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2–SUMMING OPERATORS ON $l_2(\mathscr{X})$

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Abstract. Let $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces and $l_2(\mathscr{X})$, $c_0(\mathscr{X})$ the corresponding vector valued sequence spaces. In this paper we characterize nuclear operators on $c_0(\mathscr{X})$. As an application we obtain the necessary condition for an operator on $l_2(\mathscr{X})$ to be 2-summing. In the case of multiplication operators from $l_2(\mathscr{X})$ into $l_2(\mathscr{Y})$ (respectively from $c_0(\mathscr{X})$ into $c_0(\mathscr{Y})$) we show that the sufficient condition stated by Nahoum is also necessary. We also give the necessary and sufficient conditions for a bounded linear operator from $l_2(\mathscr{H})$ into $l_2(\mathscr{K})$ to be 2-summing, where $\mathscr{H}$ and $\mathscr{K}$ are sequences of Hilbert spaces. Further we give the necessary and/or sufficient conditions that Hardy and Hilbert type operators from $l_2(\mathscr{X})$ into $l_2(Y)$ to be 2-summing.

1. Introduction and background

The concept of absolutely summing linear operator plays a key role in operator theory. We recommend in this regard the books [2, 3, 9, 10, 12, 13]. Giving its special importance, a lot of work was done in order to give the necessary and sufficient conditions for some natural operators to be absolutely summing. For example, D. J. H. Garling in [4, Theorem 9] has given an almost complete description of the summing properties for the multiplication operators from l_s to l_t. Also, E. D. Gluskin, S. V. Kisljakov, O. I. Reinov in [6] studied the same problem in a more general context. In this paper we are mainly interested in studying the case of 2-summing operators defined on the vector valued sequence space $l_2(\mathscr{X})$. We first give a characterization of the nuclear operators on $c_0(\mathscr{X})$, Theorem 1. As an application, we obtain the necessary condition for an operator on $l_2(\mathscr{X})$ to be 2-summing, Theorem 3. In the case of multiplication operators from $l_2(\mathscr{X})$ into $l_2(\mathscr{Y})$ (respectively from $c_0(\mathscr{X})$ into $c_0(\mathscr{Y})$) we show that the sufficient condition stated by Nahoum is also necessary, Corollary 1. We also give the necessary and sufficient conditions for a bounded linear operator from $l_2(\mathscr{H})$ into $l_2(\mathscr{K})$ to be 2-summing, where $\mathscr{H}$ and $\mathscr{K}$ are sequences of Hilbert spaces, Theorem 4. Further we give the necessary and/or sufficient conditions for the Hardy and Hilbert type operators from $l_2(\mathscr{X})$ into $l_2(Y)$ to be 2-summing, see Corollaries 4, 5, 6.

Next, let us fix some notations and notions.

Let $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces and we denote by $l_2(\mathscr{X})$ the Banach space of all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \|x_n\|_{X_n}^2 < \infty$,

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endowed to the norm $\left\|(x_n)_{n\in\mathbb{N}}\right\|_{l_2(\mathscr{X})} = \left(\sum\limits_{n=1}^{\infty}\|x_n\|_{X_n}^2\right)^{\frac{1}{2}}$. Similarly, $c_0(\mathscr{X})$ denotes the Banach space of all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ for all $n \in \mathbb{N}$, $\|x_n\|_{X_n} \to 0$ as $n \to \infty$, endowed to the norm $\left\|(x_n)_{n\in\mathbb{N}}\right\|_{c_0(\mathscr{X})} = \sup\limits_{n\in\mathbb{N}}\|x_n\|_{X_n}$, see [13]. When $X_n = X$ for every natural number n, we will write $l_2(X)$ respectively $c_0(X)$. We will also consider the canonical mappings $\sigma_k : X_k \to l_2(\mathscr{X})$ (respectively $\sigma_k : X_k \to c_0(\mathscr{X})$) and $p_k : l_2(\mathscr{X}) \to X_k$ (respectively $p_k : c_0(\mathscr{X}) \to X_k$) defined by $\sigma_k(x) = (0,\ldots,0,\underbrace{x}_{k^{th}},0,\ldots)$, $p_k\left((x_n)_{n\in\mathbb{N}}\right) = x_k$, where k is a natural number.

Let X, Y be Banach spaces. A bounded linear operator $T : X \to Y$ is 2-summing if there is a constant $C \geqslant 0$ such that for every $(x_k)_{1\leqslant k\leqslant n} \subset X$ the following relation holds $\left(\sum\limits_{k=1}^{n}\|T(x_k)\|^2\right)^{\frac{1}{2}} \leqslant C \sup\limits_{\|x^*\|\leqslant 1}\left(\sum\limits_{k=1}^{n}|x^*(x_k)|^2\right)^{\frac{1}{2}}$. The 2-summing norm of T is defined as $\pi_2(T) = \inf\{C \mid C \text{ as above}\}$, see [2, 3, 9, 10, 12, 13].

A bounded linear operator $T : X \to Y$ is nuclear if there exists $(x_n^*)_{n\in\mathbb{N}} \subset X^*$, $(y_n)_{n\in\mathbb{N}} \subset Y$ such that $\sum\limits_{n=1}^{\infty}\|x_n^*\|\,\|y_n\| < \infty$ and $T(x) = \sum\limits_{n=1}^{\infty}x_n^*(x)\,y_n$ for $x \in X$. Such a representation is called a nuclear representation of T. In this case $\|T\|_{nuc} = \inf\left\{\sum\limits_{n=1}^{\infty}\|x_n^*\|\,\|y_n\|\right\}$, where the infimum is taken over all nuclear representations of T, see [2, 3, 9, 12, 13]. This class is denoted by $(\mathscr{N}, \|\ \|_{nuc})$.

