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A RADON–NIKODYM TYPE THEOREM FOR α–COMPLETELY POSITIVE MAPS ON GROUPS

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Abstract. We show that an operator valued α-completely positive map on a group G is given by a unitary representation of G on a Krein space which satisfies certain conditions. Moreover, two such of unitary representations, which are unitarily equivalent, define the same α-completely positive map. Also we introduce a pre-order relation on the collection of α-completely positive maps on a group and we characterize this relation in terms of the unitary representation associated to each map.

1. Introduction

The study of completely positive maps is motivated by their applications in the theory of quantum measurements, operational approach to quantum mechanics, quantum information theory, where operator-valued completely positive maps on C^*-algebras are used as a mathematical model for quantum operations, and quantum probability [8, 7, 6]. On the other hand, the notion of locality in the Wightman formulation of gauge quantum field theory conflicts with the notion of positivity. To avoid this, Jakobczyk and Strocchi [6] introduced the concept of α-positivity. Motivated by the notions of α-positivity and P-functional [5, 1], recently, Heo, Hong and Ji [4] introduced the notion of α-completely positive map between C^*-algebras, and they provided a Kasparov-Stinespring-Gelfand-Naimark-Segal type construction for α-completely positive maps.

In [2], Heo introduced the notion of α-completely positive map from a group G to a C^*-algebra A. By analogy with the KSGNS construction for α-completely positive maps on C^*-algebras [4], he associated to an α-completely positive map φ from a group G to the C^*-algebra $L(X)$ of all adjointable operators on a Hilbert C^*-module X a quadruple $(\pi_\varphi, X_\varphi, J_\varphi, V_\varphi)$ consisting of a Krein C^*-module (X_φ, J_φ), a J_φ-unitary representation π_φ of G on X_φ and a bounded linear operator V_φ such that the linear space generated by $\{\pi_\varphi(g)V_\varphi x; g\in G, x\in X\}$ is dense in X_φ, $V_\varphi^*\pi_\varphi(g)^*\pi_\varphi(g')V_\varphi = V_\varphi^*\pi_\varphi(\alpha(g^{-1})g')V_\varphi$ for all $g,g'\in G$ and $\varphi(g) = V_\varphi^*\pi_\varphi(g)V_\varphi$ for all $g\in G$. But, in general, a such of quadruple does not define an α-completely positive map (Remark 2.6). In this paper, we consider α-completely positive maps from a group G to $L(\mathscr{H})$, the C^*-algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$, and we show

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that under some conditions, a quadruple $(\pi, \mathscr{H}, \mathscr{J}, V)$ consisting of a Krein space $(\mathscr{H}, \mathscr{J})$, a $\mathscr{J}$-unitary representation π of G on $\mathscr{H}$ and a bounded linear operator V defines an α-completely positive map and we associate to each α-completely positive map a such of quadruple, that is unique up to unitary equivalence. In Section 3, we prove a Radon-Nikodym theorem type for α-completely positive maps on groups.

2. Stinespring type theorem for α-completely positive maps

Let G be a (topological) group with an involution α (that is, a (continuous) map $\alpha : G \to G$ such that $\alpha^2 = \mathrm{id}_G, \alpha(e) = e$ and $\alpha\left(g^{-1}\right) = \alpha(g)^{-1}$ for all $g \in G$) and $\mathscr{H}$ a Hilbert space.

DEFINITION 2.1. [2, Definition 2.1] A map $\varphi : G \to L(\mathscr{H})$ is α-completely positive if:

1. $\varphi(\alpha(g_1)\alpha(g_2)) = \varphi(\alpha(g_1 g_2)) = \varphi(g_1 g_2)$ for all $g_1, g_2 \in G$;
2. for all $g_1, \dots, g_n \in G$, the matrix $\left[\varphi\left(\alpha(g_i)^{-1} g_j\right)\right]_{i,j=1}^n$ is positive in $L(\mathscr{H})$;
3. there is $K > 0$, such that
$$\left[\varphi(g_i)^* \varphi(g_j)\right]_{i,j=1}^n \leqslant K \left[\varphi\left(\alpha(g_i)^{-1} g_j\right)\right]_{i,j=1}^n$$
for all $g_1, \dots, g_n \in G$;
4. for all $g \in G$, there is $M(g) > 0$ such that
$$\left[\varphi\left(\alpha(gg_i)^{-1} gg_j\right)\right]_{i,j=1}^n \leqslant M(g) \left[\varphi\left(\alpha(g_i)^{-1} g_j\right)\right]_{i,j=1}^n$$
for all $g_1, \dots, g_n \in G$.

REMARK 2.2. Let $\varphi : G \to L(\mathscr{H})$ be an α-completely positive map. Then:

1. $\varphi(\alpha(g)) = \varphi(g)$ for all $g \in G$;
2. $\varphi\left(\alpha\left(g^{-1}\right)\right) = \varphi(g)^*$ for all $g \in G$;
3. $\varphi\left(g^{-1}\right) = \varphi(g)^*$ for all $g \in G$.

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{J}$ a bounded linear operator on $\mathscr{H}$ such that $\mathscr{J} = \mathscr{J}^* = \mathscr{J}^{-1}$. Then we can define an indefinite inner product by $[x, y] = \langle \mathscr{J} x, y \rangle$. The pair $(\mathscr{H}, \mathscr{J})$ is called a Krein space. A representation of G on the Krein space $(\mathscr{H}, \mathscr{J})$ is a morphism $\pi : G \to L(\mathscr{H})$. A $\mathscr{J}$-unitary representation of G on the Krein space $(\mathscr{H}, \mathscr{J})$ is a representation π such that $\pi\left(g^{-1}\right) = \mathscr{J} \pi(g)^* \mathscr{J}$ for all $g \in G$ and $\pi(e) = \mathrm{id}_{\mathscr{H}}$. If π is a representation of G on $\mathscr{H}$, $[\pi(G)\mathscr{H}]$ denotes the closed linear subspace of $\mathscr{H}$ generated by $\{\pi(g)\xi; g \in G, \xi \in \mathscr{H}\}$.

THEOREM 2.3. [2, Theorem 2.2] *Let $\varphi : G \to L(\mathscr{H})$ be an α-completely positive map. Then there are a Krein space $(\mathscr{H}_\varphi, \mathscr{J}_\varphi)$, a $\mathscr{J}_\varphi$-unitary representation π_φ of G on $(\mathscr{H}_\varphi, \mathscr{J}_\varphi)$ and a bounded linear operator $V_\varphi : \mathscr{H} \to \mathscr{H}_\varphi$ such that*

1. $\varphi(g) = V_\varphi^* \pi_\varphi(g) V_\varphi$ *for all* $g \in G$;
2. $[\pi_\varphi(G) V_\varphi \mathscr{H}] = \mathscr{H}_\varphi$;
3. $V_\varphi^* \pi_\varphi(g)^* \pi_\varphi(g') V_\varphi = V_\varphi^* \pi_\varphi(\alpha(g^{-1}) g') V_\varphi$ *for all* $g, g' \in G$.

