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DISCRETE DIRAC SYSTEM: RECTANGULAR WEYL FUNCTIONS, DIRECT AND INVERSE PROBLEMS

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Abstract. A transfer matrix function representation of the fundamental solution of the general-type discrete Dirac system, corresponding to rectangular Schur coefficients and Weyl functions, is obtained. Connections with Szegö recurrence, Schur coefficients and structured matrices are treated. A Borg-Marchenko-type uniqueness theorem is derived. Inverse problems on the interval and semi-axis are solved.

1. Introduction

In this paper we deal with a discrete Dirac-type (or simply Dirac) system:

$$y_{k+1}(z)=(I_m+\mathrm{i}zjC_k)y_k(z) \quad (k\in\mathbb{N}_0), \tag{1.1}$$

where $\mathbb{N}_0$ stands for the set of non-negative integers, I_m is the $m\times m$ identity matrix, "i" is the imaginary unit ($\mathrm{i}^2=-1$) and the $m\times m$ matrices $\{C_k\}$ are positive and j-unitary:

$$C_k>0, \quad C_kjC_k=j, \quad j:=\begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2}\end{bmatrix} \quad (m_1+m_2=m;\ m_1,m_2\neq 0). \tag{1.2}$$

Discrete systems are of great interest and their study is sometimes more complicated than the study of the corresponding continuous systems (see, e.g., [1, 2, 3, 7, 12] and references therein). The subcase $m_1=m_2$ of system (1.1) (satisfying (1.2)) corresponds to the self-adjoint Dirac-type systems, which were studied in [13] (and the subcase $j=I_m$ of system (1.1) corresponds to the skew-self-adjoint Dirac-type systems, an important subclass of which was investigated in [20, 23]). The analogies between system (1.1) and continuous Dirac-type systems are also discussed in [13, 20, 23] in detail. Here we follow the paper [14] on the continuous case, where m_1 does not necessarily equal m_2 and the $m_2\times m_1$ Weyl matrix functions are, correspondingly, rectangular.

It is essential that Dirac system (1.1), (1.2) is equivalent to the very well-known (see, e.g., [10, 28]) Szegö recurrence. This connection is discussed in detail in Section 2. Inverse problems for the subcase of the scalar Schur (or Verblunsky) coefficients were studied, for instance, in [5, 28] (see also various references therein), and here we deal with the rectangular matrix Schur coefficients.

In this paper Im denotes the image of a matrix (or an operator), $\sigma(A)$ stands for the spectrum of A and "span" stands for the linear span.

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2. Dirac system and Szegö recurrence

The following simple proposition sets up an important fact we will later make use of, and could be of independent interest in the theory of functions (and powers, in particular) of matrices, which is developed in a series of works (see, e.g., [6, 29] and references therein). We always assume that non-negative powers of matrices are chosen.

PROPOSITION 2.1. *Let an $m \times m$ matrix C satisfy the relations*

$$C > 0, \quad CjC = j \quad (j = j^* = j^{-1}). \tag{2.1}$$

Then the following relations hold for all $s \in \mathbb{R}$:

$$C^s > 0, \quad C^s jC^s = j. \tag{2.2}$$

Proof. Since $C > 0$, it admits a representation

$$C = u^* Du, \quad C^s = u^* D^s u, \tag{2.3}$$

where D is a diagonal matrix and

$$D > 0, \quad u^* u = uu^* = I_m. \tag{2.4}$$

We substitute (2.3) into the second equality in (2.1) to derive

$$u^* Du ju^* Du = j,$$

or, equivalently,

$$DJD = J, \quad J = J^* = J^{-1} := u ju^*. \tag{2.5}$$

Formula (2.5) yields $D^{-1} = JDJ$ and, taking power s of both parts of this equality, we obtain

$$D^{-s} = JD^s J, \qquad D^s JD^s = J. \tag{2.6}$$

Finally, using (2.4)–(2.6) we have

$$u^* D^s u ju^* D^s u = j. \quad \square \tag{2.7}$$

We substitute $s = 1/2$ and apply Proposition 2.1 to the matrices C_k in order to obtain the next proposition.

PROPOSITION 2.2. *Let matrices C_k satisfy (1.2). Then they admit representations*

$$C_k = 2\beta(k)^* \beta(k) - j, \quad \beta(k) j\beta(k)^* = I_{m_1}, \tag{2.8}$$
$$C_k = j + 2\gamma(k)^* \gamma(k), \quad \gamma(k) j\gamma(k)^* = -I_{m_2}, \tag{2.9}$$

where $\beta(k)$ and $\gamma(k)$ are $m_1 \times m$ and $m_2 \times m$ matrices given by (2.10) and (2.11), respectively.

Proof. We note that matrices C_k satisfy the conditions of Proposition 2.1, and so (2.2) holds for $C = C_k$. Next we put

$$\beta(k) := \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} C_k^{1/2} \tag{2.10}$$

and take into account the equality

$$C_k = C_k^{1/2}\Big(2\begin{bmatrix} I_{m_1} & 0 \end{bmatrix}^* \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} - j\Big)C_k^{1/2}.$$

Now, representation (2.8) is apparent from (2.2) taken with $s = 1/2$. In a similar way, formula (2.2) and equality $I_m = j + 2\begin{bmatrix} 0 & I_{m_2} \end{bmatrix}^* \begin{bmatrix} 0 & I_{m_2} \end{bmatrix}$ imply representation (2.9) for

$$\gamma(k) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} C_k^{1/2}. \quad \square \tag{2.11}$$

Now, we will consider interrelations between Dirac system (1.1), (1.2) and Szegö recurrence, which is given by the formula

$$X_{k+1}(\lambda) = \mathscr{D}_k H_k \begin{bmatrix} \lambda I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} X_k(\lambda), \tag{2.12}$$

where

$$H_k = \begin{bmatrix} I_{m_1} & \rho_k \\ \rho_k^* & I_{m_2} \end{bmatrix}, \quad \mathscr{D}_k = \mathrm{diag}\Big\{\big(I_{m_1} - \rho_k\rho_k^*\big)^{-\frac{1}{2}},\ \big(I_{m_2} - \rho_k^*\rho_k\big)^{-\frac{1}{2}}\Big\}, \tag{2.13}$$

and the $m_1 \times m_2$ matrices ρ_k are strictly contractive, that is,

$$\|\rho_k\| < 1. \tag{2.14}$$

Here $\mathrm{diag}\{d_1, d_2\}$ stands for the block diagonal matrix with the blocks d_1 and d_2 on the main diagonal.

REMARK 2.3. When $m_1 = m_2 = 1$, the factor $(1 - |\rho_k|^2)^{-1/2}$ in (2.12) can be easily removed and we obtain systems as in [4, 5], where direct and inverse problems for the case of scalar strictly pseudo-exponential potentials have been treated. The square matrix version (i.e., the version where $m_1 = m_2$) of Szegö recurrence, its connections with Schur coefficients and applications are discussed in [8, 9] (see also references therein). For the rectangular matrices ρ_k see, for instance, [10]. We note that $\mathscr{D}_k H_k$ is the Halmos extension of ρ_k (see [10, p. 167]), and that the matrices $\mathscr{D}_k$ and H_k commute (which easily follows, e.g., from [10, Lemma 1.1.12]). The matrix $\mathscr{D}_k H_k$ is j-unitary and positive, that is,

$$\mathscr{D}_k H_k j H_k \mathscr{D}_k = H_k \mathscr{D}_k j \mathscr{D}_k H_k = j, \tag{2.15}$$

$$\mathscr{D}_k H_k > 0. \tag{2.16}$$

According to [11, Theorem 1.2], any j-unitary matrix C admits a representation, which is close to the Halmos extension. More precisely, partitioning C into blocks $C=\{c_{ik}\}_{i,k=1}^2$ we see that the $m_1\times m_1$ block c_{11} and the $m_2\times m_2$ block c_{22} are invertible. Then, putting

$$\rho=c_{12}c_{22}^{-1}=(c_{11}^{-1})^*c_{21}^*,\quad u_1=\big(I_{m_1}-\rho\rho^*\big)^{1/2}c_{11},\quad u_2=\big(I_{m_2}-\rho^*\rho\big)^{1/2}c_{22},$$

we have the respresentation:

$$C=\mathscr{D}H\begin{bmatrix}u_1&0\\0&u_2\end{bmatrix},\quad u_i^*u_i=u_iu_i^*=I_{m_i};\quad H=\begin{bmatrix}I_{m_1}&\rho\\\rho^*&I_{m_2}\end{bmatrix},\tag{2.17}$$

$$\mathscr{D}=\mathrm{diag}\Big\{\big(I_{m_1}-\rho\rho^*\big)^{-\frac12},\ \big(I_{m_2}-\rho^*\rho\big)^{-\frac12}\Big\},\quad \rho^*\rho<I_{m_2}.\tag{2.18}$$

Although relations (2.15)–(2.17) are well-known, it does not appear that the converse to (2.15), (2.16) has been shown. Hence, we prove it below.

