Operators and Matrices
Volume 9, Number 1 (2015), 1–29
doi:10.7153/oam-09-01

FACTORIZATION OF SOME TRIANGULAR MATRIX FUNCTIONS AND ITS APPLICATIONS

YU. I. KARLOVICH, J. LORETO-HERNÁNDEZ AND I. M. SPITKOVSKY

(*Communicated by L. Rodman*)

Abstract. We consider defined on the real line $\mathbb{R}$ matrix functions with monomial terms of the form $ce^{i\lambda x}$ on the main diagonal and one row, and with zero entries elsewhere. The factorability of such matrices is established and, moreover, the algorithm for their factorization is provided. In particular, formulas for the partial indices are derived, and conditions for them to all equal zero (that is, for the factorization to be canonical) are stated. These results are then used to obtain Fredholmness criteria for some convolution type equations on unions of intervals.

1. Introduction

For any algebra $\mathfrak{A}$, we denote by $\mathscr{G}\mathfrak{A}$ the group of its invertible elements, and by $\mathfrak{A}_{N\times N}$ the algebra of all $N\times N$ matrices with the entries in $\mathfrak{A}$.

Let APP be the algebra of *almost periodic polynomials*, that is, the set of all finite linear combinations of elements e_λ $(\lambda\in\mathbb{R})$, with e_λ defined by

$$e_\lambda(x) = e^{i\lambda x}, \quad x\in\mathbb{R}. \tag{1.1}$$

The closure of APP with respect to the uniform norm is the C^*-algebra AP of almost periodic functions, and the closure of APP with respect to the stronger norm,

$$\|\Sigma_\lambda c_\lambda e_\lambda\|_W = \Sigma_\lambda |c_\lambda|, \quad c_\lambda\in\mathbb{C},$$

is the Banach algebra APW.

The basic information about AP functions can be found in several monographs, including [3, 7] and [16]. For our purposes, the following will suffice.

For any $f\in AP$ there exists the *Bohr mean value*

$$\mathbf{M}(f) = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} f(x)\,dx.$$

Mathematics subject classification (2010): Primary 47A68; Secondary 45E10, 47A53, 47B35.
Keywords and phrases: Almost periodic factorization, geometric mean, convolution type equations, Fredholm operators.
This work was supported by the CONACYT Project No. 168104 (México) and by PROMEP (México) via "Proyecto de Redes" [YuK], by the CONACYT Project No. 119043 (México) and by PROMEP via "Apoyo a la Incorporación de nuevos PTC" [JLH], and by the Summer Research grant of the College of William & Mary [IS]. The last author is also thankful to the NYU Abu Dhabi for the perfect working environment he enjoyed during his stay there in the Fall semester of 2013–14 academic year when the work on the paper was completed.

The functions $f \in AP$ are defined uniquely by the *Bohr-Fourier series*

$$\sum_{\lambda \in \Omega(f)} \widehat{f}(\lambda) e_\lambda$$

where $\Omega(f) := \left\{ \lambda \in \mathbb{R} : \ \widehat{f}(\lambda) \neq 0 \right\}$ is the *Bohr-Fourier spectrum* of f and the numbers $\widehat{f}(\lambda) = \mathbf{M}(f e_{-\lambda})$ are referred to as the *Bohr-Fourier coefficients* of f.

Let

$$AP^{\pm} := \left\{ f \in AP \colon \Omega(f) \subset \mathbb{R}_{\pm} \right\}, \quad APW^{\pm} := AP^{\pm} \cap APW,$$

$$APW_0^{\pm} := \left\{ f \in APW^{\pm} \colon \widehat{f}(0) = 0 \right\}, \quad APP^{\pm} := AP^{\pm} \cap APP,$$

where, as usual, $\mathbb{R}_{\pm} = \{ x \in \mathbb{R} \colon \ \pm x \geqslant 0 \}$.

A function $f \in AP$ is invertible in AP if and only if it is invertible in $L_\infty(\mathbb{R})$, that is, if and only if $\inf_{x \in \mathbb{R}} |f(x)| > 0$. For every $f \in \mathscr{G}AP$, the following limits exist, are finite, equal and independent of the choice of a continuous branch of the argument of f:

$$\kappa(f) := \lim_{T \to +\infty} \frac{1}{2T} \left\{ \arg f(x) \right\}_{-T}^{T} = \lim_{T \to \pm\infty} \frac{1}{T} \left\{ \arg f(x) \right\}_{0}^{T}.$$

Their common value is called the *mean motion* (or the *AP index*) of f.

We say that $G \in AP_{N \times N}$ admits a *canonical* left *AP factorization* if

$$G = G_+ G_- \tag{1.2}$$

with $G_{\pm} \in \mathscr{G}(AP^{\pm}_{N \times N})$. If in fact $G_{\pm} \in \mathscr{G}(APW^{\pm}_{N \times N})$, (1.2) is said to be a canonical left APW factorization of G. More generally, a left AP or APW factorization (not necessarily canonical) of G is a representation

$$G = G_+ D G_- \tag{1.3}$$

with $G_{\pm}$ as above and an extra middle factor $D = \operatorname{diag}[e_{\kappa_1}, \dots, e_{\kappa_N}]$. The parameters $\kappa_j \in \mathbb{R}$ are defined by G uniquely up to a permutation whenever the factorization exists, and are called the (left) *partial AP indices* of G. Of course, condition $G \in \mathscr{G}(AP_{N \times N})$ (resp., $G \in \mathscr{G}(APW_{N \times N})$) is necessary in order for G to admit a left AP (resp., APW) factorization, and

$$\kappa_1 + \dots + \kappa_N = \kappa(\det G) := \kappa. \tag{1.4}$$

A left APP factorization of G is introduced along the same lines, as a representation (1.3) with $G_{\pm} \in \mathscr{G}(APP^{\pm}_{N \times N})$. The latter condition implies that $\det G_{\pm}$ is constant, and so G can possibly admit an APP factorization only if

$$G \in APP_{N \times N} \text{ and } \det G = c e_\kappa \text{ with some non-zero } c \in \mathbb{C}. \tag{1.5}$$

Conversely, under conditions (1.5) an AP factorization of G is in fact its APP factorization whenever at least one of four matrix functions $G_{\pm}, G_{\pm}^{-1}$ belongs to $APP_{N \times N}$.

An important invariant of the canonical factorization (1.2), if it exists, is the product $\mathbf{d}(G) := \mathbf{M}(G_+)\mathbf{M}(G_-)$, with the Bohr means of matrices understood entry-wise. Since for $N = 1$

$$\mathbf{d}(G) = \exp \mathbf{M}(\log G),$$

$\mathbf{d}(G)$ is called the *geometric mean* of G, even when $N > 1$.

A canonical AP factorization of $G \in APW_{N\times N}$ is automatically its (naturally, also canonical) APW factorization. For $N = 1$, any $G \in \mathcal{G}APW$ admits an APW factorization, and thus AP (and even APW) factorable functions form a dense subset of AP. As was discovered recently [6], this is not the case any more if $N > 1$.

AP factorization arises in a number of applications. In particular, consideration of convolution type equations on systems of intervals yields the factorization problem for matrix functions of the form

$$G = \begin{bmatrix} e_{\lambda_1} & & & & \\ & e_{\lambda_2} & & & \\ & & \ddots & & \\ & & & e_{\lambda_{n-1}} & \\ g_1 & g_2 & \cdots & g_{n-1} & e_{\lambda_n} \end{bmatrix}, \tag{1.6}$$

where $\lambda_1, \dots, \lambda_{n-1}$ are the lengths of intervals in question,

$$\lambda_n = -(\lambda_1 + \cdots \lambda_{n-1}), \tag{1.7}$$

and g_j are in a certain way associated with the Fourier transforms of the equations' kernels. One such application is considered in the final Section 9 of this paper, while the preceding Section 8 contains the necessary function-theoretic background. The constructive results are obtained in the case of monomial g_j arising in matrices (1.6):

$$g_j = a_j e_{\gamma_j}, \quad j = 1, \dots, n-1. \tag{1.8}$$

They are based on the fact that all matrices (1.6) satisfying (1.8) are APP factorable, established in Section 2, and on the criterion of their canonical factorability derived in Section 3. Note that, due to (1.4), condition (1.7) is necessary for the factorization of G to be canonical, but will not be a priori imposed.

In the case when the APP factorization of G is not (necessarily) canonical, formulas for its partial AP indices are provided in Section 4. These results, including the factorization construction, are illustrated on matrices of low size in Section 5 (for $n = 3$) and Section 6 (for $n = 4$). Finally, a short Section 7 concerns the extension of the factorization results from Sections 2–6 to the setting of abstract ordered groups.

2. Factorization existence

The following notation will be used in this and the next section: E_{ij} is the matrix with the only non-zero entry, equal one, in the (i, j) position.

THEOREM 2.1. *Any matrix function* (1.6) *with the last row entries given by* (1.8) *admits a left APP factorization.*

Proof. The proof is by induction on the size of the matrix. The base is trivial: a function e_λ admits an obvious APP factorization (1.3) with $G_\pm = 1, D = e_\lambda$. Note that the case $n=2$ is also elementary, and has been treated in passing a long time ago (see [14, 15]). So, we may suppose that $n \geqslant 3$.