The quadruple $(\pi_\varphi, \mathscr{H}_\varphi, \mathscr{J}_\varphi, V_\varphi)$ is called the minimal Naimark -KSGNS dilation of φ [2].

REMARK 2.4. If $(\pi_\varphi, \mathscr{H}_\varphi, \mathscr{J}_\varphi, V_\varphi)$ is the minimal Naimark -KSGNS dilation of φ in the sense of Heo, then $(\pi_\varphi, \mathscr{H}_\varphi, \mathscr{J}_\varphi, W)$, where $W = \mathscr{J}_\varphi \; V_\varphi$, is a minimal Naimark -KSGNS dilation of φ too. Indeed, we have:

1. $\varphi(g) = \varphi(g^{-1})^* = (V_\varphi^* \pi_\varphi(g^{-1}) V_\varphi)^* = (V_\varphi^* \mathscr{J}_\varphi \pi_\varphi(g)^* \mathscr{J}_\varphi V_\varphi)^*$
   $= (W^* \pi_\varphi(g)^* W)^* = W^* \pi_\varphi(g) W$ for all $g \in G$.
2. Since $V_\varphi^* \pi_\varphi(g)^* \pi_\varphi(g') V_\varphi = V_\varphi^* \pi_\varphi(\alpha(g^{-1}) g') V_\varphi$ for all $g, g' \in G$, and $[\pi_\varphi(G) V_\varphi \mathscr{H}] = \mathscr{H}_\varphi$, we have
   $$V_\varphi^* \pi_\varphi(g)^* = V_\varphi^* \pi_\varphi(\alpha(g^{-1}))$$
   for all $g \in G$, and then
   $$\pi_\varphi(g) V_\varphi = \mathscr{J}_\varphi \pi_\varphi(\alpha(g)) \mathscr{J}_\varphi V_\varphi$$
   for all $g \in G$. Then
   $$\begin{aligned} [\pi_\varphi(G) W \mathscr{H}] &= \mathscr{J}_\varphi [\mathscr{J}_\varphi \pi_\varphi(\alpha(G)) V_\varphi \mathscr{H}] = \mathscr{J}_\varphi [\pi_\varphi(G) V_\varphi \mathscr{H}] \\ &= \mathscr{J}_\varphi \mathscr{H}_\varphi = \mathscr{H}_\varphi. \end{aligned}$$
3. Let $g, g' \in G$. Then
   $$\begin{aligned} W^* \pi_\varphi(g)^* \pi_\varphi(g') W &= V_\varphi^* \mathscr{J}_\varphi \pi_\varphi(g)^* \pi_\varphi(g') \mathscr{J}_\varphi V_\varphi \\ &= V_\varphi^* \pi_\varphi(g^{-1}) \mathscr{J}_\varphi \pi_\varphi(g') \mathscr{J}_\varphi V_\varphi \\ &= V_\varphi^* \pi_\varphi(g^{-1}) \pi_\varphi(\alpha(g')) V_\varphi \\ &= V_\varphi^* \pi_\varphi(g^{-1} \alpha(g')) V_\varphi = \varphi(g^{-1} \alpha(g')) = \varphi(\alpha(g^{-1}) g') \\ &= \varphi(g'^{-1} \alpha(g))^* = (V_\varphi^* \pi_\varphi(g'^{-1} \alpha(g)) V_\varphi)^* \\ &= \left(V_\varphi^* \mathscr{J}_\varphi \pi_\varphi(\alpha(g^{-1}) g')^* \mathscr{J}_\varphi V_\varphi\right)^* \\ &= \left(W^* \pi_\varphi(\alpha(g^{-1}) g')^* W\right)^* = W^* \pi_\varphi(\alpha(g^{-1}) g') W. \end{aligned}$$

REMARK 2.5. We remark that $\mathscr{J}_{\varphi}\pi_{\varphi}(g)V_{\varphi} = \pi_{\varphi}(\alpha(g))V_{\varphi}$ for all $g \in G$, if and only if, $V_{\varphi}^{*}\pi_{\varphi}(g)^{*}\pi_{\varphi}(g')V_{\varphi} = V_{\varphi}^{*}\pi_{\varphi}\left(\alpha\left(g^{-1}\right)g'\right)V_{\varphi}$ for all $g, g' \in G$ and $\mathscr{J}_{\varphi}V_{\varphi} = V_{\varphi}$.

Indeed, if $\mathscr{J}_{\varphi}\pi_{\varphi}(g)V_{\varphi} = \pi_{\varphi}(\alpha(g))V_{\varphi}$ for all $g \in G$, then $\mathscr{J}_{\varphi}V_{\varphi} = V_{\varphi}$ and

$$\begin{aligned} V_{\varphi}^{*}\pi_{\varphi}(g)^{*}\pi_{\varphi}(g')V_{\varphi} &= V_{\varphi}^{*}\mathscr{J}_{\varphi}\pi_{\varphi}\left(g^{-1}\right)\mathscr{J}_{\varphi}\pi_{\varphi}(g')V_{\varphi} \\ &= V_{\varphi}^{*}\pi_{\varphi}\left(g^{-1}\right)\pi_{\varphi}\left(\alpha\left(g'\right)\right)V_{\varphi} = V_{\varphi}^{*}\pi_{\varphi}\left(g^{-1}\alpha\left(g'\right)\right)V_{\varphi} \\ &= \varphi\left(g^{-1}\alpha\left(g'\right)\right) = \varphi\left(\alpha\left(g^{-1}\right)g'\right) \\ &= V_{\varphi}^{*}\pi_{\varphi}\left(\alpha\left(g^{-1}\right)g'\right)V_{\varphi} \end{aligned}$$

for all $g, g' \in G$.

Conversely, if $V_{\varphi}^{*}\pi_{\varphi}(g)^{*}\pi_{\varphi}(g')V_{\varphi} = V_{\varphi}^{*}\pi_{\varphi}\left(\alpha\left(g^{-1}\right)g'\right)V_{\varphi}$ for all $g, g' \in G$, then

$$\pi_{\varphi}(g)V_{\varphi} = \mathscr{J}_{\varphi}\pi_{\varphi}(\alpha(g))\mathscr{J}_{\varphi}V_{\varphi} \quad \text{(Remark 2.4 (2))}$$

for all $g \in G$, and taking into account that $\mathscr{J}_{\varphi}V_{\varphi} = V_{\varphi}$, we have

$$\mathscr{J}_{\varphi}\pi_{\varphi}(g)V_{\varphi} = \pi_{\varphi}(\alpha(g))\mathscr{J}_{\varphi}V_{\varphi} = \pi_{\varphi}(\alpha(g))V_{\varphi}$$

for all $g \in G$.

