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PERMANENTS, DETERMINANTS, AND GENERALIZED COMPLEMENTARY BASIC MATRICES

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Abstract. This paper answers the questions posed in the article "A note on permanents and generalized complementary basic matrices", *Linear Algebra Appl.* 436 (2012), by M. Fiedler and F. Hall. Determinant and permanent compound products which are intrinsic are also explored, along with extensions to total unimodularity.

1. Introduction

In [8] and [9] the complementary basic matrices, CB-matrices for short, (see [5], [6], [7]) were extended in the following way. Let A_1, A_2, ..., A_s be matrices of respective orders $k_1, k_2, \ldots, k_s$, $k_i \geqslant 2$ for all i. Denote $n = \sum_{i=1}^{s} k_i - s + 1$, and form the block diagonal matrices G_1, G_2, $\ldots$, G_s as follows:

$$G_1 = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-k_1} \end{bmatrix}, \qquad G_2 = \begin{bmatrix} I_{k_1-1} & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & I_{n-k_1-k_2+1} \end{bmatrix}, \ldots,$$

$$G_{s-1} = \begin{bmatrix} I_{n-k_{s-1}-k_s+1} & 0 & 0 \\ 0 & A_{s-1} & 0 \\ 0 & 0 & I_{k_s-1} \end{bmatrix}, \qquad G_s = \begin{bmatrix} I_{n-k_s} & 0 \\ 0 & A_s \end{bmatrix}.$$

Then, for any permutation $(i_1, i_2, \ldots, i_s)$ of $(1, 2, \ldots, s)$, we can consider the product

$$G_{i_1} G_{i_2} \cdots G_{i_s} \tag{1}$$

We call products of this form *generalized complementary basic matrices*, GCB-matrices for short. We have continued to use the notation $\prod G_k$ for these more general products. The diagonal blocks A_k are called *distinguished blocks* and the matrices G_k are called *generators* of $\prod G_k$. (In the CB-matrices, these distinguished blocks are all of order 2.) Let us also remark that strictly speaking, every square matrix can be considered as a (trivial) GCB-matrix with $s = 1$.

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Let A be an $n \times n$ real matrix. Then the *permanent* of A is defined by

$$\operatorname{per}(A) = \sum a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

where the summation extends over all the n-permutations $(i_1, i_2, \ldots, i_n)$ of the integers $1, 2, \ldots, n$. So, per(A) is the same as the determinant function apart from a factor of ± 1 preceding each of the products in the summation. As pointed out in [2], certain determinantal laws have direct analogues for permanents. In particular, the Laplace expansion for determinants has a simple counterpart for permanents. But the basic law of determinants

$$\det(AB) = \det(A)\det(B) \tag{2}$$

is flagrantly false for permanents. The latter fact is the case even for intrinsic products (see Section 3), as was observed in [9] in the example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Many striking properties of GCB-matrices have already been exhibited in [8] and [9]. In particular, in [9], it was proved that

$$\operatorname{per}(AB) = \operatorname{per}(A)\operatorname{per}(B)$$

holds for products which are GCB-matrices.

THEOREM 1.1. *Suppose the integers n, k satisfy $n > k > 1$. Let A_0 be a matrix of order k, B_0 be a matrix of order $n - k + 1$ (the sum of the orders of A_0 and B_0 thus exceeds n by one). Then, for the $n \times n$ matrix AB, where*

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & I_{n-k} \end{bmatrix} \text{ and } B = \begin{bmatrix} I_{k-1} & 0 \\ 0 & B_0 \end{bmatrix},$$

we have that

$$\operatorname{per}(AB) = \operatorname{per}(A)\operatorname{per}(B). \tag{3}$$

This result was then extended to the GCB-matrices.

COROLLARY 1.2. *Independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, we have that*

$$\operatorname{per}\left(\prod G_k\right) = \prod \operatorname{per}(G_k).$$

The purpose of this paper is to answer the questions posed in [9]. Determinant and permanent compound products which are intrinsic are considered as well, along with extensions to total unimodularity.