As the induction step, we will show how to obtain an APP factorization of $n \times n$ matrix functions of the given pattern, provided that such a factorization exists for similarly patterned matrices of a smaller size.

The reduction is obvious if $a_j = 0$ for some j, because then G splits into the direct sum of an 1×1 block e_{λ_j} and an $(n-1)\times(n-1)$ matrix function of the same type. So, without loss of generality $a_j \neq 0$ for all j. Multiplying the jth row of G by a_j while dividing its jth column by the same number for $j = 1,\dots,n-1$, we may even without loss of generality suppose that $a_1 = \dots = a_{n-1} = 1$, that is,

$$G = \begin{bmatrix} e_{\lambda_1} & & & & \\ & e_{\lambda_2} & & & \\ & & \ddots & & \\ & & & e_{\lambda_{n-1}} & \\ e_{\gamma_1} & e_{\gamma_2} & \dots & e_{\gamma_{n-1}} & e_{\lambda_n} \end{bmatrix}. \tag{2.1}$$

If $\lambda_i \leqslant \gamma_i$ for some i, then multiplication of G on the left by $F = I - e_{\gamma_i - \lambda_i} E_{ni}$ cancels out the (n,i)-element e_{γ_i} of G, leaving others unchanged. Since obviously $F \in \mathscr{G}(APP^{+}_{n\times n})$, this has no effect on the APP factorability of G while reducing the situation to already considered.

On the other hand, multiplication on the right by $I - e_{\gamma_i - \lambda_n} E_{ni}$ has a similar effect, while this matrix function belongs to $\mathscr{G}(APP^{-}_{n\times n})$ whenever $\gamma_i \leqslant \lambda_n$. So, without loss of generality we may suppose that

$$\lambda_i > \gamma_i > \lambda_n, \quad i = 1,\dots,n-1. \tag{2.2}$$

By a permutational similarity it can be arranged that, in addition,

$$\gamma_1 \leqslant \gamma_2 \leqslant \dots \leqslant \gamma_{n-1}. \tag{2.3}$$

We now consider two complementary cases under the conditions (2.3).

Case 1. For some $i \in \{2,\dots,n-1\}$, the inequalities

$$\lambda_i - \lambda_1 \geqslant \gamma_i - \gamma_1 (\geqslant 0) \tag{2.4}$$

hold. Due to (2.4), multiplication of G by $I + e_{\lambda_i - \gamma_i - \lambda_1 + \gamma_1} E_{i1}$ on the left and $I - e_{\gamma_1 - \gamma_i} E_{i1}$ on the right does not change its APP factorability properties. But upon this multiplication G becomes a direct sum of e_{λ_1} with the $(n-1)\times(n-1)$ matrix of the same structure, which is obtained from (2.1) by deleting the first row and the first column.

Case 2. It remains to consider the situation when $\lambda_i - \lambda_1 < \gamma_i - \gamma_1$ for all $i = 2,3,\dots,n-1$ or, equivalently, when

$$(0 <)\ \lambda_i - \gamma_i < \lambda_1 - \gamma_1, \quad i = 2,\dots,n-1. \tag{2.5}$$

Let us multiply G on the left by

$$\left(I - e_{\lambda_1-\gamma_1}E_{1n}\right)\prod_{i=2}^{n-1}\left(I + e_{\lambda_1-\gamma_1-\lambda_i+\gamma_i}E_{1i}\right)$$

and on the right by $I - e_{\lambda_n-\gamma_1}E_{1n}$. Conditions (2.2), (2.5) guarantee that there will be no change in APP factorability. Finally (also without influencing APP factorability) switch the first and the last columns of the resulting matrix. Simple computations show that this sequence of operations results in the direct sum of $-e_{\lambda_n+\lambda_1-\gamma_1}$ with

$$\begin{bmatrix} e_{\lambda_2} & & & & \\ & e_{\lambda_3} & & & \\ & & \ddots & & \\ & & & e_{\lambda_{n-1}} & \\ e_{\gamma_2} & e_{\gamma_3} & \dots & e_{\gamma_{n-1}} & e_{\gamma_1} \end{bmatrix}. \tag{2.6}$$

Since (2.6) has the same structure as (2.1) but its size is one less than that of the latter, we are done. □

Note that arguments used in Case 2 of the proof of Theorem 2.1 are also valid if the inequalities in (2.5) are non-strict.

3. Canonical factorization

The proof of Theorem 2.1 actually provides an algorithm allowing to construct explicitly the factorization of matrices (1.6) satisfying (1.8). Examples illustrating this point will be provided in Section 4. In the meanwhile, we would like to address the question when the resulting factorization is canonical.

PROPOSITION 3.1. *Let G be given by (1.6), with the off-diagonal entries g_j satisfying (1.8). Then for G to admit a canonical AP factorization it is necessary that $\lambda_j = 0$ whenever at least one of the conditions*

$$a_j = 0,\ \gamma_j \leqslant \lambda_n, \gamma_j \geqslant \lambda_j,\ \text{or}\ \lambda_i - \lambda_j \geqslant \gamma_i - \gamma_j\ \text{for some}\ i \neq j \tag{3.1}$$

holds, and $\lambda_j \geqslant 0$ for all other values of $j = 1,\dots,n-1$.

Proof. From the proof of Theorem 2.1 it immediately follows that, whenever one of the conditions (3.1) holds (the latter — with $j = 1$), the respective λ_j is a partial index of G. For $j \neq 1$ the reasoning for the last case of (3.1) has to be modified slightly. Namely, it suffices to observe that multiplication of G by $I + e_{\lambda_i-\gamma_i-\lambda_j+\gamma_j}E_{ij}$ on the left and $I - e_{\gamma_j-\gamma_i}E_{ij}$ on the right does not change its APP factorability properties. But upon this multiplication G becomes a direct sum of e_{λ_j} with an $(n-1)\times(n-1)$ matrix of the same structure. This proves that indeed λ_j must equal zero whenever one of the conditions (3.1) holds in order for G to admit a canonical AP factorization.

Moreover, for any j satisfying one of the conditions (3.1), the respective row and column of G can be deleted, and the elements of the resulting matrix renumbered accordingly. It remains therefore to consider matrix functions (2.1) for which (2.2) and (2.5) hold. We may (and will) also without loss of generality impose the ordering (2.3). But this puts us into the setting of Case 2 in the proof of Theorem 2.1. As was established there, $\lambda_n + \lambda_1 - \gamma_1$ then emerges as one of the partial indices. So, in order for the factorization to be canonical, we must have $\gamma_1 = \lambda_1 + \lambda_n$. When combined with the second inequality in (2.2) for $i = 1$, this implies positivity of λ_1.

Furthermore, the matrix (2.6) should admit a canonical factorization along with the given G. Then, by the part already proved, $\lambda_2 = 0$ if $\gamma_2 = \gamma_1$, and $\lambda_2 > 0$ if $\gamma_2 > \gamma_1$. The mathematical induction principle thus completes the proof. □

Observe that the last diagonal exponent λ_n of the matrix (1.6) admitting a canonical factorization must of course be non-positive, due to (1.7).

Necessary and sufficient conditions for (1.6) to admit a canonical factorization in principle can be stated in general, though for large n they get rather convoluted because of the growing number of subcases to be considered. One particular situation, having practical importance, can however be treated more easily. To simplify the statement, and without loss of generality, we will suppose that (2.3) holds.

THEOREM 3.2. *Let G be of the form* (1.6)*, with the elements of the last row given by* (1.8) *satisfying* (2.3)*, and such that $\lambda_j \neq 0$, $j = 1, \dots, n-1$. Then its AP factorization is canonical if and only if in fact $\lambda_j > 0$, $a_j \neq 0$ for $j = 1, \dots, n-1$, while λ_n satisfies* (1.7)*, and*

$$\gamma_i = \lambda_1 + \cdots + \lambda_i + \lambda_n, \quad i = 1, \dots, n-1. \tag{3.2}$$

Proof. Necessity. Positivity of λ_j and condition $a_j \neq 0$ for $j = 1, \dots, n-1$ follow immediately from Proposition 3.1; the necessity of (1.7) was mentioned earlier as well. We also see from the proof of Theorem 2.1 that Case 1 should not materialize, and (3.2) holds for $i = 1$. Moreover, the matrix (2.6) should admit a canonical factorization along with (1.6). Observe that $\gamma_2 > \gamma_1$ because otherwise $\lambda_2 \neq 0$ becomes a partial AP index of the matrix (2.6), which contradicts its canonical AP factorability. A simple recursive procedure, with an obvious notational adjustment, yields all other equalities in (3.2). In particular, taking into account the equality $\gamma_1 = \lambda_1 + \lambda_n$, we deduce for the matrix (2.6) that $\gamma_2 = \lambda_2 + \gamma_1 = \lambda_1 + \lambda_2 + \lambda_n$, and so on.

