Operators and Matrices
Volume 8, Number 3 (2014), 821–847 doi:10.7153/oam-08-46

COMMUTING TRACES AND LIE ISOMORPHISMS ON GENERALIZED MATRIX ALGEBRAS

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(*Communicated by H. Radjavi*)

Abstract. Let $\mathscr{G}$ be a generalized matrix algebra over a commutative ring $\mathscr{R}$, $\mathfrak{q}\colon \mathscr{G}\times\mathscr{G}\longrightarrow\mathscr{G}$ be an $\mathscr{R}$-bilinear mapping and $\mathfrak{T}_{\mathfrak{q}}\colon :\mathscr{G}\longrightarrow\mathscr{G}$ be a trace of $\mathfrak{q}$. We describe the form of $\mathfrak{T}_{\mathfrak{q}}$ satisfying the condition $\mathfrak{T}_{\mathfrak{q}}(G)G = G\mathfrak{T}_{\mathfrak{q}}(G)$ for all $G\in\mathscr{G}$. The question of when $\mathfrak{T}_{\mathfrak{q}}$ has the proper form is considered. Using the aforementioned trace function, we establish sufficient conditions for each Lie isomorphism of $\mathscr{G}$ to be almost standard. As applications we characterize Lie isomorphisms of full matrix algebras, of triangular algebras and of certain unital algebras with nontrivial idempotents. Some further research topics related to current work are proposed at the end of this article.

1. Introduction

Let $\mathscr{R}$ be a commutative ring with identity, $\mathscr{A}$ be a unital algebra over $\mathscr{R}$ and $\mathscr{Z}(\mathscr{A})$ be the center of $\mathscr{A}$. Let us denote the commutator or the Lie product of the elements $a,b\in\mathscr{A}$ by $[a,b]=ab-ba$. Recall that an $\mathscr{R}$-linear mapping $\mathfrak{f}:\mathscr{A}\longrightarrow\mathscr{A}$ is said to be *commuting* if $[\mathfrak{f}(a),a]=0$ for all $a\in\mathscr{A}$. When we investigate a commuting mapping, the principal task is to describe its form. The identity mapping and every mapping which has its range in $\mathscr{Z}(\mathscr{A})$ are two classical examples of commuting mappings. Furthermore, the sum and the pointwise product of commuting mappings are also commuting mappings. We encourage the reader to read the well-written survey paper [18], in which the author presented the development of the theory of commuting mappings and their applications in details.

Let n be a positive integer and $\mathfrak{q}\colon\mathscr{A}^n\longrightarrow\mathscr{A}$. We say that $\mathfrak{q}$ is *n-linear* if $\mathfrak{q}(a_1,\cdots,a_n)$ is $\mathscr{R}$-linear in each variable a_i, that is, $\mathfrak{q}(a_1,\cdots,ra_i+sb_i,\cdots,a_n)=r\mathfrak{q}(a_1,\cdots,a_i,\cdots,a_n)+s\mathfrak{q}(a_1,\cdots,b_i,\cdots,a_n)$ for all $r,s\in\mathscr{R},a_i,b_i\in\mathscr{A}$ and $i=1,2,\cdots,n$. The mapping $\mathfrak{T}_{\mathfrak{q}}\colon\mathscr{A}\longrightarrow\mathscr{A}$ defined by $\mathfrak{T}_{\mathfrak{q}}(a)=\mathfrak{q}(a,a,\cdots,a)$ is called a *trace* of $\mathfrak{q}$. We say that a commuting trace $\mathfrak{T}_{\mathfrak{q}}$ is *proper* if it is of the form

$$\mathfrak{T}_{\mathfrak{q}}(a)=\sum_{i=0}^{n}\mu_i(a)a^{n-i},\quad \forall a\in\mathscr{A},$$

Mathematics subject classification (2010): 15A78, 15A86, 16W10.
Keywords and phrases: Generalized matrix algebra, commuting trace, Lie isomorphism.
The work of the first author is supported by the National Natural Science Foundation of China (Grant No. 11226068 and No. 11301195).

where $\mu_i(0 \leqslant i \leqslant n)$ is a mapping from $\mathscr{A}$ into $\mathscr{Z}(\mathscr{A})$ and each $\mu_i(0 \leqslant i \leqslant n)$ is in fact a trace of the i-linear mapping $\mathfrak{q}_i$ from $\mathscr{A}^i$ into $\mathscr{Z}(\mathscr{A})$. Let $n = 1$ and $\mathfrak{f}\colon \mathscr{A} \longrightarrow \mathscr{A}$ be an $\mathscr{R}$-linear mapping. In this case, an arbitrary trace $\mathfrak{T}_{\mathfrak{f}}$ of $\mathfrak{f}$ exactly equals to itself. Moreover, if a commuting trace $\mathfrak{T}_{\mathfrak{f}}$ of $\mathfrak{f}$ is proper, then it has the form

$$\mathfrak{T}_{\mathfrak{f}}(a) = za + \eta(a), \quad \forall a \in \mathscr{A},$$

where $z \in \mathscr{Z}(\mathscr{A})$ and η is an $\mathscr{R}$-linear mapping from A into $\mathscr{Z}(\mathscr{A})$. Let us see the case of $n = 2$. Suppose that $\mathfrak{g}\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ is an $\mathscr{R}$-bilinear mapping. If a commuting trace $\mathfrak{T}_{\mathfrak{g}}$ of $\mathfrak{g}$ is proper, then it is of the form

$$\mathfrak{T}_{\mathfrak{g}}(a) = za^2 + \mu(a)a + \nu(a), \quad \forall a \in \mathscr{A},$$

where $z \in \mathscr{Z}(\mathscr{A})$, μ is an $\mathscr{R}$-linear mapping from A into $\mathscr{Z}(\mathscr{A})$ and ν is a trace of some bilinear mapping. It was Brešar who initiated the study of commuting traces of multilinear mappings in [16, 17], where he investigated the structure of commuting traces of (bi-)linear mappings on prime rings. It has turned out that this study is closely related to the problem of characterizing Lie isomorphisms or Lie derivations of associative rings [6]. Lee et al further generalized Brešar's results by showing that each commuting trace of an arbitrary multilinear mapping on a prime ring has the so-called proper form [35].

Cheung in [26] studied commuting mappings of triangular algebras (e.g., of upper triangular matrix algebras and nest algebras). He determined the class of triangular algebras for which every commuting mapping is proper. Xiao and Wei [60] extended Cheung's result to the generalized matrix algebra case. They established sufficient conditions for each commuting mapping of a generalized matrix algebra $\left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ to be proper. Motivated by the results of Brešar and Cheung, Benkovič and Eremita [11] considered commuting traces of bilinear mappings on a triangular algebra $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$. They gave conditions under which every commuting trace of a triangular algebra $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ is proper. It is worth to mention that the form of commuting traces of multilinear mappings of upper triangular matrix algebras was earlier described in [4]. One of the main aims of this article is to provide a sufficient condition for each commuting trace of arbitrary bilinear mapping on a generalized matrix algebra $\left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ to be proper. Consequently, this make it possible for us to characterize commuting traces of bilinear mappings on full matrix algebras, those of bilinear mappings on triangular algebras and those of bilinear mappings on certain unital algebras with a nontrivial idempotent.

Another important purpose of this article is to address the Lie isomorphisms problem of generalized matrix algebras. At his 1961 AMS Hour Talk, Herstein proposed many problems concerning the structure of Jordan and Lie mappings in associative simple and prime rings [32]. The renowned Herstein's Lie-type mapping research program was formulated since then. The involved Lie mappings mainly include Lie isomorphisms, Lie triple isomorphisms, Lie derivations and Lie triple derivations et al. Given a commutative ring $\mathscr{R}$ with identity and two associative $\mathscr{R}$-algebras $\mathscr{A}$ and $\mathscr{B}$, one can define a *Lie isomorphism* from $\mathscr{A}$ into $\mathscr{B}$ to be an $\mathscr{R}$-linear bijective mapping $\mathfrak{l}$ satisfying the condition

$$\mathfrak{l}([a,b]) = [\mathfrak{l}(a), \mathfrak{l}(b)], \quad \forall a, b \in \mathscr{A}.$$

For example, an isomorphism or the negative of an anti-isomorphism of one algebra onto another is also a Lie isomorphism. One can ask whether the converse is true in some special cases. That is, does every Lie isomorphism between certain associative algebras arise from isomorphisms and anti-isomorphisms in the sense of modulo mappings whose range is central? If $\mathfrak{m}$ is an isomorphism or the negative of an anti-isomorphism from $\mathscr{A}$ onto $\mathscr{B}$ and $\mathfrak{n}$ is an $\mathscr{R}$-linear mapping from $\mathscr{A}$ into the center $\mathscr{Z}(\mathscr{B})$ of $\mathscr{B}$ such that $\mathfrak{n}([a,b]) = 0$ for all $a, b \in \mathscr{A}$, then the mapping

$$\mathfrak{l} = \mathfrak{m} + \mathfrak{n} \qquad (\spadesuit)$$

is a Lie homomorphism. We shall say that a Lie isomorphism $\mathfrak{l}\colon A \longrightarrow B$ is *standard* in the case where it can be expressed in the preceding form $(\spadesuit)$.

The resolution of Herstein's Lie isomorphisms problem in matrix algebra background has been well-known for a long time. Hua [33] proved that every Lie automorphism of the full matrix algebra $\mathscr{M}_n(\mathscr{D})(n \geqslant 3)$ over a division ring $\mathscr{D}$ is of the standard form $(\spadesuit)$. This result was extended to the nonlinear case by Dolinar [30] and Šemrl [57] and was further refined by them. Doković [28] showed that every Lie automorphism of upper triangular matrix algebras $\mathscr{T}_n(\mathscr{R})$ over a commutative ring $\mathscr{R}$ without nontrivial idempotents has the standard form as well. Marcoux and Sourour [40] classified the linear mappings preserving commutativity in both directions (i.e., $[x,y] = 0$ if and only if $[\mathfrak{f}(x), \mathfrak{f}(y)] = 0$) on upper triangular matrix algebras $\mathscr{T}_n(\mathbb{F})$ over a field $\mathbb{F}$. Such a mapping is either the sum of an algebra automorphism of $\mathscr{T}_n(\mathbb{F})$ (which is inner) and a mapping into the center $\mathbb{F}I$, or the sum of the negative of an algebra anti-automorphism and a mapping into the center $\mathbb{F}I$. The classification of the Lie automorphisms of $\mathscr{T}_n(\mathbb{F})$ is obtained as a consequence. Benkovič and Eremita [11] directly applied the theory of commuting traces to the study of Lie isomorphisms on a triangular algebra $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$. They provided sufficient conditions under which every commuting trace of $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ is proper. This is directly applied to the study of Lie isomorphisms of $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$. It turns out that under some mild assumptions, each Lie isomorphism of $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ has the standard form $(\spadesuit)$. On the other hand, Martindale together with some of his students studied Lie isomorphisms problems of associative rings in a series of papers [13, 14, 42, 43, 45, 46, 47, 48, 56]. Speaking in a loose manner, the problems have been resolved provided that the rings in question contain certain nontrivial idempotents. Simultaneously, the treatment of the problems has been extended from simple rings to prime rings. The question whether the results on Lie isomorphisms can be obtained in rings containing no nontrivial idempotents has been open for a long time. The first idempotent free result on Lie isomorphisms was obtained in 1993 by Brešar [16]. Under some mild technical assumptions (which were removed somewhat later [19]), he described the form of a Lie isomorphism between arbitrary prime rings. This was also the first paper based on applications of the theory of functional identities. Just recently, Beidar, Brešar, Chebotar, Martindale jointly gave a final solution to the long-standing Herstein's conjecture of Lie isomorphisms of prime rings using the theory of functional identities, see the paper [5] and references therein. Simultaneously, Lie isomorphisms between rings and between (non-)self-adjoint operator algebras have received a fair amount of attention and have also been intensively studied. The involved rings and operator algebras include (semi-)prime rings, the algebra of bounded linear operators, C^*-algebras, von Neumann algebras, H^*-algebras, Banach space nest algebras, Hilbert space nest algebras, reflexive algebras, see [1, 2, 3, 21, 22, 23, 24, 39, 41, 49, 50, 51, 54, 55, 57, 58, 59, 61, 62].

