LINEAR TRANSFORMATIONS WITH CHARACTERISTIC
SUBSPACES THAT ARE NOT HYPERINVARIANT

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Abstract. If f is an endomorphism of a finite dimensional vector space over a field K then an invariant subspace $X \subseteq V$ is called hyperinvariant (respectively, characteristic) if X is invariant under all endomorphisms (respectively, automorphisms) that commute with f. According to Shoda (Math. Zeit. 31, 611–624, 1930) only if $|K| = 2$ then there exist endomorphisms f with invariant subspaces that are characteristic but not hyperinvariant. In this paper we obtain a description of the set of all characteristic non-hyperinvariant subspaces for nilpotent maps f with exactly two unrepeated elementary divisors.

1. Introduction

Let K be a field, V an n-dimensional vector space over K and $f : V \to V$ a K-linear map. A subspace $X \subseteq V$ is said to be *hyperinvariant* (under f) [9, p. 305] if it remains invariant under all endomorphisms of V that commute with f. If X is an f-invariant subspace of V and if X is invariant under all automorphisms of V that commute with f, then [1] we say that X is *characteristic* (with respect to f). Let $\mathrm{Inv}(V, f)$, $\mathrm{Hinv}(V, f)$, and $\mathrm{Chinv}(V, f)$ be sets of invariant, hyperinvariant and characteristic subspaces of V, respectively. These sets are lattices, and

$$\mathrm{Hinv}(V, f) \subseteq \mathrm{Chinv}(V, f) \subseteq \mathrm{Inv}(V, f).$$

If the characteristic polynomial of f splits over K (such that all eigenvalues of f are in K) then one can restrict the study of hyperinvariant and of characteristic subspaces to the case where f has only one eigenvalue, and to the case where f is nilpotent. Thus, throughout this paper we shall assume $f^n = 0$. Let $\Sigma(\lambda) = \mathrm{diag}(1, \ldots, 1, \lambda^{t_1}, \ldots, \lambda^{t_m}) \in K^{n \times n}[\lambda]$ be the Smith normal form of f such that $t_1 + \cdots + t_m = n$. We say that an elementary divisor λ^r is *unrepeated* if it appears exactly once in $\Sigma(\lambda)$.

The structure of the lattice $\mathrm{Hinv}(V, f)$ is well understood ([11], [6], [12], [9, p. 306]). We point out that $\mathrm{Hinv}(V, f)$ is the sublattice of $\mathrm{Inv}(V, f)$ generated by

$$\mathrm{Ker} f^k, \ \mathrm{Im} f^k, k = 0, 1, \ldots, n.$$

It is known ([13], [10, p. 63/64], [1]) that each characteristic subspace is hyperinvariant if $|K| > 2$. Hence, only if V is a vector space over the field $K = GF(2)$ one may find K-endomorphisms f of V with characteristic subspaces that are not hyperinvariant. A necessary and sufficient condition for the existence of such mappings f is due to Shoda (see also [3, Theorem 9, p. 510] and [10, p. 63/64]). It involves unrepeated elementary divisors of f.

THEOREM 1.1. [13, Satz 5, p. 619] *Let V be a finite dimensional vector space over the field $K = GF(2)$ and let $f : V \to V$ be nilpotent. The following statements are equivalent.*

(i) *There exists a characteristic subspace of V that is not hyperinvariant.*

(ii) *For some numbers R and S with $R+1 < S$ the map f has exactly one elementary divisor λ^R and exactly one of the form λ^S.*

Provided that f satisfies condition (ii) of Shoda's theorem how can one construct all characteristic subspaces of V that are not hyperinvariant? For the moment the answer to that question is open. In this paper we assume that f has exactly one pair of unrepeated elementary divisors. In that case we show how to construct the family of characteristic and non-hyperinvariant subspaces associated to f. For that purpose we prove rather general results on the structure of characteristic non-hyperinvariant subspaces and we clarify the role of unrepeated elementary divisors of f. We note that our study can be interpreted in the setting of module theory. In the context of abelian group theory [8] one would deal with characteristic subgroups of p-groups that are not fully invariant.

We first discuss an example, which displays features of characteristic subspaces that will become important later, and we introduce concepts that will allow us to state Theorem 1.3 at the end of this section.

1.1. An example and basic concepts

The inequality $R+1 < S$ in Theorem 1.1 is valid for $(R,S) = (1,3)$. In Example 1.2 below we describe a subspace that is characteristic but not hyperinvariant with respect to a map f with elementary divisors λ and λ^3. We first introduce some notation, in particular we define the concepts of exponent and height. We set $V[f^j] = \operatorname{Ker} f^j$, $j \geq 0$. Thus, $f^n = 0$ implies $V = V[f^n]$. Define $\iota = \operatorname{id}_V$ and $f^0 = \iota$. Let $x \in V$. The smallest nonnegative integer ℓ with $f^\ell x = 0$ is called the *exponent* of x. We write $e(x) = \ell$. A nonzero vector x is said to have *height* q if $x \in f^q V$ and $x \notin f^{q+1}V$. In this case we write $h(x) = q$. We set $h(0) = -\infty$. The n-tuple

$$H(x) = \big(h(x), h(fx), \ldots, h(f^{n-1}x)\big)$$

is the *indicator* [8, p. 3] or *Ulm sequence* [10] of x. Thus, if $e(x) = k$ then $H(x) = (h(x), \ldots, h(f^{k-1}x), \infty, \ldots, \infty)$. We say that $H(x)$ has a *gap* at j, if $1 \leq j < e(x)$ and $h(f^j x) > 1 + h(f^{j-1}x)$. Let $\operatorname{End}(V, f)$ be the algebra of all endomorphisms of V that

commute with f. The group of automorphisms of V that commute with f will be denoted by $\text{Aut}(V, f)$. Let

$$\langle x \rangle = \text{span}\{f^i x, i \geqslant 0\} = \{c_0 x + c_1 f x + \cdots + c_{n-1} f^{n-1} x; c_i \in K, i = 0, 1, \ldots, n-1\}$$

be the f-cyclic subspace generated by x. If $B \subseteq V$ we define $\langle B \rangle = \sum_{b \in B} \langle b \rangle$ and

$$B^c = \langle \alpha b; b \in B, \alpha \in \text{Aut}(V, f) \rangle.$$

We call B^c the *characteristic hull* of B. Clearly, if $\alpha \in \text{Aut}(V, f)$ then $\alpha(f^j x) = f^j(\alpha x)$ for all $x \in V$. Hence it is obvious that

$$e(\alpha x) = e(x) \quad \text{and} \quad h(\alpha x) = h(x) \quad \text{for all} \quad x \in V, \alpha \in \text{Aut}(V, f). \tag{1}$$

Let $e_1 = (1, 0, \ldots, 0)^T, \ldots, e_m = (0, \ldots, 0, 1)^T$ be the unit vectors of K^m, and let N_m denote the lower triangular nilpotent $m \times m$ Jordan block.

EXAMPLE 1.2. [10, p. 63/64]) Let $K = GF(2)$. Consider $V = K^4$ and let $f : V \to V$ be K-linear with elementary divisors λ and λ^3. With respect to the basis $\{e_i\}$, $i = 1, \ldots, 4$, the map f is given by $fx = Nx$ with

$$N = \text{diag}(N_1, N_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and $V = \langle e_1 \rangle \oplus \langle e_2 \rangle$ with $e(e_1) = 1$ and $e(e_2) = 3$. Then $\alpha \in \text{Aut}(V, f)$ if and only if $\alpha x = Ax$ and $A \in K^{4 \times 4}$ is a nonsingular matrix satisfying $AN = NA$, that is (see e.g. [14, p. 28]),

$$A = \begin{pmatrix} 1 & v & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & \omega & 1 & 0 \\ \kappa & \mu & \omega & 1 \end{pmatrix}, \ \kappa, \mu, v, \omega \in K.$$

Define $z = e_1 + e_3$. We show that the characteristic subspace $X = \langle z \rangle^c$ is not hyperinvariant. We have $\alpha z = (e_1 + \kappa e_4) + (e_3 + \omega e_4)$, and therefore

$$X = \langle z \rangle^c = \langle e_1 + f e_2 \rangle^c = \text{span}\{e_1 + e_3, e_1 + e_3 + e_4\} = \{0, e_1 + e_3, e_1 + e_3 + e_4, e_4\}.$$

Let $\pi_1 = \text{diag}(1, 0, 0, 0)$ be the orthogonal projection on Ke_1. Then $\pi_1 \in \text{End}(V, f)$. We have $\pi_1 z = e_1$, but $e_1 \notin X$. Therefore X is a characteristic subspace that is not hyperinvariant. From $H(z) = (0, 2, \infty, \infty)$ we see that $H(z)$ has a gap at $j = 1$. □

To make the connection with Kaplansky's exposition of Shoda's theorem [10, p. 63/64] we define the numbers

$$d(f, r) = \dim \left(V[f] \cap f^{r-1} V \, / \, V[f] \cap f^r V \right), \ r = 1, 2, \ldots, n.$$

In accordance with the terminology of abelian p-groups [7, p. 154] or p-modules [10, p.27] we call $d(f,r)$ the $(r-1)$-th *Ulm invariant* of f. Then $d(f,r)$ is equal to the number of entries λ^r in the Smith form of f, and $d(f,r) = 1$ means that λ^r is an unrepeated elementary divisor. In the following it may be convenient to write $d(r)$ instead of $d(f,r)$. We recall (see e.g. [4]) that methods or concepts of abelian group theory must be translated to modules over principal ideal domains and then specialized to $K[\lambda]$-modules before they can be applied to linear algebra.