Sufficiency. Conditions imposed on the signs of λ_j and formulas (3.2) guarantee that (2.2), (2.5) hold. Equivalently, we are in the Case 2 setting of the proof of Theorem 2.1. Consequently, one of the partial indices of G is $\lambda_n + \lambda_1 - \gamma_1 = 0$, and the others coincide with the partial indices of (2.6). But the latter matrix (once again, after an obvious notational adjustment) satisfies the alleged sufficiency conditions, along with the given one. Since the $n = 2$ case is known, this completes the proof. □

Note that the sufficiency was also proved in [21], though in a slightly different way. We provided the proof here for consistency and convenience of reference.

As a matter of fact, it is not hard to explicitly construct a canonical factorization of G in the setting of Theorem 3.2.

THEOREM 3.3. *Let G be given by*

$$G = \begin{bmatrix} e_{\lambda_1} & & & & \\ & e_{\lambda_2} & & & \\ & & \ddots & & \\ & & & e_{\lambda_{n-1}} & \\ a_1 e_{\gamma_1} & a_2 e_{\gamma_1} & \dots & a_{n-1} e_{\gamma_{n-1}} & e_{\lambda_n} \end{bmatrix}, \tag{3.3}$$

where $\lambda_j > 0$ and $a_j \neq 0$ for $j = 1, \dots, n-1$, λ_n satisfies (1.7), and (3.2) holds. Then G admits a canonical left APP factorization $G = G_+ G_-$ with

$$G_+ = \begin{bmatrix} a_1^{-1} & -a_1^{-1} e_{\lambda_1} & & & & \\ & a_2^{-1} & -a_2^{-1} e_{\lambda_2} & & & \\ & & \ddots & \ddots & & \\ & & & a_{n-2}^{-1} & -a_{n-2}^{-1} e_{\lambda_{n-2}} & \\ & & & & a_{n-1}^{-1} & a_{n-1}^{-1} e_{\lambda_{n-1}} \\ & & & & & 1 \end{bmatrix},$$

$$G_- = \begin{bmatrix} a_1^{-1} e_{-\lambda_1} & -a_1^{-1} & & & & \\ & a_2^{-1} e_{-\lambda_2} & -a_2^{-1} & & & \\ & & \ddots & \ddots & & \\ & & & a_{n-2}^{-1} e_{-\lambda_{n-2}} & -a_{n-2}^{-1} & \\ & & & & a_{n-1}^{-1} e_{-\lambda_{n-1}} & a_{n-1}^{-1} \\ -1 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}^{-1}$$

Proof. A direct computation shows in view of (1.7) and (3.2) that $GG_-^{-1} = G_+$, while $G_\pm, G_\pm^{-1} \in APP^\pm_{n\times n}$. $\square$

4. Partial indices

We now discuss in more detail how to determine the partial AP indices of the matrix function (2.1) under conditions (2.2) and (2.3). To this end, let us introduce recursively a sequence of matrix functions G_k of the same structure as (2.1) but decreasing in size. Namely, let

$$G_k = \begin{bmatrix} e_{\lambda_{1,k}} & & & & \\ & e_{\lambda_{2,k}} & & & \\ & & \ddots & & \\ & & & e_{\lambda_{n-k-1,k}} & \\ e_{\gamma_{1,k}} & e_{\gamma_{2,k}} & \dots & e_{\gamma_{n-k-1,k}} & e_{\lambda_{n-k,k}} \end{bmatrix}, \quad k = 0, \dots, n-1, \tag{4.1}$$

where

$$\lambda_{j,k} = \lambda_{j+k},\ \gamma_{j,k} = \gamma_{j+k}, \quad (j = 1, \dots, n-k-1),$$

while the right lower entry is defined recursively by the rule $\lambda_{n,0} = \lambda_n$ (so that G_0 coincides with G given by (2.1)) and, for $k = 1, \ldots, n-1$,

$$\lambda_{n-k,k} = \begin{cases} \lambda_{n-k+1,k-1} & \text{if } \lambda_{i,k-1} - \lambda_{1,k-1} \geqslant \gamma_{i,k-1} - \gamma_{1,k-1} \\ & \text{for some } i = 2, \ldots, n-k, \\ \gamma_k & \text{otherwise.} \end{cases} \tag{4.2}$$

Notice that $\lambda_{1,n-1} = \gamma_{n-1}$ in view of (4.2).

Let us also introduce a binary string $\mathscr{M} = [m_1, \ldots, m_{n-1}]$ associated with the matrix (2.1) in the following way: m_j equals 1 or 2, depending on whether the first or the second case of (4.2) takes place at the jth iteration. Clearly, $m_{n-1} = 2$ because $\lambda_{1,n-1} = \gamma_{n-1}$.

THEOREM 4.1. *Let G be given by* (2.1)*, with the associated string*

$$\mathscr{M} = [m_1, \ldots, m_{n-1}].$$

Then the partial AP indices $\mu_1, \ldots, \mu_n$ of G can be computed by the rule: for $k = 1, 2, \ldots, n-1$,

$$\mu_k = \begin{cases} \lambda_k & \text{if } m_k = 1, \\ \lambda_n + \lambda_k - \gamma_k & \text{if } m_k \text{ is the first digit } 2 \text{ in } \mathscr{M}, \\ \gamma_j + \lambda_k - \gamma_k & \text{if } m_j = m_k = 2 \text{ for some } j < k, \\ & \text{and there are no } 2\text{s between } m_j \text{ and } m_k, \end{cases} \tag{4.3}$$

and $\mu_n = \gamma_{n-1}$.

Proof. According to the procedure described in the proof of Theorem 2.1, for every $k = 1, \ldots, n-1$, the matrix function G_{k-1} via multiplication on the left (right) by an appropriate element of $\mathscr{G}(APP^+_{(n-k+1)\times(n-k+1)})$ (resp. $\mathscr{G}(APP^-_{(n-k+1)\times(n-k+1)})$) can be reduced to the direct sum of G_k with a singleton e_{ν_k}, where

$$\nu_k = \begin{cases} \lambda_{1,k-1} & \text{if } m_k = 1, \\ \lambda_{n-k+1,k-1} + \lambda_{1,k-1} - \gamma_{1,k-1} & \text{if } m_k = 2. \end{cases}$$

From (4.2) it is easy to see that in fact $\nu_k = \mu_k$ for $k = 1, \ldots, n-1$, with μ_k given by (4.3). On the other hand, since $G_{n-1} = e_{\lambda_{1,n-1}} = e_{\gamma_{n-1}}$, we deduce that $\mu_n = \gamma_{n-1}$. So, starting with $G_0 = G$ in $n-1$ steps we will end up with $\mu_1, \ldots, \mu_n$ as the set of all partial AP indices. $\square$

For example, for the 5×5 matrix function G we obtain the following possible 5-tuples of partial AP indices:

$$\begin{aligned} \{\lambda_1, \lambda_2, \lambda_3, \lambda_5 + \lambda_4 - \gamma_4, \gamma_4\} \quad & \text{if } \mathscr{M} = [1,1,1,2], \\ \{\lambda_1, \lambda_2, \lambda_5 + \lambda_3 - \gamma_3, \gamma_3 + \lambda_4 - \gamma_4, \gamma_4\} \quad & \text{if } \mathscr{M} = [1,1,2,2], \end{aligned}$$

$$\begin{aligned}
\{\lambda_1,\lambda_5+\lambda_2-\gamma_2,\lambda_3,\gamma_2+\lambda_4-\gamma_4,\gamma_4\} &\quad \text{if } \mathscr{M}=[1,2,1,2],\\
\{\lambda_1,\lambda_5+\lambda_2-\gamma_2,\gamma_2+\lambda_3-\gamma_3,\gamma_3+\lambda_4-\gamma_4,\gamma_4\} &\quad \text{if } \mathscr{M}=[1,2,2,2],\\
\{\lambda_5+\lambda_1-\gamma_1,\lambda_2,\lambda_3,\gamma_1+\lambda_4-\gamma_4,\gamma_4\} &\quad \text{if } \mathscr{M}=[2,1,1,2],\\
\{\lambda_5+\lambda_1-\gamma_1,\lambda_2,\gamma_1+\lambda_3-\gamma_3,\gamma_3+\lambda_4-\gamma_4,\gamma_4\} &\quad \text{if } \mathscr{M}=[2,1,2,2],\\
\{\lambda_5+\lambda_1-\gamma_1,\gamma_1+\lambda_2-\gamma_2,\lambda_3,\gamma_2+\lambda_4-\gamma_4,\gamma_4\} &\quad \text{if } \mathscr{M}=[2,2,1,2],\\
\{\lambda_5+\lambda_1-\gamma_1,\gamma_1+\lambda_2-\gamma_2,\gamma_2+\lambda_3-\gamma_3,\gamma_3+\lambda_4-\gamma_4,\gamma_4\} &\quad \text{if } \mathscr{M}=[2,2,2,2].
\end{aligned}$$

5. Case $n=3$

To illustrate the factorization approach described in the proof of Theorem 2.1, in this section we construct an explicit left APP factorizations (1.3) of matrix functions (1.6) satisfying (1.8) in the case when $n=3$. That is, throughout this section

$$G=\begin{bmatrix} e_{\lambda_1} & 0 & 0\\ 0 & e_{\lambda_2} & 0\\ a_1e_{\gamma_1} & a_2e_{\gamma_2} & e_{\lambda_3}\end{bmatrix} \quad \text{with} \quad a_1a_2\neq 0. \tag{5.1}$$