This is the first paper in a series of three that we are planning on this topic. The other two papers will be dedicated to studying, in more detail, centralizing traces and Lie triple isomorphisms on triangular algebras and those mappings on generalized matrix algebras [38]. The roadmap of this paper is as follows. Section 2 contains the definition of generalized matrix algebra and some classical examples. In Section 3 we provide sufficient conditions for each commuting trace of an arbitrary bilinear mapping on a generalized matrix algebra $\left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ to be proper (Theorem 3.4). And then we apply this result to describe the commuting traces of various generalized matrix algebras. In Section 4 we will give sufficient conditions under which every Lie isomorphism from a generalized matrix algebra into another one has the standard form (Theorem 4.3). As corollaries of Theorem 4.3, characterizations of Lie isomorphisms on triangular algebras, on full matrix algebras and on certain unital algebras with nontrivial idempotents are obtained. The last section contains some potential future research topics related to our current work.

2. Generalized matrix algebras and examples

Let us begin with the definition of generalized matrix algebras given by a Morita context. Let $\mathscr{R}$ be a commutative ring with identity. A *Morita context* consists of two $\mathscr{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN}: M\underset{B}{\otimes}N\longrightarrow A$ and $\Psi_{NM}: N\underset{A}{\otimes}M\longrightarrow B$ satisfying the following commutative diagrams:

$$\begin{CD} M\underset{B}{\otimes}N\underset{A}{\otimes}M @>{\Phi_{MN}\otimes I_M}>> A\underset{A}{\otimes}M \\ @V{I_M\otimes\Psi_{NM}}VV @VV{\cong}V \\ M\underset{B}{\otimes}B @>{\cong}>> M \end{CD} \qquad \text{and} \qquad \begin{CD} N\underset{A}{\otimes}M\underset{B}{\otimes}N @>{\Psi_{NM}\otimes I_N}>> B\underset{B}{\otimes}N \\ @V{I_N\otimes\Phi_{MN}}VV @VV{\cong}V \\ N\underset{A}{\otimes}A @>{\cong}>> N\,. \end{CD}$$

Let us write this Morita context as $(A,B,M,N,\Phi_{MN},\Psi_{NM})$. We refer the reader to [53] for the basic properties of Morita contexts. If $(A,B,M,N,\ \Phi_{MN},\Psi_{NM})$ is a Morita context, then the set

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| a\in A, m\in M, n\in N, b\in B \right\}$$

form an $\mathscr{R}$-algebra under matrix-like addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. Such an $\mathscr{R}$-algebra is usually called a *generalized matrix algebra* of order 2 and is denoted by

$$\mathscr{G} = \mathscr{G}(A, M, N, B) = \begin{bmatrix} A & M \\ N & B \end{bmatrix}.$$

In a similar way, one can define a generalized matrix algebra of order $n > 2$. It was shown that up to isomorphism, arbitrary generalized matrix algebra of order n $(n \geqslant 2)$ is a generalized matrix algebra of order 2 [36, Example 2.2]. If one of the modules M and N is zero, then $\mathscr{G}$ exactly degenerates to an *upper triangular algebra* or a *lower triangular algebra*. In this case, we denote the resulted upper triangular algebra (resp. lower triangular algebra) by

$$\mathscr{T}^{\mathscr{U}} = \mathscr{T}(A, M, B) = \begin{bmatrix} A & M \\ O & B \end{bmatrix} \quad \left(\text{resp. } \mathscr{T}_{\mathscr{L}} = \mathscr{T}(A, N, B) = \begin{bmatrix} A & O \\ N & B \end{bmatrix}\right)$$

Note that our current generalized matrix algebras contain those generalized matrix algebras in the sense of Brown [20] as special cases. Let $\mathscr{M}_n(\mathscr{R})$ be the full matrix algebra consisting of all $n \times n$ matrices over $\mathscr{R}$. It is worth to point out that the notion of generalized matrix algebras efficiently unifies triangular algebras with full matrix algebras together. The distinguished feature of our systematic work is to deal with all questions related to (non-)linear mappings of triangular algebras and of full matrix algebras under a unified frame, which is the admired generalized matrix algebras frame, see [36, 37, 60].

Let us list some classical examples of generalized matrix algebras which will be revisited in the sequel (Section 3 and Section 4). Since these examples have already been presented in many papers, we just state their title without any introduction. We refer the reader to [36, 60] for more details.

(a) Unital algebras with nontrivial idempotents;

(b) Full matrix algebras;

(c) Inflated algebras;

(d) Upper and lower triangular matrix algebras;

(e) Hilbert space nest algebras

3. Commuting traces of bilinear mappings on generalized matrix algebras

In this section we will establish sufficient conditions for each commuting trace of an arbitrary bilinear mapping on a generalized matrix algebra $\mathscr{G}$ to be proper (Theorem 3.4). Consequently, we are able to describe commuting traces of bilinear mappings on triangular algebras, on full matrix algebras and on certain unital algebras with nontrivial idempotents. The most important is that Theorem 3.4 will be used to characterize Lie isomorphisms from a generalized matrix algebra into another in Section 4.

Throughout this section, we denote the generalized matrix algebra of order 2 originated from the Morita context $(A, B, {}_AM_B, {}_BN_A, \Phi_{MN}, \Psi_{NM})$ by

$$\mathscr{G} = \mathscr{G}(A, M, N, B) = \begin{bmatrix} A & M \\ N & B \end{bmatrix},$$

where at least one of the two bimodules M and N is distinct from zero. We always assume that M is faithful as a left A-module and also as a right B-module, but no any constraint conditions on N. The center of $\mathscr{G}$ is

$$\mathscr{Z}(\mathscr{G}) = \{\, a \oplus b \mid am = mb,\ na = bn,\ \forall\, m \in M,\ \forall n \in N\}.$$

Indeed, by [34, Lemma 1] we know that the center $\mathscr{Z}(\mathscr{G})$ consists of all diagonal matrices $a \oplus b$, where $a \in \mathscr{Z}(A)$, $b \in \mathscr{Z}(B)$ and $am = mb$, $na = bn$ for all $m \in M, n \in N$. However, in our situation which M is faithful as a left A-module and also as a right B-module, the conditions that $a \in \mathscr{Z}(A)$ and $b \in \mathscr{Z}(B)$ become redundant and can be deleted. Indeed, if $am = mb$ for all $m \in M$, then for any $a' \in A$ we get

$$(aa' - a'a)m = a(a'm) - a'(am) = (a'm)b - a'(mb) = 0.$$

The assumption that M is faithful as a left A-module leads to $aa' - a'a = 0$ and hence $a \in \mathscr{Z}(A)$. Likewise, we also have $b \in \mathscr{Z}(B)$.

Let us define two natural $\mathscr{R}$-linear projections $\pi_A : \mathscr{G} \to A$ and $\pi_B : \mathscr{G} \to B$ by

$$\pi_A : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \longmapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \longmapsto b.$$

By the above paragraph, it is not difficult to see that $\pi_A(\mathscr{Z}(\mathscr{G}))$ is a subalgebra of $\mathscr{Z}(A)$ and that $\pi_B(\mathscr{Z}(\mathscr{G}))$ is a subalgebra of $\mathscr{Z}(B)$. Given an element $a \in \pi_A(\mathscr{Z}(\mathscr{G}))$, if $a \oplus b, a \oplus b' \in \mathscr{Z}(\mathscr{G})$, then we have $am = mb = mb'$ for all $m \in M$. Since M is faithful as a right B-module, $b = b'$. That implies there exists a unique $b \in \pi_B(\mathscr{Z}(\mathscr{G}))$, which is denoted by $\varphi(a)$, such that $a \oplus b \in \mathscr{Z}(\mathscr{G})$. It is easy to verify that the map $\varphi : \pi_A(\mathscr{Z}(\mathscr{G})) \longrightarrow \pi_B(\mathscr{Z}(\mathscr{G}))$ is an algebraic isomorphism such that $am = m\varphi(a)$ and $na = \varphi(a)n$ for all $a \in \pi_A(\mathscr{Z}(\mathscr{G})), m \in M, n \in N$.

Let $\mathscr{A}$ and $\mathscr{B}$ be algebras. Recall an $(\mathscr{A}, \mathscr{B})$-bimodule $\mathscr{M}$ is *loyal* if $a\mathscr{M}b = 0$ implies that $a = 0$ or $b = 0$ for all $a \in \mathscr{A}, b \in \mathscr{B}$. Let us first state several lemmas without proofs, since their proofs are identical with those of [11, Lemma 2.4, Lemma 2.5, Lemma 2.6].

LEMMA 3.1. *Let $\mathscr{G} = \mathscr{G}(A, M, N, B)$ be a generalized matrix algebra with a loyal (A, B)-bimodule M. For arbitrary element $\lambda \in \pi_A(\mathscr{Z}(\mathscr{G}))$ and arbitrary nonzero element $a \in A$, if $\lambda a = 0$, then $\lambda = 0$*

LEMMA 3.2. *Let $\mathscr{G} = \mathscr{G}(A, M, N, B)$ be a generalized matrix algebra with a loyal (A, B)-bimodule M. Then the center $\mathscr{Z}(\mathscr{G})$ of $\mathscr{G}$ is a domain.*

LEMMA 3.3. *The generalized matrix algebra $\mathscr{G} = \mathscr{G}(A, M, N, B)$ has no nonzero central ideals.*

We are ready to state and prove the main result of this section.

THEOREM 3.4. *Let $\mathscr{G} = \mathscr{G}(A,M,N,B)$ be a 2-torsionfree generalized matrix algebra over a commutative ring $\mathscr{R}$ and $\mathfrak{q}: \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G}$ be an $\mathscr{R}$-bilinear mapping. If*

(1) *every commuting linear mapping on A or B is proper;*

(2) *$\pi_A(\mathscr{Z}(\mathscr{G})) = \mathscr{Z}(A) \neq A$ and $\pi_B(\mathscr{Z}(\mathscr{G})) = \mathscr{Z}(B) \neq B$;*

(3) *M is loyal,*

then every commuting trace $\mathfrak{T}_{\mathfrak{q}} : \mathscr{G} \longrightarrow \mathscr{G}$ of $\mathfrak{q}$ is proper.

For convenience, let us write $A_1 = A$, $A_2 = M$, $A_3 = N$ and $A_4 = B$. Suppose that $\mathfrak{T}_{\mathfrak{q}}$ is an arbitrary trace of the $\mathscr{R}$-bilinear mapping $\mathfrak{q}$. Then there exist $\mathscr{R}$-bilinear mappings $f_{ij} : A_i \times A_j \to A_1$, $g_{ij} : A_i \times A_j \to A_2$, $h_{ij} : A_i \times A_j \to A_3$ and $k_{ij} : A_i \times A_j \to A_4$, $1 \leqslant i \leqslant j \leqslant 4$, such that

$$\begin{aligned} \mathfrak{T}_{\mathfrak{q}} &: \mathscr{G} \longrightarrow \mathscr{G} \\ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} &\longmapsto \begin{bmatrix} F(a_1,a_2,a_3,a_4) & G(a_1,a_2,a_3,a_4) \\ H(a_1,a_2,a_3,a_4) & K(a_1,a_2,a_3,a_4) \end{bmatrix}, \forall \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathscr{G} \end{aligned}$$

where

$$F(a_1,a_2,a_3,a_4) = \sum_{1 \leqslant i \leqslant j \leqslant 4} f_{ij}(a_i,a_j),$$

$$G(a_1,a_2,a_3,a_4) = \sum_{1 \leqslant i \leqslant j \leqslant 4} g_{ij}(a_i,a_j),$$

$$H(a_1,a_2,a_3,a_4) = \sum_{1 \leqslant i \leqslant j \leqslant 4} h_{ij}(a_i,a_j),$$

$$K(a_1,a_2,a_3,a_4) = \sum_{1 \leqslant i \leqslant j \leqslant 4} k_{ij}(a_i,a_j).$$

Since $\mathfrak{T}_{\mathfrak{q}}$ is commuting, we have

$$\begin{aligned} 0 &= \left[\begin{bmatrix} F & G \\ H & K \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right] \\ &= \begin{bmatrix} Fa_1 + Ga_3 - a_1F - a_2H & Fa_2 + Ga_4 - a_1G - a_2K \\ Ha_1 + Ka_3 - a_3F - a_4H & Ha_2 + Ka_4 - a_3G - a_4K \end{bmatrix} \end{aligned} \qquad (\bigstar)$$

for all $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathscr{G}$.

