}^{-1}(z)$ is one-dimensional. $\square$

Based on Corollary 2.2 and Theorem 3.1, a complete description of all multiply generated points can be given for any $A$ provided it has been obtained for its unitarily irreducible blocks.

Here is one such result, to illustrate the point.

THEOREM 3.2. *For $A\in M_3(\mathbb{C})$, there is at most one multiply generated extreme point on $\partial F(A)$. This point, when it exists, must be a normal eigenvalue of $A$.*

*Proof.* Suppose $z\in\partial F(A)$ is a multiply generated extreme point. According to Proposition 2.4 (if $z$ is round) or Corollary 2.3 (if it is a corner point), $A$ is then unitarily similar to a matrix of the form

$$\begin{pmatrix} z & 0 & * \\ 0 & z & * \\ * & * & * \end{pmatrix}.$$

In particular, $z$ is an eigenvalue of $A$. Lying on the boundary of $F(A)$, it then must be a normal eigenvalue, that is, $A$ is in fact unitarily similar to $(z)\oplus B$.

Since $z$ is a round point of $\partial F(A)$, a 2-by-2 block $B$ is not normal, and $z\in\partial F(B)$. From Theorem 3.1 we see that $z$ is then indeed a multiply generated (round) point of $\partial F(A)$ while all other points of $\partial F(A)$ are singularly generated. $\square$

So, unitarily irreducible 3-by-3 matrices $A$ do not have multiply generated round points on the boundary of $F(A)$. This fact was actually observed (and used) in the proof of [5, Theorem 10].

## 4. The 4-by-4 case

With the dimension increase, multiply generated round points start to emerge, in limited quantities.

THEOREM 4.1. *If $A$ is an unitarily irreducible 4-by-4 matrix, then there is at most one boundary round point that is multiply generated.*

*Proof.* We will show that $A\in M_4(\mathbb{C})$ having two multiply generated round points $z_1,z_2\in\partial F(A)$ must be unitarily reducible.

*Case* 1. *The points $z_1$ and $z_2$ lie on parallel supporting lines.* Considering the matrix $\alpha A+\beta I$ in place of $A$ if needed, by an appropriate choice of $\alpha,\beta\in\mathbb{C}$ ($\alpha\neq 0$) we may without loss of generality arrange for these supporting lines to be vertical. By Proposition 2.4, there exist 2-dimensional subspaces $L_j$ of the eigenspaces of $H$ corresponding to the eigenvalues $\xi_j:=\operatorname{Re} z_j$ for which $K|L_j=\eta_j I$ (here $\eta_j=\operatorname{Im} z_j$, $j=1,2$). If $\xi_1=\xi_2:=\xi$, then $L_1,L_2$ are subspaces of the same eigenspace of $H$, intersecting only at 0. Thus, the $\xi$-eigenspace of $H$ is (at least) 4-dimensional, implying that $H=\xi I$. This makes $A$ normal, and therefore unitarily reducible.

On the other hand, if $\xi_1\neq\xi_2$, then $L_1,L_2$ are the respective eigenspaces of $H$, and are therefore orthogonal. An appropriate unitary similarity then yields

$$A\cong H+iK=\begin{pmatrix}\xi_1 I & 0\\ 0 & \xi_2 I\end{pmatrix}+i\begin{pmatrix}\eta_1 I & B\\ B^* & \eta_2 I\end{pmatrix}. \tag{4.1}$$

Yet another unitary similarity of the form $U\oplus V$ with 2-by-2 blocks $U,V$ allows us to replace $B$ in (4.1) by $D=U^*BV$ without changing $H$ and the diagonal blocks in $K$. Using $U,V$ from the singular value decomposition of $B$, we may thus achieve $D=\operatorname{diag}(d_1,d_2)$, that is,

$$A\cong A'=\begin{pmatrix}\xi_1 & 0 & 0 & 0\\ 0 & \xi_1 & 0 & 0\\ 0 & 0 & \xi_2 & 0\\ 0 & 0 & 0 & \xi_2\end{pmatrix}+i\begin{pmatrix}\eta_1 & 0 & d_1 & 0\\ 0 & \eta_1 & 0 & d_2\\ d_1 & 0 & \eta_2 & 0\\ 0 & d_2 & 0 & \eta_2\end{pmatrix}.$$

From this form, it is clear that $\operatorname{span}\{e_1,e_3\}$ is a shared invariant subspace of $\operatorname{Re}A'$ and $\operatorname{Im}A'$, implying that $A'$ (and thus $A$) is unitarily reducible, a contradiction.

*Case* 2. *The supporting lines of $z_1$ and $z_2$ are not parallel.* Applying an affine equivalence as described in Section 2, we may without loss of generality suppose that $F(A)$ is located in the first quadrant while $z_1,z_2$ are, respectively, real and pure imaginary: $z_1=\lambda$ and $z_2=i\mu$. Of course, $z_1,z_2\neq 0$ (because otherwise we would be in the setting of Case 1), so $\lambda,\mu>0$.

As in Case 1, let us invoke Proposition 2.4 to find a 2-dimensional subspace $L$ the compression of $A$ onto which equals $\lambda I$, and then use a unitary similarity mapping the span of $\{e_1,e_2\}$ onto $L$ to observe that

$$A\cong H+iK=\begin{pmatrix}\lambda I & D\\ D & H_0\end{pmatrix}+i\begin{pmatrix}0 & 0\\ 0 & K_0\end{pmatrix},\ \text{where } D=\begin{pmatrix}d_1 & 0\\ 0 & d_2\end{pmatrix}. \tag{4.2}$$

The presence of a multiply generated round point on the imaginary axis implies that $\dim\ker H\geqslant 2$. From here, $H_0=\lambda^{-1}D^2$, and the vectors

$$v_1=[d_1,0,-\lambda,0]^T,\quad v_2=[0,d_2,0,-\lambda]^T$$

form an orthogonal basis of $\ker H$.

A direct computation shows that $\langle Kv_1,v_2\rangle=\lambda^2\omega$, where $\omega$ is an off diagonal entry of $K_0$ in (4.2). So, both $H_0$ and $K_0$ are diagonal, which implies that the span of $\{e_1,e_3\}$ is an invariant subspace for both $H$ and $K$, Thus, $A$ is untarily reducible. $\square$

There indeed exist unitarily irreducible 4-by-4 matrices with multiply generated round boundary points. We will give here an example of such a matrix which, in addition, has a flat portion on the boundary of $F(A)$. The proof of unitary irreducibility in this example, and some others below, is based on the following simple observation.

PROPOSITION 4.2. *Let $A = H + iK \in M_n(\mathbb{C})$ be such that $H(=\operatorname{Re} A)$ has a simple eigenvalue $\mu$, with $v$ being a corresponding eigenvector. Suppose that*

$$\{v, Kv, Q_1Kv, Q_2Kv, \ldots\}$$

*has rank $n$, where each $Q_i$ is some product of $H$ and $K$. Then $A$ is unitarily irreducible.*

*Proof.* If $L$ is a reducing subspace of $A$, then $\mu$ is an eigenvalue of the compression of $A$ onto $L$ or $L^\perp$. Consequently, $v$ must lie in one of these subspaces; switching the roles of $L$ and $L^\perp$ if needed, we may without loss of generality suppose that $v \in L$. Being invariant under both $H$ and $K$, the subspace $L$ must then also contain all the vectors $Qv$, where $Q$ is a product of $H$s and $K$s in an arbitrary order. Under the conditions imposed, this implies that $L$ is the whole space. In other words, $A$ does not have non-trivial reducing subspaces and, as such, is unitarily irreducible. □

EXAMPLE 4.3. Let $A = H + iK$, where

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1/2 \\ 0 & 0 & -1/2 & 5 \end{pmatrix}.$$

Then $\operatorname{span}\{e_1, e_2\} = \ker K$, and $H|\ker K = I$. So, 1 is a round boundary point. On the other hand, $0$ is the minimal eigenvalue of $H$, with $\ker H$ spanned by

$$v_1 = [1, 0, -1, 0]^T, \quad v_2 = [0, 2, 0, -1]^T.$$

Since $\langle Kv_1, v_2\rangle = -1/2 \neq 0$, $F(A)$ has a flat portion along the imaginary axis.