Let $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$, $\mathscr{Y} = (Y_n)_{n\in\mathbb{N}}$ be two sequences of Banach spaces and $\mathscr{V} = (V_n)_{n\in\mathbb{N}}$ a sequence of bounded linear operators $V_n : X_n \to Y_n$ with $\sup\limits_{n\in\mathbb{N}}\|V_n\| < \infty$. The multiplication operator $M_{\mathscr{V}} : l_2(\mathscr{X}) \to l_2(\mathscr{Y})$ (respectively $M_{\mathscr{V}} : c_0(\mathscr{X}) \to c_0(\mathscr{Y})$) is defined by $M_{\mathscr{V}}\left((x_n)_{n\in\mathbb{N}}\right) = (V_n(x_n))_{n\in\mathbb{N}}$.

2. The results

Our first result gives a characterization of the nuclear operators defined on $c_0(\mathscr{X})$.

THEOREM 1. *Let $T : c_0(\mathscr{X}) \to Y$ be a bounded linear operator. The following assertions are equivalent:*

(i) T is nuclear.

(ii) all $T \circ \sigma_n : X_n \to Y$ are nuclear and $\sum\limits_{n=1}^{\infty}\|T \circ \sigma_n\|_{nuc} < \infty$.

Moreover, $\|T\|_{nuc} = \sum\limits_{n=1}^{\infty}\|T \circ \sigma_n\|_{nuc}$.

Proof. Let $x \in c_0(\mathscr{X})$. Then $x = \sum\limits_{n=1}^{\infty}(\sigma_n \circ p_n)(x)$, where the series is convergent in $c_0(\mathscr{X})$. Since T is a bounded linear operator, we have

$$T(x) = \sum_{n=1}^{\infty} T((\sigma_n \circ p_n)(x)) = \sum_{n=1}^{\infty}(T \circ \sigma_n)(p_n(x)). \tag{1}$$

(i)$\Rightarrow$(ii). By (i) let $(\psi_k)_{k\in\mathbb{N}} \subset (c_0(\mathscr{X}))^*$, $(y_k)_{k\in\mathbb{N}} \subset Y$ be such that $\sum_{k=1}^{\infty} \|\psi_k\| \|y_k\| < \infty$ and

$$T(x) = \sum_{k=1}^{\infty} \psi_k(x) y_k \text{ for } x \in c_0(\mathscr{X}). \tag{2}$$

Let $n \in \mathbb{N}$ and $x \in X_n$. From (2) we deduce $(T \circ \sigma_n)(x) = \sum_{k=1}^{\infty} (\psi_k \circ \sigma_n)(x) y_k$, thus all $T \circ \sigma_n$ are nuclear and $\|T \circ \sigma_n\|_{nuc} \leqslant \sum_{k=1}^{\infty} \|\psi_k \circ \sigma_n\|_{X_n^*} \|y_k\|$. Since, as it is well known and easy to prove, $\sum_{n=1}^{\infty} \|\psi \circ \sigma_n\|_{X_n^*} = \|\psi\|$ for $\psi \in (c_0(\mathscr{X}))^*$, it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \|T \circ \sigma_n\|_{nuc} &\leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|\psi_k \circ \sigma_n\|_{X_n^*} \|y_k\| \\
&= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \|\psi_k \circ \sigma_n\|_{X_n^*} \right) \|y_k\| = \sum_{k=1}^{\infty} \|\psi_k\| \|y_k\|.
\end{aligned}$$

Taking the infimum over all the nuclear representation of T as above, we obtain $\sum_{n=1}^{\infty} \|T \circ \sigma_n\|_{nuc} \leqslant \|T\|$ i.e. (ii).

(ii)$\Rightarrow$(i). Since all $T \circ \sigma_n$ are nuclear, then all $T \circ \sigma_n \circ p_n$ are nuclear, $\|T \circ \sigma_n \circ p_n\|_{nuc} \leqslant \|T \circ \sigma_n\|_{nuc}$, thus by (ii), $\sum_{n=1}^{\infty} \|T \circ \sigma_n \circ p_n\|_{nuc} < \infty$. By a general result, see [9, Theorem 6.2.3, p. 91], it follows that the series $\sum_{n=1}^{\infty} T \circ \sigma_n \circ p_n$ is convergent in $\mathscr{N}(c_0(\mathscr{X}), Y)$ and let $S = \sum_{n=1}^{\infty} T \circ \sigma_n \circ p_n$ be its sum. Note that $\|S\|_{nuc} \leqslant \sum_{n=1}^{\infty} \|T \circ \sigma_n \circ p_n\|_{nuc} \leqslant \sum_{n=1}^{\infty} \|T \circ \sigma_n\|_{nuc}$. We get that $S(x) = \sum_{n=1}^{\infty} (T \circ \sigma_n \circ p_n)(x)$ for $x \in c_0(\mathscr{X})$ and by (1), $S = T$ i.e. T is nuclear. $\square$

We recall Nahoum's theorem, see [7, Lemme, p. 5], [13, Lemma 23, p. 274]. For the sake of completeness we include a proof different from that in [7, Lemme, p. 5].