PROPOSITION 2.4. *Let an $m\times m$ matrix C be j-unitary and positive. Then it admits a representation*

$$C=\mathscr{D}H,\tag{2.19}$$

where H and $\mathscr{D}$ are of the form (2.17) *and* (2.18) (*i.e., the last factor on the right-hand side of the first equality in* (2.17) *is removed*).

Proof. Recall that C admits representation (2.17). We fix a unitary matrix $\widetilde{U}$ such that $\mathscr{D}H=\widetilde{U}\widetilde{D}\widetilde{U}^*$, where $\widetilde{D}$ is a diagonal matrix, $\widetilde{D}>0$. Then, relations $C=C^*$ and (2.17) yield the equality

$$\widetilde{U}\widetilde{D}\widetilde{U}^*\begin{bmatrix}u_1&0\\0&u_2\end{bmatrix}=\begin{bmatrix}u_1^*&0\\0&u_2^*\end{bmatrix}\widetilde{U}\widetilde{D}\widetilde{U}^*,$$

which we rewrite in the form

$$\widetilde{D}\widehat{U}=\widehat{U}^*\widetilde{D},\quad \widehat{U}:=\widetilde{U}^*\begin{bmatrix}u_1&0\\0&u_2\end{bmatrix}\widetilde{U}.\tag{2.20}$$

According to (2.20), $\widetilde{D}\widehat{U}$ is a selfadjoint matrix, and so $\widetilde{D}^{1/2}\widehat{U}\widetilde{D}^{-1/2}$ is a selfadjoint matrix as well. Thus, there is a representation

$$\widetilde{D}^{1/2}\widehat{U}\widetilde{D}^{-1/2}=\breve{U}D_1\breve{U}^*,\tag{2.21}$$

where $\breve{U}$ and $D_1=D_1^*$ are unitary and diagonal matrices, respectively. The definition of $\widehat{U}$ in (2.20) implies that $\widehat{U}$ is unitary. Therefore, in view of (2.21), D_1 is linearly similar to a unitary matrix, that is, its entries are ±1. Moreover $D_1>0$, since $C>0$ and formulas (2.17), (2.20) and (2.21) yield

$$C=\widetilde{U}\widetilde{D}\widetilde{U}^*\begin{bmatrix}u_1&0\\0&u_2\end{bmatrix}=\widetilde{U}\widetilde{D}\widehat{U}\widetilde{U}^*=\widetilde{U}\widetilde{D}^{1/2}\breve{U}D_1\breve{U}^*\widetilde{D}^{1/2}\widetilde{U}^*.\tag{2.22}$$

From the inequality $D_1 > 0$ and the fact that the entries of D_1 equal either 1 or -1, we have $D_1 = I_m$. Thus, the last equality in (2.22) implies $C = \widetilde{U}\widetilde{D}\widetilde{U}^*$, that is, (2.19) holds. □

Proposition 2.4 completes Propositions 2.1 and 2.2 which deal with the representations and properties of C_k. Taking into account (2.15), (2.16) and Proposition 2.4, we rewrite Szegö recurrence (2.12) in an equivalent form

$$X_{k+1}(\lambda) = \widetilde{C}_k \begin{bmatrix} \lambda I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} X_k(\lambda), \quad k \in \mathbb{N}_0, \tag{2.23}$$

$$\widetilde{C}_k > 0, \quad \widetilde{C}_k j \widetilde{C}_k = j. \tag{2.24}$$

Using (2.24) we see that the matrix functions U_k, which are given by the equalities

$$U_0 := I_m, \quad U_{k+1} := \mathrm{i} U_k \widetilde{C}_k j = \prod_{r=0}^{k} (\mathrm{i}\widetilde{C}_r j) \quad (k \geqslant 0), \tag{2.25}$$

are also j-unitary. From (2.24) and (2.25) we have

$$\begin{aligned}&(\mathrm{i}+z)U_{k+1}(I_m + \mathrm{i}zj)\widetilde{C}_k \begin{bmatrix} \frac{z-\mathrm{i}}{z+\mathrm{i}} I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} (I_m + \mathrm{i}zj)^{-1} U_k^{-1} \\ &= I_m + \mathrm{i}z U_{k+1} j U_{k+1}^{-1}. \end{aligned} \tag{2.26}$$

In view of (2.26), the function y_k of the form

$$y_k(z) = (\mathrm{i}+z)^k U_k (I_m + \mathrm{i}zj) X_k\left(\frac{z-\mathrm{i}}{z+\mathrm{i}}\right) \tag{2.27}$$

satisfies (1.1), where $y_0(z) = (I_m + \mathrm{i}zj)X_0(z)$ and $C_k = jU_{k+1} j U_{k+1}^{-1}$. Since U_{k+1} is j-unitary, we rewrite C_k as

$$C_k = jU_{k+1}U_{k+1}^* j, \tag{2.28}$$

and so (1.2) holds. Because of (2.25), (2.28) and the j-unitarity of U_k, we have $jU_k^* C_k U_k j = \widetilde{C}_k^2$, that is,

$$\widetilde{C}_k = (jU_k^* C_k U_k j)^{1/2}. \tag{2.29}$$

The following theorem describes interconnections between systems (1.1) and (2.23).

THEOREM 2.5. *Dirac systems* (1.1), (1.2) *and Szegö recurrences* (2.23), (2.24) *are equivalent. The transformation $\mathfrak{M} : \{\widetilde{C}_k\} \to \{C_k\}$ of the Szegö recurrence into the Dirac system, and the transformation of their solutions, are given, respectively, by formulas* (2.28) *and* (2.27)*, where the matrices $\{U_k\}$ are defined in* (2.25)*. The mapping $\mathfrak{M}$ is bijective, and the inverse mapping is obtained by applying* (2.29) (*and substitution of the result into* (2.25)) *for the successive values of k.*

Proof. It was already proved above that the formulas (2.28) and (2.27) describe a mapping of the Szegö recurrence and its solution into the Dirac system and its solution, respectively. Moreover, the mapping $\mathfrak{M}$ is injective, since we can successively and uniquely recover $\widetilde{C}_k$ and U_{k+1} from C_k and U_k using formulas (2.29) and (2.25), respectively.

Next, we prove that $\mathfrak{M}$ is surjective. Indeed, given an arbitrary sequence $\{C_k\}$ satisfying (1.2), we apply relation (2.29) to the matrices of this sequence (and substitute the result into (2.25)) for the successive values of k. In this way we construct a sequence $\{\widetilde{C}_k\}$. Since the matrices $jU_k^*C_kU_kj$ are positive and j-unitary, we see, from (2.29) and Proposition 2.1, that the matrices $\widetilde{C}_k$ are also positive and j-unitary. Now, we apply the mapping $\mathfrak{M}$ to $\{\widetilde{C}_k\}$. Taking into account (2.25) and (2.29), we derive

$$jU_{k+1}U_{k+1}^*j = jU_k\widetilde{C}_k^2U_k^*j = jU_k(jU_k^*C_kU_kj)U_k^*j = C_k, \tag{2.30}$$

that is, $\mathfrak{M}$ maps the constructed sequence $\{\widetilde{C}_k\}$ into the initial sequence $\{C_k\}$. Recall that we started from an arbitrary $\{C_k\}$ satisfying (1.2). Hence, $\mathfrak{M}$ is surjective. □

3. Weyl theory: direct problems

In this section we introduce Weyl functions for matricial discrete Dirac systems (1.1). Next we prove the Weyl function's existence and, moreover, give a procedure to construct it (direct problems). Finally, we construct the S-node, which corresponds to system (1.1), and the transfer matrix function representation of the fundamental solution W_k. (See, e.g., [25, 26, 27] on the S-nodes and the transfer matrix functions in Lev Sakhnovich form.)