2. Permanent compounds

For an $n \times n$ matrix A and index sets $\alpha, \beta \subseteq \{1, ..., n\}$, $A(\alpha, \beta)$ denotes the submatrix of A that lies at the intersection of the rows indexed by α and the columns indexed by β. We simply let $A(\alpha)$ denote the principal submatrix of A that lies in the rows and columns indexed by α. The usual h^{th} *compound matrix* of A, denoted by $C_h(A)$, is the matrix of order $\binom{n}{h}$ whose entries are $\det(A(\alpha, \beta))$, where α and β are of cardinality h. Similarly, the h^{th} *permanent compound matrix* of A, denoted by $P_h(A)$, is the matrix of order $\binom{n}{h}$ whose entries are $\mathrm{per}(A(\alpha, \beta))$, where α and β are of cardinality h. There are many possibilities for ordering the family of index sets of cardinality h. Usually, the lexicographic ordering is preferred and this will be the understood order unless otherwise specified. When a different ordering is used, we obtain a compound matrix permutationally similar to $P_h(A)$, or $C_h(A)$ (in lexicographic order).

We also recall the multiplicativity of the usual compound matrix:

$$C_h(AB) = C_h(A)C_h(B).$$

In contrast, we do not have the same property for permanent compounds.

In [9] a number of interesting related papers, including [1], [3], and [4], were cited. Specifically, for compound matrices, the authors in [1] show that for nonnegative $n \times n$ matrices A and B

$$P_h(AB) \geqslant P_h(A)P_h(B). \tag{4}$$

Now (4) implies for nonnegative matrices that we have

$$\mathrm{per}((AB)(\alpha)) \geqslant \mathrm{per}(A(\alpha))\mathrm{per}(B(\alpha)), \tag{5}$$

for any index set $\alpha \subseteq \{1, ..., n\}$. The inequality (5) was also shown in [3].

Let the cardinality of the set α be denoted by h. As mentioned in [9] it is straightforward to show that for matrices A and B as in Theorem 1.1, and for $h = 1, 2,$ and n, we in fact have equality in (5). The result for $h = n$ actually follows from Theorem 1.1. The following question was then raised. For GCB-matrices, to what extent can we prove equality in (4) and (5) for the other values of h, namely $h = 3, \ldots, n-1$? One of the purposes of this section is to answer this question.

Regarding (5), we can answer the question in the affirmative. Referring to matrices A and B in Theorem 1.1, let us write

$$A_0 = \begin{bmatrix} a_{11} \cdots a_{1k} \\ \cdots \\ a_{k1} \cdots a_{kk} \end{bmatrix}$$

and

$$B_0 = \begin{bmatrix} b_{kk} \cdots b_{kn} \\ \cdots \\ b_{nk} \cdots b_{nn} \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1,k-1} & a_{1k}b_{kk} & \cdots & a_{1k}b_{kn} \\ a_{21} & \cdots & a_{2,k-1} & a_{2k}b_{kk} & \cdots & a_{2k}b_{kn} \\ & \cdots & & & \cdots & \\ a_{k1} & \cdots & a_{k,k-1} & a_{kk}b_{kk} & \cdots & a_{kk}b_{kn} \\ & & & b_{k+1,k} & \cdots & b_{k+1,n} \\ & & & & \cdots & \\ & & & b_{n,k} & \cdots & b_{n,n} \end{bmatrix}. \tag{6}$$

THEOREM 2.1. *In the notation of Theorem 1.1, for any index set $\alpha \subseteq \{1,...,n\}$, we have*

$$\operatorname{per}(AB(\alpha)) = \operatorname{per}(A(\alpha))\operatorname{per}(B(\alpha)). \tag{7}$$

Proof. We can divide the proof into cases, each of which is easy to prove:
(i) $\alpha \subseteq \{1,...,k\}$ with two subcases $\alpha \subseteq \{1,...,k-1\}$ and $k \in \alpha$
(ii) $\alpha \subseteq \{k,...,n\}$ with two subcases $\alpha \subseteq \{k+1,...,n\}$ and $k \in \alpha$
(iii) $\alpha \cap \{1,...,k-1\} \neq \emptyset$ and $\alpha \cap \{k+1,...,n\} \neq \emptyset$.
Here, if $k \in \alpha$, the proof follows from the result of Theorem 1.1; if $k \notin \alpha$, it is very easy.
The arguments for these cases can be done by analyzing the matrix in (6). □

We then have a variation of Corollary 1.2.