To invoke Proposition 4.2, observe that $v = [0, 1, 0, 2]^T$ is an eigenvector of $H$ corresponding to its simple eigenvalue 5. A direct computation shows that the vectors

$$v, Kv = [0, 0, -1, 10]^T, \quad HKv = [-1, 20, -1, 40]^T, \quad KHKv = [0, 0, -22, 401/2]^T$$

are linearly independent. Thus, $A$ is unitarily irreducible.

Putting pieces together with the help of Theorem 3.1, we thus obtain the following result for 4-by-4 matrices, unitarily reducible or not.

THEOREM 4.4. *For $A \in M_4(\mathbb{C})$, the set of multiply generated extreme points may consist of $0, 1, 2$, or all points of $\partial F(A)$. More than one multiply generated extreme point may occur only if $\partial F(A)$ is an ellipse.*

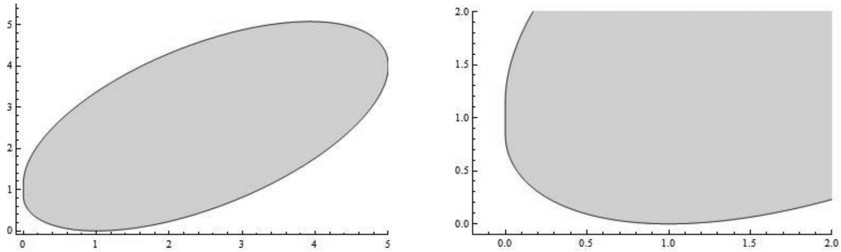

Figure 1: *$F(A)$ for Example 4.3. The magnified image on the right illustrates a short vertical flat portion containing $i$*

*Proof.* For unitarily irreducible matrices, all extreme points must be round, and for $n = 4$ the number is 0 or 1, according to Theorem 4.1 and Example 4.3. If $A \cong B \oplus (z)$, where $B \in M_3(\mathbb{C})$ is unitarily irreducible, then the only way a multiply generated extreme boundary point may occur is if $z \in \partial F(B)$ (not lying in the relative interior of its flat portion, if any), and then it is the only one. So, 0 and 1 are again the only possibilities. Similarly, for $A \cong C \oplus (z_1) \oplus (z_2)$ with a unitarily irreducible 2-by-2 block $C$, we have $0, 1$, or 2 multiply generated extreme points on $\partial F(A)$, depending on the location of $z_1, z_2$ relative to the ellipse $\partial F(C)$ and each other. Note that two multiply generated round points materialize only if $z_1, z_2 \in \partial F(C)$, in which case $F(A) = F(C)$ is an ellipse.

If $A$ is normal, then there are no round boundary points at all, multiply or singularly generated, and at most 2 multiply generated corner points.

Finally, let $A \cong A_1 \oplus A_2$, with unitarily irreducible 2-by-2 blocks $A_1, A_2$. Then there are no corner points. The ellipses $\partial F(A_1)$, $\partial F(A_2)$ may have 0,1,2 tangent points, or coincide. If the number of tangent points is zero or one, then the number of multiply generated round boundary points does not exceed one (note that it equals 0 when $\partial F(A_1)$, $\partial F(A_2)$ are tangent but lie outside of each other). On the other hand, if there are at least two multiply generated round boundary points, then one of the ellipses $F(A_1)$ and $F(A_2)$ lies inside another, say $F(A_1) \supseteq F(A_2)$. Then $F(A) = F(A_1)$ is an ellipse, and either there are exactly two multiply generated round boundary points (if $F(A_1) \neq F(A_2)$), or all the points of $\partial F(A)$ are multiply generated. $\square$

## 5. The 5-by-5 case

We start with a simple observation, concerning matrices of any size and having two multiply generated round points on the boundary.

LEMMA 5.1. *Let $z_1, z_2$ be multiply generated round points of $\partial F(A)$ for some $A \in M_n(\mathbb{C})$. Then $n \geqslant 4$, and $A$ is unitarily similar to*

$$\begin{pmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1^* & C_2^* & D \end{pmatrix}, \tag{5.1}$$

*where* $A_1, A_2 \in M_2(\mathbb{C})$ *are such that* $z_1, z_2 \in \partial F(A_1) \cap \partial F(A_2)$ *and*

$$\text{the range of } e^{-i\theta_j} B_k - e^{i\theta_j} C_k \text{ is orthogonal to } f_{A_k}^{-1}(z_j), \quad j,k = 1,2. \tag{5.2}$$

Here $\theta_j$ is the angle formed by the supporting line of $F(A)$ containing $z_j$ with the positive real axis; condition (5.2) is vacuously satisfied for $n = 4$.

*Proof.* Choose linearly independent $x_j, y_j \in f_A^{-1}(z_j)$, $j = 1,2$, and denote by $\mathscr{L}$ the span of $\{x_1, x_2, y_1, y_2\}$. Then $z_1, z_2$ remain extreme points of $F(A_0)$, where $A_0 = A|\mathscr{L}$. By construction of $\mathscr{L}$, they are also multiply generated. According to Theorem 3.2, $\dim \mathscr{L} = 4$, so in particular $n \geqslant 4$. From the considerations of Theorem 4.4 it is clear that $A_0 \cong A_1 \oplus A_2$, with $A_1, A_2$ as described in the statement. In other words, (5.1) holds.

We now turn to condition (5.2). All four combinations of $j,k$ can be treated in exactly in the same way, so without loss of generality let $j = k = 1$. Since, moreover, (5.2) is invariant under rotations and translations, it suffices to consider the situation when $\theta_1 = \pi/2$ and $z_1 = 0$. The matrix (5.1) then must have (positive) semi-definite real part, which property is thus inherited by all its compressions. Let now $x \in f_{A_1}^{-1}(0)$ while $y \in \mathbb{C}^{n-4}$ is arbitrary. The compression of (5.1) onto the span of

$$[x, \underbrace{0, \ldots, 0}_{n-2 \text{ times}}]^T \text{ and } [0,0,0,0,y]^T$$

has the real part equal

$$\begin{pmatrix} 0 & z \\ \overline{z} & \langle \operatorname{Re} D y, y \rangle \end{pmatrix}, \text{ where } z = \frac{1}{2} \langle (B_1 + C_1) y, x \rangle.$$

The semi-definiteness of the latter matrix is equivalent to $z = 0$ which, due to the arbitrariness of $y$, implies (5.1). □

Note that each of the matrices $A_j$ may be unitarily reducible, in which case $z_1, z_2$ are the eigenvalues of the respective $A_j$ (and thus the endpoints of its numerical range). For $n = 5$, this yields some important consequences.

LEMMA 5.2. *Suppose* $A \in M_5(\mathbb{C})$ *has two multiply generated round points* $z_1, z_2 \in \partial F(A)$, *and in the respective representation* (5.1) *both* $A_1, A_2$ *are unitarily reducible. Then* $A$ *itself is unitarily reducible. Moreover,*

$$A \cong B \oplus (z_1) \oplus (z_2), \text{ where } B \in M_3(\mathbb{C}) \text{ and } z_1, z_2 \in \partial F(B). \tag{5.3}$$

*Proof.* Due to the unitary reducibility of $A_1, A_2$, representation (5.1) can be simplified further as

$$\begin{pmatrix} z_1 & 0 & 0 & 0 & b_1 \\ 0 & z_2 & 0 & 0 & b_2 \\ 0 & 0 & z_1 & 0 & * \\ 0 & 0 & 0 & z_2 & * \\ \overline{c_1} & \overline{c_2} & * & * & * \end{pmatrix}.$$

Yet another unitary similarity, mapping the spans of $\{e_1, e_3\}$ and $\{e_2, e_4\}$ onto themselves, allows us to arrange in addition for $c_1 = c_2 = 0$. Then (5.2) implies that $b_1 = b_2 = 0$. Thus, (5.3) holds. Since $z_1, z_2 \in F(B)$, we also have $F(B) = F(A)$, so in fact $z_1, z_2 \in \partial F(B)$. $\square$

Now we can give a complete description of all 5-by-5 matrices with two multiply generated round points lying on the same supporting line.

