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ON FUNCTIONAL EQUATIONS RELATED TO DERIVATIONS AND BICIRCULAR PROJECTIONS

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Abstract. In this paper we investigate some functional equations on standard operator algebras and semiprime rings. We prove, for example, the following result, which is related to a classical result of Chernoff. Let X be a real or complex Banach space, let $\mathscr{L}(X)$ be the algebra of all bounded linear operators on X and let $\mathscr{A}(X) \subset \mathscr{L}(X)$ be a standard operator algebra. Suppose there exists a linear mapping $D : \mathscr{A}(X) \to \mathscr{L}(X)$ satisfying the relation $D(A^n) = D(A)A^{n-1} + AD(A^{n-2})A + A^{n-1}D(A)$ for all $A \in \mathscr{A}(X)$, where $n > 2$ is some fixed integer. In this case D is of the form $D(A) = [A,B]$ for all $A \in \mathscr{A}(X)$ and some fixed $B \in \mathscr{L}(X)$. Some functional equations related to bicircular projections are also investigated.

This research is a continuation of our recent work [26]. Throughout, R will represent an associative ring with center $Z(R)$. As usual, we write $[x,y]$ for $xy - yx$. Given an integer $n \geqslant 2$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called involution in case $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. Let A be an algebra over the real or complex field and let B be a subalgebra of A. A linear mapping $D:\ B \to A$ is called a linear derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. In case we have a ring R, an additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$, such that $D(x) = [x,a]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [15] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein theorem can be found in [4].

Cusack [8] generalized Herstein theorem to 2-torsion free semiprime rings (see [5] for an alternative proof). Herstein theorem has been fairly generalized by Beidar, Brešar, Chebotar and Martindale [1]. Let X be a real or complex Banach space and let $\mathscr{L}(X)$ and $\mathscr{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal

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Paper OaM-08-47

of all finite rank operators in $\mathscr{L}(X)$, respectively. An algebra $\mathscr{A}(X) \subset \mathscr{L}(X)$ is said to be standard in case $\mathscr{F}(X) \subset \mathscr{A}(X)$. Any standard operator algebra is prime, which is a consequence of a Hahn-Banach theorem. In case we have a real or complex Hilbert space, we denote by A^* the adjoint operator of a bounded linear operator A. Let X be a complex Banach space. A projection $P \in \mathscr{L}(X)$ is bicircular in case all mappings of the form $e^{i\alpha}P + e^{i\beta}(I-P)$, where I denotes the identity operator, are isometric for all pairs of real numbers α, β. Let us start with the following result proved by Brešar [6] (see [21] for a generalization).

THEOREM 1. *Let R be a 2-torsion free semiprime ring and let $D: R \to R$ be an additive mapping satisfying the relation*

$$D(xyx) = D(x)yx + xD(y)x + xyD(x) \tag{1}$$

for all pairs $x, y \in R$. In this case D is a derivation.

Note that in case a ring has an identity element, the proof of the result above is immediate. Namely, in this case the substitution $y = e$ in the relation (1), where e stands for the identity element, gives that D is a Jordan derivation and then it follows from Cusack's generalization of Herstein theorem that D is a derivation. An additive mapping satisfying the relation (1) on an arbitrary ring is called a Jordan triple derivation. It is easy to prove that any Jordan derivation on a 2-torsion free ring is a Jordan triple derivation, which means that Theorem 1 generalizes Cusack's generalization of Herstein theorem.

The substitution $y = x^{n-2}$ in the relation (1) gives

$$D(x^n) = D(x)x^{n-1} + xD(x^{n-2})x + x^{n-1}D(x).$$

It is our aim in this paper to prove the following result, which is related to the above relation.

THEOREM 2. *Let X be a real or complex Banach space and let $\mathscr{A}(X)$ be a standard operator algebra on X. Suppose there exists a linear mapping $D: \mathscr{A}(X) \to \mathscr{L}(X)$ satisfying the relation*

$$D(A^n) = D(A)A^{n-1} + AD(A^{n-2})A + A^{n-1}D(A)$$

for all $A \in \mathscr{A}(X)$ and some fixed integer $n > 2$. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathscr{A}(X)$ and some fixed $B \in \mathscr{L}(X)$.