REMARK 2.6. If G is a group with an involution α, π is a $\mathscr{J}$-unitary representation of G on $(\mathscr{K}, \mathscr{J})$ and V a bounded linear operator from a Hilbert space $\mathscr{H}$ to $\mathscr{K}$ such that $[\pi(G)V\mathscr{H}] = \mathscr{K}$ and $V^{*}\pi(g)^{*}\pi(g')V = V^{*}\pi\left(\alpha\left(g^{-1}\right)g'\right)V$ for all $g, g' \in G$, then the map $\varphi : G \to L(\mathscr{H})$ defined by $\varphi(g) = V^{*}\pi(g)V$ is not in general an α-completely positive map.

EXAMPLE. Let $\mathbb{Z}$ be the additive group of integers and $\alpha(n) = -n$ an involution of $\mathbb{Z}$. The map $\mathscr{J} : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $\mathscr{J}(x,y) = (y,x)$ is a bounded linear operator such that $\mathscr{J} = \mathscr{J}^{*} = \mathscr{J}^{-1}$, the map $\pi : \mathbb{Z} \to L(\mathbb{C}^2)$ defined $\pi(n)(x,y) = (e^{n}x, e^{-n}y)$ is a $\mathscr{J}$-unitary representation of $\mathbb{Z}$ on $\left(\mathbb{C}^2, \mathscr{J}\right)$, and the map $V : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $V(x,y) = (x-y, y)$ is a bounded linear operator. It is easy to verify that $\left[\pi(\mathbb{Z})V\mathbb{C}^2\right] = \mathbb{C}^2$ and $V^{*}\pi(n)^{*}\pi(m)V = V^{*}\pi(n+m)V = V^{*}\pi(\alpha(-n)+m)V$ for all $n, m \in \mathbb{Z}$, but $\varphi : \mathbb{Z} \to L(\mathbb{C}^2)$ defined by $\varphi(n) = V^{*}\pi(n)V$ is not α-completely positive, because $\varphi(n) \neq \varphi(-n) = \varphi(\alpha(n))$.

PROPOSITION 2.7. *Let G be a group with an involution α, π a $\mathscr{J}$-unitary representation of G on $(\mathscr{K}, \mathscr{J})$ and V a bounded linear operator from a Hilbert space $\mathscr{H}$ such that $[\pi(G)V\mathscr{H}] = \mathscr{K}$ and $\mathscr{J}\pi(g)V = \pi(\alpha(g))V$ for all $g \in G$. Then the map $\varphi : G \to L(\mathscr{H})$ defined by $\varphi(g) = V^{*}\pi(g)V$ is an α-completely positive map.*

Proof. It is similar to the proof of Proposition 3.1. □

THEOREM 2.8. *Let $\varphi : G \to L(\mathscr{H})$ be an α-completely positive map.*

1. *There are a Krein space $\left(\mathscr{H}_{\varphi}, \mathscr{J}_{\varphi}\right)$, a $\mathscr{J}_{\varphi}$-unitary representation π_{φ} of G on $\left(\mathscr{H}_{\varphi}, \mathscr{J}_{\varphi}\right)$ and a bounded linear operator $V_{\varphi} : \mathscr{H} \to \mathscr{H}_{\varphi}$ such that*

(a) $\varphi(g) = V_\varphi^* \pi_\varphi(g) V_\varphi$ *for all* $g \in G$;

(b) $\left[\pi_\varphi(G) V_\varphi \mathscr{H}\right] = \mathscr{H}_\varphi$;

(c) $\mathscr{J}_\varphi \pi_\varphi(g) V_\varphi = \pi_\varphi(\alpha(g)) V_\varphi$ *for all* $g \in G$.

2. *If* π *is a* $\mathscr{J}$*-unitary representation of* G *on a Krein space* $(\mathscr{K}, \mathscr{J})$ *and* $V : \mathscr{H} \to \mathscr{K}$ *is a bounded linear operator such that*

(a) $\varphi(g) = V^* \pi(g) V$ *for all* $g \in G$;

(b) $[\pi(G) V \mathscr{H}] = \mathscr{K}$;

(c) $\mathscr{J} \pi(g) V = \pi(\alpha(g)) V$ *for all* $g \in G$,

then there is a unitary operator $U : \mathscr{H}_\varphi \to \mathscr{K}$ *such that*

i. $U \mathscr{J}_\varphi = \mathscr{J} U$;

ii. $U V_\varphi = V$;

iii. $U \pi_\varphi(g) = \pi(g) U$ *for all* $g \in G$.

Proof. (1). We will give a sketch of proof (see [2, Theorem 2.2] and Remark 2.5 for the detailed proof). Let $\mathscr{F}(G, \mathscr{H})$ be the vector space of all functions from G to $\mathscr{H}$ with finite support. The map $\langle \cdot, \cdot \rangle : \mathscr{F}(G, \mathscr{H}) \times \mathscr{F}(G, \mathscr{H}) \to \mathbb{C}$ defined by

$$\langle f_1, f_2 \rangle = \sum_{g, g'} \left\langle f_1(g), \varphi\left(\alpha\left(g^{-1}\right) g'\right) f_2(g') \right\rangle$$

is a positive semi-definite sesquilinear form and $\mathscr{H}_\varphi$ is the Hilbert space obtained by the completion of the pre-Hilbert space $\mathscr{F}(G, \mathscr{H}) / \mathscr{N}_\varphi$, where $\mathscr{N}_\varphi = \{f \in \mathscr{F}(G, \mathscr{H}) / \langle f, f \rangle = 0\}$.

The linear map $\mathscr{J}_\varphi : \mathscr{F}(G, \mathscr{H}) \to \mathscr{F}(G, \mathscr{H})$ given by $\mathscr{J}_\varphi(f) = f \circ \alpha$ extends to a bounded linear operator $\mathscr{J}_\varphi : \mathscr{H}_\varphi \to \mathscr{H}_\varphi$. Moreover, $\mathscr{J}_\varphi = \mathscr{J}_\varphi^* = \mathscr{J}_\varphi^{-1}$ and $\left(\mathscr{H}_\varphi, \mathscr{J}_\varphi\right)$ is a Krein space. For each $g \in G$, the map $\pi_\varphi(g) : \mathscr{F}(G, \mathscr{H}) \to \mathscr{F}(G, \mathscr{H})$ given by $\pi_\varphi(g)(f)(g') = f\left(g^{-1} g'\right)$ extends to a bounded linear operator from $\mathscr{H}_\varphi$ to $\mathscr{H}_\varphi$, and the map $g \mapsto \pi_\varphi(g)$ is a $\mathscr{J}_\varphi$-unitary representation π_φ of G on $\left(\mathscr{H}_\varphi, \mathscr{J}_\varphi\right)$. The linear map $V_\varphi : \mathscr{H} \to \mathscr{F}(G, \mathscr{H})$ given by $V_\varphi \xi = \xi \delta_e$, where $\delta_e : G \to \mathbb{C}, \delta_e(g) = 0$ if $g \neq e$ and $\delta_e(e) = 1$.