With regard to Theorem 1.3 below we note additional definitions. Suppose $\dim \operatorname{Ker} f = m$. Let $\lambda^{t_1}, \ldots, \lambda^{t_m}$ be the elementary divisors of f such that $t_1 + \cdots + t_m = \dim V$. Then V can be decomposed into a direct sum of f-cyclic subspaces $\langle u_i \rangle$ such that

$$V = \langle u_1 \rangle \oplus \cdots \oplus \langle u_m \rangle \quad \text{and} \quad e(u_i) = t_i, i = 1, \ldots, m. \tag{2}$$

If (2) holds and if the elements in U are ordered by nondecreasing exponents such that

$$e(u_1) \leqslant \cdots \leqslant e(u_m)$$

then we call $U = (u_1, \ldots, u_m)$ a *generator tuple* of V (with respect to f). The tuple $(t_m, \ldots, t_1)$ of exponents – written in nonincreasing order – is known as *Segre characteristic* of f. The set of generator tuples of V will be denoted by $\mathscr{U}$. We call $u \in V$ a *generator* of V (see also [7, p.4]) if $u \in U$ for some $U \in \mathscr{U}$. In other words, $u \in V$ is a generator if and only if $u \neq 0$ and

$$V = \langle u \rangle \oplus V_2 \quad \text{for some} \quad V_2 \in \operatorname{Inv}(V,f). \tag{3}$$

If f has only two elementary divisors then part (ii) of the following theorem gives a description of the set $\operatorname{Chinv}(V,f) \setminus \operatorname{Hinv}(V,f)$.

THEOREM 1.3. *Assume* $|K| = 2$. *Suppose* λ^R *and* λ^S *are unrepeated elementary divisors of* f *and* $R+1 < S$. *Let* u *and* v *be corresponding generators of* V *such that* $e(u) = R$ *and* $e(v) = S$.

(i) *A subspace*

$$X = \langle f^{R-s}u + f^{S-q}v \rangle^c \tag{4}$$

is characteristic and not hyperinvariant if the integers s, q *satisfy*

$$0 < s \leqslant R, \ s < q, \ R - s < S - q. \tag{5}$$

(ii) *Suppose* $V = \langle u \rangle \oplus \langle v \rangle$. *Then an invariant subspace* $X \subseteq V$ *is characteristic and not hyperinvariant if and only if* X *is of the form* (4) *and* s, q *satisfy* (5).

The proof of Theorem 1.3(ii) will be given in Section 2, where two propositions will be proved, one dealing with sufficiency and the other one with necessity of condition (5). In Section 3 we split the space V into two complementary invariant subspaces E and G such that the unrepeated elementary divisors of f are those of $f_{|E}$ and the repeated ones are those of $f_{|G}$. It will be shown that a characteristic subspace X is hyperinvariant in V if and only if $X \cap E$ is hyperinvariant in E. An application of that approach is a proof of Theorem 1.3(i). In Section 4 we extend Theorem 1.3(ii) assuming that f has only two unrepeated elementary divisors.

2. The case of two elementary divisors

In this section we give a proof of Theorem 1.3(ii). It is based on auxiliary results on hyperinvariant subspaces and on images of generators under f-commuting automorphisms of V.

2.1. Hyperinvariant subspaces

Suppose $\dim \operatorname{Ker} f = m$. Let $U = (u_1, \ldots, u_m) \in \mathcal{U}$ be a generator tuple with $e(u_i) = t_i$, $i = 1, \ldots, m$, such that

$$0 < t_1 \leqslant \cdots \leqslant t_m.$$

Set $\vec{t} = (t_1, \ldots, t_m)$ and $t_0 = 0$. Let $\mathcal{L}(\vec{t})$ be the set of m-tuples $\vec{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m$ satisfying

$$0 \leqslant r_1 \leqslant \cdots \leqslant r_m \text{ and } 0 \leqslant t_1 - r_1 \leqslant \cdots \leqslant t_m - r_m. \tag{6}$$

We write $\vec{r} \preceq \vec{s}$ if $\vec{r} = (r_i)_{i=1}^m$, $\vec{s} = (s_i)_{i=1}^m \in \mathcal{L}(\vec{t})$ and $r_i \leqslant s_i$, $i = 1, \ldots, m$. Then $\left(\mathcal{L}(\vec{t}), \preceq \right)$ is a lattice. The following theorem is due to Fillmore, Herrero and Longstaff [6]. We refer to [9] for a proof.

THEOREM 2.1. *Let* $f : V \to V$ *be nilpotent.*

(i) *If* $\vec{r} \in \mathcal{L}(\vec{t})$, *then*

$$W(\vec{r}) = f^{r_1} V \cap V[f^{t_1 - r_1}] + \cdots + f^{r_m} V \cap V[f^{t_m - r_m}]$$

is a hyperinvariant subspace. Conversely, each $W \in \operatorname{Hinv}(V, f)$ *is of the form* $W = W(\vec{r})$ *for some* $\vec{r} \in \mathcal{L}(\vec{t})$.

(ii) *If* $\vec{r} \in \mathcal{L}(\vec{t})$ *then* $W(\vec{r}) = f^{r_1} \langle u_1 \rangle \oplus \cdots \oplus f^{r_m} \langle u_m \rangle$.

(iii) *The mapping* $\vec{r} \mapsto W(\vec{r})$ *is a lattice isomorphism from* $\left(\mathcal{L}(\vec{t}), \preceq \right)$ *onto* $(\operatorname{Hinv}(V, f), \supseteq)$.

For a given $\vec{t}$ *the number of hyperinvariant subspaces is*

$$n_H(\vec{t}) = \prod_{i=1}^m (1 + t_i - t_{i-1}). \tag{7}$$

Let X_H denote the largest hyperinvariant subspace contained in a characteristic subspace X. Using a generator tuple $U = (u_1, \ldots, u_m) \in \mathcal{U}$ one can give an explicit description of X_H. Let $x \in V$ be decomposed as

$$x = x_1 + \cdots + x_m, \ x_i \in \langle u_i \rangle, \ i = 1, \ldots, m, \tag{8}$$

and let $\pi_j : V \to V$ be the projections defined by $\pi_j x = x_j$, $i = 1, \ldots, m$. If $X \subseteq V$ then $\pi_j x \in X$ for all $x \in X$ is equivalent to

$$\pi_j X = X \cap \langle u_j \rangle. \tag{9}$$

The following theorem shows that (9) holds if $e(u_j) = t$ and λ^t is a repeated elementary divisor.

THEOREM 2.2. [2, Lemma 4.5, Lemma 4.2, Theorem 4.3] *Suppose X is a characteristic subspace of V. Let $U = (u_1,\ldots,u_m) \in \mathscr{U}$.*

(i) *If $d(t_j) > 1$ then $\pi_j X = X \cap \langle u_j \rangle$.*

(ii) *The subspace X is hyperinvariant if and only if*

$$\pi_j X = X \cap \langle u_j \rangle, \; j = 1,\ldots,m, \tag{10}$$

or equivalently,

$$X = \oplus_{i=1}^m \left(X \cap \langle u_i \rangle \right). \tag{11}$$

(iii) *The subspace*

$$X_H = \oplus_{i=1}^m \left(X \cap \langle u_i \rangle \right) \tag{12}$$

is the largest hyperinvariant subspace contained in X.

In a characteristic subspace X elements outside of X_H are of special interest (if they exist).

LEMMA 2.3. *Let X be a characteristic subspace.*

(i) *If W is a proper subspace of X then $X = \langle X \setminus W \rangle^c$.*

(ii) *If X is not hyperinvariant then $X = \langle X \setminus X_H \rangle^c$ and*

$$\langle X \setminus X_H \rangle^c = \langle X \setminus X_H \rangle. \tag{13}$$

Proof. (i) From $X \setminus W \subseteq X$ and $X^c = X$ follows $\langle X \setminus W \rangle^c \subseteq X$. Conversely, if $x \in X$ then either $x \in X \setminus W$ or $x \in W$. In the first case it is obvious that $x \in \langle X \setminus W \rangle^c$. Suppose $x \in W$. Choose an element $z \in X \setminus W$. Then $x + z \in X \setminus W$. Thus $z \in \langle X \setminus W \rangle^c$ and $x + z \in \langle X \setminus W \rangle^c$, and therefore $x \in \langle X \setminus W \rangle^c$. Hence $X \subseteq \langle X \setminus W \rangle^c$, which completes the proof.

(ii) Because of $X \supsetneqq X_H$ we can choose $W = X_H$, and obtain $X = \langle X \setminus X_H \rangle^c$. Let us show that

$$\alpha(X \setminus X_H) = X \setminus X_H \text{ for all } \alpha \in \mathrm{Aut}(V, f). \tag{14}$$

Since X_H is hyperinvariant and X is characteristic we have $\alpha(X_H) = X_H$ and $\alpha(X) = X$. Consider $x \in X \setminus X_H$. Suppose $\alpha x \in X_H$. Then $x \in \alpha^{-1}(X_H) = X_H$, which is a contradiction. It is obvious that (14) is equivalent to (13). $\quad\square$

2.2. Images under automorphisms

If $x \in V$ and $\alpha \in \mathrm{Aut}(V, f)$ then it follows from (1) that $H(\alpha x) = H(x)$. We note a converse result due to Baer.

THEOREM 2.4. (See [10], [8, p. 4]) *Let $x, y \in V$. Then $H(x) = H(y)$ if and only if $y = \alpha x$ for some $\alpha \in \mathrm{Aut}(V, f)$.*

We shall use Baer's theorem in Lemma 2.7 to determine the set $\{\alpha u; \alpha \in \text{Aut}(V, f)\}$ for generators u of V. With regard to the proof of Lemma 2.7 we put together basic facts on exponent and height.