For convenience, let us write $a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ in the sequel. Now we divide the proof of Theorem 3.4 into a series of lemmas for comfortable reading.

LEMMA 3.5. *$H(a_1,a_2,a_3,a_4)=h_{13}(a_1,a_3)+h_{23}(a_2,a_3)+h_{33}(a_3,a_3)+h_{34}(a_3,a_4)$ holds for all $a\in\mathscr{G}$.*

Proof. It follows from the matrix relation $(\bigstar)$ that

$$Ha_1+Ka_3-a_3F-a_4H=0. \tag{3.1}$$

Let us choose $a_2=0$, $a_3=0$ and $a_4=0$. Then (3.1) implies that $h_{11}(a_1,a_1)a_1=0$ for all $a_1\in A_1$. Obviously, $h_{11}(1,1)=0$. Replacing a_1 by a_1+1 and $1-a_1$ in $h_{11}(a_1,a_1)a_1=0$ in turn, we obtain

$$(h_{11}(a_1,a_1)+h_{11}(a_1,1)+h_{11}(1,a_1))(a_1+1)=0$$

and

$$(h_{11}(a_1,a_1)-h_{11}(a_1,1)-h_{11}(1,a_1))(1-a_1)=0$$

for all $a_1\in A_1$. Combining the above two equations yields that $2(h_{11}(a_1,1)+h_{11}(1,a_1))=0$. Since $\mathscr{G}$ is 2-torsion free, $h_{11}(a_1,a_1)=0$ for all $a_1\in A_1$.

Let us take $a_3=0$ and $a_4=0$ in (3.1). Then we get

$$(h_{12}(a_1,a_2)+h_{22}(a_2,a_2))a_1=0 \tag{3.2}$$

for all $a_1\in A_1,a_2\in A_2$. Substituting $-a_2$ for a_2 in (3.2) gives

$$(-h_{12}(a_1,a_2)+h_{22}(a_2,a_2))a_1=0 \tag{3.3}$$

for all $a_1\in A_1,a_2\in A_2$. By (3.2) and (3.3) we know that $2h_{22}(a_2,a_2)a_1=0$ for all $a_1\in A_1,a_2\in A_2$. Hence $h_{22}(a_2,a_2)=0$ for all $a_2\in A_2$.

Now the relation (3.2) shows that $h_{12}(a_1,a_2)a_1=0$ for all $a_1\in A_1,a_2\in A_2$. Thus $h_{12}(1,a_2)=0$. Replacing a_1 by a_1+1 in $h_{12}(a_1,a_2)a_1=0$ leads to $0=(h_{12}(a_1,a_2)+h_{12}(1,a_2))(a_1+1)=h_{12}(a_1,a_2)$.

Let us choose $a_1=0$, $a_2=0$ and $a_3=0$. Applying (3.1) yields that $a_4h_{44}(a_4,a_4)=0$ for all $a_4\in A_4$. Therefore $h_{44}(1,1)=0$. Substituting a_4+1 and $1-a_4$ for a_4 in $a_4h_{44}(a_4,a_4)=0$ in turn, we arrive at

$$(a_4+1)(h_{44}(a_4,a_4)+h_{44}(a_4,1)+h_{44}(1,a_4))=0$$

and

$$(1-a_4)(h_{44}(a_4,a_4)-h_{44}(a_4,1)-h_{44}(1,a_4))=0$$

for all $a_4\in A_4$. Combining the above two equations gives $2(h_{44}(a_4,1)+h_{44}(1,a_4))=0$. Since $\mathscr{G}$ is 2-torsion free, $h_{44}(a_4,a_4)=0$ for all $a_4\in A_4$.

If we take $a_1=0$ and $a_3=0$ into (3.1), then

$$a_4(h_{22}(a_2,a_2)+h_{24}(a_2,a_4))=0 \tag{3.4}$$

for all $a_2\in A_2,a_4\in A_4$. Note that the fact $h_{22}(a_2,a_2)=0$ for all $a_2\in A_2$. Hence (3.4) implies that $a_4h_{24}(a_2,a_4)=0$ for all $a_2\in A_2,a_4\in A_4$. So $h_{24}(a_2,1)=0$. Replacing

a_4 by a_4+1 in $a_4h_{24}(a_2,a_4)=0$, we obtain $0=(a_4+1)(h_{24}(a_2,a_4)+h_{24}(a_2,1))=h_{24}(a_2,a_4)$.

Finally let us choose $a_3=0$. Then (3.1) becomes

$$h_{14}(a_1,a_4)a_1-a_4h_{14}(a_1,a_4)=0 \tag{3.5}$$

for all $a_1\in A_1, a_4\in A_4$. Replacing a_1 by $-a_1$ in (3.5) we have

$$h_{14}(a_1,a_4)a_1+a_4h_{14}(a_1,a_4)=0 \tag{3.6}$$

for all $a_1\in A_1, a_2\in A_2$. Combining (3.5) with (3.6) yields $h_{14}(a_1,a_4)a_1=0$ for all $a_1\in A_1, a_4\in A_4$. Clearly, $h_{14}(1,a_4)=0$ for all $a_4\in A_4$. Substituting a_1+1 for a_1 in $h_{14}(a_1,a_4)a_1=0$, we get $0=(h_{14}(a_1,a_4)+h_{14}(1,a_4))(a_1+1)=h_{14}(a_1,a_4)$ and this completes the proof of the lemma. □

Similarly, we can show

LEMMA 3.6. *$G(a_1,a_2,a_3,a_4)=g_{12}(a_1,a_2)+g_{22}(a_2,a_2)+g_{23}(a_2,a_3)+g_{24}(a_2,a_4)$ holds for all $a\in\mathscr{G}$.*

LEMMA 3.7. *With notations as above, we have*

(1) $a_1\mapsto f_{11}(a_1,a_1)$ *is a commuting trace;*

(2) $a_1\mapsto f_{12}(a_1,a_2)$, $a_1\mapsto f_{13}(a_1,a_3)$, $a_1\mapsto f_{14}(a_1,a_4)$ *are commuting linear mappings for each* $a_2\in A_2, a_3\in A_3, a_4\in A_4$, *respectively;*

(3) $f_{22},f_{24},f_{33},f_{34},f_{44}$ *map into* $\mathscr{Z}(A_1)$.

Proof. It follows from the matrix relation (★) that

$$Fa_1+Ga_3-a_1F-a_2H=0. \tag{3.7}$$

Let us take $a_2=0$, $a_3=0$ and $a_4=0$ in (3.7). Thus $[f_{11}(a_1,a_1),a_1]=0$ for all $a_1\in A_1$.

Let us choose $a_3=0$ and $a_4=0$. Applying Lemma 3.5 and (3.7) yields $[F,a_1]=0$, that is

$$[f_{12}(a_1,a_2)+f_{22}(a_2,a_2),a_1]=0 \tag{3.8}$$

for all $a_1\in A_1, a_2\in A_2$. Replacing a_1 by $-a_1$ in (3.8) we obtain

$$[f_{12}(a_1,a_2)-f_{22}(a_2,a_2),a_1]=0 \tag{3.9}$$

for all $a_1\in A_1, a_2\in A_2$. Combining (3.8) with (3.9) we get $[f_{12}(a_1,a_2),a_1]=0$ and $[f_{22}(a_2,a_2),a_1]=0$ for all $a_1\in A_1, a_2\in A_2$.

If we take $a_3=0$, then (3.7) and Lemma 3.5 imply that

$$[f_{14}(a_1,a_4)+f_{24}(a_2,a_4)+f_{44}(a_4,a_4),a_1]=0 \tag{3.10}$$

for all $a_1 \in A_1, a_2 \in A_2, a_4 \in A_4$. Substituting $-a_1$ for a_1 in (3.10) we have

$$[f_{14}(a_1,a_4) - f_{24}(a_2,a_4) - f_{44}(a_4,a_4), a_1] = 0 \tag{3.11}$$

for all $a_1 \in A_1, a_2 \in A_2, a_4 \in A_4$. In view of (3.10) and (3.11), we arrive at $[f_{14}(a_1,a_4), a_1] = 0$ and $[f_{24}(a_2,a_4) + f_{44}(a_4,a_4), a_1] = 0$. Taking $a_2 = 0$ into the last equality we get $f_{44}(a_4,a_4) \in \mathscr{Z}(A_1)$ and hence $f_{24}(a_2,a_4) \in \mathscr{Z}(A_1)$ for all $a_2 \in A_2, a_4 \in A_4$.

Let us choose $a_2 = 0$. By (3.7) and Lemma 3.6 it follows that

$$[f_{13}(a_1,a_3) + f_{33}(a_3,a_3) + f_{34}(a_3,a_4), a_1] = 0 \tag{3.12}$$

for all $a_1 \in A_1, a_3 \in A_3, a_4 \in A_4$. Let us put $a_4 = 0$ in (3.12). Then

$$[f_{13}(a_1,a_3) + f_{33}(a_3,a_3), a_1] = 0 \tag{3.13}$$

for all $a_1 \in A_1, a_3 \in A_3$, which gives $f_{34}(a_3,a_4) \in \mathscr{Z}(A_1)$. Replacing a_1 by $-a_1$ in (3.13) yields

$$[f_{13}(a_1,a_3) - f_{33}(a_3,a_3), a_1] = 0 \tag{3.14}$$

for all $a_1 \in A_1, a_3 \in A_3$. Combining (3.13) with (3.14) we obtain $f_{33}(a_3,a_3) \in \mathscr{Z}(A_1)$ and $[f_{13}(a_1,a_3), a_1] = 0$ for all $a_1 \in A_1, a_3 \in A_3$. □

Using an analogous proof of Lemma 3.7 the following results hold.

LEMMA 3.8. *With notations as above, we have*

(1) $a_4 \mapsto k_{44}(a_4,a_4)$ *is a commuting trace;*

(2) $a_4 \mapsto k_{14}(a_1,a_4)$, $a_4 \mapsto k_{24}(a_2,a_4)$, $a_4 \mapsto k_{34}(a_3,a_4)$ *are commuting mappings for each* $a_1 \in A_1, a_2 \in A_2, a_3 \in A_3$, *respectively;*

(3) $k_{11}, k_{12}, k_{13}, k_{22}, k_{33}$ *map into* $\mathscr{Z}(A_4)$.

LEMMA 3.9. *We have* $f_{22}(a_2,a_2) \oplus k_{22}(a_2,a_2) \in \mathscr{Z}(\mathscr{G})$ *and* $f_{33}(a_3,a_3) \oplus k_{33}(a_3,a_3) \in \mathscr{Z}(\mathscr{G})$ *for all* $a_2 \in A_2, a_3 \in A_3$.