THEOREM 2. *Let $U : Z \to l_2(\mathscr{Y})$ (resp. $U : Z \to c_0(\mathscr{Y})$) be defined by $U(z) = (U_n(z))_{n\in\mathbb{N}}$. If all U_n are 2-summing and $\sum_{n=1}^{\infty} [\pi_2(U_n)]^2 < \infty$, then U is 2-summing and $[\pi_2(U)]^2 \leqslant \sum_{n=1}^{\infty} [\pi_2(U_n)]^2$.*

Proof. First, let us note that $\|U(z)\|^2 = \sum_{n=1}^{\infty} \|U_n(z)\|^2$ for $z \in Z$ (respectively $\|U(z)\|^2 \leqslant \sum_{n=1}^{\infty} \|U_n(z)\|^2$ for $z \in Z$). Let $(z_i)_{1\leqslant i\leqslant k} \subset Z$, then we have

$$\sum_{i=1}^{k}\|U(z_i)\|^2 \leqslant \sum_{n=1}^{\infty}\sum_{i=1}^{k}\|U_n(z_i)\|^2 \leqslant \sum_{n=1}^{\infty}[\pi_2(U_n)]^2\left[\sup_{\|x^*\|\leqslant 1}\left(\sum_{i=1}^{k}|x^*(z_i)|^2\right)^{\frac{1}{2}}\right]^2$$
$$\leqslant \left(\sum_{n=1}^{\infty}[\pi_2(U_n)]^2\right)\left[\sup_{\|x^*\|\leqslant 1}\left(\sum_{i=1}^{k}|x^*(z_i)|^2\right)^{\frac{1}{2}}\right]^2$$

and the statement follows. □

One of the main ingredients in our proof is the following result which is a completion of [8, Proposition 2.4.] and [11, Satz 8].

LEMMA 1. *Let $X \xrightarrow{V} Y$ be a bounded linear operator. The following assertions are equivalent:*

(i) V is 2-summing.

(ii) For each Banach space Z, each 2-summing operator $Z \xrightarrow{U} X$, $V \circ U$ is nuclear. Moreover, $\sup\limits_{\pi_2(U)\leqslant 1}\|V\circ U\|_{nuc} = \pi_2(V)$, where the supremum is taken over all Banach spaces Z and all 2-summing operators $Z \xrightarrow{U} X$.

Proof. (i)⇒(ii). From Grothendieck's theorem, see [3, Theorem 5.31], $V \circ U$ is nuclear and $\|V\circ U\|_{nuc} \leqslant \pi_2(V)\pi_2(U)$. Then

$$\sup_{\pi_2(U)\leqslant 1}\|V\circ U\|_{nuc} \leqslant \pi_2(V). \tag{1}$$

(ii)⇒(i). From (ii) it follows that for each Banach space Z the mapping $h_V^Z : \Pi_2(Z,X) \to \mathcal{N}(Z,Y)$ defined by $h_V^Z(U) = V\circ U$ is well defined. By the closed graph theorem h_V^Z is bounded linear, thus $\sup\limits_{\pi_2(U)\leqslant 1}\left\|h_V^Z(U)\right\|_{nuc} = C_V^Z < \infty$, where the supremum is taken over all 2-summing operators $Z \xrightarrow{U} X$. We will prove that $M_V = \sup\limits_{\pi_2(U)\leqslant 1}\|V\circ U\|_{nuc} < \infty$, where the supremum is taken now over all Banach spaces Z and all 2-summing operators $Z \xrightarrow{U} X$. Indeed, if $\sup\limits_{\pi_2(U)\leqslant 1}\|V\circ U\|_{nuc} = \infty$, where the supremum is taken over all Banach spaces Z and all 2-summing operators $Z \xrightarrow{U} X$, we deduce that there exist Banach spaces Z_n and 2-summing operators $Z_n \xrightarrow{T_n} X$ such that $\pi_2(T_n) \leqslant 1$ and $\|V\circ T_n\|_{nuc} \geqslant n\cdot 2^n$ for all natural numbers n. Then $U_n = \frac{1}{2^n}T_n : Z_n \to X$ are such that $\pi_2(U_n) \leqslant \frac{1}{2^n}$ and $\|V\circ U_n\|_{nuc} \geqslant n$ for all natural numbers n. Let us consider the sequence $\mathscr{Z} = (Z_n)_{n\in\mathbb{N}}$ and define $U : l_2(\mathscr{Z}) \to X$ by $U = \sum\limits_{n=1}^{\infty} U_n \circ p_n$. Since $\sum\limits_{n=1}^{\infty}\pi_2(U_n\circ p_n) \leqslant \sum\limits_{n=1}^{\infty}\pi_2(U_n) \leqslant 1$, by a general result, see [9, Theorem 6.2.3, p. 91] it follows that U is 2-summing and $\pi_2(U) \leqslant 1$. Then $\|V \circ U\|_{nuc} \leqslant C_V^{l_2(\mathscr{X})}$ and so $\|V \circ U \circ \sigma_n\|_{nuc} \leqslant \|V \circ U\|_{nuc} \leqslant C_V^{l_2(\mathscr{X})}$ for all natural numbers n. But $U \circ \sigma_n = U_n$ and hence $n \leqslant \|V \circ U_n\|_{nuc} \leqslant C_V^{l_2(\mathscr{X})}$ for all natural numbers n, which is impossible.