Naturally, formulas for the factors depend on the relations between the exponents γ_j,λ_j.

i) Let us start with the situations in which the partial indices coincide with the indices of the diagonal entries, that is,

$$D=\operatorname{diag}[e_{\lambda_1},e_{\lambda_2},e_{\lambda_3}].$$

a) If $\gamma_1\geqslant\lambda_1$ and $\gamma_2\geqslant\lambda_2$, then

$$G_+=\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ a_1e_{\gamma_1-\lambda_1} & a_2e_{\gamma_2-\lambda_2} & 1\end{bmatrix},\quad G_-=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

b) If $\gamma_1\geqslant\lambda_1$ and $\gamma_2\leqslant\lambda_3$, then

$$G_+=\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ a_1e_{\gamma_1-\lambda_1} & 0 & 1\end{bmatrix},\quad G_-=\begin{bmatrix}1&0&0\\0&1&0\\0&a_2e_{\gamma_2-\lambda_3}&1\end{bmatrix}.$$

c) If $\gamma_1\leqslant\lambda_3$ and $\gamma_2\geqslant\lambda_2$, then

$$G_+=\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & a_2e_{\gamma_2-\lambda_2} & 1\end{bmatrix},\quad G_-=\begin{bmatrix}1&0&0\\0&1&0\\a_1e_{\gamma_1-\lambda_3}&0&1\end{bmatrix}.$$

d) If $\gamma_1\leqslant\lambda_3$ and $\gamma_2\leqslant\lambda_3$, then

$$G_+=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad G_-=\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ a_1e_{\gamma_1-\lambda_3} & a_2e_{\gamma_2-\lambda_3} & 1\end{bmatrix}.$$

ii) Let now $\lambda_1 > \gamma_1 > \lambda_3$ and either a) $\gamma_2 \geqslant \lambda_2$, or b) $\gamma_2 \leqslant \lambda_3$.

a) If $\gamma_2 \geqslant \lambda_2$ then, disposing of $a_2 e_{\gamma_2}$ as in subcase a) of case i) and applying the factorization

$$\begin{bmatrix} e_{\lambda_1} & 0 \\ a_1 e_{\gamma_1} & e_{\lambda_3} \end{bmatrix} = \begin{bmatrix} a_1^{-1} & a_1^{-1} e_{\lambda_1-\gamma_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{\lambda_1+\lambda_3-\gamma_1} & 0 \\ 0 & e_{\gamma_1} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ a_1 & e_{\lambda_3-\gamma_1} \end{bmatrix}, \tag{5.2}$$

we obtain the following factors of a left APP factorization of G:

$$G_+ = \begin{bmatrix} a_1^{-1} & 0 & a_1^{-1} e_{\lambda_1-\gamma_1} \\ 0 & 1 & 0 \\ 0 & a_2 e_{\gamma_2-\lambda_2} & 1 \end{bmatrix}, \quad D = \begin{bmatrix} e_{\lambda_1+\lambda_3-\gamma_1} & 0 & 0 \\ 0 & e_{\lambda_2} & 0 \\ 0 & 0 & e_{\gamma_1} \end{bmatrix},$$
$$G_- = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ a_1 & 0 & e_{\lambda_3-\gamma_1} \end{bmatrix}.$$

b) If $\gamma_2 \leqslant \lambda_3$ then, disposing of $a_2 e_{\gamma_2}$ as in subcase b) of case i) and applying factorization (5.2), we obtain

$$G_+ = \begin{bmatrix} a_1^{-1} & 0 & a_1^{-1} e_{\lambda_1-\gamma_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} e_{\lambda_1+\lambda_3-\gamma_1} & 0 & 0 \\ 0 & e_{\lambda_2} & 0 \\ 0 & 0 & e_{\gamma_1} \end{bmatrix},$$
$$G_- = \begin{bmatrix} 0 & -a_2 e_{\gamma_2-\lambda_3} & -1 \\ 0 & 1 & 0 \\ a_1 & a_2 e_{\gamma_2-\gamma_1} & e_{\lambda_3-\gamma_1} \end{bmatrix}.$$

iii) Let $\lambda_2 > \gamma_2 > \lambda_3$ and either a) $\gamma_1 \geqslant \lambda_1$, or b) $\gamma_1 \leqslant \lambda_3$.

a) If $\gamma_1 \geqslant \lambda_1$ then, getting rid of $a_1 e_{\gamma_1}$ as in subcase a) of case i) and applying the factorization

$$\begin{bmatrix} e_{\lambda_2} & 0 \\ a_2 e_{\gamma_2} & e_{\lambda_3} \end{bmatrix} = \begin{bmatrix} a_2^{-1} & a_2^{-1} e_{\lambda_2-\gamma_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{\lambda_2+\lambda_3-\gamma_2} & 0 \\ 0 & e_{\gamma_2} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ a_2 & e_{\lambda_3-\gamma_2} \end{bmatrix}, \tag{5.3}$$

we obtain the following factors of a left APP factorization of G:

$$G_+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_2^{-1} & a_2^{-1} e_{\lambda_2-\gamma_2} \\ a_1 e_{\gamma_1-\lambda_1} & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} e_{\lambda_1} & 0 & 0 \\ 0 & e_{\lambda_2+\lambda_3-\gamma_2} & 0 \\ 0 & 0 & e_{\gamma_2} \end{bmatrix},$$
$$G_- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & a_2 & e_{\lambda_3-\gamma_2} \end{bmatrix}.$$

b) If $\gamma_1 \leqslant \lambda_3$ then, disposing of $a_1 e_{\gamma_1}$ as in subcase c) of case i) and applying the factorization (5.3), we obtain

$$G_+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_2^{-1} & a_2^{-1} e_{\lambda_2-\gamma_2} \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} e_{\lambda_1} & 0 & 0 \\ 0 & e_{\lambda_2+\lambda_3-\gamma_2} & 0 \\ 0 & 0 & e_{\gamma_2} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 \\ -a_1 e_{\gamma_1-\lambda_3} & 0 & -1 \\ a_1 e_{\gamma_1-\gamma_2} & a_2 & e_{\lambda_3-\gamma_2} \end{bmatrix}.$$

It remains to consider a more involved situation when $\lambda_1 > \gamma_1 > \lambda_4$ and $\lambda_2 > \gamma_2 > \lambda_3$. We summarize results for this case in the following theorem.

THEOREM 5.1. *Let G be the matrix function given by* (5.1) *with*

$$\lambda_1 > \gamma_1 > \lambda_3, \ \ \lambda_2 > \gamma_2 > \lambda_3.$$

Then G admits a left APP factorization $G = G_+ D G_-$ with factors given by:

$$G_+ = \begin{bmatrix} a_1^{-1} & 0 & 0 \\ -a_2^{-1} e_{\lambda_2-\lambda_1+\gamma_1-\gamma_2} & a_2^{-1} & a_2^{-1} e_{\lambda_2-\gamma_2} \\ 0 & 0 & 1 \end{bmatrix}, \ D = \begin{bmatrix} e_{\lambda_1} & 0 & 0 \\ 0 & e_{\lambda_2+\lambda_3-\gamma_2} & 0 \\ 0 & 0 & e_{\gamma_2} \end{bmatrix},$$
$$G_- = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & -1 \\ a_1 e_{\gamma_1-\gamma_2} & a_2 & e_{\lambda_3-\gamma_2} \end{bmatrix} \tag{5.4}$$

if $\lambda_2 - \lambda_1 \geqslant \gamma_2 - \gamma_1$ and $\gamma_2 \geqslant \gamma_1$;

$$G_+ = \begin{bmatrix} a_1^{-1} & -a_1^{-1} e_{\gamma_2-\gamma_1-\lambda_2+\lambda_1} & 0 \\ 0 & a_2^{-1} & a_2^{-1} e_{\lambda_2-\gamma_2} \\ 0 & 0 & 1 \end{bmatrix},$$
$$D = \begin{bmatrix} e_{\lambda_3+\lambda_1-\gamma_1} & 0 & 0 \\ 0 & e_{\gamma_1+\lambda_2-\gamma_2} & 0 \\ 0 & 0 & e_{\gamma_2} \end{bmatrix}, \ G_- = \begin{bmatrix} 0 & 0 & -1 \\ -a_1 & 0 & -e_{\lambda_3-\gamma_1} \\ a_1 e_{\gamma_1-\gamma_2} & a_2 & e_{\lambda_3-\gamma_2} \end{bmatrix} \tag{5.5}$$

if $\lambda_2 - \lambda_1 \leqslant \gamma_2 - \gamma_1$ and $\gamma_2 \geqslant \gamma_1$;