We normalize the fundamental $m\times m$ solution $\{W_k\}$ of (1.1) as follows:

$$W_0(z) = I_m. \tag{3.1}$$

Similar to the continuous analog of (1.1) in [14, 16] (see also the canonical system case [27, p. 7]), the Weyl functions of system (1.1) on the interval $[0, r]$ (i.e., of system (1.1) considered for $0 \leqslant k \leqslant r$) are defined by the Möbius (linear-fractional) transformation:

$$\varphi_r(z,\mathscr{P}) = \begin{bmatrix}0 & I_{m_2}\end{bmatrix} W_{r+1}(z)^{-1}\mathscr{P}(z)\Big(\begin{bmatrix}I_{m_1} & 0\end{bmatrix} W_{r+1}(z)^{-1}\mathscr{P}(z)\Big)^{-1}, \tag{3.2}$$

where $\mathscr{P}(z)$ are nonsingular $m\times m_1$ matrix functions with property-j. That is, $\mathscr{P}(z)$ are meromorphic in $\mathbb{C}_+$ matrix functions such that

$$\mathscr{P}(z)^*\mathscr{P}(z) > 0, \quad \mathscr{P}(z)^*j\mathscr{P}(z) \geqslant 0 \tag{3.3}$$

for all points in $\mathbb{C}_+$ (excluding, possibly, a discrete set). The first inequality in (3.3) means nonsingularity (nondegeneracy) of $\mathscr{P}$ and the second inequality is called property-j. Since $\mathscr{P}$ is meromorphic, property-j almost everywhere in $\mathbb{C}_+$ and the first inequality in (3.3) at some $z_0 \in \mathbb{C}_+$ suffice for the conditions on $\mathscr{P}$ to hold.

Before showing that the Möbius transformation (3.2) is indeed correctly defined for matrix functions $\mathscr{P}$ satisfying (3.3) and for $z \neq \mathrm{i}$, we show that the situation at $z = \mathrm{i}$

is somewhat more complicated. Namely, the determinant of the fundamental solution vanishes at the point $z = \mathrm{i}$. First, it is apparent from (1.1) and (3.1) that

$$W_{r+1}(z) = \prod_{k=0}^{r} (I_m + \mathrm{i}zjC_k). \tag{3.4}$$

In view of (2.9) and (3.4) we obtain

$$W_{r+1}(\mathrm{i}) = (-2)^{r+1} \prod_{k=0}^{r} \big(j\gamma(k)^*\gamma(k)\big). \tag{3.5}$$

Hence, $\det W_{r+1}(\mathrm{i}) = 0$, and we do not consider $z = \mathrm{i}$ in this section.

REMARK 3.1. We note that the behavior of Weyl functions in the neighborhood of $z = \mathrm{i}$ is essential for the inverse problems that are dealt with in the next section. Therefore, unlike the Weyl disc case (see Notation 3.4), in the definition (3.2) of the Weyl functions on the interval we assume that $\mathscr{P}$ is not only nonsingular with property-j but also has an additional property. Namely, that it is well-defined and nonsingular at $z = \mathrm{i}$. We do not use this additional property in this section, although, in important cases, it could be obtained via multiplication by a scalar function.

The lemma below shows that the transformations $\varphi_r(z, \mathscr{P})$ are well-defined.

LEMMA 3.2. *Fix any $z \in \mathbb{C}_+$ such that the inequalities* $\det W_r(z) \neq 0$ *and* (3.3) *hold. Then we have the inequality*

$$\det\Big(\big[I_{m_1}\ \ 0\big]\, W_{r+1}(z)^{-1}\mathscr{P}(z)\Big) \neq 0. \tag{3.6}$$

Proof. Using (1.2) and (2.9) we obtain

$$\begin{aligned}(I_m + \mathrm{i}zjC_k)^* j(I_m + \mathrm{i}zjC_k) &= (1 + \mathrm{i}(z-\overline{z}) + |z|^2)j + 2\mathrm{i}(z-\overline{z})\gamma(k)^*\gamma(k) \\ &\leqslant (1 - 2\Im(z) + |z|^2)j, \\ &\qquad (1 - 2\Im(z) + |z|^2) > 0 \quad \text{for} \quad z \neq \mathrm{i}.\end{aligned} \tag{3.7}$$

Since the equality (3.4) holds, formula (3.7) implies that

$$\big(W_{r+1}(z)^{-1}\big)^* jW_{r+1}(z)^{-1} \geqslant (1 - 2\Im(z) + |z|^2)^{-r-1} j \quad (z \in \mathbb{C}_+,\ z \neq \mathrm{i}). \tag{3.8}$$

Because of (3.3) and (3.8), we see that $\widetilde{\mathscr{P}} := W_{r+1}(z)^{-1}\mathscr{P}(z)$ satisfies the inequality $\widetilde{\mathscr{P}}^* j\widetilde{\mathscr{P}} \geqslant 0$. It is apparent that the same inequality holds for the matrix $\big[I_{m_1}\ \ 0\big]^*$. In other words, $\operatorname{Im} W_{r+1}(z)^{-1}\mathscr{P}(z)$ and $\operatorname{Im}\big[I_{m_1}\ \ 0\big]^*$ are maximal j-non-negative subspaces. Therefore, the inequality (3.6) follows in a standard way from j-theoretic considerations (see, e.g., the proof of (3.48) or the proof of [13, inequality (5.6)] for such considerations). $\square$

COROLLARY 3.3. *The following relations hold for the fundamental solution* W_{r+1} *of* (1.1) (*where the matrices* C_k *satisfy* (1.2)) :

$$\det W_{r+1}(z) \neq 0, \quad W_{r+1}(z)^{-1} = (1+z^2)^{-r-1} jW_{r+1}(\overline{z})^* j \qquad (z \neq \pm \mathrm{i}). \tag{3.9}$$

Proof. Relations (3.7) and (3.4) imply that

$$W_{r+1}(z)^* jW_{r+1}(z) = (1+z^2)^{r+1} j, \qquad z = \overline{z}.$$

Hence, using analyticity considerations, we obtain

$$W_{r+1}(\overline{z})^* jW_{r+1}(z) \equiv (1+z^2)^{r+1} j, \tag{3.10}$$

and (3.9) is apparent. □

NOTATION 3.4. The set of values of matrices $\varphi_r(z,\mathscr{P})$, which are given by the transformation (3.2) where parameter matrices $\mathscr{P}(z)$ satisfy (3.3), is denoted by $\mathscr{N}(r,z)$ (or, sometimes, simply $\mathscr{N}(r)$).

Usually, $\mathscr{N}(r,z)$ is called the Weyl disk. Indeed, taking into account (3.8) and following considerations from [15], we derive that $\mathscr{N}(r,z)$ is a matrix disk (ball). More precisely, putting

$$\mathfrak{A} = \{\mathfrak{A}_{ik}\}_{i,k=1}^2 = \mathfrak{A}(r,z) := W_{r+1}(z)^* jW_{r+1}(z),$$

we see that $-\mathfrak{A}_{22} > 0$, $\mathfrak{A}_{11} - \mathfrak{A}_{12}\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21} > 0$ and formulas (2.18) and (2.19) from [15] hold, that is,

$$\mathscr{N}(r,z) = \{\widehat{\varphi} : \widehat{\varphi} = \rho_l \omega \rho_r - \mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}, \quad \omega^*\omega \leqslant I_{m_2}\} \tag{3.11}$$

$$\rho_l := \big(-\mathfrak{A}_{22}\big)^{-1/2}, \quad \rho_r := (\mathfrak{A}_{11} - \mathfrak{A}_{12}\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}\big)^{1/2}, \tag{3.12}$$

where ρ_l and ρ_r are the left and right semi-radii of the Weyl disk.

COROLLARY 3.5. *The sets* $\mathscr{N}(r,z)$ *are embedded* (*i.e.,* $\mathscr{N}(r,z) \subseteq \mathscr{N}(r-1,z)$) *for all* $r > 0$ *and* $z \in \mathbb{C}_+$, $z \neq \mathrm{i}$. *Moreover, for all* φ_k $(k \geqslant 0)$ *we have*

$$\varphi_k(z)^* \varphi_k(z) \leqslant I_{m_1}. \tag{3.13}$$

Proof. It follows from Corollary 3.3 that the matrices $W_{r+1}(z)$, $W_r(z)$ and $(I_m + \mathrm{i}zjC_r)$ are invertible. Hence formulas (3.3) and (3.7) imply that $\widetilde{\mathscr{P}} := (I_m + \mathrm{i}zjC_r)^{-1}\mathscr{P}(z)$ satisfies (3.3). Therefore, we rewrite (3.2) in the form

$$\varphi_r(z,\mathscr{P}) = \big[0 \ I_{m_2}\big]\, W_r(z)^{-1}\widetilde{\mathscr{P}}(z)\Big(\big[I_{m_1} \ 0\big]\, W_r(z)^{-1}\widetilde{\mathscr{P}}(z)\Big)^{-1}, \tag{3.14}$$

and see that $\varphi_r(z) \in \mathscr{N}(r-1,z)$ $(r > 0)$. Inequality (3.13) is obtained for the matrices from $\mathscr{N}(0,z)$ via substitution of $r = 0$ into (3.14). □

Weyl functions of system (1.1) on the semi-axis $\mathbb{N}_0$ of non-negative integers are defined in a different and more traditional way (in terms of summability) – see definition below. We will show also that the definitions of Weyl functions on the interval and semi-axis are interrelated.