In case $n = 3$ the result above reduces to Theorem 1 in [29]. In the proof of Theorem 2 we use Herstein theorem and the result below.

THEOREM 3. *Let X be a real or complex Banach space, let $\mathscr{A}(X)$ be a standard operator algebra on X and let $D: \mathscr{A}(X) \to \mathscr{L}(X)$ be a linear derivation. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathscr{A}(X)$ and some fixed $B \in \mathscr{L}(X)$.*

Theorem 3 has been proved by Chernoff [7] (see also [25, 27]). We are now in the position to prove Theorem 2.

Proof of Theorem 2. We have the relation

$$D(A^n) = D(A)A^{n-1} + AD(A^{n-2})A + A^{n-1}D(A). \tag{2}$$

Let us first restrict our attention to $\mathscr{F}(X)$.

Let A be from $\mathscr{F}(X)$ and let $P \in \mathscr{F}(X)$ be a projection with $AP = PA = A$. Putting $A+P$ for A in the above relation, we obtain

$$\begin{aligned}\sum_{i=0}^{n} \tbinom{n}{i} D(A^{n-i}P^i) = D(A+P)&\left(\sum_{i=0}^{n-1} \tbinom{n-1}{i} A^{n-1-i}P^i\right)\\ &+\sum_{i=0}^{n-2} \tbinom{n-2}{i} (A+P)D(A^{n-2-i}P^i)(A+P)\\ &+\left(\sum_{i=0}^{n-1} \tbinom{n-1}{i} A^{n-1-i}P^i\right) D(A+P).\end{aligned} \tag{3}$$

Using (2) and rearranging the relation (3) in sense of collecting together terms involving equal number of factors of P, we obtain

$$\sum_{i=1}^{n-1} f_i(A,P) = 0,$$

where $f_i(A,P)$ stands for the expression of terms involving i factors of P. Replacing A by $A+2P$, $A+3P$, $\ldots$, $A+(n-1)P$ in turn in the relation (2) and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(A,P)$, $i = 1,2,\ldots,n-1$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of this matrix is different from zero, it follows that the system has only a trivial solution. In particular,

$$\begin{aligned} f_{n-1}(A,P) = {} & \tbinom{n}{n-1} D(A) - \tbinom{n-1}{n-1} D(A)P - \tbinom{n-1}{n-2} D(P)A \\ & - \tbinom{n-2}{n-2} AD(P)P - \tbinom{n-2}{n-2} PD(P)A - \tbinom{n-2}{n-3} PD(A)P \\ & - \tbinom{n-1}{n-1} PD(A) - \tbinom{n-1}{n-2} AD(P). \end{aligned}$$

The above relation reduces to

$$\begin{aligned} nD(A) = {} & (n-1)(D(P)A + AD(P)) + D(A)P + PD(A) \\ & + AD(P)P + PD(P)A + (n-2)PD(A)P. \end{aligned} \tag{4}$$

Putting P for A in (2) gives $D(P)=D(P)P+PD(P)P+PD(P)$. Left (right, two-sided) multiplication of this relation by A gives, respectively,

$$AD(P)P=PD(P)A=AD(P)A=0. \tag{5}$$

Considering above ascertainments in (4) leads to

$$nD(A)=(n-1)(D(P)A+AD(P))+D(A)P+PD(A)+(n-2)PD(A)P. \tag{6}$$

Right (left) multiplication by P in (6) gives, respectively,

$$\begin{aligned} D(A)P &= D(P)A+PD(A)P, \\ PD(A) &= AD(P)+PD(A)P. \end{aligned}$$