THEOREM 5.3. *A matrix $A \in M_5(\mathbb{C})$ has two multiply generated round points $z_1, z_2$ lying on the same supporting line of $F(A)$ if and only if (5.3) holds with $B \in M_3(\mathbb{C})$ unitarily irreducible and $[z_1, z_2]$ being the flat portion of $\partial F(B)$. All other round points of $\partial F(A)$ are singularly generated.*

*Proof. Necessity.* According to Lemma 5.1, $A$ is unitarily similar to (5.1). It is clear that $F(A_j) \subset F(A)$, and by assumption $[z_1, z_2]$ is contained in the boundary of $F(A)$. To prevent the line segment $(z_1, z_2)$ from lying in the interior of at least one of the $F(A_j)$, it must be the case that both of the matrices $A_j$ are normal. Now (5.3) follows from Lemma 5.2. In our setting, the 3-by-3 matrix $B$ has a flat portion on the boundary with the endpoints being round, not corner points. This is only possible if $B$ is unitarily irreducible.

*Sufficiency* follows immediately from Theorem 3.1. Combined with Theorem 3.2, Theorem 3.1 also implies that all round points of $\partial F(B)$ except for $z_1, z_2$ are singularly generated. $\square$

Theorem 5.3 implies in particular that matrices $A \in M_5(\mathbb{C})$ with two multiply generated round points lying on the *same* supporting line of $F(A)$ must be unitarily reducible. The following examples show that this does not have to be the case when the supporting lines are different.

EXAMPLE 5.4. Let

$$A = H + iK = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix} + i \begin{pmatrix} 1 & -i & 0 & 0 & 1 \\ i & 1 & 0 & 0 & i \\ 0 & 0 & 1 & -i & 0 \\ 0 & 0 & i & 1 & 0 \\ 1 & -i & 0 & 0 & 5 \end{pmatrix}$$

Then $\sigma(H) = \{4, 2, 1, 0, 0\}, \sigma(K) \approx \{5.56, 2, 1.44, 0, 0\}$. The eigenvector $v = [0, 0, 1, 1, 2]^T$ corresponds to a simple eigenvalue 4 of $H$, and the set

$$\{v, Kv, HKv, KHKv, HKHKv\} \tag{5.4}$$

is linearly independent, so $A$ is unitarily irreducible by Proposition 4.2. On the other hand, a basis for $\ker H$ is given by $v_1 = [0, 0, -1/\sqrt{2}, 1/\sqrt{2}, 0]^T$ and $v_2 = [-1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0]^T$ while the compression of $K$ onto $\operatorname{span}\{v_1, v_2\}$ is the identity operator. So, $i$ is a multiply generated round point of $F(A)$. A similar computation shows that 1 is as well.

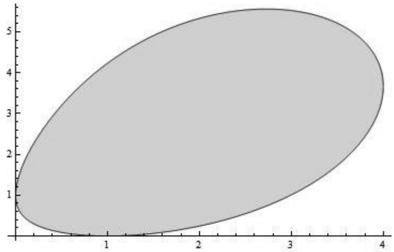


Figure 2: $F(A)$ *for Example* 5.4*. The points 1 and* $i$ *are multiply generated.*

EXAMPLE 5.5. Let

$$A = H + iK = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{pmatrix} + i \begin{pmatrix} 1 & -i & 0 & 0 & 1 \\ i & 1 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -i & 0 & 1 & 5 \end{pmatrix}.$$

The matrix $A$ is unitarily irreducible because $v = [0,0,-1+\sqrt{2},0,1]^T$ is an eigenvector of $H$ corresponding to its simple eigenvalue $2+\sqrt{2}$, and the set (5.4) is linearly independent for this choice of $v, H, K$ as well. Direct computation again shows that $A|\ker K = H|\ker K = I$ and $A|\ker H = iK|\ker H = iI$, so we have multiply generated round points $1$ and $i$ on $\partial F(A)$.

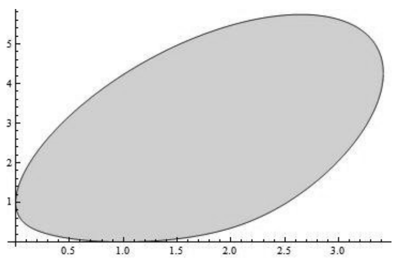


Figure 3: $F(A)$ *for Example* 5.5*. The points 1 and* $i$ *are multiply generated.*

EXAMPLE 5.6. Now let

$$A = H + iK = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{pmatrix} + i \begin{pmatrix} 1 & -i/2 & 0 & 0 & 1 \\ i/2 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -i/2 & 3 \\ 0 & 0 & i/2 & -1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Here $v = e_5$ is an eigenvector of $H$, corresponding to its simple eigenvalue $1/2$, and the vectors $v, Kv, HKv, KHKv, K^2v$ are linearly independent. Thus, $A$ is unitarily irreducible. The remaining eigenvalues of $H$ are 0 and 1, both having multiplicity two. It is easy to check that $K|\ker H = K|\ker(H-I) = 0$, which makes 0 and 1 multiply generated round points of $\partial F(A)$.

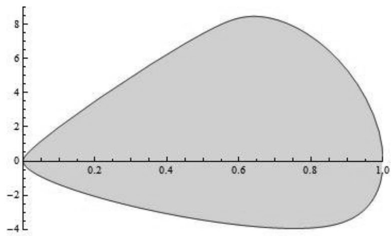


Figure 4: *$F(A)$ for Example 5.6. The points 0 and 1 are multiply generated.*

Note that in all three examples the matrices are of the form (5.1). Since they were shown to be unitarily irreducible, according to Lemma 5.2 at least one of the blocks $A_1, A_2$ in each of them should be unitarily irreducible as well. In agreement with this, both blocks are unitarliy irreducible in Examples 5.4 and 5.6, while in Example 5.5 block $A_2$ is diagonal (and thus unitarily reducible in a trivial way). Note also that the supporting lines coincide with the real and imaginary axes in Examples 5.4, 5.5 (and thus are not parallel) and are both vertical in Example 5.6.

So, there exist unitarily irreducible $A \in M_5(\mathbb{C})$ with two multiply generated round points on $\partial F(A)$. In fact, this bound is sharp:

THEOREM 5.7. *A 5-by-5 matrix $A$ with more than two multiply generated round points on $\partial F(A)$ is unitarily reducible.*

We precede the proof of Theorem 5.7 by two technical lemmas, which also provide additional useful information.

LEMMA 5.8. *Let $A \in M_5(\mathbb{C})$ have two multiply generated round points $z_1, z_2 \in \partial F(A)$ lying on parallel supporting lines, and yet another exceptional supporting line $\ell$. Then either $\ell$ contains a flat portion of $\partial F(A)$ with singularly generated endpoints, or $A$ is unitarily reducible.*

*Proof.* The case of $z_1, z_2$ lying on the same supporting line is covered by Theorem 5.3 (and the presence of $\ell$ is then irrelevant). So, let these supporting lines be different.

Passing from $A$ to its affine equivalent, we may without loss of generality suppose that the multiply generated round boundary points lie on vertical supporting lines, the left of them coinciding with the imaginary axis:

$$z_1 = i\lambda, \quad z_2 = \xi + i\mu,$$

and that $\ell$ is the real axis, supporting $F(A)$ from below.

A subsequent unitary transformation, existing due to Lemma 5.1, allows then to represent $A$ in the form

$$A = H + iK = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & x \end{pmatrix} + i \begin{pmatrix} \lambda & d_1 & 0 & 0 & y_1 \\ d_1 & \mu & 0 & 0 & y_2 \\ 0 & 0 & \lambda & d_2 & y_3 \\ 0 & 0 & d_2 & \mu & y_4 \\ \overline{y_1} & \overline{y_2} & \overline{y_3} & \overline{y_4} & y \end{pmatrix}.$$

Note that the off diagonal entries in the last row and column of $H$ equal zero because $0 \leqslant H \leqslant \xi I$.

If $\lambda = 0$ or $\mu = 0$, then $z_1$ (respectively, $z_2$) is a corner point of $F(A)$, so $A$ is reducible. Thus, we may suppose $\lambda, \mu \neq 0$. The kernel of $K$ is at least two-dimensional, so the rank of $K$ does not exceed 3. Computing the row echelon form of $K$, we conclude from here that at least one of the $d_j$ must equal $\sqrt{\lambda\mu}$, and for the respective $j$, $y_{2j} = y_{2j-1}\sqrt{\mu/\lambda}$.