COROLLARY 2.2. *Independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, for any index set $\alpha \subseteq \{1,...,n\}$, we have that*

$$\operatorname{per}((\prod G_k)(\alpha)) = \prod \operatorname{per}(G_k(\alpha)).$$

Proof. We use induction with respect to s. If $s = 2$, the result follows from Theorem 2.1. Suppose that $s > 2$ and that the result holds for $s-1$ matrices. Observe that the matrices G_i and G_k commute if $|i-k| > 1$. This means that if 1 is before 2 in the permutation $(i_1, i_2, \ldots, i_s)$, we can move G_1 into the first position without changing the product. The product Π of the remaining $s-1$ matrices G_k has the form

$$\Pi = G_{j_2} \cdots G_{j_s} = \begin{bmatrix} I_{k_1-1} & 0 \\ 0 & B_0 \end{bmatrix},$$

where $(j_2, \cdots, j_s)$ is a permutation of $(2, \cdots, s)$. By the induction hypothesis,

$$\operatorname{per}((\Pi)(\alpha)) = \operatorname{per}((G_{j_2})(\alpha)) \cdots \operatorname{per}((G_{j_s})(\alpha)),$$

where we can view $G_2, G_3, \ldots, G_s$ as $s-1$ generators of an $n \times n$ GCB-matrix. Then by Theorem 2.1,

$$\operatorname{per}((\prod G_k)(\alpha)) = \operatorname{per}((G_1\Pi)(\alpha))$$
$$= \operatorname{per}((G_1)(\alpha))\operatorname{per}((\Pi)(\alpha)) = \prod \operatorname{per}((G_k)(\alpha)).$$

If 1 is behind 2 in the permutation, we can move G_1 into the last position without changing the product. The previous proof then applies to the transpose of the product. Since the permanent of a matrix and its transpose are the same, the proof of this case can proceed as follows:

$$\operatorname{per}((\prod G_k)(\alpha)) = \operatorname{per}((\Pi G_1)(\alpha)) = \operatorname{per}([(\Pi G_1)(\alpha)]^T)$$

$$= \operatorname{per}((\Pi G_1)^T(\alpha)) = \operatorname{per}((G_1{}^T\Pi^T)(\alpha)) = \prod \operatorname{per}((G_{i_k}^T)(\alpha))$$

$$= \prod \operatorname{per}([(G_{i_k})(\alpha)]^T) = \prod \operatorname{per}((G_{i_k})(\alpha)) = \prod \operatorname{per}((G_k)(\alpha)). \quad \square$$

COROLLARY 2.3. *If all the distinguished blocks A_k have positive principal permanental minors, then independent of the ordering of the factors, the generalized complementary basic matrix $\prod G_k$ has positive principal permanental minors.*

REMARK 2.4. For equality in (4), we have a counterexample for 3×3 matrices using 2×2 distinguished diagonal blocks, ie, using CB-matrices. Specifically, using distinguished blocks of 1's, we get

$$P_2(AB) = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix},$$

while

$$P_2(A)P_2(B) = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix},$$

differing in only the $(1,3)$-entry.

Furthermore, note the block diagonal forms

$$P_2(A) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P_2(B) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Finally in this section, we give a structural characterization of $P_h(A)$ and $P_h(B)$, where A and B are as in Theorem 1.1.