With the above two relations, the relation (6) reduces to

$$D(A)=D(P)A+AD(P)+PD(A)P. \tag{7}$$

Putting A^2 for A in the above relation gives

$$D(A^2)=D(P)A^2+A^2D(P)+PD(A^2)P. \tag{8}$$

Right (left) multiplication by A in the relation (7) and considering (5) gives, respectively,

$$D(A)A=D(P)A^2+PD(A)A \quad \text{and} \quad AD(A)=A^2D(P)+AD(A)P. \tag{9}$$

From the van der Monde matrix we also obtain

$$\begin{aligned} f_{n-2}(A,P) = {} & \tbinom{n}{n-2}D(A^2)-\tbinom{n-1}{n-2}D(A)A-\tbinom{n-1}{n-3}D(P)A^2 \\ & -\tbinom{n-2}{n-2}AD(P)A-\tbinom{n-2}{n-3}AD(A)P-\tbinom{n-2}{n-3}PD(A)A \\ & -\tbinom{n-2}{n-4}PD(A^2)P-\tbinom{n-1}{n-2}AD(A)-\tbinom{n-1}{n-3}A^2D(P). \end{aligned}$$

Considering (5) and some calculations reduces the above relation to

$$\begin{aligned} n(n-1)D(A^2) = {} & 2(n-1)(D(A)A+AD(A)) \\ & +2(n-2)(AD(A)P+PD(A)A) \\ & +(n-1)(n-2)(A^2D(P)+D(P)A^2) \\ & +(n-2)(n-3)PD(A^2)P. \end{aligned}$$

By (8) we have $PD(A^2)P=D(A^2)-D(P)A^2-A^2D(P)$. Considering this in the above relation leads to

$$\begin{aligned} (2n-3)D(A^2) = {} & (n-1)(D(A)A+AD(A)) \\ & +(n-2)(AD(A)P+PD(A)A) \\ & +(n-2)(A^2D(P)+D(P)A^2). \end{aligned}$$

Because of (9) the above relation reduces to

$$D(A^2) = D(A)A + AD(A). \tag{10}$$

From the relation (7) one can conclude that D maps $\mathscr{F}(X)$ into itself. We therefore have a linear mapping D, which maps $\mathscr{F}(X)$ into itself and satisfies (10) for all $A \in \mathscr{F}(X)$. In other words, D is a Jordan derivation on $\mathscr{F}(X)$ and since $\mathscr{F}(X)$ is prime, it follows, according to Herstein theorem, that D is a derivation on $\mathscr{F}(X)$. Applying Theorem 3, one can conclude that D is of the form

$$D(A) = [A, B] \tag{11}$$

for all $A \in \mathscr{F}(X)$ and some fixed $B \in \mathscr{L}(X)$. It remains to prove that (11) holds for all $A \in \mathscr{A}(X)$ as well. For this purpose we introduce $D_1 : \mathscr{A}(X) \to \mathscr{L}(X)$ by $D_1(A) = [A, B]$, where B is from the relation (11). Let us denote $D - D_1$ by D_0. The mapping D_0 is, obviously, linear and satisfies the relation (2). Besides, D_0 vanishes on $\mathscr{F}(X)$. It is our aim to prove that D_0 vanishes on $\mathscr{A}(X)$ as well. Let A be from $\mathscr{A}(X)$, let P be a one-dimensional projection and let us introduce $S \in \mathscr{A}(X)$ by $S = A + PAP - (AP + PA)$. Since $S - A \in \mathscr{F}(X)$, we have $D_0(S) = D_0(A)$. Besides, $SP = PS = 0$. By the relation (2) we have

$$D_0(A^n) = D_0(A)A^{n-1} + AD_0(A^{n-2})A + A^{n-1}D_0(A) \tag{12}$$

for all $A \in \mathscr{A}(X)$. Therefore

$$\begin{aligned}
& D_0(S)S^{n-1} + SD_0(S^{n-2})S + S^{n-1}D_0(S) = D_0(S^n) \\
& = D_0(S^n + P) = D_0((S+P)^n) \\
& = D_0(S+P)(S+P)^{n-1} + (S+P)D_0((S+P)^{n-2})(S+P) \\
& \quad + (S+P)^{n-1}D_0(S+P) \\
& = D_0(S)(S^{n-1} + P) + (S+P)D_0(S^{n-2})(S+P) + (S^{n-1} + P)D_0(S) \\
& = D_0(S)S^{n-1} + D_0(S)P + SD_0(S^{n-2})S + SD_0(S^{n-2})P + PD_0(S^{n-2})S \\
& \quad + PD_0(S^{n-2})P + S^{n-1}D_0(S) + PD_0(S).
\end{aligned} \tag{13}$$