The rest of the reasoning depends on the number of $j$'s for which $d_j = \sqrt{\lambda\mu}$.

*Case* 1. $d_j = \sqrt{\lambda\mu}$ *for both* $j = 1,2$. Then

$$K = \begin{pmatrix} \lambda & \sqrt{\lambda\mu} & 0 & 0 & y_1 \\ \sqrt{\lambda\mu} & \mu & 0 & 0 & y_1\sqrt{\mu/\lambda} \\ 0 & 0 & \lambda & \sqrt{\lambda\mu} & y_3 \\ 0 & 0 & \sqrt{\lambda\mu} & \mu & y_3\sqrt{\mu/\lambda} \\ \overline{y_1} & \overline{y_1}\sqrt{\mu/\lambda} & \overline{y_3} & \overline{y_3}\sqrt{\mu/\lambda} & y \end{pmatrix}.$$

Let now $W$ be the subspace spanned by the vectors $v_1 = -\overline{y_3}e_1 + \overline{y_1}e_3$ and $v_2 = -\overline{y_3}e_2 + \overline{y_1}e_4$. Clearly, $W$ is invariant under $H$: the first vector is in $\ker H$ and the second is a $\xi$-eigenvector. But it is also invariant under $K$:

$$Kv_1 = \lambda v_1 + \sqrt{\lambda\mu}v_2, \quad Kv_2 = \sqrt{\lambda\mu}v_1 + \mu v_2.$$

Since $H$ and $K$ share a non-trivial proper subspace, $A$ is unitarily reducible.

*Case* 2. $d_j = \sqrt{\lambda\mu}$ *for exactly one value of* $j$. Without loss of generality (at a cost of a permutational similarity) let $d_1 = \sqrt{\lambda\mu}$ while $d := d_2 \neq \sqrt{\lambda\mu}$.

In this case condition $\dim\ker K \geqslant 2$ yields

$$y = \frac{1}{\lambda}\left(|y_1|^2 + |y_3|^2 + \frac{|\lambda y_4 - dy_3|^2}{\lambda\mu - d^2}\right). \tag{5.5}$$

Moreover, then $\dim\ker K = 2$ and the vectors

$$w_1 = \begin{bmatrix} -\sqrt{\mu/\lambda} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} -\frac{y_1}{\lambda+\mu} \\ -\frac{y_1}{\lambda+\mu}\sqrt{\mu/\lambda} \\ \frac{dy_4 - \mu y_3}{\lambda\mu - d^2} \\ \frac{dy_3 - \lambda y_4}{\lambda\mu - d^2} \\ 1 \end{bmatrix}$$

form an orthogonal basis of $\ker K$. Observe now that $\langle Hw_1, w_2\rangle = -\xi \frac{y_1}{\lambda+\mu}\sqrt{\mu/\lambda}$. So, $y_1 \neq 0$ implies that $H|\ker K$ is not a scalar multiple of the identity, yielding the flat portion of $\partial F(A)$ on $\ell$. The endpoints of this flat portion are singularly generated, because $\dim\ker K = 2$.

On the other hand, $y_1 = 0$ implies $y_2 = 0$, thus making the span of $\{e_1, e_2\}$ a common invariant subspace of $H$ and $K$. The matrix $A$ is then unitarily reducible. □

LEMMA 5.9. *Let $A \in M_5(\mathbb{C})$ be unitarily irreducible, with two multiply generated round points $z_1, z_2 \in \partial F(A)$ lying on non-parallel supporting lines $\ell_1, \ell_2$. Then there is at most one more additional direction of an exceptional supporting line.*

*Proof.* Passing from $A$ to its affine equivalent we will not change the configuration and number of multiply generated round points or flat portions on $\partial F(A)$. Unitary irreducibility of $A$ also is invariant under such transformations. So, we may without loss of generality suppose that $F(A)$ lies in the first quadrant and given multiply generated boundary points are located on the axes equidistantly from the origin.

Invoking then Lemma 5.1, we may further suppose without loss of generality that $A$ is of the form (5.1). If both blocks $A_1, A_2$ are unitarily reducible, then $A$ is also unitarily reducible (Lemma 5.2), and we are done. In the remaining situation, one of the sets $F(A_1), F(A_2)$ is an ellipse $E$ while the other is either a line segment connecting two points $z_1, z_2 \in \partial E$ or yet another ellipse, tangent to $E$ at these two points. Since two ellipses having two tangent points cannot intersect elsewhere, one of the sets $F(A_j)$ is a subset of another. Using a permutational similarity if needed, we may suppose that $F(A_1) \supseteq F(A_2)$.

Observe now that the centers of $F(A_j)$ are located on the bisector of the first quadrant, that is, can be written as $q_j(1+i)$ with $q_j > 0$. Moreover, the matrices $\operatorname{Re} A_j$, $\operatorname{Im} A_j$ must be singular, since the axes are tangent to $F(A_j)$, $j = 1,2$. Using unitary transformations to put $A_j$ in a constant diagonal form, we arrive at the representations

$$\operatorname{Re} A_j = q_j \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \operatorname{Im} A_j = q_j \begin{pmatrix} 1 & \varepsilon_j \\ \overline{\varepsilon_j} & 1 \end{pmatrix}$$

with some unimodular $\varepsilon_j$, $j = 1,2$. Along with the additional relations imposed on the 5th column of $A$ by conditions (5.2), we finally arrive at

$$A = H + iK = \begin{pmatrix} q_1 & q_1 & 0 & 0 & x_1 \\ q_1 & q_1 & 0 & 0 & x_1 \\ 0 & 0 & q_2 & q_2 & x_2 \\ 0 & 0 & q_2 & q_2 & x_2 \\ x_1 & x_1 & x_2 & x_2 & x_3 \end{pmatrix} + i \begin{pmatrix} q_1 & \varepsilon_1 q_1 & 0 & 0 & y_1 \\ \overline{\varepsilon_1} q_1 & q_1 & 0 & 0 & \overline{\varepsilon_1} y_1 \\ 0 & 0 & q_2 & \varepsilon_2 q_2 & y_2 \\ 0 & 0 & \overline{\varepsilon_2} q_2 & q_2 & \overline{\varepsilon_2} y_2 \\ \overline{y_1} & \varepsilon_1 \overline{y_1} & \overline{y_2} & \varepsilon_2 \overline{y_2} & y_3 \end{pmatrix}. \tag{5.6}$$

(Note that the off diagonal elements in the 5th row and column of $H$ can be made real via an additional diagonal unitary similarity, not changing the left upper 4-by-4 block of $H$ and $K$.)

If $x_j = y_j = 0$ for $j = 1$ or $2$, then the respective $A_j$ becomes a direct summand of $A$, making $A$ unitarily reducible in a trivial way. So,

$$\text{At least one of the entries } x_j \text{ and } y_j \text{ differs from zero for each } j = 1,2. \tag{5.7}$$

By Theorem 2.1, there is a one-to-one correspondence between the directions of additional exceptional supporting lines, not parallel to the axes, and real $t \neq 0$ such that the matrix

$$tH + K = \begin{pmatrix} (t+1)q_1 & (t+\varepsilon_1)q_1 & 0 & 0 & tx_1 + y_1 \\ (t+\overline{\varepsilon_1})q_1 & (t+1)q_1 & 0 & 0 & tx_1 + \overline{\varepsilon_1}y_1 \\ 0 & 0 & (t+1)q_2 & (t+\varepsilon_2)q_2 & tx_2 + y_2 \\ 0 & 0 & (t+\overline{\varepsilon_2})q_2 & (t+1)q_2 & tx_2 + \overline{\varepsilon_2}y_2 \\ * & * & * & * & tx_3 + y_3 \end{pmatrix} \tag{5.8}$$

has a multiple maximal or minimal eigenvalue $\lambda$. Being multiple, $\lambda$ is thus an eigenvalue of the left upper 4-by-4 block

$$Z = Z_1 \oplus Z_2 := q_1 \begin{pmatrix} t+1 & t+\varepsilon_1 \\ t+\overline{\varepsilon_1} & t+1 \end{pmatrix} \oplus q_2 \begin{pmatrix} t+1 & t+\varepsilon_2 \\ t+\overline{\varepsilon_2} & t+1 \end{pmatrix} \tag{5.9}$$

of (5.8).