If $x \in V$, $x \neq 0$, and $e(x) = k$, then $e(f^j x) = k - j$, $j = 0, 1, \ldots, k-1$. The height of $f^j x$ satisfies the inequality $h(f^j x) \geqslant j + h(x)$, $j = 0, 1, \ldots, k-1$. If $x_1, \ldots, x_m \in V$ then

$$h(x_1 + \cdots + x_m) \geqslant \min\{h(x_i); 1 \leqslant i \leqslant m\}. \tag{15}$$

In general, the inequality (15) is strict. Consider Example 1.2 with $x_1 = e_3 + e_1$, $x_2 = e_4 + e_1$, and $h(x_1) = h(x_2) = 0$ and $h(x_1 + x_2) = 1$. We have equality in (15) if the vectors x_i satisfy the assumption of the following lemma.

LEMMA 2.5. *Let* $U = (u_1, \ldots, u_m) \in \mathcal{U}$. *If* $x = \sum_{i=1}^m x_i$, $x_i \in \langle u_i \rangle$, $i = 1, \ldots, m$, $x \neq 0$, *then*

$$h(x) = \min\{h(x_i); 1 \leqslant i \leqslant m, x_i \neq 0\} \tag{16}$$

and

$$e(x) = \max\{e(x_i); 1 \leqslant i \leqslant m, x_i \neq 0\}. \tag{17}$$

Proof. To prove the identity (16) we set $\tilde{q} = \min\{h(x_i); x_i \neq 0\}$ and $q = h(x)$. Then $q \geqslant \tilde{q}$. On the other hand we have $x = f^q y$, $y = \sum y_i$, $y_i \in \langle u_i \rangle$. Hence $x_i = f^q y_i$ for all i, and therefore $\tilde{q} \geqslant q$.

With regard to (17) we set $\tilde{\ell} = \max\{e(x_i); x_i \neq 0\}$ and $\ell = e(x)$. Then $0 = f^\ell x = \sum_{i=1}^m f^\ell x_i$ implies $f^\ell x_i = 0$ for all i. Hence $\tilde{\ell} \leqslant \ell$. On the other hand

$$0 \neq f^{\ell-1} x = \sum_{i=1}^m f^{\ell-1} x_i$$

implies $f^{\ell-1} x_j \neq 0$ for some j, $1 \leqslant j \leqslant k$, and therefore $\tilde{\ell} \geqslant \ell$.   □

We remark that the preceding lemma can be deduced from results on marked subspaces in [5].

LEMMA 2.6. *Suppose* λ^t *is an elementary divisor of* f. *Then* u *is a generator of* V *with* $e(u) = t$ *if and only if* $f^t u = 0$ *and*

$$h(f^j u) = j, \ j = 0, 1, \ldots, t-1. \tag{18}$$

Proof. Suppose u is a generator and

$$V = \langle u \rangle \oplus V_2 \quad \text{and} \quad e(u) = t. \tag{19}$$

Let $h(f^{t-1} u) = (t-1) + \tau$, $\tau \geqslant 0$. Then $f^{t-1} u = f^{t-1+\tau} \tilde{w}$ for some $\tilde{w} = w + w_2$ with $w \in \langle u \rangle$, $w_2 \in V_2$. Then $\langle u \rangle \cap V_2 = 0$ implies $f^{t-1} u = f^{t-1+\tau} w$, and we obtain $\tau = 0$. Hence $h(f^{t-1} u) = t - 1$, which is equivalent to (18).

Now suppose $u \in V$ satisfies (18). Let $v \in V$ be a generator corresponding to λ^t such that $V = \langle v \rangle \oplus W_2$ and $e(v) = t$. We have shown before that $H(v) = (0, 1, \ldots, t-1, \infty, \ldots, \infty)$. Hence $H(u) = H(v)$, and therefore Theorem 2.4 implies $u = \alpha v$ for

some $\alpha \in \mathrm{Aut}(V,f)$. Then $V = \alpha(\langle v \rangle \oplus W_2) = \langle u \rangle \oplus \alpha W_2$ shows that u is also a generator. $\square$

A consequence of Lemma 2.6 is the following observation. Suppose u is a generator of V and $w \in \langle u \rangle$ and $\mathrm{h}(w) = 0$. Then $\langle w \rangle = \langle u \rangle$.

LEMMA 2.7. *Suppose* λ^t *is an unrepeated elementary divisor of* f. *Let* $U = (u_1, \ldots, u_m) \in \mathcal{U}$, *and* $\mathrm{e}(u_p) = t$. *If* $u \in V$ *then the following statements are equivalent.*

(i) *The vector* u *is a generator of* V *with* $\mathrm{e}(u) = t$.

(ii) *There exists an* $\alpha \in \mathrm{Aut}(V,f)$ *such that* $u = \alpha u_p$.

(iii) *We have*

$$u = y + g \ \ with \ \ y = cu_p + fv, \ c \neq 0, \ v \in \langle u_p \rangle, \ g \in \langle u_i; \ i \neq p \rangle[f^t]. \quad (20)$$

Proof. If u and u_ρ are generators of V then we have $\mathrm{e}(u) = \mathrm{e}(u_\rho)$ if and only if $H(u) = H(u_\rho)$. Hence it follows from Theorem 2.4 and Lemma 2.6 that (i) and (ii) are equivalent.

(i) $\Rightarrow$ (iii) Let u be decomposed such that $u = y + g$ and $y \in \langle u_p \rangle$ and $g \in \langle u_i; i \neq p \rangle$. Then (17) implies $\mathrm{e}(g) \leqslant \mathrm{e}(u) = t$, that is $g \in V[f^t]$. Let us show that $\mathrm{h}(y) = 0$, or equivalently

$$y = cu_p + fv, \ c \neq 0, \ v \in \langle u_p \rangle. \quad (21)$$

We have $g = g_< + g_>$ with $g_< \in \langle u_i; i < p \rangle$, and $g_> \in \langle u_i; i > p \rangle$. If $i < p$ then $\mathrm{e}(u_i) < \mathrm{e}(u_p) = t$. Hence $f^{t-1}g_< = 0$, and

$$f^{t-1}u = f^{t-1}y + f^{t-1}g_>. \quad (22)$$

If $i > p$ then $\mathrm{e}(u_i) > t$. Therefore

$$\langle u_i \rangle[f^t] = f^{\mathrm{e}(u_i)-t}\langle u_i \rangle \subseteq fV.$$

Hence $\mathrm{h}(g_>) \geqslant 1$ and $\mathrm{h}(f^{t-1}g_>) \geqslant t$. Now suppose $\mathrm{h}(y) \neq 0$. Then $\mathrm{h}(y) \geqslant 1$ and $\mathrm{h}(f^{t-1}y) \geqslant t$. Therefore (22) implies $\mathrm{h}(f^{t-1}u) \geqslant t$. This is a contradiction to the assumption that u is a generator with $\mathrm{h}(f^{t-1}u) = t - 1$. Hence $\mathrm{h}(y) = 0$.

(iii) $\Rightarrow$ (i) Assume (20). Then (21) implies that y is a generator with $\mathrm{e}(y) = t$. Thus $\mathrm{h}(y) = 0$ and $\mathrm{h}(f^{t-1}y) = t - 1$. Moreover, we have $\mathrm{e}(g) \leqslant t$. Hence $\mathrm{e}(u) = \max\{\mathrm{e}(y), \mathrm{e}(g)\} = t$. From (16) follows

$$t - 1 \leqslant \mathrm{h}(f^{t-1}u) = \min\{\mathrm{h}(f^{t-1}y), \mathrm{h}(f^{t-1}g)\} \leqslant \mathrm{h}(f^{t-1}y) = t - 1.$$

Hence $\mathrm{h}(f^{t-1}u) = t - 1$. Then Lemma 2.6 completes the proof. $\square$

Lemma 2.7 will be used in the proof of Proposition 2.11. We note that the assumption $|K| = 2$ implies that the vector y in (20) is of the form

$$y = u_\rho + fv, \ v \in \langle u_\rho \rangle. \quad (23)$$

If u is a generator with $e(u) = t$ then there is no gap in the indicator sequence $H(f^j u) = (j, j+1, \ldots, t-1, \infty, \ldots, \infty)$. We mention without proof that $\langle x \rangle^c$ is hyperinvariant if and only if $H(x)$ has no gap. We only need the following special case of that result.

LEMMA 2.8. *Let λ^t be an unrepeated elementary divisor of f and u a generator of V with $e(u) = t$. Then*

$$\langle f^j u \rangle^c = \operatorname{Im} f^j \cap \operatorname{Ker} f^{t-j}, \ j = 0, \ldots, t, \tag{24}$$

and $\langle f^j u \rangle^c$ is hyperinvariant.

Proof. Let $U = (u_1, \ldots, u_m) \in \mathscr{U}$ such that $e(u_p) = t$ and $u_p = u$. Then

$$\operatorname{Ker} f^t = \langle u_p \rangle \bigoplus \left(\oplus_{1 \leqslant i \leqslant m; i \neq p} \langle u_i \rangle [f^t] \right)$$

and $\langle u_p \rangle = \langle c u_p + f v; c \neq 0, \ v \in \langle u_p \rangle \rangle$. Hence

$$\operatorname{Ker} f^t = \langle c u_p + f v + g; c \neq 0, \ v \in \langle u_p \rangle, \ g \in \langle u_i; i \neq p \rangle [f^t] \rangle$$
$$= \langle \alpha u_p; \alpha \in \operatorname{Aut}(V, f) \rangle = \langle u_p \rangle^c.$$

If $0 < j < t$ then

$$\langle f^j u_p \rangle^c = f^j \langle u_p \rangle^c = f^j \operatorname{Ker} f^t = f^j \operatorname{Ker} f^{t-j} = \operatorname{Im} f^j \cap \operatorname{Ker} f^{t-j}. \quad \square$$

A general theorem that contains the following lemma can be found in [8, Lemma 65.4, p. 4].

