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SPECTRAL PROPERTIES OF NORMAL OPERATORS HAVING SYMMETRIES ARISING FROM CONJUGATIONS

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Abstract. We study the consequences of equations such as $AB = BA$ and $AB = BA^*$ on the spectrum of B when A is a normal operator that is real or complex skew-symmetric. Our main result is a spectral pairing theorem for such operators which generalizes results about normal matrices obtained in [11]. Important tools used are the properties of conjugations and complex symmetric operators. This paper continues the study initiated in [14].

0. Introduction

Let $A : \mathscr{H} \to \mathscr{H}$ be a bounded *normal operator* defined on a separable Hilbert space $\mathscr{H}$, i.e., it has the property $AA^* = A^*A$ where A^* is the adjoint of A (see [3] for more details).

A *conjugation* $J : \mathscr{H} \to \mathscr{H}$ is an anti-linear involution, i.e., J satisfies:

(i) $J(\alpha f + \beta g) = \bar{\alpha}J(f) + \bar{\beta}J(g)$, for all $f, g \in \mathscr{H}$ and $\alpha, \beta \in \mathbb{C}$.

(ii) $\langle Jf, Jg\rangle = \langle g, f\rangle$ for all $f, g \in \mathscr{H}$, where $\langle f, g\rangle$ is the inner product on $\mathscr{H}$.

(iii) $J^2 = E$ (where E is the identity operator).

DEFINITION. Let $J : \mathscr{H} \to \mathscr{H}$ be a conjugation and $A : \mathscr{H} \to \mathscr{H}$ a bounded linear operator.

(i) The operator A is said to be J-*real* if $JAJ = A$.

(ii) A is J-*symmetric* if $JA^*J = A$.

(iii) A is J-*imaginary* if $JAJ = -A$.

(iv) A is J-skew-symmetric if $JA^*J = -A$.

We say that A is *complex symmetric* (respectively *complex skew-symmetric*), if there is a conjugation J for which A is J-symmetric (respectively. J-skew-symmetric). Complex symmetric operators have recently been studied in considerable detail: see [4], [5], [6] and [17]. Of particular interest has been the question of what operators are complex symmetric. Our concern in this paper is rather to use the property of an operator being complex symmetric. Unfortunately, most of our results concern normal operators. These are known to be complex symmetric, but we believe that it may be

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possible to prove more general theorems requiring only complex symmetry, rather than normality.

Other symmetries of operators can be defined in a similar way, for example the notion of A being J-complex orthogonal.

Given an n-by-n matrix A, it is real if $\bar{A} = A$, symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$. Since $A^T = \bar{A}^*$, the above definitions are the natural generalizations to operators of these concepts.

In Section 1 we recall some properties of normal operators and conjugations. In Section 2 we consider equations such as $AB = BA$ and $AB = BA^*$, and we obtain a conclusion about the doubling of the spectrum of B.

This type of study has been done in the matrix situation by a number of authors. Goodson, Merino and Horn [11] considered the implications of the equations $AB = BA^T$, $AB = BA$ amongst others, for various types of matrices (see also [8] and [9]). In an earlier paper [13] it was shown that if both A and B are bounded real normal operators with $AB = B^*A$ (B invertible), then on the subspace $H_0 = \ker(B - B^*)^\perp$, the spectral multiplicity function of A takes only even values (so for example, the eigenvalues have to have even multiplicity). This was used to give a generalization of the notions of Hamiltonian and skew-Hamiltonian matrices to the infinite dimensional situation. Waterhouse [16] had shown the spectral doubling property of such matrices, and we were able to give infinite dimensional version of his results. In the current paper, using quite different methods we study spectral doubling type results for equations such as $AB = BA$ and $AB = BA^*$. It should be possible, but we have not yet achieved it, to use methods similar to those of this paper, to prove our earlier results.