Now let $l_2 \xrightarrow{S} X$ be a bounded linear operator. Let also $a = (a_n)_{n\in\mathbb{N}} \in l_2$. Then $c_0 \xrightarrow{M_a} l_2$ is 2-summing with $\pi_2(M_a) = \|a\|_2$, hence $c_0 \xrightarrow{S\circ M_a} X$ will be 2-summing. Then $\|V \circ S \circ M_a\|_{nuc} \leqslant M_V \pi_2(S \circ M_a)$. Since, as is well-known

$$\|V \circ S \circ M_a\|_{nuc} = \sum_{n=1}^{\infty} \|(V \circ S \circ M_a)(e_n)\| = \sum_{n=1}^{\infty} |a_n| \|V(S(e_n))\|$$

we get $\sum\limits_{n=1}^{\infty} |a_n| \|V(S(e_n))\| \leqslant M_V \|a\|_2 \|S\|$. Since $a = (a_n)_{n\in\mathbb{N}} \in l_2$ is arbitrary we deduce that $\left(\sum\limits_{n=1}^{\infty} \|V(S(e_n))\|^2\right)^{\frac{1}{2}} \leqslant M_V \|S\|$. This means that V is 2-summing and

$$\pi_2(V) \leqslant M_V = \sup_{\pi_2(U)\leqslant 1} \|V \circ U\|_{nuc}, \tag{2}$$

see [3, Proposition 2.7], i.e. (i). From (1) and (2) we get also the equality from the statement. □

If $T : l_2 \to Y$ (respectively $T : c_0 \to Y$) is a 2-summing operator, since for each natural number n we have, $\sup\limits_{\|x^*\|\leqslant 1} \left(\sum\limits_{k=1}^{n} |x^*(e_k)|^2\right)^{\frac{1}{2}} = 1$, then $\sum\limits_{k=1}^{n} \|T(e_k)\|^2 \leqslant [\pi_2(T)]^2$ and thus $\sum\limits_{n=1}^{\infty} \|T(e_n)\|^2 \leqslant [\pi_2(T)]^2$.

The following result is the main result of this paper. It gives a necessary condition that an operator on $l_2(\mathscr{X})$ (respectively $c_0(\mathscr{X})$) be 2-summing and can be regarded as a vector version of the scalar case shown above.

THEOREM 3. *Let $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces, Y a Banach space and $T : l_2(\mathscr{X}) \to Y$ (respectively $T : c_0(\mathscr{X}) \to Y$) a bounded linear operator. If T is 2-summing, then all $T \circ \sigma_n$ are 2-summing and $\sum\limits_{n=1}^{\infty} [\pi_2(T \circ \sigma_n)]^2 < \infty$. Moreover, $\sum\limits_{n=1}^{\infty} [\pi_2(T \circ \sigma_n)]^2 \leqslant [\pi_2(T)]^2$.*

Proof. Let $\mathscr{Z} = (Z_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces and $U_n : Z_n \to X_n$ be such that U_n are 2-summing and $\pi_2(U_n) \leqslant 1$ for all $n \in \mathbb{N}$. Let also $a = (a_n)_{n\in\mathbb{N}} \in l_2$. We define $U : c_0(\mathscr{Z}) \to l_2(\mathscr{X})$ (resp. $U : c_0(\mathscr{Z}) \to c_0(\mathscr{X})$) by $U(z) = (a_n U_n \circ p_n(z))_{n\in\mathbb{N}}$. From $\pi_2(U_n \circ p_n) \leqslant \pi_2(U_n) \leqslant 1$ for all $n \in \mathbb{N}$ and $a \in l_2$, by Nahoum's theorem 2, it follows that U is 2-summing and $\pi_2(U) \leqslant \|a\|_2$. Since T is 2-summing, by Grothendieck's theorem, see [3, Theorem 5.31], $T \circ U : c_0(\mathscr{Z}) \to Y$ is nuclear and $\|T \circ U\|_{nuc} \leqslant \pi_2(T)\pi_2(U) \leqslant \pi_2(T)\|a\|_2$. By Theorem 1 it follows that all $T \circ U \circ \sigma_n$ are nuclear and $\sum\limits_{n=1}^{\infty} \|T \circ U \circ \sigma_n\|_{nuc} = \|T \circ U\|_{nuc}$. By a simple calculations, we have $U \circ \sigma_n = a_n \sigma_n \circ U_n$, $T \circ U \circ \sigma_n = a_n T \circ \sigma_n \circ U_n$, so, all $a_n T \circ \sigma_n \circ U_n$ are nuclear and $\sum\limits_{n=1}^{\infty} \|a_n T \circ \sigma_n \circ U_n\|_{nuc} = \|T \circ U\|_{nuc}$. Since this is true for all $a \in l_2$ we deduce that all $T \circ \sigma_n \circ U_n$ are nuclear (take $a = e_n$, $n \in \mathbb{N}$) and $\sum\limits_{n=1}^{\infty} |a_n| \, \|T \circ \sigma_n \circ U_n\|_{nuc} = \|T \circ U\|_{nuc}$. Then

$$\sum_{i=1}^{n} |a_i| \, \|T \circ \sigma_i \circ U_i\|_{nuc} \leqslant \pi_2(T) \|a\|_2 \text{ for } n \in \mathbb{N} \tag{1}$$

Taking in (1) the supremum, first for $\pi_2(U_1) \leqslant 1$, then for $\pi_2(U_2) \leqslant 1$, ..., for $\pi_2(U_n) \leqslant 1$, from Lemma 1, we obtain $\sum\limits_{i=1}^{n} |a_i| \pi_2(T \circ \sigma_i) \leqslant \pi_2(T) \|a\|_2$ for $n \in \mathbb{N}$ i.e. $\sum\limits_{n=1}^{\infty} |a_n| \pi_2(T \circ \sigma_n) \leqslant \pi_2(T) \|a\|_2$. As it is well known, from here it follows that $\sum\limits_{n=1}^{\infty} [\pi_2(T \circ \sigma_n)]^2 < \infty$ and $\left(\sum\limits_{n=1}^{\infty} [\pi_2(T \circ \sigma_n)]^2\right)^{\frac{1}{2}} = \sup\limits_{\|a\|_2 \leqslant 1} \left(\sum\limits_{n=1}^{\infty} |a_n| \pi_2(T \circ \sigma_n)\right) \leqslant \pi_2(T)$. $\square$

In the case of multiplication operators from $l_2(\mathscr{X})$ into $l_2(\mathscr{Y})$ (respectively $c_0(\mathscr{X})$ into $c_0(\mathscr{Y})$) we show that the sufficient condition stated by Nahoum is also necessary.