LEMMA 2.9. *Suppose $V = \langle u_1 \rangle \oplus \langle u_2 \rangle$ and $e(u_1) < e(u_2)$. If $x \in V$ then there exists an automorphism $\alpha \in \operatorname{Aut}(V, f)$ such that*

$$\alpha x = f^{k_1} u_1 + f^{k_2} u_2 \tag{25}$$

for some $k_1, k_2 \in \mathbb{N}_0$.

Proof. Let $x = x_1 + x_2$, $x_i \in \langle u_i \rangle$, $i = 1, 2$. Suppose $x_1 \neq 0$ and $x_2 \neq 0$. If $h(x_i) = k_i$ then $x_i = f^{k_i} \tilde{u}_i$, $\tilde{u}_i \in \langle u_i \rangle$, $h(\tilde{u}_i) = 0$, $i = 1, 2$. Therefore $\langle \tilde{u}_i \rangle = \langle u_i \rangle$, $i = 1, 2$, such that $(\tilde{u}_1, \tilde{u}_2) \in \mathscr{U}$. Then $\alpha : (\tilde{u}_1, \tilde{u}_2) \mapsto (u_1, u_2) \in \operatorname{Aut}(V, f)$ yields (25). $\quad \square$

2.3. Proof of Theorem 1.3(ii)

In this section we assume $\dim \operatorname{Ker} f = 2$ such that $V = \langle u_1 \rangle \oplus \langle u_2 \rangle$. We prove two propositions. They provide the complete description of the set $\operatorname{Chinv}(V, f) \setminus \operatorname{Hinv}(V, f)$ in Theorem 1.3(ii). The following notation will be convenient. If we write

$$\alpha : (u_1, u_2) \mapsto (\hat{u}_1, \hat{u}_2) \in \operatorname{Aut}(V, f)$$

we assume $(\hat{u}_1, \hat{u}_2) \in \mathscr{U}$, and α denotes the automorphism in $\operatorname{Aut}(V, f)$ defined by $(\alpha u_1, \alpha u_2) = (\hat{u}_1, \hat{u}_2)$.

PROPOSITION 2.10. *Let* $|K| = 2$. *Suppose* $V = \langle u_1 \rangle \oplus \langle u_2 \rangle$ *and* $\mathrm{e}(u_1) = R$, $\mathrm{e}(u_2) = S$, *and* $R + 1 < S$. *If* X *is a characteristic non-hyperinvariant subspace of* V *then*

$$X = \langle f^{R-s}u_1 + f^{S-q}u_2 \rangle^c \quad \text{with} \quad 0 < s < q \quad \text{and} \quad 0 \leqslant R - s < S - q, \quad (26)$$

and

$$X_H = \langle f^{R-s+1}u_1, f^{S-q+1}u_2 \rangle = \operatorname{Im} f^{R-s+1} \cap \operatorname{Ker} f^{q-1} \quad (27)$$

is the largest hyperinvariant subspace contained in X.

Proof. Theorem 2.2(iii) and Theorem 2.1(ii) imply

$$X_H = (X \cap \langle u_1 \rangle) \oplus (X \cap \langle u_2 \rangle) = W(\vec{r}) = \langle f^{r_1}u_1 \rangle \oplus \langle f^{r_2}u_2 \rangle \quad (28)$$

for some pair $\vec{r} = (r_1, r_2)$ satisfying $r_1 \leqslant R$, $r_2 \leqslant S$, and

$$0 \leqslant r_1 \leqslant r_2 \quad \text{and} \quad 0 \leqslant R - r_1 \leqslant S - r_2. \quad (29)$$

Since X is not hyperinvariant we have $X \supsetneqq X_H$. Let $x \in X \setminus X_H$. By Lemma 2.9 there exists an automorphism $\alpha \in \operatorname{Aut}(V, f)$ such that

$$\alpha x = z = f^{\mu_1}u_1 + f^{\mu_2}u_2 \in X \setminus X_H. \quad (30)$$

Suppose $\mu_1 \geqslant r_1$. Then $f^{\mu_1}u_1 \in \langle f^{r_1}u_1 \rangle \subseteq X$, and because $z \in X$ also $f^{\mu_2}u_2 = z - f^{\mu_1}u_1 \in X$. Hence $f^{\mu_1}u_1 \in X \cap \langle u_1 \rangle$ and $f^{\mu_2}u_2 \in X \cap \langle u_2 \rangle$, and we would obtain $z \in X_H$. Similarly, it is impossible that $\mu_2 \geqslant r_2$. Therefore

$$\mu_1 < r_1, \ \mu_2 < r_2. \quad (31)$$

We shall see that

$$\mu_1 + 1 = r_1, \quad \mu_2 + 1 = r_2. \quad (32)$$

If $R = 1$ then $\mu_1 = 0$, and $\mu_1 + 1 = r_1 = 1$. If $R > 1$ then $fu_1 \neq 0$, and

$$\beta : (u_1, u_2) \mapsto (u_1 + fu_1, u_2) \in \operatorname{Aut}(V, f)$$

yields

$$\beta z = (f^{\mu_1}u_1 + f^{\mu_1+1}u_1) + f^{\mu_2}u_2 = z + f^{\mu_1+1}u_1 \in X.$$

Since X is characteristic and $z \in X$, we have $\beta z \in X$. Hence

$$f^{\mu_1+1}u_1 \in X \cap \langle u_1 \rangle = \langle f^{r_1}u_1 \rangle,$$

and we obtain $\mu_1 + 1 \geqslant r_1$. A similar argument yields, $\mu_2 + 1 \geqslant r_2$. Hence (31) implies the relations (32). Thus

$$z = f^{\mu_1}u_1 + f^{\mu_2}u_2, \ \mu_1 = r_1 - 1, \ \mu_2 = r_2 - 1.$$

Hence there exists a unique vector $z \in X \setminus X_H$ with a representation (30). Then (14) implies $X \setminus X_H = \{\alpha z; \alpha \in \mathrm{Aut}(V,f)\}$ and (13) yields

$$X = \langle z \rangle^c = \langle f^{\mu_1} u_1 + f^{\mu_2} u_2 \rangle^c. \tag{33}$$

According to the definitions of s and t we have $0 \leqslant \mu_1 \leqslant \mu_2$ and $0 \leqslant R - \mu_1 \leqslant S - \mu_2$. Hence it remains to show that

$$\mu_1 \neq R, \ \mu_1 \neq \mu_2 \text{ and } R - \mu_1 \neq S - \mu_2. \tag{34}$$

Suppose $R = \mu_1$. Then $z = f^{\mu_2} u_2$, and Lemma 2.8 implies $X = \langle f^{\mu_1} u_2 \rangle^c \in \mathrm{Hinv}(V,f)$. Suppose $\mu_1 = \mu_2$. Then $z = f^{\mu_1}(u_1 + u_2)$. Using

$$\gamma : (u_1, u_2) \mapsto (u_1, u_2 + u_1) \in \mathrm{Aut}(V,f)$$

we obtain $\gamma^{-1} z = f^{\mu_1} u_2 \in X$. Hence Lemma 2.8 implies $X = \langle f^{\mu_1} u_2 \rangle^c \in \mathrm{Hinv}(V,f)$. Suppose $R - \mu_1 = S - \mu_2$. Then $S - (\mu_2 - \mu_1) = R$ and

$$z = f^{\mu_1}(u_1 + f^{\mu_2 - \mu_1} u_2) = f^{\mu_1}(u_1 + f^{S-R} u_2).$$

Therefore $\sigma : (u_1, u_2) \mapsto (u_1 + f^{S-R} u_2, u_2) \in \mathrm{Aut}(V,f)$ yields $\sigma^{-1} z = f^{\mu_1} u_1$. Then $X = \langle f^{\mu_1} u_1 \rangle^c \in \mathrm{Hinv}(V,f)$. Hence the inequalities (34) are valid.

From (28) and (32) follows $X_H = \langle f^{\mu_1+1} u_1, f^{\mu_2+1} u_2 \rangle$. Moreover (34) implies $\langle f^{\mu_1+1} u_1, f^{\mu_2+1} u_2 \rangle = \mathrm{Im}\, f^{\mu_1+1} \cap \mathrm{Ker}\, f^{S-(\mu_2+1)}$.

We obtain (26) and (27) if we set $\mu_1 = R - s$ and $\mu_2 = S - q$. $\square$

PROPOSITION 2.11. *Assume $|K| = 2$. Suppose $V = \langle u_1 \rangle \oplus \langle u_2 \rangle$, and $\mathrm{e}(u_1) = R$, $\mathrm{e}(u_2) = S$ such that $R + 1 < S$. Let s, q be integers satisfying*

$$0 < s < q, \ 0 \leqslant R - s < S - q. \tag{35}$$

Then the subspace

$$X = \langle f^{R-s} u_1 + f^{S-q} u_2 \rangle^c$$

is characteristic and not hyperinvariant, and

$$X = \langle f^{R-s} u_1 + f^{S-q} u_2, \ f^{R-s+1} u_1, \ f^{S-q+1} u_2 \rangle. \tag{36}$$

We have

$$\dim X = s + q - 1. \tag{37}$$

If $s > 1$ then $f_{|X}$ has the elementary divisors λ^q and λ^{s-1}. If $s = 1$ then $X = \langle f^{R-s} u_1 + f^{S-q} u_2 \rangle$ and the corresponding elementary divisor is λ^q.