Proof. By the relation (★) we know that

$$Fa_2 + Ga_4 - a_1G - a_2K = 0. \tag{3.15}$$

Let us take $a_1 = 0$ and $a_4 = 0$. Then (3.15) implies that

$$(f_{22}(a_2,a_2) + f_{23}(a_2,a_3) + f_{33}(a_3,a_3))a_2 = a_2(k_{22}(a_2,a_2) + k_{23}(a_2,a_3) + k_{33}(a_3,a_3)) \tag{3.16}$$

for all $a_2 \in A_2, a_3 \in A_3$. Moreover, setting $a_3 = 0$ in (3.16) we get

$$f_{22}(a_2,a_2)a_2 = a_2k_{22}(a_2,a_2) \tag{3.17}$$

for all $a_2 \in A_2$. Applying Lemma 3.7, Lemma 3.8 and [60, Lemma 3.2] yields that $(f_{22}(a_2,a_2) - \varphi^{-1}(k_{22}(a_2,a_2)))a_2 = 0$. By the complete linearization we have

$$\beta(x,y)z + \beta(z,x)y + \beta(y,z)x = 0 \tag{3.18}$$

for all $x,y,z \in A_2$, where

$$\beta(x,y) = f_{22}(x,y) - \varphi^{-1}(k_{22}(x,y)) + f_{22}(y,x) - \varphi^{-1}(k_{22}(y,x)).$$

Obviously, the mapping $\beta : A_2 \times A_2 \to \mathscr{Z}(A_1)$ is bilinear and symmetric. By the hypothesis there exist $a,b \in A_1$ such that $[a,b] \neq 0$. Replacing z by az in (3.18) and subtracting the left multiplication of (3.18) by a, we get

$$(\beta(az,x) - \beta(z,x)a)y + (\beta(y,az) - \beta(y,z)a)x = 0$$

for all $x,y,z \in A_2$. It follows from [11, Lemma 2.3] that $\beta(az,x) = \beta(z,x)a$ and hence $\beta(z,x)[a,b] = 0$ for all $x,z \in A_2$. Applying Lemma 3.1 yields $\beta = 0$. In particular, $\beta(a_2,a_2) = 0$ for all $a_2 \in A_2$. Thus $f_{22}(a_2,a_2) \oplus k_{22}(a_2,a_2) \in \mathscr{Z}(\mathscr{G})$.

Now the relation (3.16) becomes

$$(f_{23}(a_2,a_3) + f_{33}(a_3,a_3))a_2 = a_2(k_{23}(a_2,a_3) + k_{33}(a_3,a_3)) \tag{3.19}$$

for all $a_2 \in A_2, a_3 \in A_3$. Substituting $-a_2$ for a_2 and applying (3.19), we arrive at $f_{33}(a_3,a_3)a_2 = a_2 k_{33}(a_3,a_3)$ for all $a_2 \in A_2, a_3 \in A_3$. In view of the fact M is faithful as a left A-module and $k_{33}(a_3,a_3) \in \mathscr{Z}(B) = \pi_B(\mathscr{Z}(\mathscr{G}))$, we assert that $f_{33}(a_3,a_3) \oplus k_{33}(a_3,a_3) \in \mathscr{Z}(\mathscr{G})$. $\square$

LEMMA 3.10. *$f_{12}(a_1,a_2) = \alpha(a_2)a_1 + \varphi^{-1}(k_{12}(a_1,a_2))$ and $k_{24}(a_2,a_4) = \varphi(\alpha(a_2))a_4 + \varphi(f_{24}(a_2,a_4))$ hold for all $a_1 \in A_1, a_2 \in A_2, a_4 \in A_4$, where $\alpha(a_2) = f_{12}(1,a_2) - \varphi^{-1}(k_{12}(1,a_2))$.*

Proof. Taking $a_4 = 0$ into (3.15) and using (3.16) we have

$$\begin{aligned}\big(f_{11}(a_1,a_1) + f_{12}(a_1,a_2) + f_{13}(a_1,a_3)\big)a_2 - a_2\big(k_{11}(a_1,a_1) + k_{12}(a_1,a_2) + k_{13}(a_1,a_3)\big)\\ -a_1\big(g_{12}(a_1,a_2) + g_{22}(a_2,a_2) + g_{23}(a_2,a_3)\big) = 0.\end{aligned} \tag{3.20}$$

Replacing a_1 by $-a_1$ in (3.20) we get

$$\begin{aligned}\big(f_{11}(a_1,a_1) - f_{12}(a_1,a_2) - f_{13}(a_1,a_3)\big)a_2 - a_2\big(k_{11}(a_1,a_1) - k_{12}(a_1,a_2) - k_{13}(a_1,a_3)\big)\\ -a_1\big(g_{12}(a_1,a_2) - g_{22}(a_2,a_2) - g_{23}(a_2,a_3)\big) = 0.\end{aligned} \tag{3.21}$$

Combining (3.20) with (3.21) yields

$$a_1 g_{12}(a_1,a_2) = f_{11}(a_1,a_1)a_2 - a_2 k_{11}(a_1,a_1), \tag{3.22}$$

$$a_1 g_{22}(a_2,a_2) = f_{12}(a_1,a_2)a_2 - a_2 k_{12}(a_1,a_2), \tag{3.23}$$

$$a_1g_{23}(a_2,a_3) = f_{13}(a_1,a_3)a_2 - a_2k_{13}(a_1,a_3). \tag{3.24}$$

In an analogous way, taking $a_1 = 0$ into (3.15) and using (3.16) we obtain

$$g_{24}(a_2,a_4)a_4 = a_2k_{44}(a_4,a_4) - f_{44}(a_4,a_4)a_2, \tag{3.25}$$

$$g_{22}(a_2,a_2)a_4 = a_2k_{24}(a_2,a_4) - f_{24}(a_2,a_4)a_2, \tag{3.26}$$

$$g_{23}(a_2,a_3)a_4 = a_2k_{34}(a_3,a_4) - f_{34}(a_3,a_4)a_2. \tag{3.27}$$

On the other hand, we have showed that $[f_{12}(a_1,a_2),a_1] = 0$ for all $a_1 \in A_1, a_2 \in A_2$. Substituting a_1+1 for a_1 leads to $f_{12}(1,a_2) \in \mathscr{Z}(A_1)$ for all $a_2 \in A_2$. By the relation (3.23) we know that

$$g_{22}(a_2,a_2) = \alpha(a_2)a_2, \tag{3.28}$$

where $\alpha(a_2) = f_{12}(1,a_2) - \varphi^{-1}(k_{12}(1,a_2)) \in \mathscr{Z}(A_1)$. Let us set $E(a_1,a_2) = f_{12}(a_1,a_2) - \alpha(a_2)a_1 - \varphi^{-1}(k_{12}(a_1,a_2))$. Then (3.23) and (3.28) jointly imply that $E(a_1,a_2)a_2 = 0$, which further gives $E(a_1,a_2)b_2 + E(a_1,b_2)a_2 = 0$ for all $a_1 \in A_1$ and $a_2,b_2 \in A_2$. By [11, Lemma 2.3] we conclude that $E(a_1,a_2) = 0$. Hence $f_{12}(a_1,a_2) = \alpha(a_2)a_1 + \varphi^{-1}(k_{12}(a_1,a_2))$. Similarly, we can show that k_{24} is of the desired form as well. □

LEMMA 3.11. *$f_{13}(a_1,a_3) = \tau(a_3)a_1 + \varphi^{-1}(k_{13}(a_1,a_3))$ and $k_{34}(a_3,a_4) = \varphi(\tau(a_3))a_4 + \varphi(f_{34}(a_3,a_4))$ hold for all $a_1 \in A_1, a_3 \in A_3, a_4 \in A_4$, where $\tau(a_3) = f_{13}(1,a_3) - \varphi^{-1}(k_{13}(1,a_3))$.*

Proof. Note that $[f_{13}(a_1,a_3),a_1] = 0$ for all $a_1 \in A_1, a_3 \in A_3$. Substituting $a_1 + 1$ for a_1 gives $f_{13}(1,a_3) \in \mathscr{Z}(A_1)$ for all $a_3 \in A_3$. Let us set $\tau(a_3) = f_{13}(1,a_3) - \varphi^{-1}(k_{13}(1,a_3))$ and $E(a_1,a_3) = f_{13}(a_1,a_3) - \tau(a_3)a_1 - \varphi^{-1}(k_{13}(a_1,a_3))$. It follows from (3.24) that $E(a_1,a_3)a_2 = 0$ for all $a_1 \in A_1, a_2 \in A_2, a_3 \in A_3$. Since $M = A_2$ is faithful as a left A-module, we obtain $E(a_1,a_3) = 0$ and hence $f_{13}(a_1,a_3) = \tau(a_3)a_1 + \varphi^{-1}(k_{13}(a_1,a_3))$. Similarly, using (3.27) one can prove that k_{34} is of the desired form as well. □

LEMMA 3.12. *There exist linear mapping $\gamma : A_4 \to \mathscr{Z}(A_1)$ and bilinear mapping $\delta : A_1 \times A_4 \to \mathscr{Z}(A_1)$ such that $f_{14}(a_1,a_4) = \gamma(a_4)a_1 + \delta(a_1,a_4)$ for all $a_1 \in A_1, a_4 \in A_4$.*

Proof. Since $a_1 \mapsto f_{14}(a_1,a_4)$ is a commuting mapping of A_1 for all $a_4 \in A_4$, there exist mappings $\gamma : A_4 \to \mathscr{Z}(A_1)$ and $\delta : A_1 \times A_4 \to \mathscr{Z}(A_1)$ such that

$$f_{14}(a_1,a_4) = \gamma(a_4)a_1 + \delta(a_1,a_4),$$

where δ is $\mathscr{R}$-linear in the first argument. Let us show that γ is $\mathscr{R}$-linear and that δ is $\mathscr{R}$-bilinear. It is easy to observe that

$$f_{14}(a_1,a_4+b_4) = \gamma(a_4+b_4)a_1 + \delta(a_1,a_4+b_4)$$

and

$$f_{14}(a_1,a_4) + f_{14}(a_1,b_4) = \gamma(a_4)a_1 + \delta(a_1,a_4) + \gamma(b_4)a_1 + \delta(a_1,b_4).$$

for all for all $a_1 \in A_1$ and $a_4, b_4 \in A_4$. Therefore

$$\big(\gamma(a_4+b_4)-\gamma(a_4)-\gamma(b_4)\big)a_1+\delta(a_1,a_4+b_4)-\delta(a_1,a_4)-\delta(a_1,b_4)=0$$

for all $a_1 \in A_1$ and $a_4, b_4 \in A_4$. Note that both γ and δ map into $\mathscr{Z}(A_1)$ and hence $(\gamma(a_4+b_4)-\gamma(a_4)-\gamma(b_4))[a_1,b_1]=0$ for all $a_1,b_1 \in A_1$ and $a_4,b_4 \in A_4$. Applying Lemma 3.1 yields that γ is $\mathscr{R}$-linear. Consequently, δ is $\mathscr{R}$-linear in the second argument. $\square$

LEMMA 3.13. *$k_{14}(a_1,a_4)=\gamma'(a_1)a_4+\varphi(\delta(a_1,a_4))$ holds for all $a_1 \in A_1, a_4 \in A_4$, where $\gamma'(a_1)=k_{14}(a_1,1)-\varphi(\delta(a_1,1))$.*