The following is a brief history of these results: Given an invertible measure preserving transformation (i.m.p.t.) $T : X \to X$ defined on a non-atomic, separable probability space $(X, \mathscr{B}, \mu)$, there is a unitary operator $U_T : L^2(X, \mathscr{B}, \mu) \to L^2(X, \mathscr{B}, \mu)$ defined by $U_T f(x) = f(Tx)$. It was shown by Halmos and von Neumann that when T is an ergodic transformation having discrete spectrum, then T is conjugate to T^{-1} (there is an i.m.p.t. S with $ST = T^{-1}S$ and all such S have the property that $S^2 = I$, the identity map). This result was generalized in [7] to show that for any i.m.p.t. T having simple spectrum (the unitary operator U_T has multiplicity one) which is conjugate to its inverse via S has the property that $S^2 = I$. The question arose as to what happens when their exists a conjugation S between T and T^{-1} with $S^2 \neq I$. It was shown in [8] that in this case the spectrum of U_T has an even multiplicity function on the orthogonal complement of the subspace $\{f \in L^2(X, \mathscr{B}, \mu) : S^2 f = f\}$. Results about the multiplicity function of S^2 were given in [10]. Looking at the finite dimensional situation led to some very general results (see [9], [11]) concerning square matrices with properties such as $AB = BA$, $AB^T = BA$, $AB^* = BA$ when A is normal and real (or some similar property). The proofs were now matrix theoretic, very different to the original proofs. The question now arose as to what extent these results could be generalized to the infinite dimensional situation, and [13] and [14] are attempts in this direction. The operator U_T has the property that it preserves real valued functions, so that if J is the natural conjugation $J(f) = \bar{f}$, it commutes with J so is J-real. It is therefore natural to try to generalize these results to the case of an operator A that is real with respect to some conjugation J. If A is also normal, it is useful to know that A is also complex symmetric with respect to some other conjugation K. With these generalizations in hand it may be possible to return to ergodic theory and apply the results to the theory of joinings. A joining of two measure preserving transformations S and T is an $S \times T$ invariant measure which projects to μ on each coordinate. However, these can be represented as positive operators which intertwine with the induced unitary operators U_S and U_T. Self-joings have shown to be particularly interesting and useful.

1. Basic properties of normal operators and conjugations

The first part of the following lemma is the operator version of Sylvester's Theorem (see Bhatia [1]). We will make use of the second part which generalizes the first part in a particular aspect (due to Conway).

LEMMA 1. (a) *If F and G are bounded operators having disjoint spectra, then the equation*

$$FX - XG = C,$$

has a unique bounded solution X. When $C = 0$, this solution is $X = 0$.

(b) (Conway [2]) *If F and G are bounded normal operators having mutually singular scalar spectral measures with $FX = XG$ for some bounded operator X, then $X = 0$.*

The following is also well known and is used later.

LEMMA 2. (The Fuglede-Putnam Theorem) *Let $A, B : \mathscr{H} \to \mathscr{H}$ be bounded operators with A normal. Then*

(a) $BA = AB$ *if and only if* $BA^* = A^*B$.

(b) $BA = A^*B$ *if and only if* $BA^* = AB$.

We also need the following results, discussed in [4], [5]. In particular they show the ubiquitousness of being complex symmetric.

PROPOSITION 1. *If C and J are conjugations on $\mathscr{H}$, then $U = CJ$ is a unitary operator which is both J-symmetric and C-symmetric.*

PROPOSITION 2. *If A is a normal operator on $\mathscr{H}$, then there is a conjugation J on $\mathscr{H}$ for which A is J-symmetric, i.e., any normal operator is complex symmetric.*

The converse of Proposition 1 is also known to hold (see [5]), and was proved by Godic and Lučenko.