(2). We consider the linear map $U : \text{span}\{\pi_\varphi(g) V_\varphi \xi; g \in G, \xi \in \mathscr{H}\} \to \text{span}\{\pi(g) V \xi;\ g \in G, \xi \in \mathscr{H}\}$ defined by

$$U\left(\pi_\varphi(g) V_\varphi \xi\right) = \pi(g) V \xi.$$

Since

$$\begin{aligned} & \left\langle U\left(\sum_{i=1}^n \pi_\varphi(g_i) V_\varphi \xi_i\right), U\left(\sum_{j=1}^m \pi_\varphi\left(g'_j\right) V_\varphi \zeta_j\right)\right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle \pi(g_i) V \xi_i, \pi\left(g'_j\right) V \zeta_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle \left(V^* \pi\left(g'_j\right)^* \pi(g_i) V \xi_i\right), \zeta_j \right\rangle \end{aligned}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle V^* \pi \left(\alpha \left(g_j'^{-1} \right) g_i \right) V \xi_i, \zeta_j \right\rangle$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle \varphi \left(\alpha \left(g_j'^{-1} \right) g_i \right) \xi_i, \zeta_j \right\rangle$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle V_\varphi^* \pi_\varphi \left(\alpha \left(g_j'^{-1} \right) g_i \right) V_\varphi \xi_i, \zeta_j \right\rangle$$
$$= \left\langle \sum_{i=1}^{n} \pi_\varphi (g_i) V_\varphi \xi_i, \sum_{j=1}^{m} \pi_\varphi \left(g_j' \right) V_\varphi \zeta_j \right\rangle$$

for all $g_1, ..., g_n, g_1', ..., g_m' \in G$ and for all $\xi_1, ..., \xi_n, \zeta_1, ..., \zeta_m \in \mathscr{H}$, U extends to a unitary operator U from $\mathscr{H}_\varphi$ to $\mathscr{K}$. Moreover, $U\pi_\varphi(g) = \pi(g)U$ for all $g \in G$ and $UV_\varphi = V$. Since

$$U \mathscr{J}_\varphi \left(\pi_\varphi (g) V_\varphi \xi \right) = U \left(\pi_\varphi (\alpha (g)) V_\varphi \xi \right) = \pi (\alpha (g)) V \xi$$
$$= \mathscr{J} (\pi (g) V \xi) = \mathscr{J} U \left(\pi_\varphi (g) V_\varphi \xi \right)$$

for all $g \in G$ and for all $\xi \in \mathscr{H}$, and since $\left[\pi_\varphi (G) V_\varphi \mathscr{H} \right] = \mathscr{H}_\varphi$, we have $U \mathscr{J}_\varphi = \mathscr{J} U$. □

If G is a topological group and φ is bounded, then the $\mathscr{J}_\varphi$-unitary representation π_φ is strictly continuous.

The triple $\left(\pi_\varphi, \left(\mathscr{H}_\varphi, \mathscr{J}_\varphi \right), V_\varphi \right)$ is called the minimal Stinespring construction associated to φ.

3. Radon-Nikodym type theorem for α-completely positive maps

Let G be a group with an involution α, $\mathscr{H}$ a Hilbert space and $\alpha - CP(G, \mathscr{H}) = \{\varphi : G \to L(\mathscr{H}); \varphi \text{ is } \alpha\text{-completely positive}\}$.

Let $\varphi \in \alpha - CP(G, \mathscr{H})$ and let $\left(\pi_\varphi, \left(\mathscr{H}_\varphi, \mathscr{J}_\varphi \right), V_\varphi \right)$ be the minimal Stinespring construction associated to φ.

PROPOSITION 3.1. *Let $T \in \pi_\varphi (G)' \subseteq L\left(\mathscr{H}_\varphi \right)$ such that $T \geqslant 0$ and $T \mathscr{J}_\varphi = \mathscr{J}_\varphi T$, where $\pi_\varphi (G)'$ is the commutant of $\pi_\varphi (G)'$ in $L(\mathscr{H}_\varphi)$. Then the map $\varphi_T : G \to L(\mathscr{H}_\varphi)$ defined by $\varphi_T (g) = V_\varphi^* T \pi_\varphi (g) V_\varphi$ is α-completely positive.*

Proof. From

$$\begin{aligned}
\varphi_T (\alpha (g_1) \alpha (g_2)) &= V_\varphi^* T \pi_\varphi (\alpha (g_1)) \pi_\varphi (\alpha (g_2)) V_\varphi \\
&= V_\varphi^* T \pi_\varphi (\alpha (g_1)) \mathscr{J}_\varphi \pi_\varphi (g_2) V_\varphi \\
&= V_\varphi^* \mathscr{J}_\varphi \pi_\varphi (\alpha (g_1)) \mathscr{J}_\varphi T \pi_\varphi (g_2) V_\varphi \\
&= V_\varphi^* \pi_\varphi \left(\alpha \left(g_1^{-1} \right) \right)^* T \pi_\varphi (g_2) V_\varphi \\
&= V_\varphi^* \pi_\varphi \left(g_1^{-1} \right)^* \mathscr{J}_\varphi T \pi_\varphi (g_2) V_\varphi \\
&= V_\varphi^* \mathscr{J}_\varphi \pi_\varphi (g_1) T \pi_\varphi (g_2) V_\varphi \\
&= V_\varphi^* T \pi_\varphi (g_1 g_2) V_\varphi = \varphi_T (g_1 g_2)
\end{aligned}$$

and

$$\begin{aligned}\varphi_T(\alpha(g_1g_2)) &= V_\varphi^* T\pi_\varphi(\alpha(g_1g_2))V_\varphi = V_\varphi^* T\mathscr{J}_\varphi \pi_\varphi(g_1g_2)V_\varphi \\ &= V_\varphi^* \mathscr{J}_\varphi T\pi_\varphi(g_1g_2)V_\varphi = V_\varphi^* T\pi_\varphi(g_1g_2)V_\varphi = \varphi_T(g_1g_2)\end{aligned}$$

for all $g_1, g_2 \in G$, we deduce that $\varphi_T(\alpha(g_1)\alpha(g_2)) = \varphi_T(\alpha(g_1g_2)) = \varphi_T(g_1g_2)$ for all $g_1, g_2 \in G$.