Proof. By (3.22) we know that $g_{12}(1,a_2)=f_{11}(1,1)a_2-a_2k_{11}(1,1)$ for all $a_2 \in A_2$. On the other hand, the equations (3.22) – (3.27) together with (3.15) imply that

$$f_{14}(a_1,a_4)a_2+g_{12}(a_1,a_2)a_4=a_1g_{24}(a_2,a_4)+a_2k_{14}(a_1,a_4) \tag{3.29}$$

for all $a_1 \in A_1, a_2 \in A_2, a_4 \in A_4$. Let us set $a_1=1$ in (3.29). Then

$$g_{24}(a_2,a_4)=a_2\big(\zeta a_4+\varphi(f_{14}(1,a_4))-k_{14}(1,a_4)\big) \tag{3.30}$$

for all $a_2 \in A_2, a_4 \in A_4$, where $\zeta=\varphi(f_{11}(1,1))-k_{11}(1,1)$. Similarly, using (3.25) and (3.29) we have

$$g_{12}(a_1,a_2)=\big(\theta a_1+\varphi^{-1}(k_{14}(a_1,1))-f_{14}(a_1,1)\big)a_2 \tag{3.31}$$

for all $a_1 \in A_1, a_2 \in A_2$, where $\theta=\varphi^{-1}(k_{44}(1,1))-f_{44}(1,1)$. Now the equations (3.29) – (3.31) and Lemma 3.12 jointly show that

$$(\gamma(a_4)a_1+\delta(a_1,a_4))a_2+\big(\theta a_1+\varphi^{-1}(k_{14}(a_1,1))-f_{14}(a_1,1)\big)a_2a_4$$
$$=a_2k_{14}(a_1,a_4)+a_1a_2\big(\zeta a_4+\varphi(f_{14}(1,a_4))-k_{14}(1,a_4)\big)$$

for all $a_1 \in A_1, a_2 \in A_2, a_4 \in A_4$. That is,

$$a_1a_2\big((\zeta+\varphi(\gamma(1)-\theta)a_4+\varphi(\delta(1,a_4))-k_{14}(1,a_4)\big)$$
$$=a_2\big(\gamma'(a_1)a_4+\varphi(\delta(a_1,a_4))-k_{14}(a_1,a_4)\big) \tag{3.32}$$

for all $a_1 \in A_1, a_2 \in A_2, a_4 \in A_4$. Replacing a_2 by b_1a_2 in (3.32) and subtracting the left multiplication of (3.32) by b_1 gives

$$[a_1,b_1]a_2\big((\zeta+\varphi(\gamma(1)-\theta)a_4+\varphi(\delta(1,a_4))-k_{14}(1,a_4)\big)=0$$

for all $a_1,b_1 \in A_1, a_2 \in A_2, a_4 \in A_4$. Note that $M=A_2$ is loyal and $A=A_1$ is noncommutative. It follows that

$$k_{14}(1,a_4)=(\zeta+\varphi(\gamma(1)-\theta)a_4+\varphi(\delta(1,a_4))$$

for all $a_4 \in A_4$. Consequently, the relation (3.32) implies that

$$A_2\big(\gamma'(a_1)a_4 + \varphi(\delta(a_1,a_4)) - k_{14}(a_1,a_4)\big) = 0$$

for all $a_1, a_4 \in A_4$. Since $A_2 = M$ is faithful as a right B-module, k_{14} is of the desired form. □

Proof of Theorem 3.4. Let us write $\varepsilon = \theta - \gamma(1)$ and $\varepsilon' = \zeta - \gamma'(1)$. By the equations (3.30) and (3.31) and the form of f_{14}, k_{14}, we have the following relations:

$$g_{12}(a_1,a_2) = \varepsilon a_1 a_2 + \varphi^{-1}(\gamma'(a_1))a_2, \quad g_{24}(a_2,a_4) = a_2(\varepsilon' a_4 + \varphi(\gamma(a_4))) \tag{3.33}$$

for all $a_1 \in A_1, a_2 \in A_2, a_4 \in A_4$. By (3.1) and those similar computational procedures we get

$$h_{13}(a_1,a_3) = a_3 \varepsilon a_1 + \gamma'(a_1)a_3, \quad h_{34}(a_3,a_4) = \varepsilon' a_4 a_3 + \varphi(\gamma(a_4))a_3 \tag{3.34}$$

for all $a_1 \in A_1, a_3 \in A_3, a_4 \in A_4$. Taking $a_1 = 1$ and $a_4 = 1$ into (3.29) and combining Lemma 3.12, Lemma 3.13 with (3.33), we conclude that $\varepsilon a_2 = a_2 \varepsilon'$ for all $a_2 \in A_2$. Note that $\varepsilon \in \mathscr{Z}(A_1) = \pi_A(\mathscr{Z}(\mathscr{G}))$ and $\varepsilon' \in \mathscr{Z}(A_4) = \pi_B(\mathscr{Z}(\mathscr{G}))$. In view of [60, Lemma 3.2] we obtain $\varepsilon \oplus \varepsilon' \in \mathscr{Z}(\mathscr{G})$.

It follows from (3.22) and (3.33) that

$$\big(f_{11}(a_1,a_1) - \varepsilon a_1^2 - \varphi^{-1}(\gamma'(a_1))a_1 - \varphi^{-1}(k_{11}(a_1,a_1))\big)a_2 = 0$$

for all $a_1 \in A_1, a_2 \in A_2$. Since $A_2 = M$ is faithful as a left A-module,

$$f_{11}(a_1,a_1) = \varepsilon a_1^2 + \varphi^{-1}(\gamma'(a_1))a_1 + \varphi^{-1}(k_{11}(a_1,a_1)) \tag{3.35}$$

for all $a_1 \in A_1$. Similarly,

$$k_{44}(a_4,a_4) = \varepsilon' a_4^2 + \varphi(\gamma(a_4))a_4 + \varphi(f_{44}(a_4,a_4)) \tag{3.36}$$

for all $a_4 \in A_4$.

Finally, let us set $\varepsilon \oplus \varepsilon'$ and define the mapping $\mu : \mathscr{G} \to \mathscr{Z}(\mathscr{G})$ by

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mapsto$$
$$\begin{bmatrix} \varphi^{-1}(\gamma'(a_1)) + \gamma(a_4) + \alpha(a_2) + \tau(a_3) & 0 \\ 0 & \gamma'(a_1) + \varphi(\gamma(a_4) + \alpha(a_2) + \tau(a_3)) \end{bmatrix}.$$

In view of all conclusions derived above, we see that

$$\begin{aligned} \nu(x) :&= \mathfrak{T}_{\mathfrak{q}}(x) - zx^2 - \mu(x)x \\ &\equiv \begin{bmatrix} f_{23}(a_2,a_3) - \varepsilon a_2 a_3 & 0 \\ 0 & k_{23}(a_2,a_3) - \varepsilon' a_3 a_2 \end{bmatrix} \pmod{\mathscr{Z}(\mathscr{G})} \end{aligned}$$

where $x \in \mathscr{G}$. Therefore we can write

$$\mathfrak{T}_{\mathfrak{q}}(x) = zx^2 + \mu(x)x + \begin{bmatrix} f_{23}(a_2,a_3) - \varepsilon a_2 a_3 & 0 \\ 0 & k_{23}(a_2,a_3) - \varepsilon' a_3 a_2 \end{bmatrix} + c$$

for some $c \in \mathscr{Z}(\mathscr{G})$. Since $\mathfrak{q}$ is a commuting mapping, we have

$$\left[\begin{bmatrix} f_{23}(a_2,a_3) - \varepsilon a_2 a_3 & 0 \\ 0 & k_{23}(a_2,a_3) - \varepsilon' a_3 a_2 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right] = 0.$$

This implies that $f_{23}(a_2,a_3) - \varepsilon a_2 a_3 \in \mathscr{Z}(A_1) = \pi_A(\mathscr{Z}(\mathscr{G}))$ and $k_{23}(a_2,a_3) - \varepsilon' a_3 a_2 \in \mathscr{Z}(A_4) = \pi_B(\mathscr{Z}(\mathscr{G}))$. Moreover, it shows that

$$(f_{23}(a_2,a_3) - \varepsilon a_2 a_3)a_2 = a_2(k_{23}(a_2,a_3) - \varepsilon' a_3 a_2)$$

and

$$a_3(f_{23}(a_2,a_3) - \varepsilon a_2 a_3) = (k_{23}(a_2,a_3) - \varepsilon' a_3 a_2)a_3.$$

for all $a_2 \in A_2, a_3 \in A_3$. For convenience, let us write $f(a_2,a_3) = f_{23}(a_2,a_3) - \varepsilon a_2 a_3$ and $k(a_2,a_3) = k_{23}(a_2,a_3) - \varepsilon' a_3 a_2$. Thus

$$\big(f(a_2,a_3) - \varphi^{-1}(k(a_2,a_3))\big)a_2 = 0$$

for all $a_2 \in A_2, a_3 \in A_3$. A linearization of the last relation gives

$$\big(f(a_2,a_3) - \varphi^{-1}(k(a_2,a_3))\big)b_2 + \big(f(b_2,a_3) - \varphi^{-1}(k(b_2,a_3))\big)a_2 = 0$$

for all $a_2, b_2 \in A_2, a_3 \in A_3$. Note that the hypothesis $A_2 = M$ is loyal as an (A,B)-bimodule. It follows from [11, Lemma 2.3] that $f(a_2,a_3) - \varphi^{-1}(k(a_2,a_3)) = 0$ for all $a_2 \in A_2, a_3 \in A_3$. Hence ν maps $\mathscr{G}$ into $\mathscr{Z}(\mathscr{G})$ and this completes the proof of the theorem. □

As a direct consequence of Theorem 3.4 we get

COROLLARY 3.14. [11, Theorem 3.1] *Let $\mathscr{T} = \mathscr{T}(A,M,B)$ be a 2-torsionfree triangular algebra over a commutative ring $\mathscr{R}$ and $\mathfrak{q}\colon \mathscr{T} \times \mathscr{T} \longrightarrow \mathscr{T}$ be an $\mathscr{R}$-bilinear mapping. If*

(1) *every commuting linear mapping on A or B is proper;*

(2) *$\pi_A(\mathscr{Z}(\mathscr{T})) = \mathscr{Z}(A) \neq A$ and $\pi_B(\mathscr{Z}(\mathscr{T})) = \mathscr{Z}(B) \neq B$;*

(3) *M is loyal,*

then every commuting trace $\mathfrak{T}_{\mathfrak{q}} : \mathscr{T} \longrightarrow \mathscr{T}$ of $\mathfrak{q}$ is proper.

In particular, we also have

COROLLARY 3.15. [11, Corollary 3.4] *Let $n \geqslant 2$ and $\mathscr{R}$ be a 2-torsionfree commutative domain. Suppose that $\mathfrak{q}\colon \mathscr{T}_n(\mathscr{R}) \times \mathscr{T}_n(\mathscr{R}) \longrightarrow \mathscr{T}_n(\mathscr{R})$ is an $\mathscr{R}$-bilinear mapping. Then every commuting trace $\mathfrak{T}_{\mathfrak{q}}\colon \mathscr{T}_n(\mathscr{R}) \longrightarrow \mathscr{T}_n(\mathscr{R})$ of $\mathfrak{q}$ is proper.*

COROLLARY 3.16. [11, Corollary 3.5] *Let $\mathscr{N}$ be a nest of a Hilbert space $\mathbf{H}$. Suppose that $\mathfrak{q}\colon \mathscr{T}(\mathscr{N})\times\mathscr{T}(\mathscr{N})\longrightarrow\mathscr{T}(\mathscr{N})$ is an $\mathscr{R}$-bilinear mapping. Then every commuting trace $\mathfrak{T}_{\mathfrak{q}}\colon \mathscr{T}(\mathscr{N})\longrightarrow\mathscr{T}(\mathscr{N})$ of $\mathfrak{q}$ is proper.*

In order to handle the commuting traces of bilinear mappings on full matrix algebras we need a technical lemma in below. Recall that an algebra $\mathscr{A}$ over a commutative ring $\mathscr{R}$ is said to be *central over $\mathscr{R}$* if $\mathscr{Z}(\mathscr{A})=\mathscr{R}1$.

PROPOSITION 3.17. *Let $\mathscr{G}=\mathscr{G}(A,M,N,B)$ be a 2-torsionfree generalized matrix algebra over a commutative ring $\mathscr{R}$, where B is a noncommutative algebra over $\mathscr{R}$ and both $\mathscr{G}$ and B are central over $\mathscr{R}$. Suppose that $\mathfrak{q}\colon \mathscr{G}\times\mathscr{G}\longrightarrow\mathscr{G}$ is an $\mathscr{R}$-bilinear mapping. If*

(1) *every commuting linear mapping of B is proper,*

(2) *for any $r\in\mathscr{R}$ and $m\in M$, $rm=0$ implies that $r=0$ or $m=0$,*

(3) *there exist $m_0\in M$ and $b_0\in B$ such that m_0b_0 and m_0 are $\mathscr{R}$-linearly independent,*

then every commuting trace $\mathfrak{T}_{\mathfrak{q}}:\mathscr{G}\longrightarrow\mathscr{G}$ of $\mathfrak{q}$ is proper.