Proof. Define $z = f^{R-s} u_1 + f^{S-q} u_2$. Then $X = \langle z \rangle^c$. Set $\tilde{q} = R + (q - s)$ and

$$\tilde{z} = u_1 + f^{S-\tilde{q}} u_2 \quad \text{and} \quad \tilde{X} = \langle \tilde{z} \rangle^c.$$

Then $(R-s)+(S-\tilde{q})=S-q$. Therefore $z=f^{R-s}\tilde{z}$ and $X=f^{R-s}\tilde{X}$, and (35) is equivalent to

$$R<\tilde{q},\ 0<S-\tilde{q}. \tag{38}$$

Let us first deal with the height-zero space $\tilde{X}=\langle\tilde{z}\rangle^c$ and then pass to $X=\langle z\rangle^c$. Let $\alpha\in\mathrm{Aut}(V,f)$. We determine $\alpha\tilde{z}$ using Lemma 2.7. Recall that $|K|=2$ implies (23), that is we have $y=u_\rho+fv$, $v\in\langle u_\rho\rangle$ in (20). If $\alpha u_1=x_1+x_2$, $x_i\in\langle u_i\rangle$, $i=1,2$, then

$$x_1=u_1+fv_1, v_1\in\langle u_1\rangle \quad\text{and}\quad x_2\in\langle u_i;i\neq1\rangle[f^R]=\langle u_2\rangle[f^R]=f^{S-R}\langle u_2\rangle.$$

Similarly, $\alpha u_2=y_1+y_2$, $y_i\in\langle u_i\rangle$, $i=1,2$, and

$$y_2=u_2+fw_2, w_2\in\langle u_2\rangle \quad\text{and}\quad y_1\in\langle u_i;i\neq2\rangle[f^S]=\langle u_1\rangle[f^S]=\langle u_1\rangle.$$

Then

$$\alpha\tilde{z}=(u_1+fv_1+x_2)+(f^{S-\tilde{q}}u_2+f^{S-\tilde{q}+1}w_2+f^{S-\tilde{q}}y_1)=$$
$$\tilde{z}+(fv_1+f^{S-\tilde{q}}y_1)+(x_2+f^{S-\tilde{q}+1}w_2). \tag{39}$$

From $S-\tilde{q}>0$ follows $f^{S-\tilde{q}}y_1\in f\langle u_1\rangle$. From $\tilde{q}>R$ follows $S-R\geqslant S-\tilde{q}+1$, and therefore $x_2\in f^{S-\tilde{q}+1}\langle u_2\rangle$. Set $\hat{v}=fv_1+f^{S-\tilde{q}}y_1$ and $\hat{w}=x_2+f^{S-\tilde{q}+1}w_2$. Then

$$\alpha\tilde{z}=\tilde{z}+\hat{v}+\hat{w},\quad\text{and}\quad\hat{v}\in f\langle u_1\rangle,\hat{w}\in f^{S-\tilde{q}+1}\langle u_2\rangle. \tag{40}$$

Define $\tilde{L}=\langle\tilde{z},f^{S-\tilde{q}+1}u_2\rangle$. Because of $f\tilde{z}=fu_1+f^{S-\tilde{q}+1}u_2$ we obtain $\tilde{L}=\langle\tilde{z},fu_1,f^{S-\tilde{q}+1}u_2\rangle$. Then (40) implies $\langle\tilde{z}\rangle^c\subseteq\tilde{L}$. If

$$\beta:(u_1,u_2)\mapsto(u_1,u_2+fu_2)\in\mathrm{Aut}(V,f)$$

then $\beta\tilde{z}=\tilde{z}+f^{S-\tilde{q}+1}u_2\in\langle\tilde{z}\rangle^c$. Hence

$$\beta\tilde{z}-\tilde{z}=f^{S-\tilde{q}+1}u_2\in\langle\tilde{z}\rangle^c,$$

and therefore $\tilde{L}\subseteq\langle\tilde{z}\rangle^c$, and we obtain

$$\tilde{L}=\langle\tilde{z}\rangle^c=\langle\tilde{z},fu_1,f^{S-\tilde{q}+1}u_2\rangle. \tag{41}$$

We determine the dimension of $\langle\tilde{z}\rangle^c$. If $R=1$ then $fu_1=0$ and $f\tilde{z}=f^{S-\tilde{q}+1}u_2$. Therefore $\langle\tilde{z}\rangle^c=\langle\tilde{z}\rangle$ and $\dim\langle\tilde{z}\rangle=\mathrm{e}(\tilde{z})=\tilde{q}$. If $R>1$ then $\langle\tilde{z}\rangle^c=\langle\tilde{z},fu_1\rangle$. Let

$$x=\sum_{\mu=0}^{R-1}c_\mu f^\mu u_1+\sum_{\mu=0}^{\tilde{q}-1}c_\mu f^{S-\tilde{q}+\mu}u_2\in\langle\tilde{z}\rangle.$$

Then $x\in\langle fu_1\rangle$ if and only if $c_0=\cdots=c_{\tilde{q}-1}=0$, that is, $x=0$. Hence $\langle\tilde{z}\rangle\cap\langle fu_1\rangle=0$, and

$$\dim\langle\tilde{z}\rangle^c=\dim\langle\tilde{z}\rangle+\dim\langle fu_1\rangle=\tilde{q}+(R-1). \tag{42}$$

At this point we go back to $X = \langle z \rangle^c = f^{R-s}\langle \tilde{z} \rangle$. Then (41) yields (36). From (42) we obtain

$$\dim X = \dim f^{R-s}\langle \tilde{z} \rangle + \dim f^{R-s}\langle f u_1 \rangle = [\tilde{q} - (R-s)] + [(R-1) - (R-s)] = s + q - 1,$$

which proves (37). If $s > 1$ then $X = \langle z \rangle \oplus \langle f^{R-s+1}u_1 \rangle$ is a direct sum of f-cyclic subspaces of dimension q and $s - 1$, respectively. Hence, in that case, the elementary divisors of $f_{|X}$ are λ^q and λ^{s-1}. It is easy to see that the case $s = 1$ leads to $X = \langle z \rangle$.

We show next that $f^{R-s}u_1 \notin X$. Suppose to the contrary that $f^{R-s}u_1 \in X$. Then $z = f^{R-s}u_1 + f^{S-q}u_2 \in X$ would imply $f^{S-q}u_2 \in X$. Hence $\langle f^{R-s}u_1 \rangle \oplus \langle f^{S-q}u_2 \rangle \subseteq X$, and therefore $\dim X \geqslant s + q$, in contradiction to (37). Hence $f^{R-s}u_1 \notin X$. Let π_1 be the projection of V on $\langle u_1 \rangle$ along $\langle u_2 \rangle$. Then $\pi_1 \in \mathrm{End}(V,f)$. But $\pi_1 z = f^{R-s}u_1 \notin X$. Hence the subspace X is not hyperinvariant. $\square$

EXAMPLE 1.2 *continued.* If $(R,S) = (1,3)$ then $(s,q) = (1,2)$ is the only solution of (35). Then $(R - s, S - q) = (0,1)$, and $X = \langle f^0 e_1 + f^1 e_2 \rangle^c$ is the only characteristic non-hyperinvariant subspace of V. According to (7) there are 6 hyperinvariant subspaces in V. $\square$

3. Separating repeated and unrepeated elementary divisors

According to Shoda's theorem only unrepeated elementary divisors are relevant for the existence of characteristic non-hyperinvariant subspaces. In this section we examine this fact in more detail. Let E and G be invariant subspaces of V and assume

$$V = E \oplus G \quad \text{and} \quad d(f,t) = d(f_{|E},t) \quad \text{if} \quad d(f,t) = 1 \quad \text{and}$$
$$d(f,t) = d(f_{|G},t) \quad \text{if} \quad d(f,t) > 1. \quad (43)$$

Thus the unrepeated elementary divisors of f are those of $f_{|E}$ and the repeated ones are those of $f_{|G}$. If (43) holds then there exists a generator tuple $U = (u_1, \ldots, u_m)$ adapted to E and G such that

$$E = \langle u_i; \mathrm{e}(u_i) = t_i; d(t_i) = 1 \rangle \quad \text{and} \quad G = \langle u_i; \mathrm{e}(u_i) = t_i; d(t_i) > 1 \rangle. \quad (44)$$

We shall see that a characteristic subspace X is hyperinvariant in V if and only if $X \cap E$ is hyperinvariant in E. We first consider a general direct sum decomposition of V.

LEMMA 3.1. *Let V_1 and V_2 be invariant subspaces of V such that $V = V_1 \oplus V_2$.*

(i) *If $X \in \mathrm{Chinv}(V,f)$ then $(X \cap V_i) \in \mathrm{Chinv}(V_i, f_{|V_i})$, $i = 1,2$.*

(ii) *If $X \in \mathrm{Hinv}(V,f)$ then $(X \cap V_i) \in \mathrm{Hinv}(V_i, f_{|V_i})$, $i = 1,2$.*

(iii) *A subspace X is hyperinvariant if and only if X is characteristic and*

$$X = (X \cap V_1) \oplus (X \cap V_2) \quad \text{and} \quad X \cap V_i \in \mathrm{Hinv}(V_i, f_{|V_i}), \ i = 1,2. \quad (45)$$

Proof. (i) Let $x_1 \in X \cap V_1$ and $g_1 \in \mathrm{Aut}(V_1, f_{|V_1})$. To show that $g_1(x_1) \in X$ we extend g_1 to an automorphism $g \in \mathrm{Aut}(V, f)$ as follows. Let ι_2 be the identity map of V_2. Then $g = g_1 + \iota_2 \in \mathrm{Aut}(V, f)$. Therefore $X \in \mathrm{Chinv}(V, f)$ implies $g(x_1) \in X$. On the other hand $g(x_1) = g_1(x_1) \in V_1$. Hence $g_1(x_1) \in X \cap V_1$.