DEFINITION 3.6. The Weyl-Titchmarsh (or simply Weyl) function of Dirac system (1.1) (which is given on the semi-axis $0 \leqslant k < \infty$ and satisfies (1.2)) is an $m_2 \times m_1$ matrix function $\varphi(z)$ $(z \in \mathbb{C}_+)$, such that the following inequality holds:

$$\sum_{k=0}^{\infty} q(z)^k \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} W_k(z)^* C_k W_k(z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} < \infty, \tag{3.15}$$

$$q(z) := (1+|z|^2)^{-1}. \tag{3.16}$$

LEMMA 3.7. *If $\varphi_r(z) \in \mathcal{N}(r,z)$, we have the inequality*

$$\sum_{k=0}^{r} q(z)^k \begin{bmatrix} I_{m_1} & \varphi_r(z)^* \end{bmatrix} W_k(z)^* C_k W_k(z) \begin{bmatrix} I_{m_1} \\ \varphi_r(z) \end{bmatrix} \leqslant \frac{1+|z|^2}{\mathrm{i}(\overline{z}-z)} \big(I_{m_1} - \varphi_r(z)^* \varphi_r(z)\big). \tag{3.17}$$

Proof. Because of (1.1) and (1.2) we have

$$\begin{aligned} W_{k+1}(z)^* j W_{k+1}(z) &= W_k(z)^* \Big(I_m - \mathrm{i}\overline{z} C_k j\Big) j \Big(I_m + \mathrm{i} z j C_k\Big) W_k(z) \\ &= q(z)^{-1} W_k(z)^* j W_k(z) + \mathrm{i}(z-\overline{z}) W_k(z)^* C_k W_k(z). \end{aligned} \tag{3.18}$$

Using (3.1) and (3.18), we derive a summation formula, which is similar to the formula for the case that $m_1 = m_2$, see [13, formula (4.2)]:

$$\sum_{k=0}^{r} q(z)^k W_k(z)^* C_k W_k(z) = \frac{1+|z|^2}{\mathrm{i}(\overline{z}-z)} \Big(j - q(z)^{r+1} W_{r+1}(z)^* j W_{r+1}(z)\Big). \tag{3.19}$$

On the other hand, it follows from (3.2) that

$$\begin{bmatrix} I_{m_1} \\ \varphi_r(z) \end{bmatrix} = W_{r+1}(z)^{-1} \mathscr{P}(z) \Big(\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_{r+1}(z)^{-1} \mathscr{P}(z) \Big)^{-1}, \tag{3.20}$$

and so formula (3.3) yields

$$\begin{bmatrix} I_{m_1} & \varphi_r(z)^* \end{bmatrix} W_{r+1}(z)^* j W_{r+1}(z) \begin{bmatrix} I_{m_1} \\ \varphi_r(z) \end{bmatrix} \geqslant 0. \tag{3.21}$$

Multiplying both sides of (3.19) by $\begin{bmatrix} I_{m_1} & \varphi_r(z)^* \end{bmatrix}$ from the left and by $\begin{bmatrix} I_{m_1} \\ \varphi_r(z) \end{bmatrix}$ from the right, and taking into account (3.21) (and inequalities $\frac{1+|z|^2}{\mathrm{i}(\overline{z}-z)} > 0$, $q(z) > 0$), we have

$$\sum_{k=0}^{r} q(z)^k \begin{bmatrix} I_{m_1} & \varphi_r(z)^* \end{bmatrix} W_k(z)^* C_k W_k(z) \begin{bmatrix} I_{m_1} \\ \varphi_r(z) \end{bmatrix} \leqslant \frac{1+|z|^2}{\mathrm{i}(\overline{z}-z)} \begin{bmatrix} I_{m_1} & \varphi_r(z)^* \end{bmatrix} j \begin{bmatrix} I_{m_1} \\ \varphi_r(z) \end{bmatrix}. \tag{3.22}$$

Finally, we use the definition of j in (1.2) and rewrite (3.22) (more precisely, the right-hand side of (3.22)), so that we obtain (3.17). □

Now, we are ready to prove the main direct theorem.

THEOREM 3.8. *There is a unique Weyl function of the discrete Dirac system* (1.1)*, which is given on the semi-axis* $0 \leqslant k < \infty$ *and satisfies* (1.2)*. This Weyl function* φ *is analytic and non-expansive* $($*i.e.,* $\varphi^*\varphi \leqslant I_{m_1})$ *in* $\mathbb{C}_+$*.*

Proof. The proof consists of 3 steps. First, we show that there is an analytic and non-expansive (contractive) function

$$\varphi_\infty(z) \in \bigcap_{r \geqslant 0} \mathscr{N}(r,z). \tag{3.23}$$

Next, we show that $\varphi_\infty(z)$ is a Weyl function. Finally, we prove the uniqueness.

Step 1. This step is similar to the corresponding part of the proof of [16, Proposition 2.2]. Indeed, from Corollary 3.5 we see that the set of functions $\varphi_r(z,\mathscr{P})$ of the form (3.2) is uniformly bounded in $\mathbb{C}_+$. So, Montel's theorem is applicable and there is an analytic matrix function, which we denote by $\varphi_\infty(z)$ and which is a uniform limit of some sequence

$$\varphi_\infty(z) = \lim_{i \to \infty} \varphi_{r_i}(z,\mathscr{P}_i) \quad (i \in \mathbb{N}, \quad r_i \uparrow, \quad \lim_{i \to \infty} r_i = \infty) \tag{3.24}$$

on all the bounded and closed subsets of $\mathbb{C}_+$. Clearly, φ_∞ is non-expansive. Since $r_i \uparrow$, the sets $\mathscr{N}(r,z)$ are embedded and equality (3.20) is valid, it follows that the matrix functions

$$\mathscr{P}_{ij}(z) := W_{r_i+1}(z) \begin{bmatrix} I_{m_1} \\ \varphi_{r_j}(z,\mathscr{P}_j) \end{bmatrix} \quad (j \geqslant i)$$

satisfy relations (3.3). Therefore, using (3.24) we derive that (3.3) holds for

$$\mathscr{P}_{i,\infty}(z) := W_{r_i+1}(z) \begin{bmatrix} I_{m_1} \\ \varphi_\infty(z) \end{bmatrix}, \tag{3.25}$$

which implies that we can substitute $\mathscr{P} = \mathscr{P}_{i,\infty}$ and $r = r_i$ into (3.2) to obtain

$$\varphi_\infty(z) \in \mathscr{N}(r_i,z). \tag{3.26}$$

Since (3.26) holds for all $i \in \mathbb{N}$, we see that (3.23) is fulfilled.

Step 2. Because of (3.23), the function φ_∞ satisfies the conditions of Lemma 3.7. Hence, (3.17) holds for any $r \geqslant 0$ and $\varphi_r = \varphi_\infty$, which implies (3.15). Therefore, φ_∞ is a Weyl function.

Step 3. It is apparent from (2.8) that

$$W_k(z)^*C_kW_k(z) \geqslant W_k(z)^*(-j)W_k(z). \tag{3.27}$$

Using (3.18) we also obtain

$$q(z)^kW_k(z)^*(-j)W_k(z) \geqslant q(z)^{k-1}W_{k-1}(z)^*(-j)W_{k-1}(z). \tag{3.28}$$

Formulas (3.1), (3.27) and (3.28) yield the basic for Step 3 inequality

$$q(z)^kW_k(z)^*C_kW_k(z) \geqslant -j. \tag{3.29}$$

Therefore, the following equality is immediate for any $g \in \mathbb{C}^{m_2}$:

$$\sum_{k=0}^{\infty} g^*[0 \quad I_{m_2}]q(z)^k W_k(z)^* C_k W_k(z) \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} g = \infty. \tag{3.30}$$

It was shown in Step 2 that $\varphi = \varphi_\infty$ satisfies (3.15). According to (3.15) and (3.30), the dimension of the subspace $L \in \mathbb{C}^m$ of vectors h such that

$$\sum_{k=0}^{\infty} h^* q(z)^k W_k(z)^* C_k W_k(z) h < \infty \tag{3.31}$$

equals m_1. Now, suppose that there is a Weyl function $\widetilde{\varphi} \neq \varphi_\infty$. Then we have

$$\operatorname{Im} \begin{bmatrix} I_{m_1} \\ \varphi_\infty(z) \end{bmatrix} \subseteq L, \quad \operatorname{Im} \begin{bmatrix} I_{m_1} \\ \widetilde{\varphi}(z) \end{bmatrix} \subseteq L.$$