The following are also needed and may not be well known (see [13] for the proof of Proposition 3(a)):

PROPOSITION 3. *Let $A : \mathscr{H} \to \mathscr{H}$ be a bounded normal operator, and $J : \mathscr{H} \to \mathscr{H}$ a conjugation.*

(a) *If A is J-real, then $\sigma(A)$, the spectrum of A is symmetric about the real axis. In addition, A is unitarily equivalent to A^*.*

(b) *If A is J-imaginary, then $\sigma(A)$ is symmetric with respect to the imaginary axis.*

Proof. (b) Let $\lambda \in \rho(A)$, the resolvent of A, then $A - \lambda I$ is invertible. Now

$$J(A - \lambda I)J = -A - \bar{\lambda} I = -(A + \bar{\lambda} I),$$

so that $-\bar{\lambda} \in \rho(A)$. $\square$

PROPOSITION 4. *(a) If Q is normal, $Q = Q_1 \oplus Q_2$, $Q_i : H_i \to H_i$, $i = 1,2$ where Q_i and $-Q_i^*$ have scalar spectral measures which are mutually singular, with $JQ = -QJ$ for some conjugation J, then $J(H_1) = H_2$, $J(H_2) = H_1$.*

In this case we can write $J\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} J_1 g \\ J_2 f \end{pmatrix}$, where $J_1 : H_2 \to H_1$ and $J_2 : H_1 \to H_2$ are conjugate linear (not conjugations, but J_1J_2 is the identity on H_1, and J_2J_1 is the identity on H_2). In addition

$$J_1Q_2 = -Q_1J_1 \quad \text{and} \quad J_2Q_1 = -Q_2J_2.$$

(b) If instead Q_i and $-Q_i$ have mutually singular scalar spectral measures, where $JQ = -Q^*J$, we get the same conclusion except that

$$J_1Q_2 = -Q_1^*J_1 \quad \text{and} \quad J_2Q_1 = -Q_2^*J_2.$$

Proof. (a) Q_1 and Q_2 are normal, so from Proposition 2 they are both complex symmetric. Suppose that C_1 and C_2 are conjugations with $C_1Q_1^*C_1 = Q_1$ and $C_2Q_2^*C_2 = Q_2$. Set $C\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} C_1(f) \\ C_2(g) \end{pmatrix}$, then we see that $CQ^*C = Q$.

We are given that $JQ = -QJ$, so

$$CJQ = -CQJ = -Q^*CJ \quad \text{or} \quad UQ = -Q^*U,$$

where $U = CJ$ is unitary.

Decompose U conformally with Q as $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, then the equation $UQ = -Q^*U$ give

$$U_{11}Q_1 = -Q_1^*U_{11},\ U_{21}Q_1 = -Q_2^*U_{21} \text{ and } U_{12}Q_2 = -Q_1^*U_{12},\ U_{22}Q_2 = -Q_2^*U_{22}.$$

The mutual singularity of the scalar spectral measures of Q_i and $-Q_i^*$ implies that $U_{11} = 0$ and $U_{22} = 0$, so that

$$J\begin{pmatrix} f \\ g \end{pmatrix} = CU\begin{pmatrix} f \\ g \end{pmatrix} = C\begin{pmatrix} U_{12}g \\ U_{21}f \end{pmatrix} = \begin{pmatrix} C_1U_{12}g \\ C_2U_{21}f \end{pmatrix} = \begin{pmatrix} J_1 g \\ J_2 f \end{pmatrix},$$

where $J_1 = C_1U_{12}$ and $J_2 = C_2U_{21}$. In addition

$$JQ\begin{pmatrix} f \\ g \end{pmatrix} = -QJ\begin{pmatrix} f \\ g \end{pmatrix} \Rightarrow \begin{pmatrix} J_1Q_2g \\ J_2Q_1f \end{pmatrix} = -\begin{pmatrix} Q_1J_1g \\ Q_2J_2f \end{pmatrix},$$

and (a) follows.