(ii) Let $x_1 \in X \cap V_1$ and $h_1 \in \mathrm{End}(V_1, f_{|V_1})$. If 0_2 is the zero map on V_2 then $h = h_1 + 0_2 \in \mathrm{End}(V, f)$. An argument as in part (i) shows that $h_1(x_1) \in X \cap V_1$.

(iii) Let $V_i = \oplus_{v=1}^{m_i} \langle u_v^{(i)} \rangle$, $i = 1, 2$, be decomposed into cyclic subspaces, and let $U = (u_j)_{j=1}^{m} \in \mathcal{U}$ contain the vectors $u_v^{(1)}$, $v = 1, \ldots, m_1$, and $u_v^{(2)}, v = 1, \ldots, m_2$. Define $X_i = X \cap V_i$, $i = 1, 2$. Suppose X is characteristic and (45) holds. Then Lemma 3.1(ii) and $V_1 \cap V_2 = 0$ imply

$$X_i = \oplus_{v=1}^{m_i} (X_i \cap \langle u_v^{(i)} \rangle) = \oplus_{j=1}^{m} (X_i \cap \langle u_j \rangle), \ i = 1, 2.$$

Hence

$$X = X_1 \oplus X_2 = \oplus_{j=1}^{m} \Big((X_1 \cap \langle u_j \rangle) + (X_2 \cap \langle u_j \rangle) \Big)$$
$$\subseteq \oplus_{j=1}^{m} [(X_1 + X_2) \cap \langle u_j \rangle] = \oplus_{j=1}^{m} (X \cap \langle u_j \rangle) \subseteq X.$$

Then X satisfies (11), and therefore X is hyperinvariant. Using (11) it is not difficult to see that $X \in \mathrm{Hinv}(V, f)$ implies (45). $\square$

We apply the preceding lemma to the decomposition $V = E \oplus G$. Let π_E be the projection of V on E along G, and let π_G denote the complementary projection such that $\pi_E + \pi_G = \iota$. Then $\pi_E f = f \pi_E$ and $\pi_G f = f \pi_G$. Let $(X \cap E)_{H|E}$ denote the largest hyperinvariant subspace (with respect to $f_{|E}$) contained in E.

LEMMA 3.2. *Let E and G be subspaces of V such that (43) holds. Suppose X is a characteristic subspace of V.*

(i) *Then $\pi_E X = X \cap E$ and $\pi_G X = X \cap G$, and*

$$X = (X \cap E) \oplus (X \cap G). \tag{46}$$

Moreover,

$$X \cap E \in \mathrm{Chinv}(E, f_{|E}) \quad and \quad X \cap G \in \mathrm{Hinv}(G, f_{|G}), \tag{47}$$

and

$$X_H = (X \cap E)_{H|E} \oplus (X \cap G) \tag{48}$$

(ii) *The subspace X is hyperinvariant in V if and only if $X \cap E$ is hyperinvariant in E.*

Proof. (i) Let $U \in \mathcal{U}$ satisfy (44). If $x \in X$ then Theorem 2.2(i) yields

$$\pi_G x = \Big(\sum_{d(t_i) > 1} \pi_i \Big) x = \sum_{d(t_i) > 1} \pi_i x \in X.$$

Hence $\pi_G X \subseteq X \cap G$, and therefore $\pi_G X = X \cap G$. Then $x = \pi_G x + \pi_E x$ implies $\pi_E x \in X$. Thus we obtain $\pi_E X = X \cap E$, and $X = \pi_E X \oplus \pi_G X$.

From Lemma 3.1(i) we conclude that $X \cap E$ and $X \cap G$ are characteristic in E, respectively in G. According to (43) the map $f_{|G}$ has only repeated elementary divisors. Hence Theorem 1.1 implies that $X \cap G$ is hyperinvariant in G. The description of X_H in (48) follows from (12).

(ii) The subspace $X \cap G$ is hyperinvariant in G. We apply Lemma 3.1(iii). □

The assumption that X is characteristic is essential for (46).

EXAMPLE 3.3. Let $V = \langle u_1, u_2, u_3 \rangle$ with $e(u_1) = 1$, $e(u_2) = e(u_3) = 2$. Then $E = \langle u_1 \rangle$ and $G = \langle u_2, u_3 \rangle$. Set $X = \langle u_1 + f u_2 \rangle$. Then $X \cap G = 0$ and $X \supsetneq (X \cap E) \oplus (X \cap G)$.

In the following we deal with invariant subspaces associated to subsets of unrepeated elementary divisors of f. Let T be an invariant subspace of V such that $f_{|T}$ has only unrepeated elementary divisors and such that

$$V = T \oplus V_2 \quad \text{for some} \quad V_2 \in \mathrm{Inv}(V, f),$$

and $f_{|T}$ and $f_{|V_2}$ have no elementary divisors in common. The subspace T can also be characterized as follows. There exists a $T_2 \in \mathrm{Inv}(V, f)$ such that

$$T \oplus T_2 = E, \text{ and } T_2 \oplus G = V_2, \ E \oplus G = V \text{ and } E \text{ and } G \text{ satisfy (43)}. \tag{49}$$

Let π_T be the projection on T along V_2 and π_{V_2} be the complementary projection. If $Y \subseteq T$ then Y^{c_T} denotes the characteristic hull of Y with respect to T,

$$Y^{c_T} = \langle \alpha_T y; \ y \in Y, \ \alpha_T \in \mathrm{Aut}(T, f_{|T}) \rangle.$$

A spin-off from the following lemma is a proof of Theorem 1.3(i).

LEMMA 3.4. *Let T be an invariant subspace such that $f_{|T}$ has only unrepeated elementary divisors and (49) holds. Suppose X is a characteristic subspace of V.*

(i) *Then $X \cap T \in \mathrm{Chinv}(T, f_{|T})$. If the subspace X is hyperinvariant in V then $X \cap T$ is hyperinvariant in T.*

(ii) *We have*
$$\mathrm{Aut}(T, f_{|T}) = \{\pi_T \alpha \pi_T; \alpha \in \mathrm{Aut}(V, f)\}. \tag{50}$$

(iii) *If $Y \subseteq T$ then*
$$Y^c \cap T = \pi_T Y^c = Y^{c_T}. \tag{51}$$

Proof. (i) Because of $V = T \oplus V_2$ one can apply Lemma 3.1.

(ii) Let $U = \{u_1, \ldots, u_m\}$ be a generator tuple of V such that a subtuple $U_T = \{u_{\tau_1}, \ldots, u_{\tau_q}\} \subseteq U$ is a generator tuple of T (with respect to $f_{|T}$). Set $I_T = \{\tau_1, \ldots, \tau_q\}$. If $i \in I_T$ then Lemma 2.7 implies

$$\alpha u_i = c_i u_i + f v_i + \sum_{j \in I_T, j \neq i} w_j + \sum_{k \notin I_T, 1 \leqslant k \leqslant m} x_k,$$

where $c_i \neq 0$, $v_i \in \langle u_i \rangle$, $w_j \in \langle u_j \rangle [f^{t_i}]$ and $x_k \in \langle u_k \rangle [f^{t_i}]$. Hence

$$(\pi_T \alpha) u_i = c_i u_i + f v_i + \sum_{j \in I_T, j \neq i} w_j.$$

Then Lemma 2.5 yields $e\big((\pi_T \alpha) u_i\big) = t_i$, and

$$h(f^{t_i-1} (\pi_T \alpha) u_i) = h(f^{t_i-1} u_i) = t_i - 1,$$

and $h((\pi_T \alpha) u_i) = 0$. By Lemma 2.6 the element $(\pi_T \alpha) u_i \in T$ is a generator. Hence $\big((\pi_T \alpha) u_{\tau_1}, \ldots, (\pi_T \alpha) u_{\tau_q}\big)$ is a generator tuple of T. Then the map given by

$$\pi_T \alpha : u_i \mapsto (\pi_T \alpha) u_i, \ i \in I_T,$$

is in $\mathrm{Aut}(T, f_{|T})$. Hence $\pi_T \alpha \pi_T \in \mathrm{Aut}(T, f_{|T})$. Now consider an automorphism $\alpha_T \in \mathrm{Aut}(T, f_{|T})$. We extend α_T to an automorphism $\tilde{\alpha} \in \mathrm{Aut}(V, f)$ defining

$$\tilde{\alpha} : u_i \mapsto \alpha_T u_i \quad \text{if} \quad i \in I_T, \quad \text{and} \quad \tilde{\alpha} : u_i \mapsto u_i \quad \text{if} \quad i \notin I_T. \tag{52}$$

Then $(\pi_T \tilde{\alpha} \pi_T) u_{\tau_i} = \alpha_T u_{\tau_i}$, $i = 1, \ldots, q$, and therefore $\alpha_T = \pi_T \tilde{\alpha} \pi_T$.