LEMMA 2.5. *For $n \times n$ matrices A and B as in Theorem 1.1 and different index sets α, β of the same cardinality we have that*

(i.) *$A(\alpha,\beta)$ has a zero line if α and β differ by at least one index in the set $\{k+1,\ldots,n\}$, and*

(ii.) *$B(\alpha,\beta)$ has a zero line if α and β differ by at least one index in the set $\{1,\ldots,k-1\}$.*

Proof. We prove $(i.)$; the proof of $(ii.)$ is similar. By assumption, without loss of generality, there exists $i \in \alpha \cap \{k+1,\dots,n\}$ such that $i \notin \beta$. So, $A(\alpha,\beta)$ cannot contain the 1 in the (i,i) position of A (since $i \notin \beta$). Hence, the corresponding row of $A(\alpha,\beta)$ is a zero row. $\square$

THEOREM 2.6. *For $n \times n$ matrices A and B as in Theorem 1.1 and any $1 \leqslant h \leqslant n$, we have the following:*

(i.) $P_h(A)$ is permutationally similar to a block diagonal matrix with $\binom{n-k}{h-i}$ diagonal blocks of order $\binom{k}{i}$, for $i = 0,1,\dots,h$, and

(ii.) $P_h(B)$ is permutationally similar to a block diagonal matrix with $\binom{k-1}{i}$ diagonal blocks of order $\binom{n-k+1}{h-i}$, for $i = 0,1,\dots,h$.

(As usual, $\binom{a}{b} = 0$ if $b > a$ or $b < 0$.)

Proof. For the purpose of this proof, we call the indices in the set $\{1,\dots,k-1\}$ green indices and indices in the set $\{k+1,\dots,n\}$ red indices. We first prove $(i.)$ and fix h, $1 \leqslant h \leqslant n$. Consider index sets α,β of the same cardinality h. Observe by Lemma 2.5 that $A(\alpha,\beta)$ has a zero line if α and β differ by at least one red index.

Choose any $i \in \{0,1,\dots,h\}$, fix some $h-i$ red indices, and then make all possible $\binom{k}{i}$ choices of non-red indices. We then obtain $\binom{k}{i}$ different index sets of cardinality h where any two of them have exactly those same red indices. Keeping these index sets together yields a diagonal submatrix of order $\binom{k}{i}$.

Next, observe that in this way we then obtain $\binom{n-k}{h-i}$ diagonal blocks of order $\binom{k}{i}$, where any two of them are associated with different subsets of red indices. This completes the proof of part $(i.)$.

Note that

$$\binom{n}{h} = \sum_{i=0}^{h} \binom{k}{i}\binom{n-k}{h-i},$$

which holds for any fixed $k \in \{0,1,\dots,n\}$ (with our matrices A and B, $k \in \{2,\dots,n-1\}$).

The proof of part $(ii.)$ is similar to the proof of $(i.)$. By Lemma 2.5, $B(\alpha,\beta)$ has a zero line if α and β differ by at least one green index. In this case, we choose any $i \in \{0,1,\dots,h\}$, fix some i green indices, and then make all possible $\binom{n-k+1}{h-i}$ choices of non-green indices, thereby obtaining $\binom{n-k+1}{h-i}$ different index sets of cardinality h where any two of them have exactly those same green indices. We thus obtain $\binom{k-1}{i}$ diagonal blocks of order $\binom{n-k+1}{h-i}$, where any two of them are associated with different subsets of green indices. That completes the proof of $(ii.)$.

Observe that

$$\binom{n}{h} = \sum_{i=0}^{h} \binom{n-k+1}{h-i}\binom{k-1}{i},$$

which also holds for any fixed $k \in \{0,1,\dots,n\}$. $\square$

OBSERVATION 2.7. *Since lexicographical ordering meets the requirements of the proof of part* (*ii.*) *of Theorem* 2.6*, ie. for each choice of* i *green indices the index sets with those same green indices are grouped together,* $P_h(B)$ *itself is a block diagonal matrix.*

3. Intrinsic products

Following [6], we say that the product of a row vector and a column vector is *intrinsic* if there is at most one non-zero product of the corresponding coordinates. Analogously we speak about the intrinsic product of two or more matrices, as well as about *intrinsic factorizations* of matrices. The entries of the intrinsic product are products of (some) entries of the multiplied matrices. Thus there is no addition; we could also call intrinsic multiplication *sum-free multiplication*.