COROLLARY 1. *Let $M_{\mathscr{V}} : l_2(\mathscr{X}) \to l_2(\mathscr{Y})$ (resp. $M_{\mathscr{V}} : c_0(\mathscr{X}) \to c_0(\mathscr{Y})$) be the multiplication operator. The following assertions are equivalent:*

(i) $M_{\mathscr{V}}$ is 2-summing.

(ii) all V_n are 2-summing and $(\pi_2(V_n))_{n \in \mathbb{N}} \in l_2$.

Moreover, $[\pi_2(M_{\mathscr{V}})]^2 = \sum\limits_{n=1}^{\infty} [\pi_2(V_n)]^2$.

Proof. We prove the case $M_{\mathscr{V}} : l_2(\mathscr{X}) \to l_2(\mathscr{Y})$; the other one is similar.

In view of Nahoum's theorem 2, we must prove only that (i)$\Rightarrow$(ii). If we write $Y = l_2(\mathscr{Y})$, then $M_{\mathscr{V}} : l_2(\mathscr{X}) \to Y$ and by a simple calculation we have $M_{\mathscr{V}} \circ \sigma_n = \sigma_n \circ V_n$ and $p_n \circ M_{\mathscr{V}} \circ \sigma_n = V_n$. From these relations we deduce that $M_{\mathscr{V}} \circ \sigma_n$ is 2-summing if and only if V_n is 2-summing and $\pi_2(V_n) = \pi_2(M_{\mathscr{V}} \circ \sigma_n)$. Then (i)$\Rightarrow$(ii) follows from Theorem 3 and the above relations. $\square$

LEMMA 2. *Let $\mathscr{X} = (X_n)_{n \in \mathbb{N}}$ be a sequence of Banach spaces, Y a Banach space and $V : l_2(\mathscr{X}) \to Y$ a bounded linear operator. We consider the assertions:*

(i) V is 2-summing.

(ii) all $V \circ \sigma_n$ are 2-summing and $\sum\limits_{n=1}^{\infty} [\pi_2(V \circ \sigma_n)]^2 < \infty$.

Then, always (i)$\Rightarrow$(ii). If moreover, $\mathscr{X} = (X_n)_{n \in \mathbb{N}}$ is a sequence of Hilbert spaces, Y a Hilbert space, then (i)$\Leftrightarrow$(ii) and in this case $[\pi_2(V)]^2 = \sum\limits_{n=1}^{\infty} [\pi_2(V \circ \sigma_n)]^2$.

Proof. (i)$\Rightarrow$(ii) was shown in Theorem 3.

(ii)$\Rightarrow$(i) in the case when $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ is a sequence of Hilbert spaces, Y a Hilbert space. In this case, as it is well known, $l_2(\mathscr{X})$ is a Hilbert space and the scalar product in $l_2(\mathscr{X})$ is defined by $\left\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \right\rangle_{l_2(\mathscr{X})} = \sum\limits_{n=1}^{\infty} \langle x_n, y_n \rangle_{X_n}$. Since $V : l_2(\mathscr{X}) \to Y$ is a bounded linear operator (between two Hilbert spaces) we can consider the adjoint of V, $V^* : Y \to l_2(\mathscr{X})$. We prove that $V^*(y) = \left((V\circ\sigma_n)^*(y)\right)_{n\in\mathbb{N}}$ for $y \in Y$. Indeed, let $y \in Y$ and write $V^*(y) = (A_n(y))_{n\in\mathbb{N}}$, where $A_n : Y \to X_n$. Then for each $x \in X_n$ we have $\langle V^*(y), \sigma_n(x)\rangle_{l_2(\mathscr{X})} = \langle y, V(\sigma_n(x))\rangle_Y$ i.e.

$$\langle A_n(y), x\rangle_{X_n} = \langle y, (V\circ\sigma_n)(x)\rangle_Y = \left\langle (V\circ\sigma_n)^*(y), x\right\rangle_{X_n}$$

thus, since $x \in X_n$ is arbitrary, $A_n(y) = (V\circ\sigma_n)^*(y)$. But, since on Hilbert spaces 2-summing operators coincide with the Hilbert-Schmidt operators, by (ii), all $V\circ\sigma_n$ are Hilbert-Schmidt, and then as it is well known all $(V\circ\sigma_n)^*$ are also Hilbert-Schmidt and $\|V\circ\sigma_n\|_{HS} = \left\|(V\circ\sigma_n)^*\right\|_{HS}$. Then $\sum\limits_{n=1}^{\infty} \left\|(V\circ\sigma_n)^*\right\|_{HS}^2 < \infty$ and by [1, Lemma 1], $V^* : Y \to l_2(\mathscr{X})$ is Hilbert-Schmidt, thus V is Hilbert-Schmidt, hence 2-summing i.e. (ii). □

In the sequel we give the necessary or/and sufficient conditions for a bounded linear operator from $l_2(\mathscr{X})$ into $l_2(\mathscr{Y})$ to be 2-summing. Further, we will use these results in order to give the necessary and/or sufficient conditions for the Hardy and Hilbert type operators from $l_2(\mathscr{X})$ into $l_2(Y)$ to be 2-summing.