Let $g_1, ..., g_n \in G$ and $\xi_1, ..., \xi_n \in \mathscr{H}$. Then

$$\begin{aligned}&\left\langle \left[\varphi_T\left(\alpha\left(g_i^{-1}\right)g_j\right)\right]_{i,j=1}^n (\xi_k)_{k=1}^n, (\xi_k)_{k=1}^n \right\rangle \\ &= \sum_{i,j=1}^n \left\langle \varphi_T\left(\alpha\left(g_i^{-1}\right)g_j\right)\xi_j, \xi_i \right\rangle \\ &= \sum_{i,j=1}^n \left\langle V_\varphi^* T\pi_\varphi\left(\alpha\left(g_i^{-1}\right)g_j\right)V_\varphi \xi_j, \xi_i \right\rangle \\ &= \sum_{i,j=1}^n \left\langle T\pi_\varphi(g_j)V_\varphi\xi_j, \pi_\varphi\left(\alpha\left(g_i^{-1}\right)\right)^* V_\varphi\xi_i \right\rangle \\ &= \sum_{i,j=1}^n \left\langle T\pi_\varphi(g_j)V_\varphi\xi_j, \mathscr{J}_\varphi\pi_\varphi(\alpha(g_i))\mathscr{J}_\varphi V_\varphi\xi_i \right\rangle \\ &= \sum_{i,j=1}^n \left\langle T\pi_\varphi(g_j)V_\varphi\xi_j, \mathscr{J}_\varphi\pi_\varphi(\alpha(g_i))V_\varphi\xi_i \right\rangle \\ &= \left\langle T\sum_{j=1}^n \pi_\varphi(g_j)V_\varphi\xi_j, \sum_{i=1}^n \pi_\varphi(g_i)V_\varphi\xi_i \right\rangle \geqslant 0\end{aligned}$$

and

$$\begin{aligned}&\left\langle \left[\varphi_T(g_i)^*\varphi_T(g_j)\right]_{i,j=1}^n (\xi_k)_{k=1}^n, (\xi_k)_{k=1}^n \right\rangle \\ &= \sum_{i,j=1}^n \left\langle V_\varphi^* T\pi_\varphi(g_j)V_\varphi\xi_j, V_\varphi^* T\pi_\varphi(g_i)V_\varphi\xi_i \right\rangle \\ &= \left\langle T^* V_\varphi V_\varphi^* T\sum_{j=1}^n \pi_\varphi(g_j)V_\varphi\xi_j, \sum_{i=1}^n \pi_\varphi(g_i)V_\varphi\xi_i \right\rangle \\ &\leqslant \left\|V_\varphi\right\|^2 \|T\| \sum_{i,j=1}^n \left\langle V_\varphi^* \mathscr{J}_\varphi \pi_\varphi\left(g_i^{-1}\right)\mathscr{J}_\varphi T\pi_\varphi(g_j)V_\varphi\xi_j, \xi_i \right\rangle \\ &= \left\|V_\varphi\right\|^2 \|T\| \sum_{i,j=1}^n \left\langle V_\varphi^* T\pi_\varphi\left(g_i^{-1}\right)\pi_\varphi(\alpha(g_j))V_\varphi\xi_j, \xi_i \right\rangle \\ &= \left\|V_\varphi\right\|^2 \|T\| \sum_{i,j=1}^n \left\langle V_\varphi^* T\pi_\varphi\left(g_i^{-1}\alpha(g_j)\right)V_\varphi\xi_j, \xi_i \right\rangle \\ &= \left\|V_\varphi\right\|^2 \|T\| \sum_{i,j=1}^n \left\langle \varphi_T\left(g_i^{-1}\alpha(g_j)\right)\xi_j, \xi_i \right\rangle \\ &= \left\|V_\varphi\right\|^2 \|T\| \sum_{i,j=1}^n \left\langle \varphi_T\left(\alpha\left(g_i^{-1}\right)g_j\right)\xi_j, \xi_i \right\rangle \\ &= \left\|V_\varphi\right\|^2 \|T\| \left\langle \left[\varphi_T\left(\alpha\left(g_i^{-1}\right)g_j\right)\right]_{i,j=1}^n (\xi_k)_{k=1}^n, (\xi_k)_{k=1}^n \right\rangle.\end{aligned}$$

From these relations, we deduce that φ_T verifies the conditions (2) and (3) from Definition 2.1.

Let $g \in G$. From

$$\begin{aligned}
&\left\langle \left[\varphi_T\left(\alpha\left(gg_i\right)^{-1} gg_j\right)\right]_{i,j=1}^n \left(\xi_k\right)_{k=1}^n, \left(\xi_k\right)_{k=1}^n \right\rangle \\
&= \sum_{i,j=1}^n \left\langle V_\varphi^* T \pi_\varphi\left(\alpha\left(gg_i\right)^{-1}\right) \pi_\varphi\left(gg_j\right) V_\varphi \xi_j, \xi_i \right\rangle \\
&= \sum_{i,j=1}^n \left\langle T \pi_\varphi\left(gg_j\right) V_\varphi \xi_j, \mathscr{J}_\varphi \pi_\varphi\left(\alpha\left(gg_i\right)\right) \mathscr{J}_\varphi V_\varphi \xi_i \right\rangle \\
&= \sum_{i,j=1}^n \left\langle T \pi_\varphi(g) \pi_\varphi\left(g_j\right) V_\varphi \xi_j, \pi_\varphi\left(gg_i\right) V_\varphi \xi_i \right\rangle \\
&= \left\langle \pi_\varphi(g)^* \pi_\varphi(g) \sum_{j=1}^n |T| \pi_\varphi\left(g_j\right) V_\varphi \xi_j, \sum_{i=1}^n |T| \pi_\varphi\left(g_i\right) V_\varphi \xi_i \right\rangle \\
&\leqslant \left\|\pi_\varphi(g)\right\|^2 \left\langle T \sum_{j=1}^n \pi_\varphi\left(g_j\right) V_\varphi \xi_j, \sum_{i=1}^n \pi_\varphi\left(g_i\right) V_\varphi \xi_i \right\rangle \\
&= \left\|\pi_\varphi(g)\right\|^2 \left\langle \left[\varphi_T\left(\alpha\left(g_i^{-1}\right) g_j\right)\right]_{i,j=1}^n \left(\xi_k\right)_{k=1}^n, \left(\xi_k\right)_{k=1}^n \right\rangle
\end{aligned}$$

for all $g_1,\ldots,g_n \in G$ and $\xi_1,\ldots,\xi_n \in \mathscr{H}$, we deduce that φ_T verifies the condition (4) from Definition 2.1. $\square$

Let φ, ψ be two α-completely positive maps. We say that $\psi \leqslant \varphi$ if $\varphi - \psi$ is an α-completely positive map, and ψ is *uniformly dominated* by φ, denoted by $\psi \leqslant_u \varphi$, if there is $\lambda > 0$ such that $\psi \leqslant \lambda\varphi$. The α-completely positive maps φ, ψ are *uniformly equivalent,* $\psi \equiv_u \varphi$, if $\psi \leqslant_u \varphi$ and $\varphi \leqslant_u \psi$.