Therefore, $\dim L > m_1$ (for those z, where $\widetilde{\varphi}(z) \neq \varphi_\infty(z)$) and we arrive at a contradiction. $\square$

Finally, let us construct representations of W_{r+1} $(r \geqslant 0)$ via S-nodes. First, recall that matrices $\{C_k\}$ generate a set $\{\gamma(k)\}$ of the $m_2 \times m$ matrices $\gamma(k)$ via formula (2.11). Using $\{\gamma(k)\}$, we introduce the $m_2(r+1) \times m$ matrices Γ_r and the $m_2(r+1) \times m_2(r+1)$ matrices K_r $(0 \leqslant r < \infty)$:

$$\Gamma_r := \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \dots \\ \gamma(r) \end{bmatrix}; \quad K_r := \begin{bmatrix} \varkappa_r(0) \\ \varkappa_r(1) \\ \dots \\ \varkappa_r(r) \end{bmatrix}, \tag{3.32}$$

$$\varkappa_r(k) := \mathrm{i}\gamma(k) j \left[\gamma(0)^* \ \dots \ \gamma(k-1)^* \ \gamma(k)^*/2 \ 0 \dots 0\right]. \tag{3.33}$$

It is apparent from (3.32) and (3.33) that the identity

$$K_r - K_r^* = \mathrm{i}\Gamma_r j \Gamma_r^* \tag{3.34}$$

holds. The $m_2(r+1) \times m_2(r+1)$ matrices A_r are introduced by the equalities:

$$A_r = \{a_{p-k}\}_{k,p=0}^r, \quad a_n = -\begin{cases} 0 & \text{for} \quad n > 0, \\ (\mathrm{i}/2) I_{m_2} & \text{for} \quad n = 0, \\ \mathrm{i} I_{m_2} & \text{for} \quad n < 0. \end{cases} \tag{3.35}$$

PROPOSITION 3.9. *Matrices K_r and A_r are linearly similar:*

$$K_r = E_r A_r E_r^{-1}. \tag{3.36}$$

Moreover, the similarity transformations E_r can be constructed so that

$$E_r = \begin{bmatrix} E_{r-1} & 0 \\ X_r & e_r^- \end{bmatrix} \quad (r > 0), \quad E_r^{-1}\Gamma_{r,2} = \Phi_{r,2}, \quad \Phi_{r,2} := \begin{bmatrix} I_{m_2} \\ \dots \\ I_{m_2} \end{bmatrix}, \tag{3.37}$$

$$E_0 = e_0^- = \gamma_2(0), \tag{3.38}$$

where $\Gamma_{r,p}$ *are* $m_2(r+1)\times m_p$ *blocks of* $\Gamma_r=\begin{bmatrix}\Gamma_{r,1} & \Gamma_{r,2}\end{bmatrix}$ *and* $\gamma_p(k)$ *are* $m_2\times m_p$ *blocks of* $\gamma(k)=\begin{bmatrix}\gamma_1(k) & \gamma_2(k)\end{bmatrix}$.

Proof. It follows from (2.9), (3.32), (3.33) and (3.35) that

$$K_0=A_0=-(\mathrm{i}/2)I_{m_2}, \quad \det\gamma_2(0)\neq 0, \tag{3.39}$$

$$\varkappa_r(r)=\mathrm{i}\begin{bmatrix}\gamma(r)j\gamma(0)^* & \dots & \gamma(r)j\gamma(r-1)^* & -I_{m_2}/2\end{bmatrix}. \tag{3.40}$$

We see that (3.38) and (3.39) imply (3.36) for $r=0$. Next, we prove (3.36) by induction. Assume that $K_{r-1}=E_{r-1}A_{r-1}E_{r-1}^{-1}$ and let E_r have the form (3.37), where $\det e_r^-\neq 0$. Then we obtain

$$E_r^{-1}=\begin{bmatrix}E_{r-1}^{-1} & 0\\ -(e_r^-)^{-1}X_rE_{r-1}^{-1} & (e_r^-)^{-1}\end{bmatrix}, \tag{3.41}$$

and, in view of (3.32), (3.35), (3.37), (3.40), it is necessary and sufficient (for (3.36) to hold) that

$$\begin{aligned}&\Big(\begin{bmatrix}X_rA_{r-1} & -(\mathrm{i}/2)e_r^-\end{bmatrix}-\mathrm{i}e_r^-\begin{bmatrix}I_{m_2} & \dots & I_{m_2} & 0\end{bmatrix}\Big)\begin{bmatrix}I_{rm_2}\\ -(e_r^-)^{-1}X_r\end{bmatrix}E_{r-1}^{-1}\\ &=\mathrm{i}\gamma(r)j\begin{bmatrix}\gamma(0)^* & \dots & \gamma(r-1)^*\end{bmatrix}.\end{aligned} \tag{3.42}$$

We can rewrite (3.42) in the form

$$\begin{aligned}X_r\big(A_{r-1}+(\mathrm{i}/2)I_{rm_2}\big)=&\mathrm{i}\gamma(r)j\begin{bmatrix}\gamma(0)^* & \dots & \gamma(r-1)^*\end{bmatrix}E_{r-1}\\ &+\mathrm{i}e_r^-\begin{bmatrix}I_{m_2} & \dots & I_{m_2}\end{bmatrix}.\end{aligned} \tag{3.43}$$

Next, we partition X_r $(r>1)$ into $m_2\times m_2$ and $m_2\times(r-1)m_2$ blocks

$$X_r=\begin{bmatrix}x_r^- & \widetilde{X}_r\end{bmatrix}, \tag{3.44}$$

and we will also need partitions of the matrices $A_{r-1}+(\mathrm{i}/2)I_{rm_2}$ and E_{r-1}, which follow (for $r>1$) from (3.35) and (3.37):

$$\big(A_{r-1}+(\mathrm{i}/2)I_{rm_2}\big)=\begin{bmatrix}0 & 0\\ \big(A_{r-2}-(\mathrm{i}/2)I_{(r-1)m_2}\big) & 0\end{bmatrix}, \quad E_{r-1}\begin{bmatrix}0\\ I_{m_2}\end{bmatrix}=\begin{bmatrix}0\\ e_{r-1}^-\end{bmatrix}. \tag{3.45}$$

Using (3.44) and (3.45) we see that (3.43) is equivalent to the relations

$$e_r^-=-\gamma(r)j\gamma(r-1)^*e_{r-1}^- \quad \text{for} \quad r\geqslant 1; \tag{3.46}$$

$$\begin{aligned}\widetilde{X}_r=&\mathrm{i}\Big(\gamma(r)j\begin{bmatrix}\gamma(0)^* & \dots & \gamma(r-1)^*\end{bmatrix}E_{r-1}+e_r^-\begin{bmatrix}I_{m_2} & \dots & I_{m_2}\end{bmatrix}\Big)\\ &\times\begin{bmatrix}\big(A_{r-2}-(\mathrm{i}/2)I_{(r-1)m_2}\big)^{-1}\\ 0\end{bmatrix} \quad \text{for} \quad r>1.\end{aligned} \tag{3.47}$$

Hence, if e_r^- and X_r satisfy (3.46) and (3.47), respectively, and $\det e_r^- \neq 0$, the similarity relation (3.36) holds. The inequalities $\det e_r^- \neq 0$ are apparent (by induction) from (3.38), (3.46) and the inequalities

$$\det(\gamma(r)j\gamma(r-1)^*) \neq 0, \tag{3.48}$$

and it remains to prove (3.48). Indeed, let $\gamma(r)j\gamma(r-1)^*g = 0$, $g \neq 0$. Then, the subspaces $\operatorname{Im}\gamma(r)^*$ and $\operatorname{span}\gamma(r-1)^*g$ are j-orthogonal. The second equality in (2.9) (taken for $k = r$ and $k = r-1$) implies that these subspaces are also j-negative, have zero intersection and have dimensions m_2 and 1, respectively. Thus, $\operatorname{span}\big(\gamma(r-1)^*g \cup \operatorname{Im}\gamma(r)^*\big)$ is an m_2+1-dimensional j-negative subspace, which does not exist. Therefore, the relation (3.48), and so also the equality (3.36), are proved.