The way we proceed from here varies, depending on whether the sets $F(A_j)$ coincide.

*Case* 1. *$F(A_2)$ is a proper subset of $F(A_1)$.* In particular, $A_1$ is not normal, that is, $\varepsilon_1$ is not real. Moreover, since two ellipses tangent at two points do not intersect elsewhere, we have

$$\partial F(A_1) \cap \partial F(A_2) = \{z_1, z_2\}. \tag{5.10}$$

Due to (5.10), $\lambda$ is then a simple eigenvalue of $Z_1$, and not an eigenvalue of $Z_2$. Consequently, diagonalizing $Z_1$ forces the respective element in the 5-th column of (5.8) to vanish. Equivalently,

$$[tx_1 + \varepsilon_1\overline{y_1}, -(tx_1 + \overline{y_1})]^T \tag{5.11}$$

is an eigenvector of $Z_1$ corresponding to the eigenvalue $\lambda$. A direct computation then yields

$$(t+\varepsilon_1)(tx_1 + \overline{y_1})^2 = (t+\overline{\varepsilon_1})(tx_1 + \varepsilon_1\overline{y_1})^2. \tag{5.12}$$

If one (and then, according to (5.7), exactly one) of the entries $x_1, y_1$ equals zero, then (5.12) implies $\varepsilon_1 \in \mathbb{R}$, which is not true. So, $x_1 y_1 \neq 0$. Moreover, from (5.12) it follows that

$$|tx_1 + \overline{y_1}| = |tx_1 + \overline{\varepsilon_1}y_1|.$$

Once again using $\varepsilon_1 \notin \mathbb{R}$, we conclude

$$\varepsilon_1\overline{y_1} = y_1. \tag{5.13}$$

With this in mind, (5.12) can be rewritten as

$$(t+\varepsilon_1)\zeta \in \mathbb{R}, \tag{5.14}$$

where

$$\zeta = (tx_1+\overline{y_1})/(tx_1+y_1).$$

Represent $y_1/x_1$ in the polar form as $s\omega$, where $s>0$ and $|\omega|=1$. Then

$$\zeta = (t+s\overline{\omega})/(t+s\omega), \tag{5.15}$$

while (5.13) means simply that $\varepsilon_1=\omega^2$. In particular, $\operatorname{Im}\omega\neq 0$. Condition (5.14) is thus equivalent to

$$t = -\operatorname{Im}(\omega^2\zeta)/\operatorname{Im}\zeta \tag{5.16}$$

(note that $\operatorname{Im}\zeta\neq 0$ since otherwise (5.13) would imply $\varepsilon_1\in\mathbb{R}$). Plugging (5.15) into (5.16) and canceling the joint multiple $4it\operatorname{Im}\omega\neq 0$ yields

$$t = s\frac{s\operatorname{Re}\omega-1}{\operatorname{Re}\omega-s}.$$

Thus, there is at most one additional potential direction of an exceptional supporting line.

*Case* 2. $F(A_1)=F(A_2)$ *is a non-degenerate ellipse.* Then in (5.6), and thus in (5.8), we have

$$q_1=q_2=:q, \quad \varepsilon_1=\varepsilon_2=:\varepsilon\notin\mathbb{R}. \tag{5.17}$$

So, in (5.9), $Z_1=Z_2$. We will show that the existence of an additional exceptional supporting line $\ell$ then implies the unitary reducibility of $A$.

Suppose such $\ell$ exists. Diagonalizing for the respective value of $t$ each of $Z_j$ to $\operatorname{diag}[\lambda,\mu]$, $\lambda\neq\mu$, via the same 2-by-2 unitary similarity will force both $(1,5)$ and $(3,5)$ entries of $tH+K$ to disappear, due to the semi-definiteness of $tH+K-\lambda I$. This is only possible if

$$\frac{tx_1+y_1}{tx_1+\overline{\varepsilon}y_1}=\frac{tx_2+y_2}{tx_2+\overline{\varepsilon}y_2}.$$

A direct computation shows that then

$$x_1y_2=x_2y_1. \tag{5.18}$$

Let now

$$v_1=[1,0,s,0,0]^T, \quad v_2=[0,1,0,s,0]^T,$$

where

$$s=\begin{cases}-x_1/x_2 & \text{if } x_1\neq 0,\\ -y_1/y_2 & \text{if } x_2=0, y_2\neq 0.\end{cases}$$

From (5.6), taking (5.17) and (5.18) into account, we conclude:

$$Hv_1=Hv_2=q(v_1+v_2), \quad Kv_1=q(v_1+\overline{\varepsilon}v_2),\ Kv_2=q(\varepsilon v_1+v_2).$$

In other words, the span of $v_1, v_2$ is an invariant subspace both for $H$ and $K$, and thus a reducing subspace for $A$. $\square$

*Proof of Theorem* 5.7. If two multiply generated round points have parallel (in particular, coinciding) supporting lines, there cannot be any other multiply generated round points by Lemma 5.8.

It remains thus to consider the case when $\partial F(A)$ contains (at least) three multiply generated round points, with pairwise non-parallel supporting lines. If $A$ is unitarily irreducible, the situation reduces to that of Lemma 5.9. In particular, $A$ is of the form (5.6), two of the multiply generated round points lie on the axes, and the third one is the value $z$ of $f_A$ on the unit sphere of $L = \ker(tH + K - \lambda I)$, where $\lambda$ is an eigenvalue of $Z_1$ from (5.9). An additional piece of information, compared with the setting of Lemma 5.9, is that now the compression of $A$ onto $L$ must equal $zI$.

Now observe that the eigenvector (5.11) of $Z_1$ differs only by a scalar multiple from $[1, -\zeta]^T$. So, the unitary 2-by-2 matrix $U$ diagonalizing $Z_1$ can be chosen as

$$U = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -\zeta & \zeta \end{pmatrix}.$$

Let $W = U \oplus I_3$. A direct computation shows that $W^*(tH+K)W - \lambda I_5 = (0) \oplus M$, where

$$M = \begin{pmatrix} 2\zeta(t+\varepsilon_1)q_1 & 0\ 0 & \sqrt{2}(tx_1+y_1) \\ 0 & \multirow{2}{*}{$Z_2 - \lambda I_2$} & tx_2 + y_2 \\ 0 & & tx_2 + \overline{\varepsilon_2} y_2 \\ * & *\ * & tx_3 + y_3 - \lambda \end{pmatrix}.$$

Since $Z_2 - \lambda I$ is invertible, $\ker M$ is (at most) one dimensional. On the other hand, $\dim L = \dim\ker M + 1$, so $\ker M$ must be non-trivial in order for $z$ to be a multiply generated point.

Thus, $L = W\operatorname{span}\{e_1, v\}$, with $v = [0, v_1, v_2, v_3, v_4]^T$ and $[v_1, v_2, v_3, v_4]^T$ being a non-zero vector in $\ker M$. From the structure of $M$ it follows that all $v_j$ are non-zero and, moreover, $v_1 = -\frac{\sqrt{2}(tx_1+y_1)}{2\zeta(t+\varepsilon_1)q_1} v_4$.

The condition $A|L = zI$ implies in particular that $\langle HWv, We_1\rangle = 0$. But

$$\langle HWv, We_1\rangle = \langle W^*HWv, e_1\rangle = \frac{\sqrt{2}}{2\zeta} v_4(\overline{y_1} - y_1)\left(\frac{x_1}{tx_1+y_1} - \frac{tx_1 + \operatorname{Re} y_1}{(tx_1+\overline{y_1})(t+\varepsilon_1)}\right).$$

Observe that $y_1 \notin \mathbb{R}$, since otherwise (5.13) would imply $\varepsilon_1 = 1$. So,

$$\frac{x_1}{tx_1+y_1} = \frac{tx_1 + \operatorname{Re} y_1}{(tx_1+\overline{y_1})(t+\varepsilon_1)},$$

or equivalently

$$t(g + i\operatorname{Im} g - \varepsilon_1) = \overline{g}\varepsilon_1 - g\operatorname{Re} g,$$

where $g = y_1/x_1$. Due to (5.13) we may replace $\overline{g}\varepsilon_1$ by $g$, so that finally

$$t(g + i\operatorname{Im} g - \varepsilon_1) = g(1 - \operatorname{Re} g). \tag{5.19}$$

If $\operatorname{Re} g = 1$, then the right hand side of (5.19) is zero. Since $t \neq 0$, from here we obtain $g + i\operatorname{Im} g - \varepsilon_1 = 0$. Taking the real part and recalling that $\operatorname{Re} g = 1$ yields $\operatorname{Re}\varepsilon_1 = 1$. But $|\varepsilon_1| = 1$, so in fact $\varepsilon_1 = 1$ — a contradiction.