PROPOSITION 3.2. *Let φ, ψ be two α-completely positive maps from G to $L(\mathscr{H})$. If $\psi \leqslant_u \varphi$, then there is $T \in \pi_\varphi(G)' \subseteq L\left(\mathscr{H}_\varphi\right)$, $T \geqslant 0$ and $T\mathscr{J}_\varphi = \mathscr{J}_\varphi T$ such that $\psi = \varphi_T$. Moreover, T is unique.*

Proof. Let $\left(\pi_\psi, \left(\mathscr{H}_\psi, \mathscr{J}_\psi\right), V_\phi\right)$ be the minimal Stinespring construction associated to ψ. From

$$\begin{aligned}
&\left\langle \sum_{i=1}^n \pi_\psi\left(g_i\right) V_\psi \xi_i, \sum_{i=1}^n \pi_\psi\left(g_i\right) V_\psi \xi_i \right\rangle \\
&= \sum_{i,j=1}^n \left\langle V_\psi^* \pi_\psi\left(g_j\right)^* \pi_\psi\left(g_i\right) V_\psi \xi_i, \xi_j \right\rangle \\
&= \sum_{i,j=1}^n \left\langle V_\psi^* \pi_\psi\left(\alpha\left(g_j^{-1}\right) g_i\right) V_\psi \xi_i, \xi_j \right\rangle \\
&= \left\langle \left[\psi\left(\alpha\left(g_j^{-1}\right) g_i\right)\right]_{i,j=1}^n \left(\xi_k\right)_{k=1}^n, \left(\xi_k\right)_{k=1}^n \right\rangle \\
&\leqslant \lambda \left\langle \left[\varphi\left(\alpha\left(g_j^{-1}\right) g_i\right)\right]_{i,j=1}^n \left(\xi_k\right)_{k=1}^n, \left(\xi_k\right)_{k=1}^n \right\rangle \\
&= \lambda \left\langle \sum_{i=1}^n \pi_\varphi\left(g_i\right) V_\varphi \xi_i, \sum_{i=1}^n \pi_\varphi\left(g_i\right) V_\varphi \xi_i \right\rangle
\end{aligned}$$

we deduce that there is a bounded linear operator $S:\mathscr{H}_\varphi \to \mathscr{H}_\psi$ such that $S\left(\pi_\varphi(g)V_\varphi\xi\right) = \pi_\psi(g)V_\psi\xi$. Clearly, $S\pi_\varphi(g) = \pi_\psi(g)S$ for all $g \in G$, and $SV_\varphi = V_\psi$. Moreover, $S\mathscr{J}_\varphi = \mathscr{J}_\psi S$, since

$$\begin{aligned} S\mathscr{J}_\varphi\left(\pi_\varphi(g)V_\varphi\xi\right) &= S\left(\pi_\varphi(\alpha(g))V_\varphi\xi\right) = \pi_\psi(\alpha(g))V_\psi\xi \\ &= \mathscr{J}_\psi\pi_\psi(g)V_\psi\xi = \mathscr{J}_\psi S\left(\pi_\varphi(g)V_\varphi\xi\right) \end{aligned}$$

for all $g \in G$ and for all $\xi \in \mathscr{H}$.

Let $T = S^*S$. Then $T\mathscr{J}_\varphi = \mathscr{J}_\varphi T$ and $T\pi_\varphi(g) = \pi_\varphi(g)T$ for all $g \in G$. Moreover,

$$\varphi_T(g) = V_\varphi^* T\pi_\varphi(g)V_\varphi = V_\varphi^* S^*\pi_\psi(g)SV_\varphi = V_\psi^*\pi_\psi(g)V_\psi = \psi(g)$$

for all $g \in G$.

Suppose that there is another $T_1 \in \pi_\varphi(G)' \subseteq L\left(\mathscr{H}_\varphi\right)$, $T_1 \geqslant 0$ and $T_1\mathscr{J}_\varphi = \mathscr{J}_\varphi T_1$ such that $\psi = \varphi_{T_1}$. Then

$$\begin{aligned} &\left\langle (T-T_1)\left(\pi_\varphi(g)V_\varphi\xi\right), \pi_\varphi\left(g'\right)V_\varphi\eta\right\rangle \\ &= \left\langle V_\varphi^*\mathscr{J}_\varphi\pi_\varphi\left(g'^{-1}\right)\mathscr{J}_\varphi(T-T_1)\left(\pi_\varphi(g)V_\varphi\xi\right), \eta\right\rangle \\ &= \left\langle V_\varphi^*(T-T_1)\mathscr{J}_\varphi\pi_\varphi\left(g'^{-1}\right)\left(\pi_\varphi(\alpha(g))V_\varphi\xi\right), \eta\right\rangle \\ &= \left\langle V_\varphi^*(T-T_1)\pi_\varphi\left(\alpha\left(g'^{-1}\right)g\right)V_\varphi\xi, \eta\right\rangle \\ &= \left\langle \varphi_T\left(\alpha\left(g'^{-1}\right)g\right)\xi - \varphi_{T_1}\left(\alpha\left(g'^{-1}\right)g\right)\xi, \eta\right\rangle = 0 \end{aligned}$$

for all $g, g' \in G$, and for all $\xi, \eta \in \mathscr{H}$, and since $\left[\pi_\varphi(G)V_\varphi\mathscr{H}\right] = \mathscr{H}_\varphi$, we have $T = T_1$. $\square$

From the proof of Proposition 3.1, we obtain the following corollary.

COROLLARY 3.3. *If φ, ψ are two α-completely positive maps from G to $L(\mathscr{H})$ and $\psi \leqslant \varphi$, then there is a unique positive operator T in $\pi_\varphi(G)' \subseteq L\left(\mathscr{H}_\varphi\right)$ such that $T \leqslant id_{\mathscr{H}_\varphi}$, $T\mathscr{J}_\varphi = \mathscr{J}_\varphi T$ and $\psi = \varphi_T$.*

Let φ, ψ be two α-completely positive maps from G to $L(\mathscr{H})$ such that $\psi \leqslant_u \varphi$. A positive operator $T \in \pi_\varphi(G)' \subseteq L\left(\mathscr{H}_\varphi\right)$ with $T\mathscr{J}_\varphi = \mathscr{J}_\varphi T$ and such that $\psi = \varphi_T$, denoted by $\Delta_\varphi(\psi)$, is called *the Radon-Nikodym derivative of ψ with respect to φ.*

REMARK 3.4. If $\psi \leqslant_u \varphi$, then the minimal Stinespring construction associated to ψ can be recovered by the minimal Stinespring construction associated to φ.