Formula (3.38) shows that the second equality in (3.37) holds for $r = 0$. Now, we choose X_r (for $r = 1$) and x_r^- (for $r > 1$) so that the second equality in (3.37) holds in the case that $r > 0$. Taking into account (3.41), (3.44) and using induction, we see that this equality is valid when

$$X_1 = \gamma_2(1) - e_1^-, \qquad x_r^- = \gamma_2(r) - e_r^- - \widetilde{X}_r\Phi_{r-2,2} \quad (r > 1). \quad \square \tag{3.49}$$

We note that inequalities, which are similar to (3.6) and (3.48), are often required in the study of completion problems and Weyl theory. Therefore, the next proposition, which is easily proved using the same considerations as in the proof of (3.48), could be of more general interest.

PROPOSITION 3.10. *Let the $m \times m$ matrix J satisfy equalities $J = J^* = J^{-1}$ and have $m_1 > 0$ positive eigenvalues. Let $m \times m_1$ matrices ϑ and $\widetilde{\vartheta}$ satisfy inequalities*

$$\vartheta^*\vartheta > 0, \quad \vartheta^*J\vartheta > 0, \quad \widetilde{\vartheta}^*\widetilde{\vartheta} > 0, \quad \widetilde{\vartheta}^*J\widetilde{\vartheta} \geqslant 0. \tag{3.50}$$

Then we have

$$\det \vartheta^*J\widetilde{\vartheta} \neq 0. \tag{3.51}$$

Let us substitute (3.36) into (3.34) to derive

$$E_rA_rE_r^{-1} - \big(E_r^*\big)^{-1}A_r^*E_r^* = \mathrm{i}\Gamma_r j\Gamma_r^*. \tag{3.52}$$

Multiplying both sides of (3.52) by E_r^{-1} and $\big(E_r^*\big)^{-1}$ from the left and right, respectively, we obtain the operator identity

$$A_rS_r - S_rA_r^* = \mathrm{i}\Pi_r j\Pi_r^* = \mathrm{i}(\Phi_{r,1}\Phi_{r,1}^* - \Phi_{r,2}\Phi_{r,2}^*), \tag{3.53}$$

where

$$S_r := E_r^{-1}\big(E_r^*\big)^{-1}, \quad \Pi_r := E_r^{-1}\Gamma_r = \big[\Phi_{r,1} \ \Phi_{r,2}\big]. \tag{3.54}$$

DEFINITION 3.11. The triple of matrices $\{A_r, S_r, \Pi_r\}$ forms a symmetric S-node if the operator (matrix) identity (3.53) holds, $S_r = S_r^*$ and $\det S_r \neq 0$.

The transfer matrix function (in Lev Sakhnovich form), which corresponds to this S-node, is given by the formula

$$w_A(r,\lambda) = I_m - \mathrm{i}j\Pi_r^* S_r^{-1}\big(A_r - \lambda I_{(r+1)m_2}\big)^{-1}\Pi_r. \tag{3.55}$$

REMARK 3.12. A symmetric S-node corresponding to Dirac system (1.1) (which satisfies (1.2)) on the interval $0 \leqslant k \leqslant r$ is constructed using formulas (3.35) and (3.54), where Γ_r is given in (3.32).

Recall that S-nodes, transfer matrix functions w_A and the method of operator identities are introduced and studied in [24, 25, 26, 27] (see also references therein).

Let us introduce $rm_2 \times (r+1)m_2$ and $m_2 \times (r+1)m_2$, respectively, matrices (projectors):

$$P_1 := \begin{bmatrix} I_{rm_2} & 0 \end{bmatrix} \quad (r>0), \quad P_2 = P := \begin{bmatrix} 0 & \dots & 0 & I_{m_2} \end{bmatrix}. \tag{3.56}$$

Since E_r^{-1} is a block lower triangular matrix, we easily derive from (3.41) and (3.54) that

$$P_1 S_r P_1^* = E_{r-1}^{-1}\big(E_{r-1}^*\big)^{-1} = S_{r-1}, \quad P_1\Pi_r = \Pi_{r-1}. \tag{3.57}$$

It is apparent that

$$\det S_{r-1} \neq 0, \quad P_1 A_r P_1^* = A_{r-1}. \tag{3.58}$$

In view of (3.57) and (3.58), the factorization Theorem 4 from [25] (see also [27, p. 188]) yields

$$\begin{aligned} w_A(r,\lambda) = &\Big(I_m - \mathrm{i}j\Pi_r^* S_r^{-1}P^*\big(PA_rP^* - \lambda I_{m_2}\big)^{-1}\big(PS_r^{-1}P^*\big)^{-1}PS_r^{-1}\Pi_r\Big) \\ &\times w_A(r-1,\lambda). \end{aligned} \tag{3.59}$$

PROPOSITION 3.13. *The fundamental solution W of the system* (1.1)*, where W is normalized by the condition* (3.1) *and the* potential $\{C_k\}$ *satisfies* (1.2)*, admits reprezentation*

$$W_{r+1}(z) = (1+\mathrm{i}z)^{r+1} w_A\big(r,(2z)^{-1}\big). \tag{3.60}$$

Proof. Formulas (1.1) and (2.9) imply the following equalities

$$W_{r+1}(z) = (1+\mathrm{i}z)\big(I_m + 2\mathrm{i}z(1+\mathrm{i}z)^{-1}j\gamma(r)^*\gamma(r)\big)W_r(z) \quad (r \geqslant 0). \tag{3.61}$$

On the other hand, we easily derive from (3.32), (3.35), (3.37) and (3.54) that

$$\big(PA_rP^* - \lambda I_{m_2}\big)^{-1} = -\big(\lambda + \mathrm{i}/2\big)^{-1} I_{m_2}, \quad S_r^{-1} = E_r^* E_r, \tag{3.62}$$

$$PS_r^{-1}P^* = (e_r^-)^* e_r^-, \quad PS_r^{-1}\Pi_r = PE_r^*\Gamma_r = (e_r^-)^*\gamma(r). \tag{3.63}$$

We substitute (3.62) and (3.63) into (3.59) to obtain

$$w_A(r,\lambda) = \Big(I_m + \frac{2\mathrm{i}}{2\lambda+\mathrm{i}} j\gamma(r)^*\gamma(r)\Big) w_A(r-1,\lambda) \quad (r \geqslant 1). \tag{3.64}$$

In a similar way, we rewrite (3.55) (for the case that $r=0$) in the form

$$w_A(0,\lambda) = I_m + \frac{2\mathrm{i}}{2\lambda+\mathrm{i}} j\gamma(0)^*\gamma(0). \tag{3.65}$$

Finally, we compare (3.61) with (3.64) and (3.65) (and take into account (3.1)) to see that $W_1(z) = (1+\mathrm{i}z)w_A\big(0,(2z)^{-1}\big)$ and iterative relations for the left- and right-hand sides of (3.60)) coincide. $\square$

4. Weyl theory: inverse problems

The values of φ and its derivatives at $z=\mathrm{i}$ will be of interest in this section. Therefore, using (3.9) we rewrite (3.2) in the form

$$\varphi_r(z,\mathscr{P}) = -\begin{bmatrix}0 & I_{m_2}\end{bmatrix} W_{r+1}(\overline{z})^*\mathscr{P}(z)\Big(\begin{bmatrix}I_{m_1} & 0\end{bmatrix} W_{r+1}(\overline{z})^*\mathscr{P}(z)\Big)^{-1}, \tag{4.1}$$

where $\mathscr{P}$ in (4.1) differs from $\mathscr{P}$ in (3.2) by the factor j (and so this $\mathscr{P}$ is also a nonsingular matrix function with property-j).

DEFINITION 4.1. Weyl functions of Dirac system (1.1) $\big($which is given on the interval $0 \leqslant k \leqslant r$ and satisfies (1.2)$\big)$ are $m_2 \times m_1$ matrix functions $\varphi(z)$ of the form (4.1), where $\mathscr{P}$ are nonsingular matrix functions with property-j such that $\mathscr{P}(\mathrm{i})$ are well-defined and nonsingular.