From the above relation it follows that

$$D_0(S)P + SD_0(S^{n-2})P + PD_0(S^{n-2})S + PD_0(S^{n-2})P + PD_0(S) = 0$$

and since $D_0(S) = D_0(A)$, we obtain

$$D_0(A)P + SD_0(A^{n-2})P + PD_0(A^{n-2})S + PD_0(A^{n-2})P + PD_0(A) = 0. \tag{14}$$

Two-sided multiplication of the above relation by P gives

$$2PD_0(A)P + PD_0(A^{n-2})P = 0. \tag{15}$$

Putting $2A$ for A in the above relation, we obtain

$$4PD_0(A)P + 2^{n-2}PD_0(A^{n-2})P = 0. \tag{16}$$

In case $n=3$, the relation (15) gives

$$PD_0(A)P=0. \tag{17}$$

In case $n>3$, the relations (15) and (16) give (17). Considering the above relation in the relation (15), we obtain

$$PD_0(A^{n-2})P=0.$$

The above relation reduces (14) to

$$D_0(A)P+SD_0(A^{n-2})P+PD_0(A^{n-2})S+PD_0(A)=0.$$

Putting $2A$ for A (in this case S becomes $2S$) and comparing such obtained relation with the above relation leads to

$$D_0(A)P+PD_0(A)=0.$$

Right multiplication of the above relation by P gives

$$D_0(A)P+PD_0(A)P=0$$

and the relation (17) reduces the above relation to

$$D_0(A)P=0.$$

Since P is an arbitrary one-dimensional projection, if follows from the above relation that $D_0(A)=0$ for any $A\in\mathscr{A}(X)$. The proof of the theorem is therefore complete. □

In our recent paper [26] one can find the following result.

THEOREM 4. *Let X be a real or complex Banach space and let $\mathscr{A}(X)$ be a standard operator algebra on X. Suppose there exists an additive mapping $T:\mathscr{A}(X)\to\mathscr{L}(X)$ satisfying the relation*

$$T(A^n)=T(A)A^{n-1}-AT(A^{n-2})A+A^{n-1}T(A)$$

for all $A\in\mathscr{A}(X)$ and some fixed integer $n>2$. In this case T is of the form $T(A)=AC+CA$ for all $A\in\mathscr{A}(X)$ and some fixed $C\in\mathscr{L}(X)$.

Let us point out that in Theorem 2 there is an assumption that D is linear, while in Theorem 4 T is additive. The question arises, whether Theorem 2 can be proved under weaker assumption that D is additive. The answer is in general negative. Namely, the proof of Theorem 2 depends on Theorem 3, which in general can not be proved in case the mapping D is additive, as shown by Šemrl [25].

THEOREM 5. *Let X be a real or complex Banach space and let $\mathscr{A}(X)$ be a standard operator algebra on X. Suppose there exist linear mappings $F,G:\mathscr{A}(X)\to\mathscr{L}(X)$ satisfying relations*

$$\begin{aligned}F(A^n)&=F(A)A^{n-1}+AG(A^{n-2})A+A^{n-1}F(A),\\G(A^n)&=G(A)A^{n-1}+AF(A^{n-2})A+A^{n-1}G(A)\end{aligned}$$

for all $A\in\mathscr{A}(X)$ and some fixed integer $n>2$. In this case $F(A)=AB-CA$ and $G(A)=AC-BA$ for all $A\in\mathscr{A}(X)$ and some fixed $B,C\in\mathscr{L}(X)$.

Proof. Adding both of the relations gives

$$D(A^n) = D(A)A^{n-1} + AD(A^{n-2})A + A^{n-1}D(A), \tag{18}$$

where D stands for $F+G$. Similarly we obtain

$$T(A^n) = T(A)A^{n-1} - AT(A^{n-2})A + A^{n-1}T(A), \tag{19}$$

where T denotes $F-G$. By Theorem 2, D is of the form $D(A) = [A,B]$ for all $A \in \mathscr{A}(X)$ and some fixed $B \in \mathscr{L}(X)$. Theorem 4 implies that T is of the form $T(A) = AC + CA$ for all $A \in \mathscr{A}(X)$ and some fixed $C \in \mathscr{L}(X)$. Obviously, $D+T = 2F$ and $D-T = 2G$. We have

$$\begin{aligned} D(A)+T(A) &= AB - BA + AC + CA = A(B+C) - (B-C)A, \\ D(A)-T(A) &= AB - BA - AC - CA = A(B-C) - (B+C)A. \end{aligned}$$