$$G_+ = \begin{bmatrix} a_1^{-1} & -a_1^{-1} e_{\gamma_2-\gamma_1-\lambda_2+\lambda_1} & a_1^{-1} e_{\lambda_1-\gamma_1} \\ 0 & a_2^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ D = \begin{bmatrix} e_{\lambda_1+\lambda_3-\gamma_1} & 0 & 0 \\ 0 & e_{\lambda_2} & 0 \\ 0 & 0 & e_{\gamma_1} \end{bmatrix},$$
$$G_- = \begin{bmatrix} 0 & 0 & -1 \\ 0 & a_2 & 0 \\ a_1 & a_2 e_{\gamma_2-\gamma_1} & e_{\lambda_3-\gamma_1} \end{bmatrix} \tag{5.6}$$

if $\lambda_2 - \lambda_1 \leqslant \gamma_2 - \gamma_1$ and $\gamma_2 \leqslant \gamma_1$;

$$G_+ = \begin{bmatrix} a_1^{-1} & 0 & a_1^{-1} e_{\lambda_1-\gamma_1} \\ -a_2^{-1} e_{\lambda_2-\lambda_1-\gamma_2+\gamma_1} & a_2^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$D = \begin{bmatrix} e_{\gamma_2+\lambda_1-\gamma_1} & 0 & 0 \\ 0 & e_{\lambda_3+\lambda_2-\gamma_2} & 0 \\ 0 & 0 & e_{\gamma_1} \end{bmatrix}, \ G_- = \begin{bmatrix} 0 & -a_2 & -e_{\lambda_3-\gamma_2} \\ 0 & 0 & -1 \\ a_1 & a_2 e_{\gamma_2-\gamma_1} & e_{\lambda_3-\gamma_1} \end{bmatrix} \tag{5.7}$$

if $\lambda_2 - \lambda_1 \geqslant \gamma_2 - \gamma_1$ and $\gamma_2 \leqslant \gamma_1$.

Proof. We will treat separately the following four cases a)–d).

a) If $\lambda_2-\lambda_1\geqslant\gamma_2-\gamma_1\geqslant 0$, then in view of Theorem 2.1 we obtain

$$G=A^{-1}\begin{bmatrix}1 & 0 & 0\\ -e_{\lambda_2-\lambda_1+\gamma_1-\gamma_2} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\widetilde{G}\begin{bmatrix}1 & 0 & 0\\ e_{\gamma_1-\gamma_2} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A, \tag{5.8}$$

where

$$A:=\operatorname{diag}\{a_1,a_2,1\},\quad \widetilde{G}:=\begin{bmatrix}e_{\lambda_1} & 0 & 0\\ 0 & e_{\lambda_2} & 0\\ 0 & e_{\gamma_2} & e_{\lambda_3}\end{bmatrix}. \tag{5.9}$$

Applying (5.3), we obtain the left APP factorization

$$\widetilde{G}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & e_{\lambda_2-\gamma_2}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}e_{\lambda_1} & 0 & 0\\ 0 & e_{\lambda_2+\lambda_3-\gamma_2} & 0\\ 0 & 0 & e_{\gamma_2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & e_{\lambda_3-\gamma_2}\end{bmatrix},$$

which together with (5.8) gives the left APP factorization $G=G_+DG_-$ with factors defined by (5.4).

b) If $\lambda_2-\lambda_1\leqslant\gamma_2-\gamma_1$ and $\gamma_2\geqslant\gamma_1$, then in view of Theorem 2.1 we get

$$G=A^{-1}\begin{bmatrix}1 & -e_{\gamma_2-\gamma_1-\lambda_2+\lambda_1} & e_{\lambda_1-\gamma_1}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\widetilde{G}\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & e_{\lambda_3-\gamma_1}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A, \tag{5.10}$$

where A is given by (5.9) and

$$\widetilde{G}:=\begin{bmatrix}e_{\lambda_3+\lambda_1-\gamma_1} & 0 & 0\\ 0 & e_{\lambda_2} & 0\\ 0 & e_{\gamma_2} & e_{\gamma_1}\end{bmatrix}.$$

Further, by analogy with (5.3), we infer that

$$\widetilde{G}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & e_{\lambda_2-\gamma_2}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}e_{\lambda_3+\lambda_1-\gamma_1} & 0 & 0\\ 0 & e_{\gamma_1+\lambda_2-\gamma_2} & 0\\ 0 & 0 & e_{\gamma_2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & e_{\gamma_1-\gamma_2}\end{bmatrix},$$

which together with (5.10) gives the left APP factorization $G=G_+DG_-$ with factors defined by (5.5).

The remaining two cases c) $\lambda_2-\lambda_1\leqslant\gamma_2-\gamma_1$ and $\gamma_2\leqslant\gamma_1$, as well as d) $\lambda_2-\lambda_1\geqslant\gamma_2-\gamma_1$ and $\gamma_2\leqslant\gamma_1$ are reduced to cases a) and b) by the simultaneous transposition of the the first and second columns and rows. This gives the factors of the left APP factorization of G defined by (5.6) in case c) and by (5.7) in case d). $\square$

Theorems 3.2 and 5.1 imply the following.

COROLLARY 5.2. *If the matrix function G is given by (5.1) with $a_1a_2\neq 0$ and $\lambda_1\lambda_2\neq 0$, then G admits a canonical left AP factorization if and only if $\lambda_1,\lambda_2>0$,*

$\lambda_3 = -\lambda_1 - \lambda_2$, *and either* $\gamma_1 = -\lambda_2$ *and* $\gamma_2 = 0$, *or* $\gamma_1 = 0$ *and* $\gamma_2 = -\lambda_1$. *In these cases the geometric mean* $\mathbf{d}(G)$ *is given, respectively, by*

$$\mathbf{d}(G) = \begin{bmatrix} 0 & 0 & -a_1^{-1} \\ -a_1 a_2^{-1} & 0 & 0 \\ 0 & a_2 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{d}(G) = \begin{bmatrix} 0 & -a_1^{-1}a_2 & 0 \\ 0 & 0 & -a_2^{-1} \\ a_1 & 0 & 0 \end{bmatrix}.$$

Proof. Since $\lambda_1\lambda_2 \neq 0$, from the proof of Theorem 2.1 it follows that the canonical left AP (equivalently, APP) factorization does not occur in the situations i)–iii) treated earlier in this Section. Hence, Theorems 3.2 and 5.1 immediately imply all conditions of the theorem on λ_i $(i = 1,2,3)$ and γ_i $(i = 1,2)$, which yield zero partial AP indices. The canonical left AP factorization can occur only with factors given by (5.5) or by (5.7), which correspond to the cases a) $\gamma_1 = -\lambda_2$ and $\gamma_2 = 0$, and b) $\gamma_1 = 0$ and $\gamma_2 = -\lambda_1$, respectively. Applying these equalities, we infer from (5.5) and (5.7) that the geometric mean $\mathbf{d}(G) = M(G_+)M(G_-)$ of G in case a) is determined by

$$\mathbf{d}(G) = \begin{bmatrix} a_1^{-1} & 0 & 0 \\ 0 & a_2^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_1^{-1} \\ -a_1 a_2^{-1} & 0 & 0 \\ 0 & a_2 & 0 \end{bmatrix},$$

and in case b) is determined by

$$\mathbf{d}(G) = \begin{bmatrix} a_1^{-1} & 0 & 0 \\ 0 & a_2^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -a_2 & 0 \\ 0 & 0 & -1 \\ a_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a_1^{-1}a_2 & 0 \\ 0 & 0 & -a_2^{-1} \\ a_1 & 0 & 0 \end{bmatrix},$$

which completes the proof. $\square$

6. Case $n = 4$

In this section we consider the triangular 4×4 matrix function G given by

$$G = \begin{bmatrix} e_{\lambda_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_2} & 0 & 0 \\ 0 & 0 & e_{\lambda_3} & 0 \\ a_1 e_{\gamma_1} & a_2 e_{\gamma_2} & a_3 e_{\gamma_3} & e_{\lambda_4} \end{bmatrix} \quad \text{with} \quad a_1 a_2 a_3 \neq 0. \tag{6.1}$$

We first dispose of the cases
1a) $\gamma_1 \geqslant \lambda_1$ or 1b) $\gamma_1 \leqslant \lambda_4$;
2a) $\gamma_2 \geqslant \lambda_2$ or 2b) $\gamma_2 \leqslant \lambda_4$;
3a) $\gamma_3 \geqslant \lambda_1$ or 3b) $\gamma_3 \leqslant \lambda_4$
in which the reduction to a smaller size is easy. In each of these cases we will produce matrix functions $\widetilde{G}_\pm \in \mathscr{G}(APP^\pm_{4\times 4})$ such that $\widetilde{G} = \widetilde{G}_+ G \widetilde{G}_-$ splits into the direct sum of a 1-by-1 block and the matrix

$$\widehat{G} = \begin{bmatrix} e_{\widetilde{\lambda}_1} & 0 & 0 \\ 0 & e_{\widetilde{\lambda}_2} & \\ \widetilde{a}_1 e_{\widetilde{\gamma}_1} & \widetilde{a}_2 e_{\widetilde{\gamma}_2} & e_{\widetilde{\lambda}_3} \end{bmatrix}, \tag{6.2}$$

of the form considered in Section 5. More specifically:

Case 1).