It is apparent that (4.1) is equivalent to

$$\begin{bmatrix} I_{m_1} \\ \varphi_r(z,\mathscr{P})\end{bmatrix} = jW_{r+1}(\overline{z})^*\mathscr{P}(z)\Big(\begin{bmatrix}I_{m_1} & 0\end{bmatrix} W_{r+1}(\overline{z})^*\mathscr{P}(z)\Big)^{-1}. \tag{4.2}$$

LEMMA 4.2. *Let $\mathscr{P}$ satisfy the conditions of Definition 4.1. Then we have inequality*

$$\det\Big(\begin{bmatrix}I_{m_1} & 0\end{bmatrix} W_{r+1}(-\mathrm{i})^*\mathscr{P}(\mathrm{i})\Big) \neq 0. \tag{4.3}$$

Proof. First note that in view of (2.8) we obtain

$$I_m + C_k j = 2\beta(k)^*\beta(k)j. \tag{4.4}$$

Formulas (3.4) and (4.4) imply

$$\begin{aligned}\begin{bmatrix}I_{m_1} & 0\end{bmatrix} W_{r+1}(-\mathrm{i})^*\mathscr{P}(\mathrm{i}) =& 2^{r+1}\big(\begin{bmatrix}I_{m_1} & 0\end{bmatrix}\beta(0)^*\big)\big(\beta(0)j\beta(1)^*\big)\dots \\ &\times\big(\beta(r-1)j\beta(r)^*\big)\big(\beta(r)j\mathscr{P}(\mathrm{i})\big).\end{aligned} \tag{4.5}$$

Using Proposition 3.10 and the second equality in (2.8) and putting (in Proposition 3.10) $\vartheta = \beta(k)^*$ and $\widetilde{\vartheta} = \beta(k+1)^*$ or $\widetilde{\vartheta} = \mathscr{P}(\mathrm{i})$, we derive, respectively, inequalities

$$\det(\beta(k)j\beta(k+1)^*) \neq 0 \quad \text{and} \quad \det(\beta(r)j\mathscr{P}(\mathrm{i})) \neq 0. \tag{4.6}$$

In the same way we obtain $\det\left(\begin{bmatrix} I_{m_1} & 0 \end{bmatrix}\beta(0)^*\right) \neq 0$. Now, inequality (4.3) follows from (4.5). □

The proof of our next proposition is similar to the proof of Corollary 3.5.

PROPOSITION 4.3. *Suppose φ is a Weyl function of Dirac system (1.1) on the interval $0 \leqslant k \leqslant r$, where the potential $\{C_k\}$ satisfies (1.2). Then φ is a Weyl function of the same system on all of the intervals $0 \leqslant k \leqslant \widetilde{r}$ $(\widetilde{r} \leqslant r)$.*

Proof. Clearly, it suffices to show that the statement of the proposition holds for $\widetilde{r} = r-1$ (if $r > 0$). That is, in view of Definition 4.1, we should prove that $\widetilde{\mathscr{P}}(z) := (I_m - \mathrm{i}zC_r j)\mathscr{P}(z)$ has property-j, that $\widetilde{\mathscr{P}}(\mathrm{i})$ is well-defined and that the first inequality in (3.3) written for $\widetilde{\mathscr{P}}$ at $z = \mathrm{i}$ always holds (i.e., $\widetilde{\mathscr{P}}(\mathrm{i})$ is nonsingular), if only $\mathscr{P}$ has these properties.

Indeed, since we have

$$(I_m - \mathrm{i}zC_r j)^* j(I_m - \mathrm{i}zC_r j) = (1+|z|^2)j + \mathrm{i}(\overline{z}-z)jC_r j \geqslant (1+|z|^2)j, \tag{4.7}$$

the matrix function $\widetilde{\mathscr{P}}$ has property-j. The non-singularity of $\widetilde{\mathscr{P}}(\mathrm{i}) = (I_m + C_r j)\mathscr{P}(\mathrm{i})$ is apparent from (4.4) and (4.6). □

THEOREM 4.4. *Suppose φ is a Weyl function of Dirac system (1.1) on the interval $0 \leqslant k \leqslant r$, where the potential $\{C_k\}$ satisfies (1.2). Then the set (potential) $\{C_k\}$ $(0 \leqslant k \leqslant r)$ is uniquely recovered from the first $r+1$ Taylor coefficients of $\varphi\left(\mathrm{i}\frac{1-z}{1+z}\right)$ at $z = 0$ via the following procedure.*

If $\varphi\left(\mathrm{i}\frac{1-z}{1+z}\right) = \sum_{k=0}^{r}\phi_k z^k + O(z^{r+1})$, then matrices $\Phi_{k,1}$ are recovered via the formula

$$\Phi_{k,1} = -\begin{bmatrix} \phi_0 \\ \phi_0 + \phi_1 \\ \dots \\ \phi_0 + \phi_1 + \dots + \phi_k \end{bmatrix}. \tag{4.8}$$

Using $\Phi_{k,1}$ we easily recover consecutively $\Pi_k = \begin{bmatrix} \Phi_{k,1} & \Phi_{k,2} \end{bmatrix}$ (where $\Phi_{k,2}$ is given in (3.37)) and S_k, which is the unique solution of the matrix identity

$$A_k S_k - S_k A_k^* = \mathrm{i}\Pi_k j \Pi_k^*.$$

Next, we construct

$$\gamma(k)^*\gamma(k) = \Pi_k^* S_k^{-1} P^* (P S_k^{-1} P^*)^{-1} P S_k^{-1} \Pi_k, \quad P = \begin{bmatrix} 0 & \dots & 0 & I_{m_2} \end{bmatrix}. \tag{4.9}$$

Finally, we use $\gamma(k)^*\gamma(k)$ to recover C_k via (2.9).

Proof. Put

$$\mathscr{A}(z) := |1+z^2|^{-2(r+1)} \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} W_{r+1}(z)^* j W_{r+1}(z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix}. \tag{4.10}$$

According to (3.9) and (4.2) we have

$$\begin{aligned}\mathscr{A}(z) =& \Big(\Big(\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_{r+1}(\overline{z})^* \mathscr{P}(z) \Big)^{-1} \Big)^* \mathscr{P}(z)^* j \mathscr{P}(z) \\ & \times \Big(\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_{r+1}(\overline{z})^* \mathscr{P}(z) \Big)^{-1}. \end{aligned}\tag{4.11}$$

From (4.3) and (4.11) we see that $\mathscr{A}$ is bounded in the neighbourhood of $z = \mathrm{i}$:

$$\|\mathscr{A}(z)\| = O(1) \quad \text{for} \quad z \to \mathrm{i}. \tag{4.12}$$

We now make use of the S-node (corresponding to Dirac system), which is constructed in accordance with Remark 3.12. Substitute (3.60) into (4.10) to obtain

$$\begin{aligned}\mathscr{A}(z) =& \big((1-\mathrm{i}z)(1+\mathrm{i}\overline{z})\big)^{-r-1} \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} \\ & \times \Big(j - \frac{\Im(z)}{|z|^2} \Pi_r^* \Big(A_r^* - \frac{1}{2\overline{z}} I\Big)^{-1} S_r^{-1} \Big(A_r - \frac{1}{2z} I\Big)^{-1} \Pi_r \Big) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix},\end{aligned}\tag{4.13}$$

where $I = I_{(r+1)m_2}$. Here we used the important equality

$$w_A(r,\lambda)^* j w_A(r,\widetilde{\lambda}) = j - \mathrm{i}(\widetilde{\lambda} - \overline{\lambda}) \Pi_r^* (A_r^* - \overline{\lambda} I)^{-1} S_r^{-1} (A_r - \widetilde{\lambda} I)^{-1} \Pi_r, \tag{4.14}$$

which follows from (3.53) and (3.55) (see, e.g., [21, 25]).

Notice that $S_r > 0$. Hence, formulas (3.13), (4.12) and (4.13) imply that

$$\left\| \Big(A_r - \frac{1}{2z} I\Big)^{-1} \Pi_r \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} \right\| = O(1) \quad \text{for} \quad z \to \mathrm{i}. \tag{4.15}$$

Using the block representation $\Pi_r = \begin{bmatrix} \Phi_{r,1} & \Phi_{r,2} \end{bmatrix}$ from (3.54) and multiplying both sides of (4.15) by $\left\| \Big(\Phi_{r,2}^* \big(A_r - \frac{1}{2z} I\big)^{-1} \Phi_{r,2} \Big)^{-1} \Phi_{r,2}^* \right\|$ we rewrite the result:

$$\begin{aligned}& \left\| \varphi(z) + \Big(\Phi_{r,2}^* \Big(A_r - \frac{1}{2z} I\Big)^{-1} \Phi_{r,2} \Big)^{-1} \Phi_{r,2}^* \Big(A_r - \frac{1}{2z} I\Big)^{-1} \Phi_{r,1} \right\| \\ & = O\left(\left\| \Big(\Phi_{r,2}^* \Big(A_r - \frac{1}{2z} I\Big)^{-1} \Phi_{r,2} \Big)^{-1} \right\| \right) \quad \text{for} \quad z \to \mathrm{i}. \end{aligned}\tag{4.16}$$