Let B denote $\frac{1}{2}(B+C)$ and C denote $\frac{1}{2}(B-C)$. Mappings F and G can now be presented as $F(A) = AB - CA$ and $G(A) = AC - BA$ for all $A \in \mathscr{A}(X)$ and some fixed operators $B, C \in \mathscr{L}(X)$, which was our aim to prove. $\square$

Stachó and Zalar [23, 24] investigated bicircular projections on the C^*-algebra $\mathscr{L}(H)$, where H is a complex Hilbert space. According to Proposition 3.4 in [23] every bicircular projection $P : \mathscr{L}(H) \to \mathscr{L}(H)$, where H is a complex Hilbert space, satisfies the functional equation

$$P(ABA) = P(A)BA - AP(B^*)^*A + ABP(A) \tag{20}$$

for all pairs $A, B \in \mathscr{L}(H)$, where B^* stands for the adjoint operator of $B \in \mathscr{L}(H)$. M. Fošner and Ilišević [11] investigated the above functional equation on 2-torsion free semiprime *-ring. They expressed the solution of the identity (20) in terms of derivation and so-called double centralizers. Bicircular projections and related functional equations have been extensively investigated during the last few years (see [2, 3, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 28]). Putting $B = A^{n-2}$ in the relation (20) we obtain

$$P(A^n) = P(A)A^{n-1} - AP((A^*)^{n-2})^*A + A^{n-1}P(A), \tag{21}$$

which leads to the following result.

THEOREM 6. *Let X be a real or complex Hilbert space space and let $\mathscr{A}(X)$ be a standard operator algebra, which is closed under the adjoint operation. Let $P, Q : \mathscr{A}(X) \to \mathscr{L}(X)$ be linear mappings satisfying*

$$\begin{aligned} P(A^n) &= P(A)A^{n-1} + AQ((A^*)^{n-2})^*A + A^{n-1}P(A), \\ Q(A^n) &= Q(A)A^{n-1} + AP((A^*)^{n-2})^*A + A^{n-1}Q(A) \end{aligned}$$

for all $A \in \mathscr{A}(X)$ and some fixed integer $n > 2$. In this case P and Q are of the form

$$\begin{aligned} P(A) &= A(B+C+D+E) + (C+E-B-D)A, \\ Q(A) &= A(B+C-D-E) + (C+D-B-E)A \end{aligned}$$

for all $A \in \mathscr{A}(X)$, where $B, C, D, E \in \mathscr{L}(X)$ are some fixed operators. Besides, $B^* = -B + \lambda I$, $C^* = -C$, $D^* = D + \mu I$ and $E^* = E$, where λ, μ are fixed real or complex numbers and I stands for the identity operator.

Proof. The proof goes through in three steps.

First step. Let us first assume that $P = Q$ and let F denote P. In this case we have the relation

$$F(A^n) = F(A)A^{n-1} + AF((A^*)^{n-2})^*A + A^{n-1}F(A) \tag{22}$$

for all $A \in \mathscr{A}(X)$. It is our aim to prove that $2F(A) = A(B+C) + (C-B)A$ for all $A \in \mathscr{A}(X)$ and some fixed operators $B, C \in \mathscr{L}(X)$. We will also show that $B^* = -B + \lambda I$ for some fixed real or complex number λ and $C^* = -C$. For this purpose let us introduce mappings $d, f : \mathscr{A}(X) \to \mathscr{L}(X)$ by

$$d(A) = F(A) + F(A^*)^* \tag{23}$$

and

$$f(A) = F(A) - F(A^*)^* \tag{24}$$

for all $A \in \mathscr{A}(X)$. One can easily show that the relation (22) implies