If $\operatorname{Re} g \neq 1$, then in order for the (unique) solution of (5.19) to be real we must have $(g + i\operatorname{Im} g - \varepsilon_1)\overline{g} \in \mathbb{R}$. From here, using $\varepsilon_1 \overline{g} = g$ again: $i\overline{g}\operatorname{Im} g - g \in \mathbb{R}$. Since $\operatorname{Re} g \neq 1$, this is only possible if $g$ is real. But then again $\varepsilon_1 = 1$, also a contradiction. $\square$

EXAMPLE 5.10. It is possible for a unitarily irreducible 5-by-5 matrix to have two multiply generated round boundary points and a flat portion not parallel to either one of them. Take $H$ to be

$$\begin{pmatrix} 3 & 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & \frac{1}{120}\left(1675 - 263\sqrt{30}\right) \end{pmatrix}$$

and $K$ to be

$$\begin{pmatrix} 3 & 2 - i\sqrt{5} & 0 & 0 & -\frac{5 - i\sqrt{5} - i\sqrt{6} + \sqrt{30}}{6+\sqrt{30}} \\ 2 + i\sqrt{5} & 3 & 0 & 0 & -\frac{5 + i\sqrt{5} + i\sqrt{6} + \sqrt{30}}{6+\sqrt{30}} \\ 0 & 0 & 1 & -i & -\frac{5}{2} \\ 0 & 0 & i & 1 & -\frac{5i}{2} \\ -\frac{5 + i\sqrt{5} + i\sqrt{6} + \sqrt{30}}{6+\sqrt{30}} & -\frac{5 - i\sqrt{5} - i\sqrt{6} + \sqrt{30}}{6+\sqrt{30}} & -\frac{5}{2} & \frac{5i}{2} & 8 \end{pmatrix}$$

We compute $\sigma(H) \approx \{6.445, 2, 1.509, 0, 0\}$ and $\sigma(K) \approx \{10.047, 5.653, .299, 0, 0\}$. It is easy to compute $\ker H$, $\ker K$ and verify that $H|\ker K = I_2$, whereas $K|\ker H = iI_2$. Thus, 1 and $i$ are multiply generated boundary round points. We can also compute $\sigma(H+K) = \{6+\sqrt{30}, 6+\sqrt{30}, 2.233, .534, .232\}$. Since the largest eigenvalue of $H+K$ is repeated, $\theta = \pi/4$ is an exceptional angle for $A$. Two linearly independent $(6+\sqrt{30})$-eigenvectors of $H+K$ can be chosen as

$$v_1 = \begin{pmatrix} \frac{1}{30}\left(-5 + i\sqrt{5} + \sqrt{30}\right) \\ 0 \\ \frac{-5((5+i)+\sqrt{30})}{88+16\sqrt{30}} \\ \frac{-5i((5-i)+\sqrt{30})}{88+16\sqrt{30}} \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{-i+\sqrt{5}}{\sqrt{6}} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Observe that $\langle (K-H)v_2, v_2\rangle = 0$ while $\langle (K-H)v_1, v_1\rangle = \frac{-17}{6} + \frac{263}{4\sqrt{30}} \neq 0$. So, the compression of $K-H$ onto the $(6+\sqrt{30})$-eigenspace of $H+K$ is not a scalar multiple of the identity, and thus there is a flat portion of $\partial F(A)$ on the supporting line $\ell_\theta$.

To show that $H+iK$ is unitarily irreducible, consider $v=[r,r,0,0,1]^T$, where

$$r=\frac{1}{480}\left(-955+263\sqrt{30}+\sqrt{3102295-502330\sqrt{30}}\right).$$

This is an eigenvector of $H$ corresponding to its simple eigenvalue

$$\frac{1}{240}\left(2395-263\sqrt{30}+\sqrt{3102295-502330\sqrt{30}}\right)$$

Yet another computation yields

$$\det\left(v\ Kv\ HKv\ KHKv\ HKHKv\right)\approx -1469.46i\neq 0.$$

Thus we have a linearly independent set, so $H+iK$ is unitarily irreducible.

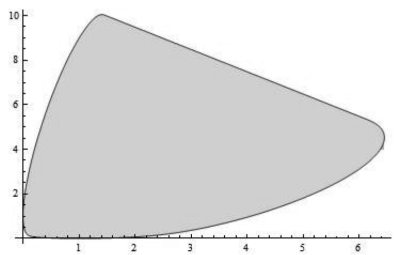


Figure 5: *$F(A)$ for example 5.10. The points 1 and $i$ are multiply generated, and there is a flat portion forming an angle of $\frac{3\pi}{4}$ with the positive real axis.*

Armed with Theorems 4.4 and 5.7, we can easily give a complete description of the multiply generated round boundary points of $\partial F(A)$ for all $A\in M_5(\mathbb{C})$, unitarily reducible or not.

THEOREM 5.11. *For $A\in M_5(\mathbb{C})$, the set of multiply generated round points consists of $0,1,2,3$ elements, is an elliptical arc, or the whole $\partial F(A)$, which in the latter case must be an ellipse.*

*Proof. Case* 1. *$A$ is unitarily irreducible.* The result follows directly from Theorems 5.7.

*Case* 2. *$A\cong B\oplus(z)$ for some $B\in M_4(\mathbb{C})$.* Denote by $\mathscr{M}(X)$ the set of all multiply generated round boundary points for the matrix $X(=A,B)$. Then $\mathscr{M}(A)=\mathscr{M}(B)$ if $z$ lies in the interior of $F(B)$, $\mathscr{M}(A)=\mathscr{M}(B)\cup\{z\}$ if $z\in\partial F(B)$, and $\mathscr{M}(A)$ coincides with the exposed portion of $\mathscr{M}(B)$ in $\operatorname{conv}\{F(B),z\}$ if $z\notin F(B)$. It remains to invoke Theorem 4.4.

*Case* 3. *$A$ is unitarily reducible but has no one-dimensional blocks.* This can only happen if $A\cong B_1\oplus B_2$ with some unitarily irreducible $B_1\in M_3(\mathbb{C})$, $B_2\in M_2(\mathbb{C})$. Then both $B_1$ and $B_2$ have no multiply generated round boundary points, according to

Theorem 3.2. Thus, multiply generated round boundary points of $A$ can possibly occur only where $\partial F(B_1)$ and $\partial F(B_2)$ are tangent. Note that $\Gamma := \partial F(B_2)$ is an ellipse.

In order to achieve the maximal number of tangent points, $\Gamma$ should be either subscribed about or inscribed in the curve $\partial F(B_1)$. In the former case, Anderson's theorem (see, e.g., [25] or Lemma 6 in [23] and historical comments therein) implies that either $\partial F(B_1) \cap \partial F(B_2)$ consists of at most two points or the two sets coincide. On the other hand, in the latter case Theorem 2.5 from [8] guarantees that either again $\partial F(B_1) \cap \partial F(B_2)$ consists of at most two points or $\partial F(B_1)$ contains an elliptical arc. Based on Kippenhahn's classification of the numerical ranges of $3\times 3$ unitarily irreducible matrices ([14], see also [13]), if $\partial F(B_1)$ contains an elliptical arc, it **is** an ellipse, that is, $F(B_1) = F(B_2)$. $\square$

## 6. Beyond $M_5(\mathbb{C})$

Based on the consideration of $n = 3,4,5$ cases above, it is natural to guess that an $n$-by-$n$ unitarily irreducible matrix $A$ can have not more than $n-3$ multiply generated round boundary points. However, starting with $n = 6$ this is no longer the case. This phenomenon is closely related with the failure of Kippenhahn's conjecture (see, e.g., [15, 21, 20] and references therein), also starting with $n = 6$. We adapt the counterexample from [15, Theorem 3.2] to our purposes of studying multiply generated boundary round points.