(b) $JQ = -Q^*J \Rightarrow CJQ = -CQ^*J = -QCJ$, or $UQ = -QU$, and proceed as in (a). $\square$

LEMMA 3. *Let $J : H_1 \oplus H_2 \to H_1 \oplus H_2$ be a conjugation of the form $J\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} J_1g \\ J_2f \end{pmatrix}$, where $J_1 : H_2 \to H_1$ and $J_2 : H_1 \to H_2$ are anti-linear operators.*

If $B = B_1 \oplus B_2$ where $B_i : H_i \to H_i$, $i = 1,2$ then

(a) *$JB = BJ$ implies $J_1B_2 = B_1J_1$ and $J_2B_1 = B_2J_2$,*

(b) *$JB^* = BJ$ implies $J_1B_2^* = B_1J_1$ and $J_2B_1^* = B_2J_2$,*

(c) *$JB = -BJ$ implies $J_1B_2 = -B_1J_1$ and $J_2B_1 = -B_2J_2$,*

(d) *$JB^* = -BJ$ implies $J_1B_2^* = -B_1J_1$ and $J_2B_1^* = -B_2J_2$,*

Proof. (a) We have $JB\begin{pmatrix} f \\ g \end{pmatrix} = BJ\begin{pmatrix} f \\ g \end{pmatrix} \Rightarrow J\begin{pmatrix} B_1f \\ B_2g \end{pmatrix} = B\begin{pmatrix} J_1g \\ J_2f \end{pmatrix} \Rightarrow \begin{pmatrix} J_1B_2g \\ J_2B_1f \end{pmatrix} = \begin{pmatrix} B_1J_1g \\ B_2J_2f \end{pmatrix}$ and (a) follows. The other statements are proved in a similar way. $\square$

LEMMA 4. *Let $J : H_1 \oplus H_2 \to H_1 \oplus H_2$ be a conjugation of the form $J\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} J_1g \\ J_2f \end{pmatrix}$, where $J_1 : H_2 \to H_1$ and $J_2 : H_1 \to H_2$ are anti-linear operators.*

If $B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$ where $B_1 : H_2 \to H_1$, $B_2 : H_1 \to H_2$, then

(a) *$JB = BJ$ implies $J_1B_2 = B_1J_2$ and $J_2B_1 = B_2J_1$,*

(b) *$JB^* = BJ$ implies $J_1B_1^* = B_1J_2$ and $J_2B_2^* = B_2J_1$,*

(c) *$JB = -BJ$ implies $J_1B_2 = -B_1J_2$ and $J_2B_1 = -B_2J_1$,*

(d) *$JB^* = -BJ$ implies $J_1B_1^* = -B_1J_2$ and $J_2B_2^* = -B_2J_1$.*

Proof. The proofs follow the same method as Lemma 3. $\square$

2. The spectrum pairing theorems

Our main theorems list different possibilities which arise for the spectrum of an operator B which commutes or "skew-commutes" with a normal operator A which is real or complex skew-symmetric. These theorems give detailed information about the structure of the operator B when B is real, complex symmetric or satisfies some similar condition.

THEOREM 1. *Let $A,B:\mathscr{H}\to\mathscr{H}$ be bounded normal operators on a complex Hilbert space and $J:\mathscr{H}\to\mathscr{H}$ a conjugation.*

If either (1) $AB=BA$ *or* (2) $AB=-BA^*$, *then the subspace*

$$H_0=\ker(A-A^*)^{\perp}=\overline{\operatorname{ran}(A-A^*)},$$

is both A and B reducing. Set $B_0=B|H_0$, the restriction of B to H_0.