$$d(A^n) = d(A)A^{n-1} + Ad(A^{n-2})A + A^{n-1}d(A) \tag{25}$$

and

$$f(A^n) = f(A)A^{n-1} - Af(A^{n-2})A + A^{n-1}f(A) \tag{26}$$

for all $A \in \mathscr{A}(X)$. We therefore have

$$d(A^*)^* = (F(A^*) + F(A)^*)^* = F(A) + F(A^*)^* = d(A).$$

Hence, $d(A^*)^* = d(A)$. By Theorem 2 it follows from (25) that d is of the form $d(A) = [A, B]$ for all $A \in \mathscr{A}(X)$ and some fixed operator $B \in \mathscr{L}(X)$. Since $d(A^*)^* = d(A)$, we obtain $AB - BA = B^*A - AB^*$, which gives $[A, B^* + B] = 0$. Therefore $B^* + B \in Z(\mathscr{A}(X))$, which means that $B^* + B = \lambda I$ for some fixed real or complex number λ.

The relation (26) together with Theorem 4 implies that f is of the form $f(A) = AC + CA$ for all $A \in \mathscr{A}(X)$ and some fixed $C \in \mathscr{L}(X)$. From (24) we obtain

$$F(A) - F(A^*)^* = AC + CA. \tag{27}$$

Putting A^* for A in the above relation leads to $F(A^*) - F(A)^* = A^*C + CA^*$. We now have $\big(F(A^*) - F(A)^*\big)^* = \big(A^*C + CA^*\big)^*$ and therefore

$$F(A^*)^* - F(A) = C^*A + AC^*.$$

The above relation together with (27) gives $A(C^* + C) + (C^* + C)A = 0$. Since every standard operator algebra is prime, it follows from the last relation that $C^* = -C$. Combining relations (23) and (27) we obtain $2F(A) = AB - BA + AC + CA$ and therefore

$2F(A) = A(B+C) + (C-B)A$ for all $A \in \mathscr{A}(X)$. The proof of the first step is now complete.

Second step. We will now assume that $P = -Q$ and let H denote P. In this case we have the relation

$$H(A^n) = H(A)A^{n-1} - AH((A^*)^{n-2})^*A + A^{n-1}H(A) \tag{28}$$

for all $A \in \mathscr{A}(X)$. In this step one has to prove that $2H(A) = A(D+E) + (E-D)A$ for all $A \in \mathscr{A}(X)$ and some fixed operators $D, E \in \mathscr{L}(X)$. Besides, $D^* = D + \mu I$ for some fixed real or complex number μ and $E^* = E$. The proof of the second step goes through using the same arguments as in the first step and will therefore be omitted.

Third step. We are now in a position to prove the theorem in its full generality. We have the relations

$$\begin{aligned} P(A^n) &= P(A)A^{n-1} + AQ((A^*)^{n-2})^*A + A^{n-1}P(A), \\ Q(A^n) &= Q(A)A^{n-1} + AP((A^*)^{n-2})^*A + A^{n-1}Q(A) \end{aligned}$$

for all $A \in \mathscr{A}(X)$. Adding (subtracting) the above relations gives, respectively,

$$\begin{aligned} F(A^n) &= F(A)A^{n-1} + AF((A^*)^{n-2})^*A + A^{n-1}F(A), \\ H(A^n) &= H(A)A^{n-1} - AH((A^*)^{n-2})^*A + A^{n-1}H(A), \end{aligned}$$

where F denotes $P+Q$ and H stands for $P-Q$. Due to results regarding (22) and (28) in first and second step, we obtain from the above relations that

$$\begin{aligned} 2P(A) + 2Q(A) &= A(B+C) + (C-B)A, \\ 2P(A) - 2Q(A) &= A(D+E) + (E-D)A \end{aligned}$$

for all $A \in \mathscr{A}(X)$, where $B, C, D, E \in \mathscr{L}(X)$ are some fixed operators with properties $B^* = -B + \lambda I$, $C^* = -C$, $D^* = D + \mu I$ and $E^* = E$. The last two relations imply that

$$\begin{aligned} 4P(A) &= A(B+C+D+E) + (C+E-B-D)A, \\ 4Q(A) &= A(B+C-D-E) + (C+D-B-E)A \end{aligned}$$

for all $A \in \mathscr{A}(X)$. Let B denote $\frac{1}{4}B$, C denote $\frac{1}{4}C$, D denote $\frac{1}{4}D$ and E denote $\frac{1}{4}E$. Then we have

$$\begin{aligned} P(A) &= A(B+C+D+E) + (C+E-B-D)A, \\ Q(A) &= A(B+C-D-E) + (C+D-B-E)A \end{aligned}$$

for all $A \in \mathscr{A}(X)$, which completes the proof of the theorem. $\square$