OBSERVATION 3.1. *Let* A*,* B*,* C *be matrices such that the product* ABC *is intrinsic in the sense that in every entry* $(ABC)_{i\ell}$ *(of the form* $\sum_{j,k} a_{ij}b_{jk}c_{k\ell}$*) there is at most one non-zero term. If* A *has no zero column and* C *no zero row, then both products* AB *and* BC *are intrinsic.*

REMARK 3.2. In general, when ABC,AB, and BC are all intrinsic, we say that the product ABC is *completely intrinsic*, and this will be used even for more than three factors.

As was already observed in [8], independent of the ordering of the factors, the GCB-matrices $\prod G_k$ are completely intrinsic.

We now return to compound matrices.

THEOREM 3.3. *For* $n \times n$ *matrices* A *and* B *as in Theorem* 1.1 *and any* $1 \leqslant h \leqslant n$*, the product* $C_h(A)C_h(B)$ *is intrinsic.*

Proof. Let

$$\alpha = \{i_1, \dots, i_{s-1}, i_s, i_{s+1}, \dots, i_h\},$$

where

$$\{i_1, \dots, i_s\} \subseteq \{1, \dots, k\}, \ \{i_{s+1}, \dots, i_h\} \subseteq \{k+1, \dots, n\}$$

and

$$\beta = \{j_1, \dots, j_t, j_{t+1}, j_{t+2}, \dots, j_h\},$$

where

$$\{j_1, \dots, j_t\} \subseteq \{1, \dots, k-1\}, \ \{j_{t+1}, \dots, j_h\} \subseteq \{k, \dots, n\}.$$

We are looking for index sets γ of cardinality h which satisfy two conditions:

(i.) $A(\alpha, \gamma)$ does not necessarily have a zero line, and

(ii.) $B(\gamma, \beta)$ does not necessarily have a zero line.

Now, by Lemma 2.5, $(i.)$ implies that γ and α have the same indices in the set $\{k+1,\dots,n\}$, and $(ii.)$ implies that γ and β have the same indices in the set $\{1,\dots,k-1\}$. Hence, $\{j_1,\dots,j_t,i_{s+1},\dots,i_h\}\subseteq\gamma$ and index k may or may not be in γ.

If $k\notin\gamma$, then γ is uniquely determined as $\gamma=\{j_1,\dots,j_t,i_{s+1},\dots,i_h\}$, which also implies that $t=s$.

If $k\in\gamma$, then γ is uniquely determined as $\gamma=\{j_1,\dots,j_t,k,i_{s+1},\dots,i_h\}$, which implies that $t=s-1$.

Since we cannot have both $t=s$ and $t=s-1$, there exists a unique γ which satisfies both $(i.)$ and $(ii.)$. Hence, the (α,β)-entry has at most one nonzero term, namely $[C_h(A)]_{(\alpha,\gamma)}[C_h(B)]_{(\gamma,\beta)}$. $\square$

As in previous cases, this result can be extended to the product $\prod G_k$.

COROLLARY 3.4. *For any $1\leqslant h\leqslant n$, independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, we have that the product $\prod C_h(G_k)$ is completely intrinsic.*

REMARK 3.5. Since square matrices which have a zero line have both determinant and permanent equal to zero, Theorem 3.3 also holds for permanent compounds: For $n\times n$ matrices A and B as in Theorem 1.1 and any $1\leqslant h\leqslant n$, the product $P_h(A)P_h(B)$ is intrinsic.

We next formulate a generalization of intrinsic products. Let A and B be $n\times n$ matrices. We say that the product AB is *totally intrinsic* if the determinant of every square submatrix of AB is either zero, or a product of two determinants, one of a square submatrix of A, the second of a square submatrix of B.

Since $C_h(AB)=C_h(A)C_h(B)$, by Theorem 3.3 we immediately have the following:

THEOREM 3.6. *For $n\times n$ matrices A and B as in Theorem 1.1, the product AB is totally intrinsic.*

COROLLARY 3.7. *Independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, the determinant of every square submatrix of $\prod G_k$ is either zero, or a product of some determinants of submatrices of the G_k, in fact, at most one determinant from each G_k.*

Next, we recall a definition, see [2]. An $m\times n$ integer matrix A is *totally unimodular* if the determinant of every square submatrix is $0,1$ or -1. The last corollary then implies that total unimodularity is an inherited property:

COROLLARY 3.8. *Independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, if each of the distinguished blocks A_k is totally unimodular, then $\prod G_k$ is totally unimodular.*

We can note that this inheritance works in a more general sense: if all $\det(A_k)$ are in a sub-semi-group $\mathscr{S}$ of the complex numbers, then $\prod G_k$ is totally unimodular with respect to $\mathscr{S}$.