Suppose (i) *A is J-real, or* (ii) *A is J-skew-symmetric, then*

(a) *if B is J-symmetric, $B_0\cong B_1\oplus B_1$ for some operator B_1,*

(b) *if B is J-real, $B_0\cong B_1\oplus B_1^*$, for some operator B_1,*

(c) *if B is J-skew-symmetric, $B_0\cong B_1\oplus -B_1$ for some operator B_1,*

(d) *if B is J-imaginary, $B_0\cong B_1\oplus -B_1^*$ for some operator B_1.*

EXAMPLES. 1. Suppose that $A:L^2(X,\mu)\to L^2(X,\mu)$ preserves real valued functions, then $A(\overline{f})=\overline{A(f)}$, so that A commutes with the complex conjugation map $J(f)=\overline{f}$, and A is J-real. Such operators A arise from measure preserving transformations $T:X\to X$ defined by $Af(x)=f(Tx)$, and are unitary when T is invertible. Although normal operators are always complex symmetric with respect to some conjugation, it seems to be very natural to study the conjugation of complex conjugation arising from such operators as this is most closely related to the finite dimensional situation where the natural conjugation is ordinary complex conjugation of a matrix. The above theorem gives information about the spectrum of a normal operator B that commutes with A when it preserves real functions or has some other type of symmetry property.

2. Let A and B be n-by-n real normal matrices with $AB=BA$ or $AB=-BA^*$. If $B=B^T$, then the eigenvalues of B occur with even multiplicity on the orthogonal complement of the subspace $\{x\in\mathbb{C}^n : Ax=A^Tx\}$.

REMARKS. 1. The normality of A in the above theorem is required so that we can use the Fuglede-Putnam Theorem to show $A^*B=BA^*$, and could be replaced by this condition. The normality of B is used to show that if $B=B_1\oplus B_2$, then each of B_1 and B_2 is normal and hence complex symmetric, so there exist conjugations $J_i:\mathscr{H}\to\mathscr{H}$ such that $J_iB_i^*J_i=B_i$, $i=1,2$. It is possible that the condition of normality of B could be replaced by B being complex symmetric.

2. It follows from the above theorem that if $AB=BA$ or $AB=-BA^*$, where A is J-real and normal, and B is normal, J-symmetric and having multiplicity one, then $A=A^*$.

3. If A and B are real normal matrices, Theorem 1 tells us that there is a doubling of the eigenvalues of B (all of the eigenvalues occur with even multiplicity) on the subspace H_0. This generalizes results from [11] in that they apply to any conjugation. However, although the results of [11] only apply to the usual complex conjugation, they are more general in other respects. For example, it is not required that B be a normal matrix (and in fact the Jordan form of B is doubled).

In order to prove Theorem 1, we need the following lemma:

LEMMA 5. *Let* $A,B : \mathscr{H} \to \mathscr{H}$ *be bounded and normal, and* $J : \mathscr{H} \to \mathscr{H}$ *a conjugation.*

If either (1) $AB = BA$ *or* (2) $AB = -BA^*$, *then* $H_0 = \ker(A-A^*)^\perp$ *is both* A *and* B *reducing.*

Suppose also that (i) A *is* J*-real, or* (ii) A *is* J*-skew-symmetric, then there is a decomposition* $H_0 = H_1 \oplus H_2$ *with operators* $B_1 : H_1 \to H_1$ *and* $B_2 : H_2 \to H_2$, $B_0 \cong B_1 \oplus B_2$, $(B_0 = B|H_0)$, *where* $J(H_1) = H_2$, $J(H_2) = H_1$.

Proof of Lemma 5. We first note that H_0 is A reducing since it is both A and A^*-invariant. Suppose that $f \in H_0^\perp$, then $(A-A^*)Bf = (AB-A^*B)f = (BA-BA^*)f = 0$, so $B(H_0^\perp) \subseteq H_0^\perp$ (using Fuglede-Putnam). Similarly $B^*(H_0^\perp) \subseteq H_0^\perp$.

We split the rest of the proof into different cases:

Case 1. Suppose that $AJ = JA$ and $AB = BA$. Write $A = P + iQ$ where

$$P = \frac{A+A^*}{2}, \quad Q = \frac{A-A^*}{2i},$$

then $P = P^*$ and $Q = Q^*$, so the spectra $\sigma(P), \sigma(Q)$ are contained in the real axis. We can check that P is J-real, and

$$JQ = \frac{JA - JA^*}{-2i} = -\frac{AJ - A^*J}{2i} = -\frac{A-A^*}{2i}J = -QJ,$$

so that Q is J-imaginary. It follows from Proposition 3 that the spectrum of Q, $\sigma(Q)$, is symmetric about the origin and contained in $\mathbb{R}$.