(iii) Let us show first that $\pi_T Y^c = Y^c \cap T$. If $\alpha \in \mathrm{Aut}(V, f)$ then $\alpha_T = \pi_T \alpha \pi_T \in \mathrm{Aut}(T, f_{|T})$. Let $\tilde{\alpha} \in \mathrm{Aut}(V, f)$ be the extension of α_T given by (52). If $y \in Y \subseteq T$ then $\pi_T \alpha y = \pi_T \alpha \pi_T y = \alpha_T y = \tilde{\alpha} y \in Y^c$. Hence $\pi_T Y^c \subseteq Y^c$, which suffices to prove $\pi_T Y^c = Y^c \cap T$. Then

$$Y^c \cap T = \pi_T Y^c \langle (\pi_T \alpha) y; \alpha \in \mathrm{Aut}(V, f), y \in Y \rangle = \langle (\pi_T \alpha \pi_T) y; \alpha \in \mathrm{Aut}(V, f), y \in Y \rangle$$
$$= \langle (\alpha_T) y; \alpha_T \in \mathrm{Aut}(T, f_{|T}), y \in Y \rangle = Y^{cT}. \quad \square$$

Proof of Theorem 1.3 *(i).* We apply Lemma 3.4. Let $T = \langle u_\rho, u_\tau \rangle$. Set $z_{s,q} = f^{R-s} u_\rho + f^{S-q} u_\tau$, and $Y = \langle z_{s,q} \rangle$. Then $Y \subseteq T$, and $Y^{cT} = Y^c \cap T$. It follows from Proposition 2.11 that Y^{cT} is not hyperinvariant in T. Hence $Y^c = \langle z_{s,q} \rangle^c$ is not hyperinvariant in V. $\quad \square$

Suppose $Y \subseteq E$ is characteristic and non-hyperinvariant with respect to $f_{|E}$ How can one extend Y to a subspace X that has these properties in the entire space V?

THEOREM 3.5. *Let E and G be subspaces of V such that (43) holds. Let $Y \in \mathrm{Chinv}(E, f_{|E})$. If Y_s is a subspace of Y and W is a subspace of G such that $Y_s + W$ is characteristic in V then*

$$(Y + W)^c \cap E = Y. \tag{53}$$

The subspace $(Y + W)^c$ is hyperinvariant in V only if Y is hyperinvariant in E.

Proof. Set $X = (Y + W)^c$. We are going to show that $\pi_E X = Y$. Because of $Y \subseteq E$ and $Y \subseteq X$ we see that

$$Y \subseteq (X \cap E) \subseteq \pi_E X. \tag{54}$$

It is obvious that $(Y + W)^c \subseteq Y^c + W^c$. From $Y, W \subseteq (Y + W)^c$ follows $Y^c + W^c \subseteq (Y + W)^c$. Hence $X = Y^c + W^c$. Therefore

$$\pi_E X = \pi_E Y^c + \pi_E W^c. \tag{55}$$

Since $Y_s + W$ is characteristic we have $W^c \subseteq (Y_s + W)^c = Y_s + W$. Hence $Y_s \subseteq E$ and $W \subseteq G$, or equivalently $\pi_E W = 0$, imply $\pi_E W^c \subseteq Y_s \subseteq Y$. Since Y is characteristic in E it follows from Lemma 3.4 that $Y^{cE} = Y = \pi_E Y^c$. Hence (55) implies $\pi_E X \subseteq Y$, and (54) yields $Y = X \cap E = \pi_E X$. Thus we proved (53). If Y is not hyperinvariant in E then it follows from Lemma 3.2 that X is not hyperinvariant in V. $\square$

4. The main theorem

In this section we drop the assumption that V is generated by two vectors. Hence the map f can have more than two elementary divisors. We assume that only two of them are unrepeated. In that case we obtain a description of the family of characteristic non-hyperinvariant subspaces which extends Theorem 1.3(ii).

We first consider the case where f has no unrepeated elementary divisors. In terms of a decomposition $V = E \oplus G$ in (43) that assumption is equivalent to $V = G$.

LEMMA 4.1. *Let* $U = (u_1, \ldots, u_m) \in \mathcal{U}$, $e(u_i) = t_i$, $i = 1, \ldots, m$. *Suppose f has no unrepeated elementary divisors, that is*

$$d(t_i) > 1 \quad for \ all \quad i = 1, \ldots, m. \tag{56}$$

Then $X \subseteq V$ is hyperinvariant if and only if

$$X = \langle f^{r_1} u_1 + \cdots + f^{r_m} u_m \rangle^c \tag{57}$$

for some $\vec{r} = (r_1, \ldots, r_m)$ satisfying (6).

Proof. By Shoda's theorem the assumption (56) implies that each characteristic subspace of V is hyperinvariant. Hence, if X is of the form (57) then $X \in \mathrm{Hinv}(V, f)$. Now suppose X is hyperinvariant. According to Theorem 2.1 we have

$$X = W(\vec{r}) = \langle f^{r_1} u_1 \rangle \oplus \cdots \oplus \langle f^{r_m} u_m \rangle \tag{58}$$

for some $\vec{r}$ satisfying (6). Define $w = f^{r_1} u_1 + \cdots + f^{r_m} u_m$. Then $w \in W(\vec{r})$. Since $W(\vec{r})$ is hyperinvariant, it is obvious that $\langle w \rangle^c \subseteq W(\vec{r})$. To prove the converse inclusion we apply Theorem 2.2(ii) to the hyperinvariant subspace $\langle w \rangle^c$. We obtain $\pi_i w = f^{r_i} u_i \in \langle w \rangle^c$, $i = 1, \ldots, m$, and therefore $W(\vec{r}) \subseteq \langle w \rangle^c$. Hence $X = W(\vec{r}) = \langle w \rangle^c$. $\square$

The notation in Theorem 4.2 below will be the following. We write $\vec{\mu}' = \vec{\mu} + (\vec{e}_\rho + \vec{e}_\tau)$ if

$$(\mu'_1,\ldots,\mu'_\rho,\ldots,\mu'_\tau,\ldots,\mu'_m) =$$
$$(\mu_1,\ldots,\mu_\rho,\ldots,\mu_\tau,\ldots,\mu_m) + (0,\ldots,1,\ldots,1,\ldots,0), \quad (59)$$

that is,

$$\mu'_i = \mu_i \quad \text{if} \quad i \neq \rho, \tau, \quad \text{and} \quad \mu'_\rho = \mu_\rho + 1, \ \mu'_\tau = \mu_\tau + 1. \quad (60)$$

Then $\vec{\mu}' \in \mathcal{L}(\vec{t})$ means

$$\mu'_i \leqslant t_i, i = 1,\ldots,m, \text{ and } \mu'_1 \leqslant \cdots \leqslant \mu'_m, \ t_1 - \mu'_1 \leqslant \cdots \leqslant t_m - \mu'_m. \quad (61)$$

THEOREM 4.2. *Let $|K| = 2$. Suppose that among the elementary divisors of f there are exactly two unrepeated ones, namely λ^R and λ^S. Let $U = (u_1,\ldots,u_m) \in \mathcal{U}$ and $\mathrm{e}(u_\rho) = R$, $\mathrm{e}(u_\tau) = S$. A subspace $X \subseteq V$ is characteristic and not hyperinvariant if and only if*

$$X = \langle f^{\mu_1}u_1 + \cdots + f^{\mu_m}u_m \rangle^c, \quad (62)$$

such that the entries μ_ρ and μ_τ of $\mu = (\mu_1,\ldots,\mu_m)$ satisfy

$$0 \leqslant \mu_\rho < \mu_\tau \quad \text{and} \quad 0 < R - \mu_\rho < S - \mu_\tau, \quad (63)$$

and such that $\vec{\mu} + (\vec{e}_\rho + \vec{e}_\tau) \in \mathcal{L}(\vec{t})$.

Proof. Set $E = \langle u_\rho \rangle \oplus \langle u_\tau \rangle$ and $G = \langle u_i; \mathrm{e}(u_i) = t_i; d(t_i) > 1 \rangle$. Then $V = E \oplus G$, as in (43). Suppose $X \in \mathrm{Chinv}(V,f) \setminus \mathrm{Hinv}(V,f)$. Lemma 3.2 implies

$$X = (X \cap E) \oplus (X \cap G).$$

The subspace $X \cap G$ is hyperinvariant in G, whereas $X \cap E$ is characteristic but not hyperinvariant in E. By assumption E is generated by two elements. Hence it follows from Proposition 2.10 that $X \cap E = \langle z \rangle^{c_E}$. Referring to (33) and (34) we obtain $z = f^{\mu_\rho}u_\rho + f^{\mu_\tau}u_\tau$ for some integers μ_ρ, μ_τ satisfying (63). According to Lemma 4.1 we have $X \cap G = \langle w \rangle^{c_G}$ for some

$$w = \sum_{1 \leqslant i \leqslant m; \ i \neq \rho, \tau} f^{\mu_i}u_i \in X \cap G.$$

Set $\hat{z} = z + w$. Let us show that $X = \langle \hat{z} \rangle^c$. If $\alpha_E \in \mathrm{Aut}(E, f_{|E})$ and $\alpha_G \in \mathrm{Aut}(f_{|G}, V)$ then $\alpha_V = \alpha_E + \alpha_G \in \mathrm{Aut}(f,V)$. Hence $\alpha_E z + \alpha_G w = \alpha_V(z + w) \in \langle \hat{z} \rangle^c$, and we obtain

$$X = \langle z \rangle^{c_E} \oplus \langle w \rangle^{c_G} \subseteq \langle \hat{z} \rangle^c. \quad (64)$$

Since X is characteristic it is obvious that $\langle \hat{z} \rangle^c \subseteq X$. Therefore $X = \langle \hat{z} \rangle^c$, and X is of the form (62). From (27) follows

$$\left(\langle z \rangle^{c_E} \right)_{H|E} = \langle f^{\mu_\rho + 1}u_\rho \rangle \oplus \langle f^{\mu_\tau + 1}u_\tau \rangle.$$

The proof of Lemma 4.1 shows that

$$\langle w \rangle^{cG} = \bigoplus_{1 \leqslant i \leqslant m; \, i \neq \rho, \tau} \langle f^{\mu_i} u_i \rangle.$$

Then (48) yields

$$X_H = \left(\langle z \rangle^{cE} \right)_{H|E} \oplus \langle w \rangle^{cG} = \bigoplus_{i=1,\dots,m} \langle f^{\mu_i'} u_i \rangle, \tag{65}$$

with μ_i', $i = 1,\dots,m$, defined by (60). The subspace X_H is hyperinvariant. Hence Theorem 2.1 implies $\vec{\mu}' = (\mu_1',\dots,\mu_m') \in \mathscr{L}(\vec{t})$.