Let $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$, $\mathscr{Y} = (Y_n)_{n\in\mathbb{N}}$ be two sequences of Banach spaces.

Let $V : l_2(\mathscr{X}) \to l_2(\mathscr{Y})$ be a bounded linear operator and if we define the bounded linear operators $V_{nk} : X_k \to Y_n$ by $V_{nk} = p_n \circ V \circ \sigma_k$, then

$$V\left((x_n)_{n\in\mathbb{N}}\right) = \left(\sum_{k=1}^{\infty} V_{nk}(x_k)\right)_{n\in\mathbb{N}} \quad \text{for } (x_n)_{n\in\mathbb{N}} \in l_2(\mathscr{X}).$$

The matrix of operators $\mathscr{V} = (V_n)_{n,\in\mathbb{N}}$ is called the operator representing matrix of V. Indeed, let us define $L_n = p_n \circ V : l_2(\mathscr{X}) \to Y_n$ and then note that L_n are bounded linear. Next $V(x) = (L_n(x))_{n\in\mathbb{N}}$ for $x = (x_n)_{n\in\mathbb{N}} \in l_2(\mathscr{X})$. We have $x = \sum\limits_{k=1}^{\infty} \sigma_k(x_k)$ from where, $L_n(x) = \sum\limits_{k=1}^{\infty} L_n(\sigma_k(x_k)) = \sum\limits_{k=1}^{\infty} (L_n\circ\sigma_k)(x_k) = \sum\limits_{k=1}^{\infty} V_{nk}(x_k)$.

COROLLARY 2. *Let $V : l_2(\mathscr{X}) \to l_2(\mathscr{Y})$ be a bounded linear operator and $\mathscr{V} = (V_{nk})_{n,k\in\mathbb{N}}$ its operator representing matrix. If V is 2-summing, then all operators $C_k : X_k \to l_2(\mathscr{Y})$ defined by $C_k(x) = (V_{nk}(x))_{n\in\mathbb{N}}$ are 2-summing and $\sum\limits_{k=1}^{\infty} [\pi_2(C_k)]^2 < \infty$.*

Moreover, $\sum\limits_{k=1}^{\infty} [\pi_2(C_k)]^2 \leqslant [\pi_2(V)]^2$.

Proof. Since V is 2-summing from Theorem 3, all $V\circ\sigma_k$ are 2-summing, $\sum_{k=1}^{\infty}[\pi_2(V\circ\sigma_k)]^2<\infty$ and $\sum_{k=1}^{\infty}[\pi_2(V\circ\sigma_k)]^2\leqslant[\pi_2(V)]^2$. For $x\in X_k$ we have

$$(V\circ\sigma_k)(x)=\left(\sum_{j=1}^{\infty}V_{nj}(p_j(\sigma_k(x)))\right)_{n\in\mathbb{N}}=(V_{nk}(x))_{n\in\mathbb{N}}=C_k(x).$$

Thus all C_k are 2-summing, $\pi_2(V\circ\sigma_k)=\pi_2(C_k)$ and the conclusion follows. □

In the case of Hilbert spaces we can prove the following result, perhaps well known, but for which we do not know any reference. Note that this result extend the well known characterization of 2-summing operators from l_2 into l_2.

THEOREM 4. *Let $\mathscr{H}=(H_n)_{n\in\mathbb{N}}$, $\mathscr{K}=(K_n)_{n\in\mathbb{N}}$ be two sequences of Hilbert spaces, $V:l_2(\mathscr{H})\to l_2(\mathscr{K})$ a bounded linear operator and $\mathscr{V}=(V_{nk})_{n,k\in\mathbb{N}}$ its operator representing matrix. The following assertions are equivalent:*

(i) V is 2-summing.

(ii) all operators $C_k:X_k\to l_2(\mathscr{K})$ defined by $C_k(x)=(V_{nk}(x))_{n\in\mathbb{N}}$ are 2-summing and $\sum_{k=1}^{\infty}[\pi_2(C_k)]^2<\infty$.

(iii) all operators $L_n:l_2(\mathscr{H})\to K_n$ defined by $L_n(x)=\sum_{k=1}^{\infty}V_{nk}(x_k)$ are 2-summing and $\sum_{n=1}^{\infty}[\pi_2(L_n)]^2<\infty$.

(iv) all $V_{nk}:H_k\to K_n$ are 2-summing and $\sum_{n,k=1}^{\infty}[\pi_2(V_{nk})]^2<\infty$.

Moreover, $[\pi_2(V)]^2=\sum_{k=1}^{\infty}[\pi_2(C_k)]^2=\sum_{n=1}^{\infty}[\pi_2(L_n)]^2=\sum_{n,k=1}^{\infty}[\pi_2(V_{nk})]^2$.

Proof. (i)⇒(ii) is a particular case of Corollary 2.

(ii)⇔(iv). Since H_k and K_n are Hilbert spaces and since every 2-summing operator defined on Hilbert spaces coincides with the Hilbert-Schmidt one, from Lemma 1 in [1], C_k is 2-summing if and only if all V_{nk} are 2-summing, $\sum_{n=1}^{\infty}[\pi_2(V_{nk})]^2<\infty$ and moreover, $[\pi_2(C_k)]^2=\sum_{n=1}^{\infty}[\pi_2(V_{nk})]^2$. The equivalence (ii)⇔(iv) follows.