Without loss of generality, we may assume that

$$H_0^\perp = \ker(A-A^*) = \ker(Q) = \{0\},$$

so the range of Q is dense (since $\overline{\operatorname{ran}(Q)} = \ker(Q)^\perp$) and the singleton $\{0\}$ has spectral measure zero.

It follows that we can assume $Q = Q_1 \oplus Q_2$, where $\sigma(Q_1) \subseteq (0,\infty)$ and $\sigma(Q_2) \subseteq (-\infty, 0)$.

Since $J(Q_1 \oplus Q_2) = -(Q_1 \oplus Q_2)J$, J maps the space associated with Q_1 onto the space associated with Q_2, so we must have (using Proposition 4 and Lemma 3), $J\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} J_1 g \\ J_2 f \end{pmatrix}$, so that

$$J_1 Q_2 = -Q_1 J_1, \quad J_2 Q_1 = -Q_2 J_2.$$

Now $AB = BA$ implies that $PB = BP$ and $QB = BQ$ (using Fuglede-Putnam). Decompose B conformally with Q, say $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, then

$$\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

This gives

$$Q_1 B_{11} = B_{11} Q_1, \quad Q_1 B_{12} = B_{12} Q_2, \quad Q_2 B_{21} = B_{21} Q_1, \quad Q_2 B_{22} = B_{22} Q_2,$$

Since Q_1 and Q_2 have mutually singular spectral measures, we must have $B_{12}=0$, $B_{21}=0$ (from Conway's Lemma), so $B_0\cong B_1\oplus B_2$ say, where $B_1=B_{11}:H_1\to H_1$ and $B_2=B_{22}:H_2\to H_2$. This completes the proof in Case 1.

Case 2. Assume that $AB=BA$ and $JA^*J=-A$. As before H_0 is both A and B reducing and we write $A=P+iQ$ where $P=P^*$, $Q=Q^*$. Also

$$JQ=\frac{JA-JA^*}{-2i}=-\frac{-A^*J+AJ}{2i}=-\frac{A-A^*}{2i}J=-QJ,$$

and $QB=BQ$, so the rest of the proof continues as before.

Case 3. In this case we assume $AB=-BA^*$ and $AJ=JA$. Again H_0 is both A and B reducing and if we set $A=P+iQ$, then again $JQ=-QJ$.

In this case $AB=-BA^*$ implies that

$$QB=\frac{(A-A^*)B}{2i}=\frac{AB-A^*B}{2i}=\frac{-BA^*+AB}{2i}=BQ,$$

so the same reasoning as previously applies.

Case 4. If $AB=-BA^*$ and $JA^*J=-A$, then we again have $QB=BQ$ and $JQ=-QJ$ as above. This completes the proof of the lemma. □

Proof of Theorem 1. (a) We have shown that $B_0\cong B_1\oplus B_2$ where $B_i:H_i\to H_i$, $i=1,2$. Assume B is J-symmetric: $JB^*J=B$, then from Proposition 4(b) we have

$$J_1B_2^*=B_1J_1 \quad\text{and}\quad J_2B_1^*=B_2J_2,$$

If B is also normal, then B_0 is normal and each of B_1 and B_2 are normal and so they are complex-symmetric. From Proposition 2 there exists conjugations J_0 and J_0' with

$$J_0B_1=B_1^*J_0 \quad\text{and}\quad J_0'B_2=B_2^*J_0'.$$

Combining these facts gives

$$J_0J_1B_2^*=J_0B_1J_1=B_1^*J_0J_1,$$

where $J_0J_1:H_2\to H_1$ is unitary, $B_1^*\cong B_2^*$ so in particular $B_0\cong B_1\oplus B_1$.