EXAMPLE 6.1. Let

$$A = \begin{pmatrix} 0 & x & 0 & cy & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -cx & 0 & \sqrt{1-c^2}\xi & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.1}$$

where $x, y, \xi, \eta, c > 0$, $x^2 + y^2 = \xi^2 + \eta^2 = 4$, $c < 1$.

According to [15], $F(A)$ is the unit disk and $A$ is unitarily irreducible. Also, the eigenvalues of $\mathrm{Re}(e^{-i\theta}A)$ do not depend on $\theta$ and equal $\pm 1$ (each having multiplicity 2) and $\pm c\eta/2$. So, each point of $\partial F(A)$ is multiply generated (and they are all round).

## 7. Tridiagonal matrices

Imposing additional algebraic structure on $A$ may force the number of multiply generated boundary points to be finite, independent of the matrix size. We consider here the so called *tridiagonalizable* matrices, that is, matrices unitarily similar to tridiagonal ones.

In its turn, a matrix $A \in M_n(\mathbb{C})$ is *tridiagonal* if $a_{ij} = 0$ whenever $|i-j| > 1$. To simplify the notation, we will record such matrices as

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & c_{n-1} & a_n \end{pmatrix}. \tag{7.1}$$

To put things in perspective, recall that for $n \leqslant 4$ all matrices $A \in M_n(\mathbb{C})$ are tridiagonalizable. This is trivial for $n = 1, 2$ and relatively easy for $n = 3$. For $n = 4$, the result was established only recently [18] and the proof uses some machinery from complex algebraic geometry. The existence of non-tridiagonalizable $n$-by-$n$ matrices with $n \geqslant 6$ was proved in [16, 22]. Later [6] it was established that, starting with $n = 5$, the non-tridiagonalizable matrices form a dense, second category subset of $M_n(\mathbb{C})$.

Following [2], we say that a tridiagonal matrix (7.1) is *improper* if it contains a zero pair of corresponding off-diagonal entries: $b_j = c_j = 0$ for some $j = 1, \ldots, n-1$, and *proper* otherwise. The flat portions on the boundary of $F(A)$ for proper tridiagonal matrices $A$ were described in [2, Theorem 10]. As part of the proof, it was shown there that the eigenvalues of $\mathrm{Re}(e^{-i\theta}A)$ are simple, unless for some $j$ either $b_j = c_j = 0$, or

$$|b_j| = |c_j| \neq 0, \theta \equiv \frac{1}{2}(\arg b_j + \arg c_j) + \frac{\pi}{2} \mod \pi \tag{7.2}$$

From here and Theorem 2.1 we immediately obtain:

LEMMA 7.1. *A tridiagonal matrix* (7.1) *is generic if* $|b_j| \neq |c_j|$ *for all* $j = 1, \ldots, n-1$. *If this is not the case but* (7.1) *is proper, then its exceptional angles are contained among* $\frac{1}{2}(\arg b_j + \arg c_j) \pm \frac{\pi}{2}$ *where* $j$ *is such that* $|b_j| = |c_j|$ *holds.*

In particular, any Jordan block $J$ is generic. It is well known that $F(J)$ is a circular disk; Lemma 7.1 implies that in addition all the points of $\partial F(J)$ are singularly generated.

THEOREM 7.2. *A proper tridiagonal matrix* (7.1) *has at most* $n-2$ *multiply generated boundary round points, and only* $\lfloor \frac{n-1}{2} \rfloor$ *of them can be located on pair-wise non-parallel supporting lines.*

*Proof.* For an exceptional $\theta$, denote by $m$ the number of $j$'s satisfying (7.2) and say that they form the cluster $J_\theta$ corresponding to $\theta$. Then $\mathrm{Re}(e^{-i\theta}A)$ is the direct sum of $m+1$ proper tridiagonal Hermitian (and thus normal) matrices, say $H_1, \ldots, H_m$. By [2, Corollary 7], each of them has simple eigenvalues while their eigenvectors have non-zero first and last entries. Denote by $\mu$ the minimal eigenvalue of $\mathrm{Re}(e^{-i\theta}A)$, and let $H_{k_1}, \ldots, H_{k_s}$ be the blocks $H_k$ having $\mu$ as their (also minimal) eigenvalue; according to Theorem 2.1 we must have $s \geqslant 2$ for a multiply generated boundary round point or flat portion to appear. Finally, let $x_1,\dots,x_s$ be the respective unit eigenvectors which thus form an orthonormal basis of the eigenspace $L$ of $\mathrm{Re}(e^{-i\theta}A)$ corresponding to the eigenvalue $\mu$.

The compression $K_0$ of $\mathrm{Im}(e^{-i\theta}A)$ onto $L$ has a tridiagonal matrix the $(i,i+1)$-entry of which is non-zero if and only if the blocks $H_{k_i},H_{k_{i+1}}$ are contiguous in $\mathrm{Re}(e^{-i\theta}A)$.

Since $s$ does not exceed $m+1$, in the case $m=1$ we are forced to have $s=m+1=2$ which in turn implies the contiguity of $H_1,H_2$. Thus, $K_0$ is not a scalar multiple of the identity and $\ell_\theta$ contains a flat portion of $\partial F(A)$ but no multiply generated round points.

So, we must have $m\geqslant 2$ in order for multiply generated round boundary points to materialize on the supporting lines with a given slope. This proves that there are at most $\lfloor\frac{n-1}{2}\rfloor$ clusters $J_\theta$ which can correspond to multiply generated boundary round points.

If $m\geqslant 2$, then a multiply generated boundary round point potentially may materialize. There are never more than two such points on the same line (since they then have to be endpoints of a flat portion), and in order for there to be two, $K_0$ has to have two multiple eigenvalues. Thus, in this case $s\geqslant 4$. Moreover, the blocks $H_{k_1},\dots,H_{k_s}$ cannot all be contiguous, since otherwise the matrix of $K_0$ would be proper and therefore could not have multiple eigenvalues. So, $s<m+1$, implying that we must have $m\geqslant 4$ if there are two multiply generated round points on at least one supporting line with the given slope.

The conclusion is that we need $m=|J_\theta|\geqslant k$ in order to have $k$ multiply generated round boundary points on the supporting lines with the slope $-\cot\theta$, and the equality is possible only if $k=2$ or $4$. So, the total number of multiply generated round boundary points is strictly less than the total number of indices $\{1,\dots,n-1\}$ (and thus does not exceed $n-2$), unless the clusters form a partition of $\{1,\dots,n-1\}$ with each $J_{\theta_k}$ of this partition containing exactly 2 or 4 elements, with exactly $\left|J_{\theta_k}\right|$ multiply generated round boundary points lying on the supporting lines with the slope $-\cot\theta_k$. This completes the proof for $n$ even. For $n$ odd it is needed to observe in addition that the cluster $J_{\theta_0}$ containing 1 or $n-1$ corresponds to only one supporting line and therefore generates less than $\left|J_{\theta_0}\right|$ points. $\square$

COROLLARY 7.3. *For $n$ odd, let $A\in M_n(\mathbb{C})$ be a proper tridiagonal matrix with $(n-1)/2$ multiply generated round points located on pair-wise non-parallel supporting lines. Then $\partial F(A)$ contains at most $(n-3)/2$ flat portions, and each of them is parallel to one of these lines.*

Indeed, for such $A$ the set $\{1,\dots,n-1\}$ must be partitioned in exactly $(n-1)/2$ clusters, each consisting of two indices and corresponding to one multiply generated round portion. The flat portions, if they exist, must therefore correspond to some of these clusters and thus be parallel to the respective supporting lines. Moreover, the cluster(s) containing 1 and $n-1$ cannot generate additional exceptional supporting lines.