Next, we shall use Remark 3.5 and a version of the Cauchy-Binet theorem (see [10]) to establish a final result on permanent compounds.

LEMMA 3.9. *If an $n \times n$ matrix A contains a $p \times q$ block of zeros with $p+q>n$, then* $\operatorname{per}(A)=0$.

Proof. Since A has a $p \times q$ block of zeros with $p+q>n$, the minimum number of lines that cover all the nonzero enties in A is less then or equal to $n-p+n-q$, which is less than n. So, by the Theorem of Konig, see [2], the maximum number of nonzero entries in A with no two of the nonzero entries on a line is less than n. Hence, $\operatorname{per}(A)=0$. $\square$

THEOREM 3.10. *For $n \times n$ matrices A and B as in Theorem 1.1 and any index sets α and β of the same cardinality h, where $1 \leqslant h \leqslant n$, we have the following:*

$$[P_h(A)P_h(B)]_{(\alpha,\beta)} = \begin{cases} [P_h(AB)]_{(\alpha,\beta)} = 0, & \text{if } \sigma_{\alpha,\beta} > h; \\ [P_h(AB)]_{(\alpha,\beta)}, & \text{if } \sigma_{\alpha,\beta} = h-1 \text{ or } h; \\ 0, & \text{if } \sigma_{\alpha,\beta} < h-1, \end{cases}$$

where $\sigma_{\alpha,\beta}$ is the number of indices in the set

$$(\alpha \cap \{k+1,\ldots,n\}) \cup (\beta \cap \{1,\ldots,k-1\}).$$

In the third case where $\sigma_{\alpha,\beta} < h-1$, $[P_h(AB)]_{(\alpha,\beta)}$ may or may not be equal to 0.

Proof. For the proof we will use the the Binet-Cauchy Theorem for permanents (see [10]). First, we introduce a new family of index sets, $G_{h,n}$, which consists of all nondecreasing sequences of h integers chosen from $\{1,\ldots,n\}$. We will also use the previous family of strictly increasing sequences of h integers chosen from $\{1,\ldots,n\}$. We will denote this latter set by $Q_{h,n}$.

Now, since $[AB](\alpha,\beta) = A(\alpha,\{1,\ldots,n\})B(\{1,\ldots,n\},\beta)$, by the Binet-Cauchy Theorem for permanents, we get

$$[P_h(AB)]_{(\alpha,\beta)} = \sum_{\gamma \in G_{h,n}} \frac{[P_h(A)]_{(\alpha,\gamma)}\,[P_h(B)]_{(\gamma,\beta)}}{\mu(\gamma)},$$

where $\mu(\gamma)$ is the product of factorials of the multiplicities of distinct integers appearing in the sequence γ.

On the other hand, the (α,β) entry of $[P_h(A)P_h(B)]$ can be written as

$$[P_h(A)P_h(B)]_{(\alpha,\beta)} = \sum_{\gamma \in Q_{h,n}} [P_h(A)]_{(\alpha,\gamma)}\,[P_h(B)]_{(\gamma,\beta)}.$$

We will denote by γ^* the set of indices in $G_{h,n}$ or $Q_{h,n}$, such that both $[P_h(A)]_{(\alpha,\gamma^*)}$ and $[P_h(B)]_{(\gamma^*,\beta)}$ do not equal to zero.

Next, let $\alpha = \{i_1,\dots,i_s,i_{s+1},\dots,i_h\}$, where

$$\{i_{s+1},\dots,i_h\} = \alpha \cap \{k+1,\dots,n\}$$

and $\beta = \{j_1,\dots,i_t,j_{t+1},\dots,j_h\}$, where

$$\{j_1,\dots,j_t\} = \beta \cap \{1,\dots,k-1\},$$

which implies $\sigma_{\alpha,\beta} = h-s+t$.