In order to obtain (4.16) we also applied the matrix (operator) norm inequality

$$\|X_1 X_2\| \leqslant \|X_1\| \|X_2\|.$$

The resolvent $(A - \lambda I)^{-1}$ is easily constructed explicitly (see, for instance, formula (1.10) in [22]). In particular, we derive

$$\Phi_{r,2}^* \Big(A_r - \frac{1}{2z} I\Big)^{-1} = -\frac{2z}{1+\mathrm{i}z} \begin{bmatrix} \widehat{q}(z)^r & \widehat{q}(z)^{r-1} & \ldots & I_{m_2} \end{bmatrix}, \quad \widehat{q} := \frac{1-\mathrm{i}z}{1+\mathrm{i}z} I_{m_2}. \tag{4.17}$$

From (4.17) we see that

$$\Phi_{r,2}^{*}\Big(A_r-\frac{1}{2z}I\Big)^{-1}\Phi_{r,2}=\mathrm{i}\Big(1-\Big(\frac{1-\mathrm{i}z}{1+\mathrm{i}z}\Big)^{r+1}\Big)I_{m_2}. \tag{4.18}$$

Partitioning $\Phi_{r,1}$ into $m_2\times m_1$ blocks $\Phi_{r,1}(k)$ and using (4.16)–(4.18) we obtain

$$\varphi\left(\mathrm{i}\frac{1-z}{1+z}\right)+\frac{1-z}{1-z^{r+1}}\sum_{k=0}^{r}z^k\Phi_{r,1}(k)=O(z^{r+1})\quad \text{for}\quad z\to 0,$$

which can be easily transformed into

$$\varphi\left(\mathrm{i}\frac{1-z}{1+z}\right)+(1-z)\sum_{k=0}^{r}z^k\Phi_{r,1}(k)=O(z^{r+1})\quad \text{for}\quad z\to 0, \tag{4.19}$$

and (4.8) follows for $k=r$.

Since $\sigma(A_r)\cap\sigma(A_r^*)=\emptyset$ the matrix S_r is uniquely recovered from the matrix identity (3.53). Finally, (4.9) for the case, where $k=r$, is apparent from (3.63). From Proposition 4.3, we see that φ is a Weyl function of our Dirac system on all of the intervals $0\leqslant k\leqslant\widetilde{r}$ $(\widetilde{r}\leqslant r)$ and so all $C_{\widetilde{r}}$ are recovered in the same way as C_r. $\square$

The next corollary is a discrete version of Borg-Marchenko-type uniqueness theorems. The active study of such theorems was triggered by the seminal papers by F. Gesztesy and B. Simon [18, 19].

COROLLARY 4.5. *Suppose φ and $\widetilde{\varphi}$ are Weyl functions of two Dirac systems with potentials $\{C_k\}$ and $\{\widetilde{C}_k\}$, which are given on the intervals $0\leqslant k\leqslant r$ and $0\leqslant k\leqslant\widetilde{r}$, respectively. We suppose that the matrices $\{C_k\}$ and $\{\widetilde{C}_k\}$ are positive and j-unitary. Moreover, we assume that*

$$\varphi\left(\mathrm{i}\frac{1-z}{1+z}\right)-\widetilde{\varphi}\left(\mathrm{i}\frac{1-z}{1+z}\right)=O(z^{p+1}),\quad z\to\infty,\quad p\in\mathbb{N}_0,\quad p\leqslant\min(r,\widetilde{r}). \tag{4.20}$$

Then we have $C_k=\widetilde{C}_k$ for all $0\leqslant k\leqslant p$.

Proof. According to Proposition 4.3 both functions φ and $\widetilde{\varphi}$ are Weyl functions of the corresponding Dirac systems on the same interval $[0,p]$. From (4.20) we see that the first $p+1$ Taylor coefficients of $\varphi\left(\mathrm{i}\frac{1-z}{1+z}\right)$ and $\widetilde{\varphi}\left(\mathrm{i}\frac{1-z}{1+z}\right)$ coincide. Hence, the uniqueness of the potential recovered from the Taylor coefficients in Theorem 4.4 yields $C_k=\widetilde{C}_k$ $(0\leqslant k\leqslant p)$. $\square$

Taking into account (4.8), we derive that the first $r+1$ Taylor coefficients of $\varphi_r\left(\mathrm{i}\frac{1-z}{1+z}\right)$ at $z=0$ (for any Weyl function φ_r of a fixed Dirac system) can be uniquely and in the same way recovered from the matrix $\Phi_{r,1}$, which, in turn, can be constructed as proposed in Remark 3.12. Therefore, the next theorem is apparent.

THEOREM 4.6. *Let Dirac system* (1.1)*, where matrices* C_k *satisfy* (1.2)*, be given on the interval* $0 \leqslant k \leqslant r$*. Then all functions* $\varphi_d(z) = \varphi_r\left(\mathrm{i}\frac{1-z}{1+z}, \mathscr{P}\right)$*, where* φ_r *are Weyl functions of this Dirac system, are non-expansive in the unit disk and have the same first* $r+1$ *Taylor coefficients* $\{\phi_k\}_0^r$ *at* $z=0$*.*

Step 1 in the proof of Theorem 3.8 shows that the Weyl function φ_∞ of Dirac system on the semi-axis can be constructed as a uniform limit of Weyl functions φ_r on increasing intervals. Hence, using Theorem 4.6 we obtain the following corollary.

COROLLARY 4.7. *Let* $\varphi(z)$ *be the Weyl function of some Dirac system* (1.1)*, which is given on the semi-axis and satisfies* (1.2)*. Assume that* φ_r *is a Weyl function of the same system on the finite interval* $0 \leqslant k \leqslant r$*. Then the first* $r+1$ *Taylor coefficients of* $\varphi\left(\mathrm{i}\frac{1-z}{1+z}\right)$ *and* $\varphi_r\left(\mathrm{i}\frac{1-z}{1+z}\right)$ *coincide. Therefore, the system is uniquely recovered from* φ *using the procedure given in Theorem* 4.4*.*

5. Operator identities and interpolation problems

One can easily derive (see, e.g, [17, p. 474]) that the equality

$$s_{k+1,p+1} - s_{kp} = Q_{kp} + Q_{k+1,p+1} - Q_{k+1,p} - Q_{k,p+1}, \quad -1 \leqslant k,p \leqslant r-1 \tag{5.1}$$

holds for the blocks s_{kp} and Q_{kp} of the block matrices $S_r = \{s_{kp}\}_{k,p=0}^r$ and $Q_r = \{Q_{kp}\}_{k,p=0}^r$, respectively, which satisfy the operator identity

$$A_r S_r - S_r A_r^* + \mathrm{i}Q = 0, \tag{5.2}$$

where A_r is given by (3.35). Here we write sometimes commas between the indices of blocks and also set

$$s_{-1,p} = s_{k,-1} = Q_{-1,p} = Q_{k,-1} = 0. \tag{5.3}$$

For the case that S_r corresponds to Dirac system, we rewrite (see below) formula (5.1) in an equivalent form and obtain the structure of S_r.

PROPOSITION 5.1. *Let* S_r *satisfy* (3.53)*, where* A_r*,* $\Phi_{r,1}$ *and* $\Phi_{r,2}$ *are given by* (3.35)*,* (4.8) *and the last equality in* (3.37)*, respectively. Then* S_r *has the following structure:*

$$s_{00} = I_{m_2} - \phi_0\phi_0^* \quad \text{and} \quad s_{k+1,p+1} - s_{kp} = \phi_{k+1}\phi_{p+1}^* \tag{5.4}$$

for $-1 \leqslant k,p \leqslant r-1, \quad k+p+2>0.$

The following statement is immediate from Theorem 4.6 and Proposition 5.1.

THEOREM 5.2. *Let Dirac system* (1.1)*, where matrices* C_k *satisfy* (1.2)*, be given on the interval* $0 \leqslant k \leqslant r$*.Then all the functions*

$$\varphi_d(z) = \varphi_r\left(\mathrm{i}\frac{1-z}{1+z}, \mathscr{P}\right),$$

where φ_r are given by (3.2), matrix functions $\mathscr{P}(z)$ in (3.2) have property-j and matrices $\mathscr{P}(\mathrm{i})$ are non-singular, are non-expansive in the unit disk and have the same first $r+1$ Taylor coefficients $\{\phi_k\}_0^r$ at $z=0$. The matrix S_r determined by these coefficients via (5.4) is positive.

On the other hand, if we assume only that the coefficients $\{\phi_k\}_0^r$ are fixed and S_r given (5.4) is positive, two related interpolation problems appear.

Interpolation problem I. Describe all the analytic and non-expansive in the unit disk matrix functions φ_d such that the coefficients $\{\phi_k\}_0^r$ are their first $r+1$ Taylor coefficients.

Interpolation problem II. Describe all the positive continuations of S_r, which preserve the structure given by (5.4).

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