$$\widetilde{G}_+ = \begin{bmatrix} 1 & 0\,0\,0 \\ 0 & 1\,0\,0 \\ 0 & 0\,1\,0 \\ -\varepsilon_+ a_1 e_{\gamma_1-\lambda_1} & 0\,0\,1 \end{bmatrix}, \quad \widetilde{G}_- = \begin{bmatrix} 1 & 0\,0\,0 \\ 0 & 1\,0\,0 \\ 0 & 0\,1\,0 \\ -\varepsilon_- a_1 e_{\gamma_1-\lambda_4} & 0\,0\,1 \end{bmatrix}, \tag{6.3}$$

with $\varepsilon_+ = 1$ and $\varepsilon_- = 0$ in subcase 1a), $\varepsilon_+ = 0$ and $\varepsilon_- = 1$ in subcase 1b), while in (6.2)

$$\widetilde{\lambda}_1 = \lambda_2,\ \widetilde{\lambda}_2 = \lambda_3,\ \widetilde{\lambda}_3 = \lambda_4,\ \widetilde{\gamma}_1 = \gamma_2,\ \widetilde{\gamma}_2 = \gamma_3,\ \widetilde{a}_1 = a_2,\ \widetilde{a}_2 = a_3.$$

Case 2).

$$\widetilde{G}_+ = \begin{bmatrix} 1 & 0 & 0\,0 \\ 0 & 1 & 0\,0 \\ 0 & 0 & 1\,0 \\ 0 & -\varepsilon_+ a_2 e_{\gamma_2-\lambda_2} & 0\,1 \end{bmatrix}, \quad \widetilde{G}_- = \begin{bmatrix} 1 & 0 & 0\,0 \\ 0 & 1 & 0\,0 \\ 0 & 0 & 1\,0 \\ 0 & -\varepsilon_- a_2 e_{\gamma_2-\lambda_4} & 0\,1 \end{bmatrix}, \tag{6.4}$$

with $\varepsilon_+ = 1$ and $\varepsilon_- = 0$ in subcase 2a), $\varepsilon_+ = 0$ and $\varepsilon_- = 1$ in subcase 2b), while in (6.2)

$$\widetilde{\lambda}_1 = \lambda_1,\ \ \widetilde{\lambda}_2 = \lambda_3,\ \ \widetilde{\lambda}_3 = \lambda_4.\ \ \widetilde{\gamma}_1 = \gamma_1,\ \ \widetilde{\gamma}_2 = \gamma_3,\ \widetilde{a}_1 = a_1,\ \widetilde{a}_2 = a_3.$$

Case 3).

$$\widetilde{G}_+ = \begin{bmatrix} 1\,0 & 0 & 0 \\ 0\,1 & 0 & 0 \\ 0\,0 & 1 & 0 \\ 0\,0 & -\varepsilon_+ a_3 e_{\gamma_3-\lambda_3} & 1 \end{bmatrix}, \quad \widetilde{G}_- = \begin{bmatrix} 1\,0 & 0 & 0 \\ 0\,1 & 0 & 0 \\ 0\,0 & 1 & 0 \\ 0\,0 & -\varepsilon_- a_3 e_{\gamma_3-\lambda_4} & 1 \end{bmatrix}, \tag{6.5}$$

where $\varepsilon_+ = 1$ and $\varepsilon_- = 0$ in subcase 3a), $\varepsilon_+ = 0$ and $\varepsilon_- = 1$ in subcase 3b), and in (6.2)

$$\widetilde{\lambda}_1 = \lambda_1,\ \ \widetilde{\lambda}_2 = \lambda_2,\ \ \widetilde{\lambda}_3 = \lambda_4,\ \ \widetilde{\gamma}_1 = \gamma_1,\ \ \widetilde{\gamma}_2 = \gamma_2,\ \widetilde{a}_1 = a_1,\ \widetilde{a}_2 = a_2.$$

The factorization process can thus be completed by applying the results of Section 5 to $\widehat{G}$ given by (6.2).

Let us now pass to the remaining, more interesting, situation, when in (6.1) we have

$$\lambda_i > \gamma_i > \lambda_4,\ i = 1,2,3. \tag{6.6}$$

Without loss of generality we will also suppose that

$$\gamma_1 \leqslant \gamma_2 \leqslant \gamma_3. \tag{6.7}$$

THEOREM 6.1. *Let G be the matrix function given by* (6.1) *and satisfying* (6.6), (6.7)*. Then G admits a left APP factorization $G = G_+ D G_-$ with the factors given by:*

$$G_+ = \begin{bmatrix} a_1^{-1} & 0 & 0 & 0 \\ -a_2^{-1} e_{\lambda_2-\lambda_1-\gamma_2+\gamma_1} & a_2^{-1} & 0 & 0 \\ 0 & -a_3^{-1} e_{\lambda_3-\lambda_2+\gamma_2-\gamma_3} & a_3^{-1} & a_3^{-1} e_{\lambda_3-\gamma_3} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} e_{\lambda_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_2} & 0 & 0 \\ 0 & 0 & e_{\lambda_3+\lambda_4-\gamma_3} & 0 \\ 0 & 0 & 0 & e_{\gamma_3} \end{bmatrix},$$

$$G_- = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_1 e_{\gamma_1-\gamma_2} & a_2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ a_1 e_{\gamma_1-\gamma_3} & a_2 e_{\gamma_2-\gamma_3} & a_3 & e_{\lambda_4-\gamma_3} \end{bmatrix} \tag{6.8}$$

if $\lambda_2 - \lambda_1 \geqslant \gamma_2 - \gamma_1$ *and* $\lambda_3 - \lambda_2 \geqslant \gamma_3 - \gamma_2$*;*

$$G_+ = \begin{bmatrix} a_1^{-1} & 0 & 0 & 0 \\ -a_2^{-1} e_{\lambda_2-\lambda_1-\gamma_2+\gamma_1} & a_2^{-1} & -a_2^{-1} e_{\gamma_3-\gamma_2-\lambda_3+\lambda_2} & 0 \\ 0 & 0 & a_3^{-1} & a_3^{-1} e_{\lambda_3-\gamma_3} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} e_{\lambda_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_4+\lambda_2-\gamma_2} & 0 & 0 \\ 0 & 0 & e_{\gamma_2+\lambda_3-\gamma_3} & 0 \\ 0 & 0 & 0 & e_{\gamma_3} \end{bmatrix},$$

$$G_- = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -a_1 e_{\gamma_1-\gamma_2} & -a_2 & 0 & -e_{\lambda_4-\gamma_2} \\ a_1 e_{\gamma_1-\gamma_3} & a_2 e_{\gamma_2-\gamma_3} & a_3 & e_{\lambda_4-\gamma_3} \end{bmatrix} \tag{6.9}$$

if $\lambda_2 - \lambda_1 \geqslant \gamma_2 - \gamma_1$ *and* $\lambda_3 - \lambda_2 \leqslant \gamma_3 - \gamma_2$*;*

$$G_+ = \begin{bmatrix} a_1^{-1} & 0 & 0 & 0 \\ 0 & a_2^{-1} & 0 & 0 \\ -a_3^{-1} e_{\lambda_3-\lambda_1-\gamma_3+\gamma_1} & -a_3^{-1} e_{\lambda_3-\lambda_2+\gamma_2-\gamma_3} & a_3^{-1} & a_3^{-1} e_{\lambda_3-\gamma_3} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} e_{\lambda_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_2} & 0 & 0 \\ 0 & 0 & e_{\lambda_3+\lambda_4-\gamma_3} & 0 \\ 0 & 0 & 0 & e_{\gamma_3} \end{bmatrix},$$

$$G_- = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ a_1 e_{\gamma_1-\gamma_3} & a_2 e_{\gamma_2-\gamma_3} & a_3 & e_{\lambda_4-\gamma_3} \end{bmatrix} \tag{6.10}$$

if $\lambda_3 - \lambda_1 \geqslant \gamma_3 - \gamma_1$ *and* $\lambda_3 - \lambda_2 \geqslant \gamma_3 - \gamma_2$*;*

$$G_+ = \begin{bmatrix} a_1^{-1} & 0 & 0 & 0 \\ 0 & a_2^{-1} & -a_2^{-1} e_{\gamma_3-\gamma_2-\lambda_3+\lambda_2} & 0 \\ -a_3^{-1} e_{\lambda_3-\lambda_1-\gamma_3+\gamma_1} & 0 & a_3^{-1} & a_3^{-1} e_{\lambda_3-\gamma_3} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} e_{\lambda_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_4+\lambda_2-\gamma_2} & 0 & 0 \\ 0 & 0 & e_{\gamma_2+\lambda_3-\gamma_3} & 0 \\ 0 & 0 & 0 & e_{\gamma_3} \end{bmatrix},$$

$$G_- = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -a_2 & 0 & -e_{\lambda_4-\gamma_2} \\ a_1 e_{\gamma_1-\gamma_3} & a_2 e_{\gamma_2-\gamma_3} & a_3 & e_{\lambda_4-\gamma_3} \end{bmatrix} \tag{6.11}$$