(b) B is J-real, so $JB=BJ$. Proposition 4(i)(a) then gives

$$J_1B_2=B_1J_1 \text{ and } J_2B_1=B_2J_2.$$

We also have

$$J_0B_1=B_1^*J_0 \quad\text{and}\quad J_0'B_2=B_2^*J_0',$$

so combining these gives $J_0J_1B_2=J_0B_1J_1=B_1^*J_0J_1$, or $B_1\cong B_2^*$ and $B_0\cong B_1\oplus B_1^*$.

(c) $JB^*=-BJ$, so by Proposition 4(i)(d) we have

$$J_1B_2^*=-B_1J_1 \text{ and } J_2B_1^*=-B_2J_2,$$

so that $J_0'J_2B_1^* = -J_0'B_2J_2 = B_2^*J_0'J_2$, giving $B_1^* \cong -B_2^*$ and $B_1 \cong -B_2$, so $B_0 \cong B_1 \oplus -B_1$.

(d) $JB = -BJ$, so Proposition 4(i)(c) gives

$$J_1B_2 = -B_1J_1 \text{ and } J_1B_1 = -B_2J_2,$$

and in this case we obtain $J_0J_1B_2 = -J_0B_1J_1 = -B_1^*J_0J_1$, so that $B_2 \cong -B_1^*$ or $B_0 \cong B_1 \oplus -B_1^*$. □

THEOREM 2. *Let $A, B : \mathscr{H} \to \mathscr{H}$ be bounded normal operators on a complex Hilbert space and $J : \mathscr{H} \to \mathscr{H}$ a conjugation.*

Suppose that (1) $AB = -BA$ *or* (2) $AB = BA^*$, *then*

$$H_0 = \ker(A - A^*)^{\perp} = \overline{\operatorname{ran}(A - A^*)},$$

is both an A and B reducing subspace of $\mathscr{H}$. Set $B_0 = B|H_0$, the restriction of B to H_0.

Suppose

(i) *A is J-real or* (ii) *A is J-skew-symmetric then*

(a) *if B is J-real, then* $B_0 \cong \begin{pmatrix} 0 & B_1 \\ J_1B_1J_2 & 0 \end{pmatrix}$, *where $H_0 = H_1 \oplus H_2$, $B_1 : H_2 \to H_1$,*

(b) *with no conditions on B,* $B_0 = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$, $B_0^2 \cong \begin{pmatrix} B_1B_2 & 0 \\ 0 & B_2B_1 \end{pmatrix} = F \oplus G$, *on $H_1 \oplus H_2$, where $\sigma(F) \setminus \{0\} = \sigma(G) \setminus \{0\}$, and if B is invertible, J-symmetric or J-skew-symmetric, then $B_0^2 \cong F \oplus F$,*

(c) *if B is Hermitian,* $B_0 \cong \begin{pmatrix} 0 & B_1 \\ B_1^* & 0 \end{pmatrix}$, *for some operator B_1,*

(d) *if B is J-imaginary, then* $B_0 \cong \begin{pmatrix} 0 & B_1 \\ -J_1B_1J_2 & 0 \end{pmatrix}$, *for some operator B_1.*

To prove Theorem 2 we need the following lemma:

LEMMA 6. *Suppose that $A, B : \mathscr{H} \to \mathscr{H}$ are bounded normal operators, and $J : \mathscr{H} \to \mathscr{H}$ is a conjugation. If either* (i) $AB = -BA$ *or* (ii) $AB = BA^*$, *then the subspace $H_0 = \ker(A - A^*)^{\perp}$ is both A and B reducing. Set $B_0 = B|H_0$.*

(a) *If in addition A is J-real, or* (b) *A is J-skew-symmetric, then there is a decomposition $H_0 = H_1 \oplus H_2$ with bounded operators $B_1 : H_2 \to H_1$ and $B_2 : H_1 \to H_2$,* $B_0 \cong \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$.