(iv)⇒(iii). Since $\sum_{k=1}^{\infty}[\pi_2(V_{nk})]^2<\infty$ from Lemma 2 (the implication (ii)⇒(i)), all L_n are 2-summing and $[\pi_2(L_n)]^2=\sum_{k=1}^{\infty}[\pi_2(V_{nk})]^2$. Also $\sum_{n=1}^{\infty}[\pi_2(L_n)]^2=\sum_{n,k=1}^{\infty}[\pi_2(V_{nk})]^2<\infty$ and (iii) follows.

(iii)⇒(i). Since as we have already observed, $V(x)=(L_n(x))_{n\in\mathbb{N}}$ and $\sum_{n=1}^{\infty}[\pi_2(L_n)]^2<\infty$, by Nahoum's Theorem 2, we get that V is 2-summing and $[\pi_2(V)]^2\leqslant\sum_{n=1}^{\infty}[\pi_2(L_n)]^2$ i.e. (i). □

In the rest of the paper, $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces, Y a Banach space, $V_k : X_k \to Y$ are bounded linear operators, $(a_{nk})_{n,k\in\mathbb{N}}$ is a matrix of scalars (real or complex numbers) such that the scalar matrix $(|a_{nk}|\,\|V_k\|)_{n,k\in\mathbb{N}}$ defines a bounded linear operator from l_2 into l_2. Note that this means that the operator $U : l_2 \to l_2$ defined by $U\left((\xi_n)_{n\in\mathbb{N}}\right) = \left(\sum\limits_{k=1}^{\infty} |a_{nk}|\,\|V_k\|\,\xi_k\right)_{n\in\mathbb{N}}$ is bounded linear. Under these assumptions, the operator $V : l_2(\mathscr{X}) \to l_2(Y)$ defined by $V\left((x_n)_{n\in\mathbb{N}}\right) = \left(\sum\limits_{k=1}^{\infty} a_{nk}V_k(x_k)\right)_{n\in\mathbb{N}}$ is bounded linear. Indeed, let $(x_n)_{n\in\mathbb{N}} \in l_2(\mathscr{X})$. First let us note that $\sum\limits_{k=1}^{\infty} \|a_{nk}V_k(x_k)\| \leqslant \sum\limits_{k=1}^{\infty} |a_{nk}|\,\|V_k\|\,\|x_k\|$ for $n\in\mathbb{N}$. Since U takes its values in l_2 and $(\|x_n\|)_{n\in\mathbb{N}} \in l_2$, the series from the right member is convergent, hence $\sum\limits_{k=1}^{\infty} a_{nk}V_k(x_k)$ is absolutely convergent, thus convergent and then we can write $y_n = \sum\limits_{k=1}^{\infty} a_{nk}V_k(x_k) \in Y$. From $\|y_n\| \leqslant \sum\limits_{k=1}^{\infty} \|a_{nk}V_k(x_k)\| \leqslant \sum\limits_{k=1}^{\infty} |a_{nk}|\,\|V_k\|\,\|x_k\|$ for $n\in\mathbb{N}$, the fact that U takes its values in l_2 and $(\|x_k\|)_{k\in\mathbb{N}} \in l_2$ we get $(y_n)_{n\in\mathbb{N}} \in l_2(\mathscr{Y})$.

COROLLARY 3. *Let $V : l_2(\mathscr{X}) \to l_2(Y)$ be the operator defined by $V\left((x_n)_{n\in\mathbb{N}}\right) = \left(\sum\limits_{k=1}^{\infty} a_{nk}V_k(x_k)\right)_{n\in\mathbb{N}}$. We consider the following assertions:*

(i) V is 2-summing.

(ii) all V_k are 2-summing and $\sum\limits_{k=1}^{\infty} [\pi_2(V_k)]^2 c_k^2 < \infty$, where $c_k = \sqrt{\sum\limits_{n=1}^{\infty} |a_{nk}|^2}$.

Then, always (i)⇒(ii). If moreover, $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ is a sequence of Hilbert spaces and Y is a Hilbert space then, (i)⇔(ii) and in this case $[\pi_2(V)]^2 = \sum\limits_{k=1}^{\infty} [\pi_2(V_k)]^2 c_k^2$.

Proof. (i)⇒(ii). The operator matrix of the operator V is $V_{nk} = a_{nk}V_k$. In this case $C_k : X_k \to l_2(Y)$ is defined by $C_k(x) = (a_{nk}V_k(x))_{n\in\mathbb{N}}$. We have $\|C_k(x)\| = c_k\|V_k(x)\|$. Since V is 2-summing, then by Corollary 2 all C_k are 2-summing, $\sum\limits_{k=1}^{\infty} [\pi_2(C_k)]^2 < \infty$ and $\sum\limits_{k=1}^{\infty} [\pi_2(C_k)]^2 \leqslant [\pi_2(V)]^2$. Thus all V_k are 2-summing, $\sum\limits_{k=1}^{\infty} [\pi_2(V_k)]^2 c_k^2 < \infty$ and $\sum\limits_{k=1}^{\infty} [\pi_2(V_k)]^2 c_k^2 \leqslant [\pi_2(V)]^2$ i.e. (ii).

(ii)⇒(i). We have

$$\sum_{n,k=1}^{\infty} [\pi_2(a_{nk}V_k)]^2 = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} [\pi_2(a_{nk}V_k)]^2\right) = \sum_{k=1}^{\infty} [\pi_2(V_k)]^2 c_k^2 < \infty$$

by (ii). Then (i) follows from Theorem 4. □

In Corollaries 4, 5 and 6, $\mathscr{V} = (V_n)_{n\in\mathbb{N}}$ is a sequence of bounded linear operators $V_n : X_n \to Y$ such that $\sup\limits_{n\in\mathbb{N}} \|V_n\| < \infty$. In these cases, by the classical Hardy and Hilbert theorems, see [5], the operators are well defined.