Proof. For the proof of this lemma, we shall follow the proof of Theorem 3.4 step by step and hence use the same notations. However, we have to make explicit changes in some necessary places. All changes take place from the Lemma 3.9 to the end.

Step 1. $f_{22}(a_2,a_2)\oplus k_{22}(a_2,a_2)\in\mathscr{R}1$ and $f_{33}(a_3,a_3)\oplus k_{33}(a_3,a_3)\in\mathscr{R}1$. By (3.17) we know that

$$\big(f_{22}(a_2,a_2)-\varphi^{-1}(k_{22}(a_2,a_2))\big)a_2=0$$

for all $a_2\in A_2=M$. Note that the fact $A_1=\mathscr{R}$ in our context. Then the assumption (2) deduces that $f_{22}(a_2,a_2)=\varphi^{-1}(k_{22}(a_2,a_2))$. Using the same proof of Lemma 3.9 one easily obtain $f_{22}(a_2,a_2)\oplus k_{22}(a_2,a_2)\in\mathscr{R}1$. On the other hand, $f_{33}(a_3,a_3)\oplus k_{33}(a_3,a_3)\in\mathscr{R}1$ follows from the second paragraph of the proof of Lemma 3.9.

Step 2. $f_{12}(a_1,a_2)=\alpha(a_2)a_1+\varphi^{-1}(k_{12}(a_1,a_2))$ and $k_{24}(a_2,a_4)=\varphi(\alpha(a_2))a_4+\varphi(f_{24}(a_2,a_4))$, where $\alpha(a_2)=f_{12}(1,a_2)-\varphi^{-1}(k_{12}(1,a_2))$. If only we show that M is loyal as an (A_1,A_4)-bimodule, then the corresponding form of f_{12} can be obtained by copying the proof of Lemma 3.10. Let $rMb=0$ for all $r\in\mathscr{R}$ and $b\in B$. Suppose that $b\neq0$. Since M is faithful as a right B-module, there exists a $m\in M$ such that $mb\neq0$. However $0=rmb=r(mb)$, the assumption (2) implies that $r=0$. And hence M is a loyal (A_1,A_4)-bimodule.

It is necessary for us to characterize the form of k_{24}. By equations (3.26) and (3.28) we see that

$$a_2\big(k_{24}(a_2,a_4)-\varphi(\alpha(a_2))a_4-\varphi(f_{24}(a_2,a_4))\big)=0 \tag{3.37}$$

for all $a_i \in A_i$ with $i = 1,2,4$. Since $a_4 \mapsto k_{24}(a_2,a_4)$ is a commuting linear mapping on A_4, there exist mappings $\psi : A_2 \longrightarrow \mathscr{R}1$ and $\omega : A_2 \times A_4 \longrightarrow \mathscr{R}1$ such that

$$k_{24}(a_2,a_4) = \psi(a_2)a_4 + \omega(a_2,a_4),$$

where ω is $\mathscr{R}$-linear in the second argument. Let us prove that ψ is an $\mathscr{R}$-linear mappings and that ω is an $\mathscr{R}$-bilinear mapping. It is straightforward to check that

$$k_{24}(a_2+b_2,a_4) = \psi(a_2+b_2)a_4 + \omega(a_2+b_2,a_4)$$

and

$$k_{24}(a_2,a_4)+k_{24}(b_2,a_4) = \psi(a_2)a_4+\omega(a_2,a_4)+\psi(b_2)a_4+\omega(b_2,a_4).$$

for all $a_2,b_2 \in A_2$ and $a_4 \in A_4$. Therefore

$$\big(\psi(a_2+b_2)-\psi(a_2)-\psi(b_2)\big)a_4+\omega(a_2+b_2,a_4)-\omega(a_2,a_4)-\omega(b_2,a_4) = 0 \quad (3.38)$$

for all $a_2,b_2 \in A_2$ and $a_4 \in A_4$. Note that both ψ and ω map into $\mathscr{Z}(A_4)$. Commuting (3.38) with $b_4 \in A_4$ we get

$$(\psi(a_2+b_2)-\psi(a_2)-\psi(b_2))[a_4,b_4] = 0$$

for all $a_2,b_2 \in A_2$ and $a_4,b_4 \in A_4$. Let us choose $a_4,b_4 \in A_4$ such that $[a_4,b_4] \neq 0$. Since M is faithful as a right A_4-module, there exists $m \in M$ such that $m[a_4,b_4] \neq 0$. Thus

$$\varphi^{-1}\big(\psi(a_2+b_2)-\psi(a_2)-\psi(b_2)\big)m[a_4,b_4] = 0$$

for all $a_2,b_2 \in A_2$. The assumption (2) implies that ψ is an $\mathscr{R}$-linear mapping. Consequently, ω is $\mathscr{R}$-linear in the first argument. Rewrite (3.37) as

$$a_2\big((\psi(a_2)-\varphi(\alpha(a_2)))a_4+\omega(a_2,a_4)-\varphi(f_{24}(a_2,a_4))\big) = 0 \quad (3.39)$$

for all $a_2 \in A_2$ and $a_4 \in A_4$. Setting $a_2 = m_0$ and $a_4 = b_0$ we obtain

$$\big(\varphi^{-1}(\psi(m_0))-\alpha(m_0)\big)m_0b_0+\big(\varphi^{-1}(\omega(m_0,b_0))-f_{24}(m_0,b_0)\big)m_0 = 0.$$

So $\alpha(m_0) = \varphi^{-1}(\psi(m_0))$ and $f_{24}(m_0,b_0) = \varphi^{-1}(\omega(m_0,b_0))$ by the condition (3). Substituting a_2+m_0 for a_2 and b_0 for a_4 in (3.39) yields

$$\big(\varphi^{-1}(\psi(a_2))-\alpha(a_2)\big)m_0b_0+\big(\varphi^{-1}(\omega(a_2,b_0))-f_{24}(a_2,b_0)\big)m_0 = 0.$$

Therefore $\alpha(a_2) = \varphi^{-1}(\psi(a_2))$ for all $a_2 \in A_2$. Then it follows from (3.39) that $\omega(a_2,a_4) = \varphi(f_{24}(a_2,a_4))$ for all $a_2 \in A_2, a_4 \in A_4$. Hence k_{24} has also the desired form.

Since M is loyal, we only need to change the places in the proof of Theorem 3.4, where the noncommutativity of A is involved. However, the proof of Lemma 3.11 does not involve the noncommutativity of A and hence it still works in our context.

Step 3. f_{14} (resp. k_{14}) is of the form as in Lemma 3.12 (resp. Lemma 3.13). Note that $a_4 \mapsto k_{14}(a_1,a_4)$ is a commuting $\mathscr{R}$-linear mapping on A_4. Then there exist mappings $\gamma' : A_1 \to \mathscr{R}1_B$ and $\delta' : A_1 \times A_4 \to \mathscr{R}1_B$ such that

$$k_{14}(a_1,a_4) = \gamma'(a_1)a_4 + \delta'(a_1,a_4), \tag{3.40}$$

where δ' is $\mathscr{R}$-linear in the second argument. Here we denote 1_B the identity of B to avoid confusion in the following discussion. We assert that γ' is an $\mathscr{R}$-linear mapping and δ' is an $\mathscr{R}$-bilinear mapping. In fact, $k_{14}(1,a_4) = \gamma'(1)a_4 + \delta'(1,a_4)$ and hence $k_{14}(a_1,a_4) = a_1\gamma'(1)a_4 + a_1\delta'(1,a_4)$. Therefore

$$(\gamma'(a_1) - a_1\gamma'(1))a_4 + \delta'(a_1,a_4) - a_1\delta'(1,a_4) = 0 \tag{3.41}$$

for all $a_1 \in \mathscr{R}$, $a_4 \in A_4$. Commuting (3.41) with $b_4 \in A_4$ we obtain

$$(\gamma'(a_1) - a_1\gamma'(1))[a_4,b_4] = 0$$

for all $a_1 \in \mathscr{R}$, $a_4,b_4 \in A_4$. Moreover,

$$\varphi^{-1}(\gamma'(a_1) - a_1\gamma'(1))m[a_4,b_4] = 0$$

for all $a_1 \in \mathscr{R}$, $a_4,b_4 \in A_4$ and $m \in M$. Since M is loyal and B is noncommutative, we have $\gamma'(a_1) = a_1\gamma'(1)$. This implies that γ' is $\mathscr{R}$-linear and hence δ' is $\mathscr{R}$-bilinear.

It would be helpful to point out here that each of the mappings f_{ij} takes its values in $\mathscr{R}$. Now the identities $(3.29), (3.30)$ and (3.31) jointly yield that

$$f_{14}(a_1,a_4)a_2 + \big(\theta a_1 + \varphi^{-1}(k_{14}(a_1,1_B)) - f_{14}(a_1,1_B)\big)a_2a_4$$

$$= a_2k_{14}(a_1,a_4) + a_1a_2\big(\eta a_4 + \varphi(f_{14}(1,a_4)) - k_{14}(1,a_4)\big)$$

and hence (taking into account the relation (3.40))

$$a_2\left\{\varphi\big(a_1\varphi^{-1}(\eta) + \varphi^{-1}(\gamma'(a_1-1) - k_{14}(a_1,1_B)) - \theta a_1 + f_{14}(a_1,1_B)\big)a_4\right.$$

$$\left. + \varphi\big((f_{14}(1,a_4) - \varphi^{-1}(\delta'(1,a_4)))a_1 + \varphi^{-1}(\delta'(a_1,a_4)) - f_{14}(a_1,a_4)\big)\right\} = 0 \tag{3.42}$$

for all $a_i \in A_i$ with $i = 1,2,4$. Let us choose $a_4,b_4 \in A_4$ such that $[a_4,b_4] \neq 0$. Then the fact A_2 is faithful as a right A_4-module and the relation (3.42) deduce that

$$\varphi\big(a_1\varphi^{-1}(\eta) + \varphi^{-1}(\gamma'(a_1-1) - k_{14}(a_1,1_B)) - \theta a_1 + f_{14}(a_1,1_B)\big)[a_4,b_4] = 0.$$

for all $a_1 \in A_1$. Thus

$$\big(a_1\varphi^{-1}(\eta) + \varphi^{-1}(\gamma'(a_1-1) - k_{14}(a_1,1_B)) - \theta a_1 + f_{14}(a_1,1_B)\big)M[a_4,b_4] = 0$$

for all $a_1 \in A_1$. Since M is faithful as a right B-module, there exists a $m \in M$ such that $m[a_4,b_4] \neq 0$. Therefore the condition (2) implies that

$$a_1\varphi^{-1}(\eta) + \varphi^{-1}(\gamma'(a_1-1) - k_{14}(a_1,1_B)) - \theta a_1 + f_{14}(a_1,1_B) = 0$$

for all $a_1 \in A_1$. Then the relation (3.42) shows

$$(f_{14}(1,a_4) - \varphi^{-1}(\delta'(1,a_4)))a_1 + \varphi^{-1}(\delta'(a_1,a_4)) = f_{14}(a_1,a_4)$$

for all $a_1 \in A_1, a_4 \in A_4$. Let us $\gamma(a_4) := f_{14}(1,a_4) - \varphi^{-1}(\delta'(1,a_4))$ and $\delta(a_1,a_4) := \varphi^{-1}(\delta'(a_1,a_4))$. Then $f_{14}(a_1,a_4) = \gamma(a_4)a_1 + \delta(a_1,a_4)$ and $k_{14}(a_1,a_4) = \gamma'(a_1)a_4 + \varphi(\delta(a_1,a_4))$

Finally, following the rest part of the proof of Theorem 3.4 we can obtain the required result. □

COROLLARY 3.18. *Let $\mathscr{R}$ be a 2-torsionfree commutative domain and $\mathscr{M}_n(\mathscr{R})$ be the full matrix algebra over $\mathscr{R}$. Suppose that $\mathfrak{q}\colon \mathscr{M}_n(\mathscr{R}) \times \mathscr{M}_n(\mathscr{R}) \longrightarrow \mathscr{M}_n(\mathscr{R})$ is an $\mathscr{R}$-bilinear mapping. Then every commuting trace $\mathfrak{T}_{\mathfrak{q}}\colon \mathscr{M}_n(\mathscr{R}) \longrightarrow \mathscr{M}_n(\mathscr{R})$ of $\mathfrak{q}$ is proper.*

Proof. If $n > 3$, then $\mathscr{M}_n(\mathscr{R}) = \begin{bmatrix} M_{2\times 2}(\mathscr{R}) & M_{2\times(n-2)}(\mathscr{R}) \\ M_{(n-2)\times 2}(\mathscr{R}) & M_{(n-2)\times(n-2)}(\mathscr{R}) \end{bmatrix}$. By [60, Corollary 4.1] we know that each commuting linear mapping on $\mathscr{M}_2(\mathscr{R})$ and $\mathscr{M}_{n-2}(\mathscr{R})$ is proper. The assumptions (2) and (3) in Theorem 3.4 clearly holds for $\mathscr{M}_n(\mathscr{R})$ $(n > 3)$. Applying Theorem 3.4 yields the desired conclusion.