Proof. As before, H_0 is A and B reducing. Set $A = P + iQ$ where

$$P = \frac{A + A^*}{2}, \quad Q = \frac{A - A^*}{2i},$$

so that $P = P^*$, $Q = Q^*$ and $JQ = -QJ$, and $\sigma(Q)$ is real and is symmetrical about the origin and $Q = Q_1 \oplus Q_2$ where $\sigma(Q_1) \subseteq (0, \infty)$, $\sigma(Q_2) \subseteq (-\infty, 0)$ (where we again may assume that $\ker(Q) = \ker(A - A^*) = \{0\}$.

Case 1. $JA = AJ$ and $AB = -BA$ implies that $PB = -BP$ and $QB = -BQ$, so we proceed as before, decomposing B conformally with $Q = Q_1 \oplus Q_2$, but this time (using Q_i and $-Q_i$, $i = 1,2$ having mutually singular scalar spectral measures), the equation $QB = -BQ$ gives B_0 of the form $B_0 = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$.

Case 2. $JA = AJ$ and $AB = BA^*$ gives $JQ = -QJ$ and $QB = -BQ$ and we proceed as in Case 1.

Case 3. $JA^*J = -A$ and $AB = -BA$, then again we get $JQ = -QJ$ and $AB = -BA$. Similarly in the last case where $JA^*J = -A$ and $AB = BA^*$. $\square$

Proof of Theorem 2. From Lemma 6 we have $B_0 = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$.

(a) If $JB = BJ$, then Lemma 4 gives

$$J_1B_2 = B_1J_2 \text{ and } J_2B_1 = B_2J_1.$$

In particular $B_2 = J_1B_1J_2$, so that $B_0 = \begin{pmatrix} 0 & B_1 \\ J_1B_1J_2 & 0 \end{pmatrix}$.

(b) $B_0^2 = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}^2 = \begin{pmatrix} B_1B_2 & 0 \\ 0 & B_2B_1 \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix} = F \oplus G$ say. Now B is normal, so B_0^2 is normal and in particular $F = B_1B_2$ and $G = B_2B_1$ are normal. It is well known that $\sigma(F) \setminus \{0\} = \sigma(G) \setminus \{0\}$. If B is invertible, $FB_1 = (B_1B_2)B_1 = B_1(B_2B_1) = B_1G$ and B_1 invertible and B_1B_2, B_2B_1 normal implies $F \cong G$, so $B_0^2 \cong F \oplus F$.

In the case that B is J-symmetric, $JB^* = BJ$ and $B^*J = JB$, then we see that

$$J_1B_1^* = B_1J_2, \;\; J_2B_2^* = B_2J_1 \;\text{ and }\; B_2^*J_2 = J_1B_2, \;\; B_1^*J_1 = J_2B_1.$$

It follows that they are complex symmetric, so there are conjugations C_1 and C_2:

$$C_1(B_1B_2)^* = (B_1B_2)C_1, \;\text{ and }\; C_2(B_2B_1)^* = (B_2B_1)C_2.$$

Then

$$J_2C_1(B_1B_2)^* = J_2(B_1B_2)C_1 = B_1^*J_1B_2C_1 = B_1^*B_2^*J_2C_1 = (B_2B_1)^*J_2C_1.$$

This shows that B_1B_2 and B_2B_1 are unitarily equivalent via $U = J_2C_1$, and the result follows. The case where B is J-skew-symmetric is similar.

(c) If $B = B^*$, then clearly $B_1 = B_2^*$ and $B_2 = B_1^*$ and the result follows.

(d) This is similar to (a). $\square$

REMARKS. In the finite dimensional case of the previous theorem, for (a) and (d) it can be shown that B_0 is similar to a matrix of the form $-R \oplus R$ where R is a real. For (c), B_0 is similar to $\Sigma \oplus -\Sigma$ for some matrix Σ arising from a singular value decomposition (see [11]). We conjecture that there are infinite dimensional versions of these results.

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