Observe further that although Lemma 2.5 was formulated for index sets from $Q_{h,n}$, the similar assertions are true for index sequences from $G_{h,n}$, as well. Hence, γ^* must contain $\{j_1,\dots,j_t\}$ and $\{i_{s+1},\dots,i_h\}$ together in both cases of $Q_{h,n}$ and $G_{h,n}$.

Next, we observe that if $\gamma \in G_{h,n}$ contains a repeating index from the set $\{i_{s+1},\dots,i_h\}$, then $A(\alpha,\gamma)$ has a $p\times q$ block of zeros with $p+q>h$. Similarly, if $\gamma\in G_{h,n}$ contains a repeating index from the set $\{j_1,\dots,j_t\}$, then $B(\gamma,\beta)$ has a $p\times q$ block of zeros with $p+q>h$. By Lemma 3.9, this implies that $\mathrm{per}(\mathrm{A}(\alpha,\gamma))=0$ or $\mathrm{per}(\mathrm{B}(\gamma,\beta))=0$. Hence, γ^* cannot contain repeating indices other than k.

Now, we consider all possible cases for the values of $\sigma_{\alpha,\beta}$ and exhibit the explicit form for a γ^* index sequence.

Case 1. $\sigma_{\alpha,\beta}>h$. In this case there are no γ^* index sequences in either $Q_{h,n}$ or $G_{h,n}$, which implies $[P_h(A)P_h(B)]_{(\alpha,\beta)} = [P_h(AB)]_{(\alpha,\beta)} = 0$.

Case 2.

Subcase 2.1 $\sigma_{\alpha,\beta}=h$. Here, γ^* is uniquely determined as $\gamma^* = \{j_1,\dots,j_t,i_{s+1},\dots,i_h\}$ in both $Q_{h,n}$ and $G_{h,n}$, with $\mu(\gamma^*)=1$.

Subcase 2.2 $\sigma_{\alpha,\beta}=h-1$. Here, γ^* is uniquely determined as $\gamma^* = \{j_1,\dots,j_t,k,i_{s+1},\dots,i_h\}$ in both $Q_{h,n}$ and $G_{h,n}$, with $\mu(\gamma^*)=1$.

Hence, for any α and β which satisfy $\sigma_{\alpha,\beta}=h-1$ or h, we get

$$[P_h(A)P_h(B)]_{(\alpha,\beta)} = [P_h(AB)]_{(\alpha,\beta)} = \mathrm{per}(\mathrm{A}(\alpha,\gamma^*))\mathrm{per}(\mathrm{B}(\gamma^*,\beta)).$$

Case 3. $\sigma_{\alpha,\beta}<h-1$. In this case there are no γ^* index sequences in $Q_{h,n}$ and there is a unique $\gamma^* = \{j_1,\dots,j_t,k,\dots,k,i_{s+1},\dots,i_h\}$ in $G_{h,n}$ where index k appears $h-\sigma_{\alpha,\beta}$ times. Hence, $[P_h(A)P_h(B)]_{(\alpha,\beta)}=0$ while

$$[P_h(AB)]_{(\alpha,\beta)} = \frac{\mathrm{per}(\mathrm{A}(\alpha,\gamma^*))\mathrm{per}(\mathrm{B}(\gamma^*,\beta))}{\mu(\gamma^*)}$$

which is not equal to zero in general. □

OBSERVATION 3.11. *We note that Theorem 2.1 is a special case of Theorem 3.10.*

REMARK 3.12. We recall that in Remark 2.4, $P_2(A)P_2(B)$ and $P_2(AB)$ differed only in the second super-diagonal position. With the use of Theorem 3.10, one can extend this fact to $n\times n$ matrices A and B as in Theorem 1.1 and any $1\leqslant h<n$ and obtain the following. With respect to a certain hierarchical ordering of the index sets, $P_h(A)P_h(B)-P_h(AB)$ is permutationally similar to a block upper-triangular matrix with

both the block diagonal and first block super-diagonal consisting entirely of zero blocks. An even more explicit determination of $P_h(\prod G_k)$ appears to be formidable in general, even for just three generators.

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