If $n = 3$, then $\mathscr{M}_3(\mathscr{R}) = \begin{bmatrix} M_{1\times 1}(\mathscr{R}) & M_{1\times 2}(\mathscr{R}) \\ M_{2\times 1}(\mathscr{R}) & M_{2\times 2}(\mathscr{R}) \end{bmatrix}$. Therefore there exist elements

$$m_0 = [1,0] \in \mathscr{M}_{1\times 2}(\mathscr{R}) \quad \text{and} \quad b_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathscr{M}_2(\mathscr{R})$$

such that $m_0 b_0$ and m_0 are linearly independent over $\mathscr{R}$. By [60, Corollary 4.1] and Proposition 3.17 we conclude that $\mathfrak{T}_{\mathfrak{q}}$ has the proper form.

If $n = 2$, the result follows from [19, Theorem 3.1].

Finally, if $n = 1$, the conclusion is obvious. □

COROLLARY 3.19. *Let $\mathscr{R}$ be a 2-torsionfree commutative domain, V be an $\mathscr{R}$-linear space and $B(\mathscr{R},V,\gamma)$ be the inflated algebra of $\mathscr{R}$ along V. Suppose that $\mathfrak{q}\colon B(\mathscr{R},V,\gamma) \times B(\mathscr{R},V,\gamma) \longrightarrow B(\mathscr{R},V,\gamma)$ is an $\mathscr{R}$-bilinear mapping. Then every commuting trace $\mathfrak{T}_{\mathfrak{q}}\colon B(\mathscr{R},V,\gamma) \longrightarrow B(\mathscr{R},V,\gamma)$ of $\mathfrak{q}$ is proper.*

Let us see the commuting traces of bilinear mappings of several unital algebras with nontrivial idempotents.

COROLLARY 3.20. *Let $\mathscr{A}$ be a 2-torsionfree unital prime algebra over a commutative ring $\mathscr{R}$. Suppose that $\mathscr{A}$ contains a nontrivial idempotent e and that $f = 1-e$. If $e\mathscr{Z}(\mathscr{A})e = \mathscr{Z}(e\mathscr{A}e) \neq e\mathscr{A}e$ and $f\mathscr{Z}(\mathscr{A})f = \mathscr{Z}(f\mathscr{A}f) \neq f\mathscr{A}f$, then every commuting trace of an arbitrary bilinear mappings on $\mathscr{A}$ is proper.*

Proof. Let us write $\mathscr{A}$ as a natural generalized matrix algebra $\begin{bmatrix} eAe & eAf \\ fAe & fAf \end{bmatrix}$. It is clear that $e\mathscr{A}e$ and $f\mathscr{A}f$ are prime algebras. By [17, Theorem 3.2] it follows that each commuting additive mapping on $e\mathscr{A}e$ and $f\mathscr{A}f$ is proper. On the other hand, if $(eae)e\mathscr{A}f(fbf)=0$ holds for all $a,b\in\mathscr{A}$, then the primeness of $\mathscr{A}$ implies that $eae=0$ or $fbf=0$. This shows $e\mathscr{A}f$ is a loyal $(e\mathscr{A}e,f\mathscr{A}f)$-bimodule. Applying Theorem 3.4 yields that each commuting trace of an arbitrary bilinear mappings on $\mathscr{A}$ is proper. □

COROLLARY 3.21. *Let X be a Banach space over the real or complex field $\mathbb{F}$, $\mathscr{B}(X)$ be the algebra of all bounded linear operators on X. Then every commuting trace of an arbitrary bilinear mapping on $\mathscr{B}(X)$ is proper.*

Proof. Note that $\mathscr{B}(X)$ is a centrally closed prime algebra. If X is infinite dimensional, the result follows from Corollary 3.20. If X is of dimension n, then $\mathscr{B}(X)=\mathscr{M}_n(\mathbb{F})$. In this case the result follows from Corollary 3.18. □

4. Lie isomorphisms on generalized matrix algebras

In this section we shall use the main result in Section 3 (Theorem 3.4) to describe the form of an arbitrary Lie isomorphism of a certain class of generalized matrix algebras (Theorem 4.3). As applications of Theorem 4.3, we characterize Lie isomorphisms of certain generalized matrix algebras. The involved algebras include upper triangular matrix algebras, nest algebras, full matrix algebras, inflated algebras, prime algebras with nontrivial idempotents.

LEMMA 4.1. *Let $\mathscr{G}=\mathscr{G}(A,M,N,B)$ be a 2-torsionfree generalized matrix algebra over a commutative ring $\mathscr{R}$. Then $\mathscr{G}$ satisfies the polynomial identity $[[x^2,y],[x,y]]$ if and only if both A and B are commutative.*

Proof. If A and B are commutative, then we can prove that $\mathscr{G}$ satisfies the polynomial identity $[[x^2,y],[x,y]]$ by a direct but rigorous procedure.

The necessity can be obtained by a similar proof of [11, Lemma 2.7]. □

The following proposition is a much more common generalization of [11, Lemma 4.1].

PROPOSITION 4.2. *Let $\mathscr{G}=\mathscr{G}(A,M,N,B)$ and $\mathscr{G}'=\mathscr{G}'(A',M',N',B')$ be generalized matrix algebras over $\mathscr{R}$ with $1/2\in\mathscr{R}$. Let $\mathfrak{l}\colon\mathscr{G}\longrightarrow\mathscr{G}'$ be a Lie isomorphism. If*

(1) *every commuting trace of an arbitrary bilinear mapping on $\mathscr{G}'$ is proper,*

(2) *at least one of A,B and at least one of A',B' are noncommutative,*

(3) *M' is loyal,*

then $\mathfrak{l}=\mathfrak{m}+\mathfrak{n}$, where $\mathfrak{m}:\mathscr{G}\longrightarrow\mathscr{G}'$ is a homomorphism or the negative of an anti-homomorphism, $\mathfrak{m}$ is injective, and $\mathfrak{n}\colon\mathscr{G}\longrightarrow\mathscr{Z}(\mathscr{G}')$ is a linear mapping vanishing on each commutator. Moreover, if $\mathscr{G}'$ is central over $\mathscr{R}$, then $\mathfrak{m}$ is surjective.

The proof of this proposition is almost a copy of the proof of Lemma [11, Lemma 4.1] and is left out.

THEOREM 4.3. *Let $\mathscr{G} = \mathscr{G}(A,M,N,B)$ and $\mathscr{G}' = \mathscr{G}'(A',M',N',B')$ be generalized matrix algebras over $\mathscr{R}$ with $1/2 \in \mathscr{R}$. Let $\mathfrak{l}\colon \mathscr{G} \longrightarrow \mathscr{G}'$ be a Lie isomorphism. If*

(1) *every commuting linear mapping on A' or B' is proper,*

(2) $\pi_{A'}(\mathscr{Z}(\mathscr{G}')) = \mathscr{Z}(A') \neq A'$ *and* $\pi_{B'}(\mathscr{Z}(\mathscr{G}')) = \mathscr{Z}(B') \neq B'$,

(3) *either A or B is noncommutative,*

(4) *M' is loyal,*

then $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}\colon \ :\mathscr{G} \longrightarrow \mathscr{G}'$ is a homomorphism or the negative of an antihomomorphism, $\mathfrak{m}$ is injective, and $\mathfrak{n}\colon \mathscr{G} \longrightarrow \mathscr{Z}(\mathscr{G}')$ is a linear mapping vanishing on each commutator. Moreover, if $\mathscr{G}'$ is central over $\mathscr{R}$, then $\mathfrak{m}$ is surjective.

Proof. It follows from Theorem 3.4 and Proposition 4.2 directly. □

As a direct consequence of Theorem 4.3 we have

COROLLARY 4.4. [11, Theorem 4.3] *Let $\mathscr{T} = \mathscr{T}(A,M,B)$ and $\mathscr{T}' = \mathscr{T}'(A',M',B')$ be triangular algebras over $\mathscr{R}$ with $1/2 \in \mathscr{R}$. Let $\mathfrak{l}\colon \mathscr{T} \longrightarrow \mathscr{T}'$ be a Lie isomorphism. If*

(1) *every commuting linear mapping on A' or B' is proper,*

(2) $\pi_{A'}(\mathscr{Z}(\mathscr{T}')) = \mathscr{Z}(A') \neq A'$ *and* $\pi_{B'}(\mathscr{Z}(\mathscr{T}')) = \mathscr{Z}(B') \neq B'$,

(3) *either A or B is noncommutative,*

(4) *M' is loyal,*

then $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}\colon \ :\mathscr{T} \longrightarrow \mathscr{T}'$ is a homomorphism or the negative of an antihomomorphism, $\mathfrak{m}$ is injective, and $\mathfrak{n}\colon \mathscr{T} \longrightarrow \mathscr{Z}(\mathscr{T}')$ is a linear mapping vanishing on each commutator. Moreover, if $\mathscr{T}'$ is central over $\mathscr{R}$, then $\mathfrak{m}$ is surjective.

In particular, we also have

COROLLARY 4.5. [11, Corollary 4.4] *Let $n \geqslant 2$ and $\mathscr{R}$ be a commutative domain with $\frac{1}{2} \in \mathscr{R}$. If $\mathfrak{l}\colon \mathscr{T}_n(\mathscr{R}) \longrightarrow \mathscr{T}_n(\mathscr{R})$ is a Lie isomorphism, then $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}\colon \mathscr{T}_n(\mathscr{R}) \longrightarrow \mathscr{T}_n(\mathscr{R})$ is an isomorphism or the negative of an antiisomorphism and $\mathfrak{n}\colon \mathscr{T}_n(\mathscr{R}) \longrightarrow \mathscr{R}1$ is a linear mapping vanishing on each commutator.*

COROLLARY 4.6. [11, Corollary 4.5] *Let $\mathscr{N}$ and $\mathscr{N}'$ be nests on a Hilbert space $\mathbf{H}$, $\mathscr{T}(\mathscr{N})$ and $\mathscr{T}(\mathscr{N}')$ be the nest algebras associated with $\mathscr{N}$ and $\mathscr{N}'$, respectively. If $\mathfrak{l}\colon \mathscr{T}(\mathscr{N}) \longrightarrow \mathscr{T}(\mathscr{N}')$ is a Lie isomorphism, then $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}\colon \mathscr{T}(\mathscr{N}) \longrightarrow \mathscr{T}(\mathscr{N}')$ is an isomorphism or the negative of an antiisomorphism and $\mathfrak{n}\colon \mathscr{T}(\mathscr{N}) \longrightarrow \mathbb{C}1'$ is a linear mapping vanishing on each commutator.*

For the Lie isomorphisms of full matrix algebras, we have similar characterizations.