Now consider a subspace

$$X = \left\langle \sum_{i=1}^m f^{\mu_i} u_i \right\rangle^c$$

assuming that the inequalities (63) hold and $\vec{\mu}' = \vec{\mu} + (\vec{e}_\rho + \vec{e}_\tau) \in \mathscr{L}(\vec{t})$. Define

$$z = f^{\mu_\rho} u_\rho + f^{\mu_\tau} u_\tau, \quad w = \sum_{1 \leqslant i \leqslant m; \, i \neq \rho, \tau} f^{\mu_i} u_i, \quad \text{and} \quad \hat{z} = z + w.$$

Then $X = \langle \hat{z} \rangle^c$. Since X is characteristic it follows from Lemma 3.2 that $\pi_E \hat{z} = z \in X \cap E$, $\pi_G \hat{z} = w \in X \cap G$. Set $Y = \langle z \rangle^{cE}$ and $W = \langle w \rangle^{cG}$. Then Lemma 3.4(iii) implies $Y = \langle z \rangle^c \cap E$. The inequalities (61) ensure that W is hyperinvariant in G and that $W = \oplus_{i \neq \rho, \tau} \langle f^{\mu_i} u_i \rangle$. Let us show first that $X = (Y + W)^c$. Since X is characteristic we have $(Y + W)^c \subseteq X$. Conversely, $z + w \in Y + W$ yields $X = \langle z + w \rangle^c \subseteq (Y + W)^c$. It follows from (63) that $\langle z \rangle^{cE} = Y$ is not hyperinvariant in E. Hence

$$Y_{H|E} = \langle f^{\mu_\rho + 1} u_\rho, f^{\mu_\tau + 1} u_\tau \rangle = \langle f^{\mu_\rho'} u_\rho, f^{\mu_\tau'} u_\tau \rangle.$$

Then $\vec{\mu}' \in \mathscr{L}(\vec{t})$ implies that $Y_{H|E} + W$ is hyperinvariant in V. We apply Theorem 3.5 choosing $Y_s = Y_{H|E}$ and conclude that X is not hyperinvariant. $\square$

EXAMPLE 4.3. We consider a map f with elementary divisors $\lambda, \lambda^3, \lambda^7, \lambda^7$ and $V = \oplus_{i=1}^4 \langle u_i \rangle$ such that $\vec{t} = (e(u_i)) = (1,3,7,7)$. We know that

$$0 \leqslant \mu_1 < 1, 0 \leqslant \mu_2 < 3 \quad \text{and} \quad \mu_1 < \mu_2, \; 1 - \mu_1 < 3 - \mu_2,$$

has the unique solution $(\mu_1, \mu_2) = (0, 1)$. Hence

$$\vec{\mu} = (0, 1, \mu_3, \mu_4) \quad \text{and} \quad \vec{\mu}' = (1, 2, \mu_3, \mu_4).$$

Then $\vec{\mu}' \in \mathscr{L}(\vec{t})$ if and only if $(\mu_3, \mu_4) = (j, j)$, $j = 2,\dots,6$. Define $g_j = u_1 + fu_2 + f^j u_3 + f^j u_4$ and $X_j = \langle g_j \rangle^c$, $j = 2,\dots,6$, and $X = \langle u_1 + fu_2 \rangle^c$. Then

$$e(u_2 + f^{j-1} u_3 + f^{j-1} u_4) = e(u_2), \; j = 5, 6,$$

implies $X_5 = X_6 = X$. We obtain 4 different characteristic not hyperinvariant subspaces, namely X, X_2, X_3, X_4. There are $n_H = 30$ hyperinvariant subspaces in V.

4.1. Concluding remarks

We have restricted our study to the case of two unrepeated elementary divisors. Using methods of our paper one can prove more general results. For example one can extend Theorem 1.3(i) as follows.

THEOREM 4.4. *Let* $|K| = 2$. *Suppose* $\lambda^{t_{\rho_1}}, \ldots, \lambda^{t_{\rho_k}}$ *are unrepeated elementary divisors of* f *such that* $t_{\rho_1} < \cdots < t_{\rho_k}$. *Let* $U = (u_1, \ldots, u_m) \in \mathscr{U}$ *be a generator tuple with* $\mathrm{e}(u_{\rho_j}) = t_{\rho_j}$, $j = 1, \ldots, k$. *Suppose* $(\mu_{\rho_1}, \ldots, \mu_{\rho_k}) \in \mathbb{N}_0^k$ *and set* $z = f^{\mu_{\rho_1}} u_{\rho_1} + \cdots + f^{\mu_{\rho_k}} u_{\rho_k}$. *If*

$$\mu_{\rho_j} < t_{\rho_j}, j = 1, \ldots, k, \tag{66}$$

and

$$\mu_{\rho_1} < \mu_{\rho_2} < \cdots < \mu_{\rho_k} \quad and \quad 0 < t_{\rho_1} - \mu_{\rho_1} < \cdots < t_{\rho_k} - \mu_{\rho_k}, \tag{67}$$

then $X = \langle z \rangle^c$ *is a characteristic and not hyperinvariant subspace of* V.

In Example 4.5 below we illustrate Theorem 4.4. We also construct a non-hyperinvariant subspace that is not the characteristic hull of a single element. Such subspaces will be studied in a subsequent paper.

EXAMPLE 4.5. *Let* $|K| = 2$. Suppose f has elementary divisors $\lambda, \lambda^3, \lambda^5$ such that $V = \langle u_1 \rangle \oplus \langle u_2 \rangle \oplus \langle u_3 \rangle$ and

$$(t_1, t_2, t_2) = \big(\mathrm{e}(u_1), \mathrm{e}(u_2), \mathrm{e}(u_3)\big) = (1, 3, 5).$$

Then $(\mu_1, \mu_2, \mu_3) = (0, 1, 2)$ is the only solution of the set of inequalities

$$0 \leqslant \mu_1 < 1, 0 \leqslant \mu_2 < 3, 0 \leqslant \mu_3 < 5; 0 \leqslant \mu_1 < \mu_2 < \mu_3, 0 < 1 - \mu_1 < 3 - \mu_2 < 5 - \mu_3.$$

Set

$$z = u_1 + f u_2 + f^2 u_3. \tag{68}$$

Then $X = \langle z \rangle^c \in \mathrm{Chinv}(V, f) \setminus \mathrm{Hinv}(V, f)$. The fact that X is not hyperinvariant can be verified as follows. We have $\mathrm{e}(z) = 3$ and the indicator sequence of z is $H(z) = (0, 2, 4, \infty, \infty)$. Define $Y = \{x \in V \mid H(x) = H(z)\}$. Then

$$Y = \{z + v_2 + v_3; v_2 \in f^2 \langle u_2 \rangle; v_3 \in f^3 \langle u_3 \rangle\}.$$

Hence

$$\langle z \rangle^c = \langle Y \rangle = \langle z, f^2 u_2, f^3 u_3 \rangle. \tag{69}$$

Then $\pi_3 z = f^2 u_3 \notin \langle Y \rangle$. Therefore $\langle Y \rangle$ is not hyperinvariant in V.

The following subspace W is also in $\mathrm{Chinv}(V, f) \setminus \mathrm{Hinv}(V, f)$. We shall see that it is not the characteristic hull of a single element. Let

$$W = \langle z_1, z_2 \rangle^c \quad and \quad z_1 = u_1 + f u_2, z_2 = f^2 u_3.$$

Then $\langle z_1, f^3 u_3 \rangle^c = \langle z_1 \rangle$ and $\langle z_2 \rangle^c = \mathrm{Im} f^2 \cap \mathrm{Ker} f^3 = \mathrm{Im} f^2 = \langle f^2 u_2, f^2 u_3 \rangle$. Therefore $W = \langle z_1 \rangle \oplus \langle f^2 u_3 \rangle$. We have $\pi_1 z_1 = u_1 \notin W$. Hence W is not hyperinvariant. Suppose

$W = \langle w \rangle^c$ for some $w \in W$. Then $w = x + y$ with $x \in \langle z_1 \rangle$, $y \in \langle f^2 u_3 \rangle$. Because of $z_1 \in W$ and $\mathrm{h}(z_1) = 0$ it is necessary that $\mathrm{h}(w) = 0$. Hence $\mathrm{h}(x) = 0$. Therefore $\langle x \rangle = \langle z_1 \rangle$, and $x = \gamma z_1$ for some $\gamma \in \mathrm{Aut}(f, V)$. Hence we can assume $w = z_1 + y$. If $\mathrm{h}(y) \geqslant 3$ then $\mathrm{e}(w) = \mathrm{e}(z_1)$ and $\langle w \rangle^c = \langle z_1 \rangle^c \subsetneqq W$. If $\mathrm{h}(y) = 2$ then $y = \beta(f^2 u_3)$ for some $\beta \in \mathrm{Aut}(f, V)$, and therefore

$$\langle w \rangle^c = \langle z_1 + f^2 u_3 \rangle^c = \langle z \rangle^c,$$

where z is given by (68). We have seen before that $\pi_3 z = f^2 u_3 \notin \langle z \rangle^c$. Hence $\langle w \rangle^c = \langle z_1 + z_2 \rangle^c \subsetneqq W$. Therefore $\langle w \rangle^c \neq W$ for all $w \in W$. $\square$

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