if $\lambda_3 - \lambda_1 \geqslant \gamma_3 - \gamma_1$ *and* $\lambda_3 - \lambda_2 \leqslant \gamma_3 - \gamma_2$*;*

$$G_+ = \begin{bmatrix} a_1^{-1} & 0 & -a_1^{-1} e_{\gamma_3-\gamma_1-\lambda_3+\lambda_1} & 0 \\ 0 & a_2^{-1} & 0 & 0 \\ 0 & -a_3^{-1} e_{\lambda_3-\lambda_2+\gamma_2-\gamma_3} & a_3^{-1} & a_3^{-1} e_{\lambda_3-\gamma_3} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} e_{\lambda_4+\lambda_1-\gamma_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_2} & 0 & 0 \\ 0 & 0 & e_{\lambda_3+\gamma_1-\gamma_3} & 0 \\ 0 & 0 & 0 & e_{\gamma_3} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & a_2 & 0 & 0 \\ -a_1 & 0 & 0 & -e_{\lambda_4-\gamma_1} \\ a_1 e_{\gamma_1-\gamma_3} & a_2 e_{\gamma_2-\gamma_3} & a_3 & e_{\lambda_4-\gamma_3} \end{bmatrix} \tag{6.12}$$

if $\lambda_2 - \lambda_1 \leqslant \gamma_2 - \gamma_1$, $\lambda_3 - \lambda_1 \leqslant \gamma_3 - \gamma_1$ *and* $\lambda_3 - \lambda_2 \geqslant \gamma_3 - \gamma_2$*;*

$$G_+ = \begin{bmatrix} a_1^{-1} & -a_1^{-1} e_{\gamma_2-\gamma_1-\lambda_2+\lambda_1} & 0 & 0 \\ 0 & a_2^{-1} & -a_2^{-1} e_{\gamma_3-\gamma_2-\lambda_3+\lambda_2} & 0 \\ 0 & 0 & a_3^{-1} & a_3^{-1} e_{\lambda_3-\gamma_3} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} e_{\lambda_4+\lambda_1-\gamma_1} & 0 & 0 & 0 \\ 0 & e_{\gamma_1+\lambda_2-\gamma_2} & 0 & 0 \\ 0 & 0 & e_{\gamma_2+\lambda_3-\gamma_3} & 0 \\ 0 & 0 & 0 & e_{\gamma_3} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & -1 \\ -a_1 & 0 & 0 & -e_{\lambda_4-\gamma_1} \\ -a_1 e_{\gamma_1-\gamma_2} & -a_2 & 0 & -e_{\lambda_4-\gamma_2} \\ a_1 e_{\gamma_1-\gamma_3} & a_2 e_{\gamma_2-\gamma_3} & a_3 & e_{\lambda_4-\gamma_3} \end{bmatrix} \tag{6.13}$$

if

$$\lambda_2-\lambda_1 \leqslant \gamma_2-\gamma_1,\ \lambda_3-\lambda_1 \leqslant \gamma_3-\gamma_1 \text{ and } \lambda_3-\lambda_2 \leqslant \gamma_3-\gamma_2. \tag{6.14}$$

Proof. a) If $\lambda_2-\lambda_1 \geqslant \gamma_2-\gamma_1$, then in view of Theorem 2.1 we obtain

$$G = A^{-1}\begin{bmatrix} 1 & 0\,0\,0 \\ -e_{\lambda_2-\lambda_1-\gamma_2+\gamma_1} & 1\,0\,0 \\ 0 & 0\,1\,0 \\ 0 & 0\,0\,1 \end{bmatrix} \widetilde{G} \begin{bmatrix} 1 & 0\,0\,0 \\ e_{\gamma_1-\gamma_2} & 1\,0\,0 \\ 0 & 0\,1\,0 \\ 0 & 0\,0\,1 \end{bmatrix} A \tag{6.15}$$

where

$$A := \operatorname{diag}\{a_1,a_2,a_3,1\}, \quad \widetilde{G} := \begin{bmatrix} e_{\lambda_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_2} & 0 & 0 \\ 0 & 0 & e_{\lambda_3} & 0 \\ 0 & e_{\gamma_2} & e_{\gamma_3} & e_{\lambda_4} \end{bmatrix}. \tag{6.16}$$

If $\lambda_3-\lambda_2 \geqslant \gamma_3-\gamma_2$, applying (5.4) with an obvious notational adjustment we obtain the left APP factorization

$$\widetilde{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -e_{\lambda_3-\lambda_2-\gamma_3+\gamma_2} & 1 & e_{\lambda_3-\gamma_3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_{\lambda_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_2} & 0 & 0 \\ 0 & 0 & e_{\lambda_3+\lambda_4-\gamma_3} & 0 \\ 0 & 0 & 0 & e_{\gamma_3} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & e_{\gamma_2-\gamma_3} & 1 & e_{\lambda_4-\gamma_3} \end{bmatrix}, \tag{6.17}$$

which together with (6.15) gives the left APP factorization $G = G_+DG_-$ with factors defined by (6.8).

If $\lambda_3-\lambda_2 \leqslant \gamma_3-\gamma_2$, applying (5.5) with an obvious notational adjustment we obtain the left APP factorization

$$\widetilde{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -e_{\gamma_3-\gamma_2-\lambda_3+\lambda_2} & 0 \\ 0 & 0 & 1 & e_{\lambda_3-\gamma_3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_{\lambda_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_4+\lambda_2-\gamma_2} & 0 & 0 \\ 0 & 0 & e_{\gamma_2+\lambda_3-\gamma_3} & 0 \\ 0 & 0 & 0 & e_{\gamma_3} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -e_{\lambda_4-\gamma_2} \\ 0 & e_{\gamma_2-\gamma_3} & 1 & e_{\lambda_4-\gamma_3} \end{bmatrix}, \tag{6.18}$$

which together with (6.15) gives the left APP factorization $G = G_+DG_-$ with factors defined by (6.9).

b) If $\lambda_3-\lambda_1 \geqslant \gamma_3-\gamma_1$, then in view of Theorem 2.1 we obtain

$$G = A^{-1}\begin{bmatrix} 1 & 0\,0\,0 \\ 0 & 1\,0\,0 \\ -e_{\lambda_3-\lambda_1-\gamma_3+\gamma_1} & 0\,1\,0 \\ 0 & 0\,0\,1 \end{bmatrix} \widetilde{G} \begin{bmatrix} 1 & 0\,0\,0 \\ 0 & 1\,0\,0 \\ e_{\gamma_1-\gamma_3} & 0\,1\,0 \\ 0 & 0\,0\,1 \end{bmatrix} A, \tag{6.19}$$

where A and $\widetilde{G}$ are given by (6.16).

If $\lambda_3-\lambda_2 \geqslant \gamma_3-\gamma_2$, combining (6.19) with (6.17) we obtain the left APP factorization $G=G_+DG_-$ with factors defined by (6.10).

If $\lambda_3-\lambda_2 \leqslant \gamma_3-\gamma_2$, combining (6.19) with (6.18) we obtain the left APP factorization $G=G_+DG_-$ with factors defined by (6.11).

c) If $\lambda_2-\lambda_1 \leqslant \gamma_2-\gamma_1$ and $\lambda_3-\lambda_1 \leqslant \gamma_3-\gamma_1$, then in view of Theorem 2.1 we obtain

$$G=A^{-1}\begin{bmatrix}1 & -e_{\gamma_2-\gamma_1-\lambda_2+\lambda_1} & -e_{\gamma_3-\gamma_1-\lambda_3+\lambda_1} & e_{\lambda_1-\gamma_1}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\widetilde{G}$$

$$\times\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & e_{\lambda_4-\gamma_1}\end{bmatrix}A, \tag{6.20}$$

where

$$A:=\operatorname{diag}\{a_1,a_2,a_3,1\}, \quad \widetilde{G}:=\begin{bmatrix}-e_{\lambda_4+\lambda_1-\gamma_1} & 0 & 0 & 0\\ 0 & e_{\lambda_2} & 0 & 0\\ 0 & 0 & e_{\lambda_3} & 0\\ 0 & e_{\gamma_2} & e_{\gamma_3} & e_{\gamma_1}\end{bmatrix}.$$

If $\lambda_3-\lambda_2 \geqslant \gamma_3-\gamma_2$, applying (5.4) with an obvious notational adjustment we obtain the left APP factorization

$$\widetilde{G}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -e_{\lambda_3-\lambda_2+\gamma_2-\gamma_3} & 1 & e_{\lambda_3-\gamma_3}\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-e_{\lambda_4+\lambda_1-\gamma_1} & 0 & 0 & 0\\ 0 & e_{\lambda_2} & 0 & 0\\ 0 & 0 & e_{\lambda_3+\gamma_1-\gamma_3} & 0\\ 0 & 0 & 0 & e_{\gamma_3}\end{bmatrix}$$

$$\times\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & e_{\gamma_2-\gamma_3} & 1 & e_{\gamma_1-\gamma_3}\end{bmatrix}, \tag{6.21}$$

which together with (6.20) gives the left APP factorization $G=G_+DG_-$ with factors defined by (6.12).