COROLLARY 4. *Let* $H_{\mathscr{V}} : l_2(\mathscr{X}) \to l_2(Y)$ *be the Hardy operator defined by* $H_{\mathscr{V}}\left((x_n)_{n\in\mathbb{N}}\right) = \left(\frac{V_1(x_1)+\cdots+V_n(x_n)}{n}\right)_{n\in\mathbb{N}}$. *We consider the following assertions:*

(i) $H_{\mathscr{V}}$ *is* 2*-summing.*

(ii) all V_k *are* 2*-summing and* $\sum\limits_{k=1}^{\infty}[\pi_2(V_k)]^2 c_k^2 < \infty$, *where* $c_k = \sqrt{\sum\limits_{n=k}^{\infty}\frac{1}{n^2}}$.

(iii) all V_n *are* 2*-summing and* $\sum\limits_{k=1}^{\infty}\frac{[\pi_2(V_k)]^2}{k} < \infty$.

Then, always (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii). If moreover, $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ *is a sequence of Hilbert spaces and* Y *is a Hilbert space then, (i)$\Leftrightarrow$(ii) and in this case* $[\pi_2(H_{\mathscr{V}})]^2 = \sum\limits_{k=1}^{\infty}[\pi_2(V_k)]^2 c_k^2$.

Proof. In view of Corollary 3 only the equivalence (ii)$\Leftrightarrow$(iii) needs a proof. Since $\sum\limits_{n=k}^{\infty}\frac{1}{n^2} \backsim \frac{1}{k}$ as $k \to \infty$ we get $c_k \backsim \frac{1}{\sqrt{k}}$ as $k \to \infty$. Thus $\sum\limits_{k=1}^{\infty}[\pi_2(V_k)]^2 c_k^2 < \infty$ if and only if $\sum\limits_{k=1}^{\infty}\frac{[\pi_2(V_k)]^2}{k} < \infty$. $\square$

The following result is a particular case of Corollary 3.

COROLLARY 5. *Let* $H_{\mathscr{V}} : l_2(\mathscr{X}) \to l_2(Y)$ *be the Hardy operator defined by* $H_{\mathscr{V}}\left((x_n)_{n\in\mathbb{N}}\right) = \left(\sum\limits_{k=n}^{\infty}\frac{V_k(x_k)}{k}\right)_{n\in\mathbb{N}}$. *We consider the following assertions:*

(i) $H_{\mathscr{V}}$ *is* 2*-summing.*

(ii) all V_k *are* 2*-summing and* $\sum\limits_{k=1}^{\infty}\frac{[\pi_2(V_k)]^2}{k} < \infty$.

Then, always (i)$\Rightarrow$(ii). If moreover, $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ *is a sequence of Hilbert spaces and* Y *is a Hilbert space then, (i)$\Leftrightarrow$(ii) and in this case* $[\pi_2(H_{\mathscr{V}})]^2 = \sum\limits_{k=1}^{\infty}\frac{[\pi_2(V_k)]^2}{k}$.

COROLLARY 6. *Let* $H_{\mathscr{V}} : l_2(\mathscr{X}) \to l_2(Y)$ *be the Hilbert operator defined by* $H_{\mathscr{V}}\left((x_n)_{n\in\mathbb{N}}\right) = \left({\sum\limits_{k=1}^{\infty}}'\frac{V_k(x_k)}{n-k}\right)_{n\in\mathbb{N}}$ *where the dash indicates that the sum ranges over all* k *except* $k=n$. *We consider the following assertions:*

(i) $H_{\mathscr{V}}$ *is* 2*-summing.*

(ii) all V_k *are* 2*-summing and* $\sum\limits_{k=1}^{\infty}[\pi_2(V_k)]^2 < \infty$.

Then, always (i)$\Rightarrow$(ii). If moreover, $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ *is a sequence of Hilbert spaces and* Y *is a Hilbert space then, (i)$\Leftrightarrow$(ii) and in this case* $[\pi_2(H_{\mathscr{V}})]^2 = [\pi_2(V_1)]^2\frac{\pi^2}{3} + \sum\limits_{k=2}^{\infty}[\pi_2(V_k)]^2 c_k^2$, *where* $c_k^2 = \frac{\pi^2}{6} + \sum\limits_{n=1}^{k-1}\frac{1}{n^2}$ *for* $k \geqslant 2$.

Proof. If we take in Corollary 3 $a_{nk} = \frac{1}{n-k}$ for $n \neq k$ and $a_{nn} = 0$, then $c_1^2 = \sum_{n=1}^{\infty}{}' \frac{1}{(n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $c_k^2 = \sum_{n=1}^{\infty}{}' \frac{1}{(n-k)^2} = \sum_{n=1}^{k-1} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} + \sum_{n=1}^{k-1} \frac{1}{n^2}$ for $k \geqslant 2$. Since $c_k^2 \to \frac{\pi^2}{3}$ as $k \to \infty$, $\sum_{k=1}^{\infty} [\pi_2(V_k)]^2 c_k^2 < \infty$ if and only if $\sum_{k=1}^{\infty} [\pi_2(V_k)]^2 < \infty$. $\square$

REMARK 1. Unfortunatlely, we do not know if the implication (ii)$\Rightarrow$(i) in Corollaries 3, 4, 5 and 6, is true for each sequence $\mathscr{X} = (X_n)_{n \in \mathbb{N}}$ of Banach spaces and Y a Banach space.

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