EXAMPLE 7.4. In general, tridiagonal matrices (proper or not) can achieve $n-2$ multiply generated boundary round points: a matrix of the form

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \oplus \bigoplus_{k=0}^{n-3} e^{2\pi ik/(n-2)} \tag{7.3}$$

(which is a slight modification of Example 38 from [2], suggested by C. K. Li) has $n-2$ multiply generated boundary round points. Even if the matrix in question is required to be unitarily similar to a proper tridiagonal matrix, examples still persist. Consider, e.g., the matrix (7.3) for $n=3$, and $U = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $U^*AU = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$, a proper tridiagonal matrix.

Since every 4-by-4 matrix is tridiagonalizable, Theorem 7.2 provides an alternative justification for the Case 2 in the proof of Theorem 4.1. Note however that the one given there is more elementary and direct.

On the other hand, from Corollary 7.3 it is clear that for a tridiagonal $A \in M_5(\mathbb{C})$ with two multiply generated round boundary points lying on non-parallel supporting lines a flat portion of $\partial F(A)$, if it exists, must be parallel to one of these supporting lines. Thus, the matrix from Example 5.10 is not tridiagonalizable. We are not aware of other constructive examples of such kind. Of course, the matrices $A \in M_6(\mathbb{C})$ from Example 6.1 are not tridiagonalizable either.

## 8. Critical curves and multiply generated boundary points

In this section, we offer an alternative characterization of the multiply generated boundary points of $F(A)$.

For $A \in M_n(\mathbb{C})$, the matrix valued function $\operatorname{Re}(e^{-i\theta}A)$ is analytic in the real variable $\theta$. A result of Rellich [19] (see also [1, Section 3.5.4, Corollary 2]) implies that there is a family of analytic functions $x_k : [0,2\pi] \to \mathbb{C}^n$, $k \in \{1,\dots,n\}$, such that $\{x_1(\theta),\dots,x_n(\theta)\}$ is an orthonormal basis of eigenvectors of $\operatorname{Re}(e^{-i\theta}A)$ for each $\theta \in [0,2\pi]$. The corresponding eigenvalue functions $\lambda_k(\theta)$ are also analytic and the number of distinct eigenvalues $s \leqslant n$ remains constant for all but finitely many values of $\theta$ [1, Section 3.2.2, Theorem 2]. Those $\theta \in [0,2\pi]$ for which $\operatorname{Re}(e^{-i\theta}A)$ has less than $s$ distinct eigenvalues are called *exceptional points*.

For each $k \in \{1,\dots,n\}$, let $z_k(\theta) = \langle x_k(\theta), Ax_k(\theta)\rangle$. Each $z_k(\theta)$ is an analytic curve in $\mathbb{C}$, and we follow [12] in referring to these as the *critical curves* of $A$. A famous result of Kippenhahn [14] asserts that the numerical range of $A$ is the convex hull of a family of plane algebraic curves $C_A$. It is shown in [12] that the union of the curves $z_k(\theta)$ coincides with $C_A$.

A characterization of $z_k$ is given in [12]. It is shown there that

$$z_k(\theta) = e^{i\theta}(\lambda_k(\theta) + i\lambda_k'(\theta)), \tag{8.1}$$

where $\lambda_k'(\theta)$ denotes the derivative of $\lambda_k(\theta)$. Note that the boundary points of $F(A)$ correspond to the maximum and minimum eigenvalues $\lambda_k$. Multiply generated points on the boundary occur when two or more eigenvalues $\lambda_k$ coincide with $\max\lambda_k$ or $\min\lambda_k$, and have the same slope $\lambda_k'$. If the eigenvalue curves cross with differing slopes, then the angle $\theta$ where the intersection occurs corresponds to a flat portion of $\partial F(A)$.

Since $\mathrm{Re}(e^{-i(\theta+\pi)}A) = -\mathrm{Re}(e^{-i\theta}A)$, there is a permutation $\tau$ of the set $\{1,\dots,n\}$ such that $\lambda_k(\theta+\pi) = -\lambda_{\tau(k)}(\theta)$ and therefore $z_k(\theta+\pi) = z_{\tau(k)}(\theta)$ [7, Section 5]. For this reason, it is sufficient to look for intersections of the critical curves only on the interval $[0,\pi)$ (or on any other interval of length $\pi$).

PROPOSITION 8.1. *For $A \in M_n(\mathbb{C})$ and $z$ a round point of $\partial F(A)$, $z$ is multiply generated if and only if $z$ is contained in more than one critical curve $z_k(\theta)$ with domain $\theta \in [0,\pi)$.*

*Proof.* If $z$ is a multiply generated round boundary point, then the angle $\theta$ corresponding to $z$ must be exceptional by Theorem 2.1. From the proof of Theorem 2.1, it follows that there must be more than one eigenvector $x_k(\theta)$ of $\mathrm{Re}(e^{-i\theta}A)$ such that $z = f_A(x_k(\theta))$. Thus $z$ is contained in more than one curve $z_k$. The converse is true by definition. □

The following proposition is a restatement of Proposition 8.1 using (8.1).

PROPOSITION 8.2. *Let $A \in M_n(\mathbb{C})$ and suppose that $l_\theta$ is a supporting line of $F(A)$. If $l_\theta$ contains a multiply generated round boundary point, then there exists $k_1 \neq k_2 \in \{1,\dots,n\}$ such that $\min_k\{\lambda_k(\theta)\} = \lambda_{k_1}(\theta) = \lambda_{k_2}(\theta)$ and $\lambda_{k_1}'(\theta) = \lambda_{k_2}'(\theta)$. The supporting line $l_\theta$ contains a flat portion of $\partial F(A)$ if and only if there exists $k_1 \neq k_2 \in \{1,\dots,n\}$ such that $\min_k\{\lambda_k(\theta)\} = \lambda_{k_1}(\theta) = \lambda_{k_2}(\theta)$ and $\lambda_{k_1}'(\theta) \neq \lambda_{k_2}'(\theta)$.*

In Figure 8 we revisit Example 4.3. Here the supporting line $l_\pi$ contains a multiply generated point as the two minimum eigenvalue curves of $\mathrm{Re}(e^{-i\theta}A)$ intersect tangentially at $\theta = \pi/2$. The supporting line $l_{\pi/2}$ contains a flat portion of the boundary due to the crossing of the two minimal eigenvalue curves when $\theta = 0$.

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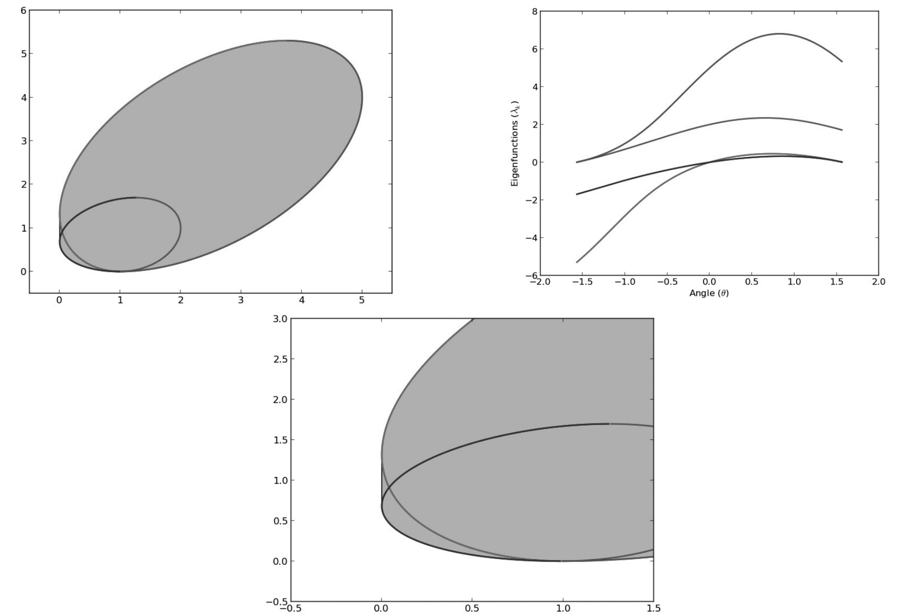


Figure 6: *Two views of the numerical range including critical curves and the corresponding $\lambda_k(\theta)$ for $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1+2i & -i/2 \\ 0 & 2 & -i/2 & 4+5i \end{pmatrix}$. The zoomed in view at the bottom shows a flat portion of the boundary.*

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