Let us point out that in Theorem 2, Theorem 5 and Theorem 6 we obtain as a result the continuity of the mappings under purely algebraic assumptions concerning these mappings, which means that these results might be of some interest from the automatic continuity point of view. For results concerning automatic continuity we refer the reader to [22].

We proceed with the following conjecture.

CONJECTURE 7. *Let R be a semiprime ring with suitable torsion restrictions and let $D: R \to R$ be an additive mapping. Suppose that*

$$D(x^n) = D(x)x^{n-1} + xD(x^{n-2})x + x^{n-1}D(x)$$

holds for all $x \in R$ and some fixed integer $n > 2$. In this case D is a derivation.

We are going to conclude the paper by proving the above conjecture in case a semiprime ring has the identity element.

THEOREM 8. *Let $n > 2$ be some fixed integer, let R be a $(2n)!$-torsion free semiprime ring with the identity element and let $D: R \to R$ be an additive mapping satisfying the relation*

$$D(x^n) = D(x)x^{n-1} + xD(x^{n-2})x + x^{n-1}D(x) \tag{29}$$

for all $x \in R$. In this case D is a derivation.

Proof. Let e be the identity element. Putting e for x in the relation (29) gives

$$D(e) = 0. \tag{30}$$

Let y be any element of $Z(R)$. Putting $x+y$ for x in the relation (29), we obtain

$$\begin{aligned}\sum_{i=0}^{n} \tbinom{n}{i} D(x^{n-i}y^i) &= D(x+y)\left(\sum_{i=0}^{n-1} \tbinom{n-1}{i} x^{n-1-i}y^i\right)\\ &+ \sum_{i=0}^{n-2} \tbinom{n-2}{i} (x+y)D(x^{n-2-i}y^i)(x+y) + \left(\sum_{i=0}^{n-1} \tbinom{n-1}{i} x^{n-1-i}y^i\right)D(x+y).\end{aligned}$$

Using (29) and rearranging the above relation in sense of collecting together terms involving equal number of factors of y, we obtain

$$\sum_{i=1}^{n-1} f_i(x,y) = 0,$$

where $f_i(x,y)$ stands for the expression of terms involving i factors of y. Replacing x by $x+2y$, $x+3y$, $\dots$, $x+(n-1)y$ in turn in the relation (29) and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(x,y)$, $i = 1,2,\dots,n-1$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of this matrix is different from zero, it follows that the system has only a trivial solution. In particular,

$$\begin{aligned} f_{n-2}(x,e) = &-\binom{n}{n-2}D(x^2)+\binom{n-1}{n-3}D(e)x^2+\binom{n-1}{n-2}D(x)x \\ &+\binom{n-2}{n-4}D(x^2)+\binom{n-2}{n-3}(D(x)x+xD(x))+\binom{n-2}{n-2}xD(e)x \\ &+\binom{n-1}{n-3}x^2D(e)+\binom{n-1}{n-2}xD(x), \end{aligned}$$

where y is replaced with the identity element e. Since (30) holds, the above relation reduces to

$$\left(\binom{n}{n-2}-\binom{n-2}{n-4}\right)D(x^2)=\left(\binom{n-1}{n-2}+\binom{n-2}{n-3}\right)(D(x)x+xD(x)).$$

After few calculations the above relation reduces to

$$(2n-3)D(x^2)=(2n-3)(D(x)x+xD(x)).$$

Since R is $(2n)!$-torsion free, it follows from the above relation that

$$D(x^2)=D(x)x+xD(x)$$

for all $x\in R$. In other words, D is a Jordan derivation on R. According to Cusack's generalization of Herstein theorem, one can conclude that D is a derivation, which completes the proof of the theorem. □

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