COROLLARY 4.7. *Let $\mathscr{R}$ be a commutative domain with $\frac{1}{2}\in\mathscr{R}$. If $\mathfrak{l}\colon \mathscr{M}_n(\mathscr{R})\to \mathscr{M}_n(\mathscr{R})$ ($n\geqslant 3$) is a Lie isomorphism, then $\mathfrak{l}=\mathfrak{m}+\mathfrak{n}$, where $\mathfrak{m}\colon \mathscr{M}_n(\mathscr{R})\to\mathscr{M}_n(\mathscr{R})$ is an isomorphism or the negative of an anti-isomorphism and $\mathfrak{n}\colon \mathscr{M}_n(\mathscr{R})\to\mathscr{R}1$ is a linear mapping vanishing on each commutator.*

Proof. We write $\mathscr{M}_n(\mathscr{R})=\begin{bmatrix} M_{1\times 1}(\mathscr{R}) & M_{1\times(n-1)}(\mathscr{R})\\ M_{(n-1)\times 1}(\mathscr{R}) & M_{(n-1)\times(n-1)}(\mathscr{R})\end{bmatrix}$. Corollary 3.16 shows that each commuting trace of arbitrary bilinear mapping on $\mathscr{M}_n(\mathscr{R})$ is proper. Moreover, $\mathscr{M}_{(n-1)\times(n-1)}(\mathscr{R})$ is noncommutative and $\mathscr{M}_{1\times(n-1)}(\mathscr{R})$ is a loyal $(\mathscr{R},\mathscr{M}_{(n-1)\times(n-1)}(\mathscr{R}))$-bimodule. Hence Proposition 4.2 implies the conclusion. □

COROLLARY 4.8. *Let $\mathscr{R}$ be a commutative domain with $\frac{1}{2}\in\mathscr{R}$, V be an $\mathscr{R}$-linear space and $B(\mathscr{R},V,\gamma)$ be the inflated algebra of $\mathscr{R}$ along V. If $\mathfrak{l}\colon B(\mathscr{R},V,\gamma)\longrightarrow B(\mathscr{R},V,\gamma)$ is a Lie isomorphism, then $\mathfrak{l}=\mathfrak{m}+\mathfrak{n}$, where $\mathfrak{m}\colon B(\mathscr{R},V,\gamma)\longrightarrow B(\mathscr{R},V,\gamma)$ is an isomorphism or the negative of an anti-isomorphism and $\mathfrak{n}\colon B(\mathscr{R},V,\gamma)\longrightarrow\mathscr{R}1$ is a linear mapping vanishing on each commutator.*

Let us consider the Lie isomorphisms of several unital algebras with nontrivial idempotents.

COROLLARY 4.9. *Let $\mathscr{A}$ be a unital prime algebra with nontrivial idempotents. Then every Lie isomorphism is of the standard form (♠).*

COROLLARY 4.10. *Let X be a Banach space over the real or complex field $\mathbb{F}$, $\mathscr{B}(X)$ be the algebra of all bounded linear operators on X. Then every Lie isomorphism has the standard form (♠).*

5. Potential topics for further research

Although the main goal of the current article is to consider commuting traces and Lie isomorphisms on generalized matrix algebras, there are more interesting mappings related to our current work on generalized matrix algebras. These mappings are still considerable interest and will draw more people's attention. In this section we will propose several potential topics for future further research.

Let $\mathscr{R}$ be a commutative ring with identity, $\mathscr{A}$ be a unital algebra over $\mathscr{R}$ and $\mathscr{Z}(\mathscr{A})$ be the center of $\mathscr{A}$. Recall that an $\mathscr{R}$-linear mapping $\mathfrak{f}:\mathscr{A}\longrightarrow\mathscr{A}$ is said to be *centralizing* if $[\mathfrak{f}(a),a]\in\mathscr{Z}(\mathscr{A})$ for all $a\in\mathscr{A}$. Let n be a positive integer and $\mathfrak{q}\colon\mathscr{A}^n\longrightarrow\mathscr{A}$ be an n-linear mapping. The mapping $\mathfrak{T}_{\mathfrak{q}}\colon\mathscr{A}\longrightarrow\mathscr{A}$ defined by $\mathfrak{T}_{\mathfrak{q}}(a)=\mathfrak{q}(a,a,\cdots,a)$ is called a *trace* of $\mathfrak{q}$. We say that a centralizing trace $\mathfrak{T}_{\mathfrak{q}}$ is *proper* if it can be written as

$$\mathfrak{T}_{\mathfrak{q}}(a)\equiv\sum_{i=0}^{n-1}\mu_i(a)a^{n-i}\ \mathrm{mod}\ \mathscr{Z}(\mathscr{A}),\quad \forall a\in\mathscr{A},$$

where $\mu_i(0 \leqslant i \leqslant n)$ is a mapping from $\mathscr{A}$ into $\mathscr{Z}(\mathscr{A})$ and every $\mu_i(0 \leqslant i \leqslant n)$ is in fact a trace of an i-linear mapping $\mathfrak{q}_i$ from $\mathscr{A}^i$ into $\mathscr{Z}(\mathscr{A})$. Let $n = 1$ and $\mathfrak{f}\colon A \longrightarrow A$ be an $\mathscr{R}$-linear mapping. In this case, an arbitrary trace $\mathfrak{T}_{\mathfrak{f}}$ of $\mathfrak{f}$ exactly equals to itself. Moreover, if a centralizing trace $\mathfrak{T}_{\mathfrak{f}}$ of $\mathfrak{f}$ is proper, then it has the form

$$\mathfrak{T}_{\mathfrak{f}}(a) \equiv za \mod \mathscr{Z}(\mathscr{A}), \quad \forall a \in \mathscr{A},$$

where $z \in \mathscr{Z}(A)$. Let us see the case of $n = 2$. Suppose that $\mathfrak{g}\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ is an $\mathscr{R}$-bilinear mapping. If a centralizing trace $\mathfrak{T}_{\mathfrak{g}}$ of $\mathfrak{g}$ is proper, then it is of the form

$$\mathfrak{T}_{\mathfrak{g}}(a) \equiv za^2 + \mu(a)a \mod \mathscr{Z}(\mathscr{A}), \quad \forall a \in \mathscr{A},$$

where $z \in \mathscr{Z}(\mathscr{A})$ and μ is an $\mathscr{R}$-linear mapping from $\mathscr{A}$ into $\mathscr{Z}(\mathscr{A})$. Brešar started the study of commuting and centralizing traces of multilinear mappings in his series of works [15, 16, 17, 18], where he investigated the structure of centralizing traces of (bi-)linear mappings on prime rings. It has turned out that in certain rings, in particular, prime rings of characteristic different from 2 and 3, every centralizing trace of a biadditive mapping is commuting. Moreover, every centralizing mapping of a prime ring of characteristic not 2 is of the proper form and is actually commuting. Lee et al further generalized Brešar's results by showing that each commuting trace of an arbitrary multilinear mapping on a prime ring also has the proper form [35]. An exciting discovery is that every centralizing trace of arbitrary bilinear mapping on triangular algebras is commuting in some additional conditions.

THEOREM 5.1. [38, Theorem 3.4] *Let $\mathscr{T} = \mathscr{T}(A,M,B)$ be a 2-torsionfree triangular algebras over a commutative ring $\mathscr{R}$ and $\mathfrak{q}\colon \mathscr{T} \times \mathscr{T} \longrightarrow \mathscr{T}$ be an $\mathscr{R}$-bilinear mapping. If*

(1) every commuting linear mapping on A or B is proper,

(2) $\pi_A(\mathscr{Z}(\mathscr{T})) = \mathscr{Z}(A) \neq A$ and $\pi_B(\mathscr{Z}(\mathscr{T})) = \mathscr{Z}(B) \neq B$,

(3) M is loyal,

then every centralizing trace $\mathfrak{T}_{\mathfrak{q}} : \mathscr{T} \longrightarrow \mathscr{T}$ of $\mathfrak{q}$ is proper. Moreover, each centralizing trace $\mathfrak{T}_{\mathfrak{q}}$ of $\mathfrak{q}$ is commuting.

It is natural to formulate the following question

QUESTION 5.2. Let $\mathscr{T} = \mathscr{T}(A,M,B)$ be a 2-torsionfree triangular algebra over a commutative ring $\mathscr{R}$ and $\mathfrak{q}\colon \mathscr{T} \times \mathscr{T} \times \cdots \times \mathscr{T} \longrightarrow \mathscr{T}$ be an n-linear mapping. Suppose that the following conditions are satisfied

(1) each commuting linear mapping on A or B is proper;

(2) $\pi_A(\mathscr{Z}(\mathscr{T})) = \mathscr{Z}(A) \neq A$ and $\pi_B(\mathscr{Z}(\mathscr{T})) = \mathscr{Z}(B) \neq B$;

(3) M is loyal.

Is any centralizing trace $\mathfrak{T}_{\mathfrak{q}}: \mathscr{T} \longrightarrow \mathscr{T}$ of $\mathfrak{q}$ proper? Furthermore, what can we say about the centralizing traces of multilinear mappings on a generalized matrix algebra $\mathscr{G} = \mathscr{G}(A, M, N, B)$?

Let $\mathscr{R}$ be a commutative ring with identity, $\mathscr{A}$ and $\mathscr{B}$ be associative $\mathscr{R}$-algebras. We define a *Lie triple isomorphism* from $\mathscr{A}$ into $\mathscr{B}$ to be an $\mathscr{R}$-linear bijective mapping $\mathfrak{l}$ satisfying the condition

$$\mathfrak{l}([[a,b],c]) = [[\mathfrak{l}(a), \mathfrak{l}(b)], \mathfrak{l}(c)], \quad \forall a,b,c \in \mathscr{A}.$$

Obviously, every Lie isomorphism is a Lie triple isomorphism. The converse is, in general, not true. In [38] we apply Theorem 5.1 to the study of Lie triple isomorphisms on triangular algebras. It is shown that every Lie triple isomorphism between triangular algebras also has an *approximate standard decomposition expression* under some additional conditions. That is

THEOREM 5.3. [38, Theorem 4.3] *Let $\mathscr{T} = \mathscr{T}(A, M, B)$ and $\mathscr{T}' = \mathscr{T}'(A', M', B')$ be triangular algebras over $\mathscr{R}$ with $1/2 \in \mathscr{R}$. Let $\mathfrak{l}\colon \mathscr{T} \longrightarrow \mathscr{T}'$ be a Lie triple isomorphism. If*

(1) *every commuting linear mapping on A' or B' is proper,*

(2) $\pi_{A'}(\mathscr{Z}(\mathscr{T}')) = \mathscr{Z}(A') \neq A'$ *and* $\pi_{B'}(\mathscr{Z}(\mathscr{T}')) = \mathscr{Z}(B') \neq B'$,

(3) *either A or B is noncommutative,*

(4) *M' is loyal,*

then $\mathfrak{l} = \pm\mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}\colon \mathscr{T} \to \mathscr{T}'$ is a Jordan homomorphism, $\mathfrak{m}$ is injective, and $\mathfrak{n}\colon \mathscr{T} \longrightarrow \mathscr{Z}(\mathscr{T}')$ is a linear mapping vanishing on each second commutator. Moreover, if $\mathscr{T}'$ is central over $\mathscr{R}$, then $\mathfrak{m}$ is surjective.

A question closely related to the above theorem is

QUESTION 5.4. Let $\mathscr{G} = \mathscr{G}(A, M, N, B)$ and $\mathscr{G}' = \mathscr{G}'(A', M', N', B')$ be generalized matrix algebras over $\mathscr{R}$ with $1/2 \in \mathscr{R}$. Let $\mathfrak{l}\colon \mathscr{G} \longrightarrow \mathscr{G}'$ be a Lie triple isomorphism. Under what conditions does $\mathfrak{l}$ has a similar decomposition expression?

Acknowledgements. The authors express their sincere gratitude to Professor Heydar Radjavi for his kind considerations and to the anonymous referees for their substantial and insightful comments which significantly help us improve the final presentation of this article. The authors would like to thank Professor Ajda Fošner and Professor Vincenzo de Filippis for their careful reading of a preliminary version of this manuscript.

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(Received March 9, 2013)

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