Let $P_{\ker\Delta_\varphi(\psi)}$ and $P_{\mathscr{H}_\varphi\ominus\ker\Delta_\varphi(\psi)}$ be the orthogonal projections on $\ker\Delta_\varphi(\psi)$, respectively $\mathscr{H}_\varphi \ominus \ker\Delta_\varphi(\psi)$. Since $\Delta_\varphi(\psi) \in \pi_\varphi(G)' \subseteq L\left(\mathscr{H}_\varphi\right)$ and $\Delta_\varphi(\psi)\mathscr{J}_\varphi = \mathscr{J}_\varphi\Delta_\varphi(\psi)$, $P_{\ker\Delta_\varphi(\psi)}$, $P_{\mathscr{H}_\varphi\ominus\ker\Delta_\varphi(\psi)} \in \pi_\varphi(G)' \subseteq L\left(\mathscr{H}_\varphi\right)$, $P_{\ker\Delta_\varphi(\psi)}\mathscr{J}_\varphi = \mathscr{J}_\varphi P_{\ker\Delta_\varphi(\psi)}$ and $P_{\mathscr{H}_\varphi\ominus\ker\Delta_\varphi(\psi)}\mathscr{J}_\varphi = \mathscr{J}_\varphi P_{\mathscr{H}_\varphi\ominus\ker\Delta_\varphi(\psi)}$. Then $\left(\mathscr{H}_\varphi \ominus \ker\Delta_\varphi(\psi), \mathscr{J}_\varphi|_{\mathscr{H}_\varphi\ominus\ker\Delta_\varphi(\psi)}\right)$ is a Krein space and it is easy to check that

$$\left(\pi_\varphi|_{\mathscr{H}_\varphi\ominus\ker\Delta_\varphi(\psi)}\left(\mathscr{H}_\varphi \ominus \ker\Delta_\varphi(\psi), \mathscr{J}_\varphi|_{\mathscr{H}_\varphi\ominus\ker\Delta_\varphi(\psi)}\right), P_{\mathscr{H}_\varphi\ominus\ker\Delta_\varphi(\psi)}\Delta_\varphi(\psi)^{\frac{1}{2}}V_\varphi\right)$$

is unitarily equivalent to the minimal Stinespring construction associated to ψ.

PROPOSITION 3.5. *Let $\varphi,\psi \in \alpha - CP(G,\mathscr{H})$. If $\varphi \equiv_u \psi$, then the Stinespring construction associated to φ is unitarily equivalent to the Stinespring construction associated to φ.*

Proof. If $\varphi \equiv_u \psi$, then $\psi \leqslant_u \varphi$ and $\varphi \leqslant_u \psi$, and by Proposition 3.2, there are two bounded linear operators $S_1 : \mathscr{H}_\varphi \to \mathscr{H}_\psi$ such that $S_1\left(\pi_\varphi(g)V_\varphi\xi\right) = \pi_\psi(g)V_\psi\xi$ and $S_2 : \mathscr{H}_\psi \to \mathscr{H}_\varphi$ such that $S_2\left(\pi_\psi(g)V_\psi\xi\right) = \pi_\varphi(g)V_\varphi\xi$. From $S_2S_1\left(\pi_\varphi(g)V_\varphi\xi\right) = \pi_\varphi(g)V_\varphi\xi$, $S_1S_2\left(\pi_\psi(g)V_\psi\xi\right) = \pi_\psi(g)V_\psi\xi$, and taking into account that $\left[\pi_\varphi(G)V_\varphi\mathscr{H}\right] = \mathscr{H}_\varphi$ and $\left[\pi_\psi(G)V_\psi\mathscr{H}\right] = \mathscr{H}_\psi$, we deduce that S_1 is invertible. Then $\Delta_\varphi(\psi) = S_1^*S_1$ is invertible, and so there is a unitary operator $U : \mathscr{H}_\varphi \to \mathscr{H}_\psi$ such that $S_1 = U\Delta_\varphi(\psi)^{\frac{1}{2}}$. It is easy to check that $U\mathscr{J}_\varphi = \mathscr{J}_\psi U, UV_\varphi = V_\psi$ and $U\pi_\varphi(g) = \pi_\psi(g)U$ for all $g \in G$. $\square$

THEOREM 3.6. *Let φ be an α-completely positive map from G to $L(\mathscr{H})$. Then the map $\psi \mapsto \Delta_\varphi(\psi)$ is an affine bijective map from $\{\psi \in \alpha - CP(G,\mathscr{H}); \psi \leqslant_u \varphi\}$ onto $\{T \in \pi_\varphi(G)' \subseteq L(\mathscr{H}_\varphi); T\mathscr{J}_\varphi = \mathscr{J}_\varphi T, T \geqslant 0\}$ which preserves the pre-order relation.*

Proof. By Propositions 3.1 and 3.2, the map $\psi \mapsto \Delta_\varphi(\psi)$ is well defined and bijective, its inverse is given by $T \mapsto \varphi_T$. Let $t \in [0,1], \psi_1 \leqslant_u \varphi$ and $\psi_2 \leqslant_u \varphi$. Then $t\psi_1 + (1-t)\psi_2 \leqslant_u \varphi$ and so

$$\begin{aligned}\varphi_{\Delta_\varphi(t\psi_1+(1-t)\psi_2)}(g) &= t\psi_1(g) + (1-t)\psi_2(g) = t\varphi_{\Delta_\varphi(\psi_1)}(g) + (1-t)\varphi_{\Delta_\varphi(\psi_2)}(g)\\ &= V_\varphi^*\left(t\Delta_\varphi(\psi_1) + (1-t)\Delta_\varphi(\psi_2)\right)\pi_\varphi(g)V_\varphi\end{aligned}$$

for all $g \in G$, whence we deduce that $\Delta_\varphi(t\psi_1 + (1-t)\psi_2) = t\Delta_\varphi(\psi_1) + (1-t)\Delta_\varphi(\psi_2)$. Therefore, the map $\psi \mapsto \Delta_\varphi(\psi)$ is affine.

Let $\psi_1 \leqslant_u \psi_2 \leqslant_u \varphi$. Then there is $\lambda \geqslant 0$ such that $\lambda\psi_2 - \psi_1$ is α-completely positive. From

$$\begin{aligned}0 &\leqslant \left\langle \left[(\lambda\psi_2 - \psi_1)\left(\alpha(g_i)^{-1}g_j\right)\right]_{i,j=1}^n (\xi_i)_{i=1}^n, (\xi_i)_{i=1}^n \right\rangle\\ &= \lambda\sum_{i,j=1}^n \left\langle \varphi_{\Delta_\varphi(\psi_2)}\left(\alpha(g_i)^{-1}g_j\right)\xi_j,\xi_i\right\rangle - \sum_{i,j=1}^n \left\langle \varphi_{\Delta_\varphi(\psi_1)}\left(\alpha(g_i)^{-1}g_j\right)\xi_j,\xi_i\right\rangle\\ &= \sum_{i,j=1}^n \left\langle V_\varphi^*\left(\lambda\Delta_\varphi(\psi_2) - \Delta_\varphi(\psi_1)\right)\pi_\varphi\left(\alpha(g_i)^{-1}g_j\right)V_\varphi\xi_j,\xi_i\right\rangle\\ &= \sum_{i,j=1}^n \left\langle \left(\lambda\Delta_\varphi(\psi_2) - \Delta_\varphi(\psi_1)\right)\pi_\varphi(g_j)V_\varphi\xi_j, \pi_\varphi\left(\alpha(g_i)^{-1}\right)^* V_\varphi\xi_i\right\rangle\\ &= \sum_{i,j=1}^n \left\langle \left(\lambda\Delta_\varphi(\psi_2) - \Delta_\varphi(\psi_1)\right)\pi_\varphi(g_j)V_\varphi\xi_j, \mathscr{J}_\varphi\pi_\varphi(\alpha(g_i))\mathscr{J}_\varphi V_\varphi\xi_i\right\rangle\\ &= \sum_{i,j=1}^n \left\langle \left(\lambda\Delta_\varphi(\psi_2) - \Delta_\varphi(\psi_1)\right)\pi_\varphi(g_j)V_\varphi\xi_j, \pi_\varphi(g_i)V_\varphi\xi_i\right\rangle\\ &= \left\langle \left(\lambda\Delta_\varphi(\psi_2) - \Delta_\varphi(\psi_1)\right)\sum_{j=1}^n \pi_\varphi(g_j)V_\varphi\xi_j, \sum_{i=1}^n \pi_\varphi(g_i)V_\varphi\xi_i\right\rangle\end{aligned}$$

for all $g_1,...,g_n \in G$, for all $\xi_1,...,\xi_n \in \mathscr{H}$, and taking into account that $\left[\pi_\varphi(G)V_\varphi\mathscr{H}\right] = \mathscr{H}_\varphi$, we conclude that $\lambda\Delta_\varphi(\psi_2) - \Delta_\varphi(\psi_1) \geqslant 0$. Therefore the map $\psi \mapsto \Delta_\varphi(\psi)$ preserves the pre-order relation. □

COROLLARY 3.7. *The map $\psi \mapsto \Delta_\varphi(\psi)$ is an affine bijective map from $\{\psi \in \alpha - CP(G,\mathscr{H}); \psi \leqslant \varphi\}$ onto $\{T \in \pi_\varphi(G)' \subseteq L(\mathscr{H}_\varphi); T\mathscr{J}_\varphi = \mathscr{J}_\varphi T, 0 \leqslant T \leqslant id_{\mathscr{H}_\varphi}\}$ which preserves the order relation.*

Let G be a discrete group. If π is a bounded $\mathscr{J}$-unitary representation of G on $(\mathscr{H},\mathscr{J})$, then the map $\widetilde{\pi}: \mathscr{F}(G) \to L(\mathscr{H})$ given by

$$\widetilde{\pi}\left(\sum_{i=1}^{n}\lambda_i\delta_{g_i}\right) = \sum_{i=1}^{n}\lambda_i\pi(g_i)$$

extends to a bounded $\mathscr{J}$- representation of $C^*(G)$. Moreover, the map $\pi \mapsto \widetilde{\pi}$ is a bijective correspondence between the collection of bounded unitary representations on Krein spaces and the collection of bounded representations of $C^*(G)$ on Krein spaces.

Let G be a discrete group and $\varphi \in \alpha - CP(G,\mathscr{H})$. Then α extends to a linear hermitian involution $\widetilde{\alpha}$ on $C^*(G)$, $\widetilde{\alpha}(f) = f \circ \alpha$ for all $f \in \mathscr{F}(G)$. If φ is bounded, then the map $\Phi: \mathscr{F}(G) \to L(\mathscr{H})$ given by $\Phi\left(\sum_{k=1}^{n}\lambda_k\delta_{g_k}\right) = \sum_{k=1}^{n}\lambda_k\varphi(g_k)$ extends to a linear hermitian bounded $\widetilde{\alpha}$-completely positive $\widetilde{\varphi}: C^*(G) \to L(\mathscr{H})$ (see [2, Theorem 2.5]). We denote by $\alpha - bCP(G,\mathscr{H})$ the collection of all bounded α-completely positive maps from G to $L(\mathscr{H})$.

REMARK 3.8. Let $\varphi \in \alpha - bCP(G,\mathscr{H})$. If $(\pi_\varphi,(\mathscr{H}_\varphi,\mathscr{J}_\varphi),V_\varphi)$ is the minimal Stinespring construction associated to φ, then it is easy to check that $(\widetilde{\pi_\varphi},(\mathscr{H}_\varphi,\mathscr{J}_\varphi),V_\varphi)$ is unitarily equivalent to the minimal Stinespring construction associated to $\widetilde{\varphi}$ ([4, Theorems 4.4 and 4.6]).

THEOREM 3.9. *Let G be a discrete group. Then the map $\varphi \mapsto \widetilde{\varphi}$ is an affine bijective map from $\alpha - bCP(G,\mathscr{H})$ to $\alpha - bCP(C^*(G),\mathscr{H})$ which preserves the order (pre-order) relation. Moreover, if $\psi \leqslant_u \varphi$ then $\Delta_\varphi(\psi) = \Delta_{\widetilde{\varphi}}(\widetilde{\psi})$.*

Proof. It is clear that the map $\varphi \mapsto \widetilde{\varphi}$ from $\alpha - bCP(G,\mathscr{H})$ to $\alpha - bCP(C^*(G),\mathscr{H})$ is well defined and injective. Let $\phi \in \alpha - bCP(C^*(G),\mathscr{H})$. Then the map $\varphi: G \to L(\mathscr{H})$ given $\varphi(g) = \phi(\delta_g)$ is a bounded α-completely positive. Moreover, $\widetilde{\varphi} = \phi$, and so the map $\varphi \mapsto \widetilde{\varphi}$ is surjective.

Clearly, $\widetilde{\varphi_1+\varphi_2} = \widetilde{\varphi_1} + \widetilde{\varphi_2}$ and $\widetilde{\lambda\varphi} = \lambda\widetilde{\varphi}$ for all $\varphi_1,\varphi_2,\varphi \in \alpha - bCP(G,\mathscr{H})$ and for all positive numbers λ. Let $\varphi,\psi \in \alpha - bCP(G,\mathscr{H})$ with $\psi \leqslant \varphi$ and $\left(\pi_\varphi,\left(\mathscr{H}_\varphi,\mathscr{J}_\varphi\right),V_\varphi\right)$ the minimal Stinespring construction associated to φ. Then $\widetilde{\psi}(f) = V_\varphi^*\Delta_\varphi(\psi)\widetilde{\pi_\varphi}(f)V_\varphi$ for all $f \in C^*(G)$. Since the minimal Stinespring construction associated to $\widetilde{\varphi}$ is unitarily equivalent to $(\widetilde{\pi_\varphi},(\mathscr{H}_\varphi,\mathscr{J}_\varphi),V_\varphi)$, and taking into account that $\Delta_\varphi(\psi) \in \widetilde{\pi_\varphi}(G)' \subseteq L(\mathscr{H}_\varphi)$, $\Delta_\varphi(\psi)\mathscr{J}_\varphi = \mathscr{J}_\varphi\Delta_\varphi(\psi)$ and $0 \leqslant \Delta_\varphi(\psi) \leqslant \mathrm{id}_{\mathscr{H}_\varphi}$, we conclude that $\widetilde{\psi} \leqslant \widetilde{\varphi}$ and $\Delta_\varphi(\psi) = \Delta_{\widetilde{\varphi}}(\widetilde{\psi})$. □

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