If $\lambda_3-\lambda_2 \leqslant \gamma_3-\gamma_2$, applying (5.5) with an obvious notational adjustment we obtain the left APP factorization

$$\widetilde{G}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -e_{\gamma_3-\gamma_2-\lambda_3+\lambda_2} & 0\\ 0 & 0 & 1 & e_{\lambda_3-\gamma_3}\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\times\begin{bmatrix}-e_{\lambda_4+\lambda_1-\gamma_1} & 0 & 0 & 0\\ 0 & e_{\gamma_1+\lambda_2-\gamma_2} & 0 & 0\\ 0 & 0 & e_{\gamma_2+\lambda_3-\gamma_3} & 0\\ 0 & 0 & 0 & e_{\gamma_3}\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & -1 & 0 & -e_{\gamma_1-\gamma_2}\\ 0 & e_{\gamma_2-\gamma_3} & 1 & e_{\gamma_1-\gamma_3}\end{bmatrix}, \tag{6.22}$$

which together with (6.20) gives the left APP factorization $G = G_+DG_-$ with factors defined by (6.13). □

Theorems 3.2 and 6.1 imply the following.

COROLLARY 6.2. *If the matrix function G is given by (6.1) with $\lambda_1\lambda_2\lambda_3 \neq 0$ and γ_j satisfying (6.7), then G admits a canonical left AP factorization if and only if*

$$\lambda_1, \lambda_2, \lambda_3 > 0,\ \lambda_4 = -\lambda_1 - \lambda_2 - \lambda_3,\ \textit{and}\ \gamma_1 = -\lambda_2 - \lambda_3,\ \gamma_2 = -\lambda_3,\ \gamma_3 = 0. \quad (6.23)$$

If this is the case, then

$$G := \begin{bmatrix} e_{\lambda_1} & 0 & 0 & 0 \\ 0 & e_{\lambda_2} & 0 & 0 \\ 0 & 0 & e_{\lambda_3} & 0 \\ a_1 e_{-\lambda_2-\lambda_3} & a_2 e_{-\lambda_3} & a_3 & e_{-\lambda_1-\lambda_2-\lambda_3} \end{bmatrix}$$
$$= \begin{bmatrix} a_1^{-1} & -a_1^{-1}e_{\lambda_1} & 0 & 0 \\ 0 & a_2^{-1} & -a_2^{-1}e_{\lambda_2} & 0 \\ 0 & 0 & a_3^{-1} & a_3^{-1}e_{\lambda_3} \\ 0 & 0 & 0 & 1, \end{bmatrix}$$
$$\times \begin{bmatrix} 0 & 0 & 0 & -1 \\ -a_1 & 0 & 0 & -e_{-\lambda_1} \\ -a_1 e_{-\lambda_2} & -a_2 & 0 & -e_{-\lambda_1-\lambda_2} \\ a_1 e_{-\lambda_2-\lambda_3} & a_2 e_{-\lambda_3} & a_3 & e_{-\lambda_1-\lambda_2-\lambda_3} \end{bmatrix} \quad (6.24)$$

is a canonical left APP factorization of G, and the geometric mean $\mathbf{d}(G)$ is given by

$$\mathbf{d}(G) = \begin{bmatrix} 0 & 0 & 0 & -a_1^{-1} \\ -a_1a_2^{-1} & 0 & 0 & 0 \\ 0 & -a_2a_3^{-1} & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{bmatrix}.$$

Proof. The equivalence of (6.23) to the canonical factorability of G is immediate from Theorem 3.2. Since (6.23) implies (6.6), Theorem 6.1 is applicable.

Moreover, conditions (6.23) imply

$$\gamma_2 - \gamma_1 = \lambda_2 > \lambda_2 - \lambda_1, \gamma_3 - \gamma_1 = \lambda_2 + \lambda_3 > \lambda_3 - \lambda_1, \gamma_3 - \gamma_2 = \lambda_3 > \lambda_3 - \lambda_2,$$

so that (6.14) holds. Consequently, a factorization of G can be obtained according to formulas (6.13). The latter turn into (6.24) when simplified with the use of (6.23). Consequently, the geometric mean of G is determined by

$$\mathbf{d}(G) = \begin{bmatrix} a_1^{-1} & 0 & 0 & 0 \\ 0 & a_2^{-1} & 0 & 0 \\ 0 & 0 & a_3^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ -a_1 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -a_1^{-1} \\ -a_1 a_2^{-1} & 0 & 0 & 0 \\ 0 & -a_2 a_3^{-1} & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{bmatrix},$$

which completes the proof. $\square$

7. Intermezzo

Before passing to the applications, we observe that the factorization formulas obtained in this paper are purely algebraic, and make no use of the intrinsic nature of AP functions. Therefore, the results can be reinterpreted for the following abstract setting. Let Γ be a linearly ordered additively written abelian group, with the order denoted $\preceq$, and let $\exp\Gamma$ be its (multiplicatively written) isomorphic copy. Denote by e_λ the element of $\exp\Gamma$ corresponding to $\lambda \in \Gamma$, so that $e_\lambda e_\mu = e_{\lambda+\mu}$, and introduce the algebra $\mathfrak{A}$ generated by $\exp\Gamma$. The elements of $\mathfrak{A}$ have the form

$$f = \sum c_j e_{\lambda_j}, \text{ where } c_j \in \mathbb{C}, \lambda_j \in \Gamma. \tag{7.1}$$

Moreover, let $\mathfrak{A}^\pm$ consist of such $f \in \mathfrak{A}$ for which in the representation (7.1) $0 \preceq \pm\lambda_j$ for all j. Finally, we will say that (1.3) is a (left) $\mathfrak{A}$-factorization of $G \in \mathfrak{A}_{N\times N}$ if $G_\pm \in \mathscr{G}(\mathfrak{A}^\pm_{N\times N})$. With these adjustments in mind, Theorem 2.1, for example, implies that matrices (1.6) with g_j as in (1.8) and $\lambda_j, \gamma_j \in \Gamma$ are $\mathfrak{A}$-factorable. All other results can be recast along the same lines.

Note that linearly ordered Γ arise naturally as character groups of connected compact abelian groups; we refer an interested reader to e.g. [18, 17] for some other aspects of the factorization theory in this setting.

8. Some function algebras and their maximal ideals

This section contains necessary background information on some functional classes appearing in the forthcoming description of convolution type equations. The results are not new, and are therefore stated with pertinent references but without proofs.

A measurable function $w : \mathbb{R} \to [0,\infty]$ is called a *weight* if $w^{-1}(\{0,\infty\})$ has Lebesgue measure zero. Given $1 < p < \infty$, we denote by $L^p(\mathbb{R},w)$ the weighted Lebesgue space with the norm

$$\|f\|_{p,w} := \left(\int_{\mathbb{R}} |f(x)|^p w^p(x) dx \right)^{1/p}.$$

In what follows we assume that $1 < p < \infty$ and w is a *Muckenhoupt weight* (notation: $w \in A_p(\mathbb{R})$), that is (see [9] and also [8], [4]):

$$\sup_I \left(\frac{1}{|I|} \int_I w^p(x)\, dx \right)^{1/p} \left(\frac{1}{|I|} \int_I w^{-q}(x)\, dx \right)^{1/q} < \infty,$$

where $1/p+1/q=1$, I ranges over all bounded intervals $I\subset\mathbb{R}$, and $|I|$ is the length of I.

Let $\mathscr{F}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ denote the *Fourier transform*,

$$(\mathscr{F}f)(x):=\int_{\mathbb{R}}e^{-itx}f(t)dt,\quad x\in\mathbb{R}. \tag{8.1}$$

A function $a\in L^\infty(\mathbb{R})$ is called a *Fourier multiplier* on $L^p(\mathbb{R},w)$ if the convolution operator $W^0(a):=\mathscr{F}^{-1}a\mathscr{F}$ maps $L^2(\mathbb{R})\cap L^p(\mathbb{R},w)$ into itself and extends to a bounded linear operator on $L^p(\mathbb{R},w)$. Let $M_{p,w}$ stand for the Banach algebra of all Fourier multipliers a on $L^p(\mathbb{R},w)$ equipped with the norm

$$\|a\|_{M_{p,w}}:=\|W^0(a)\|_{\mathscr{B}(L^p(\mathbb{R},w))},$$

where $\mathscr{B}(L^p(\mathbb{R},w)$ is the Banach algebra of all bounded linear operators acting on the space $L^p(\mathbb{R},w)$. Denote by $A_p^0(\mathbb{R})$ the set of all weights $w\in A_p(\mathbb{R})$ for which the functions $e_\lambda\in M_{p,w}$ for all $\lambda\in\mathbb{R}$. According to [10, Subsection 2.2], there exists a weight $\omega\in A_p^0(\mathbb{R})\cap C(\mathbb{R})$ such that $w/\omega,\omega/w\in L^\infty(\mathbb{R})$.

Clearly, $APP\subset M_{p,w}$ for any $w\in A_p^0(\mathbb{R})$. The closure $AP_{p,w}$ of APP in $M_{p,w}$ is a Banach subalgebra of $M_{p,w}$.

Let $\dot{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ and $\overline{\mathbb{R}}=[-\infty,+\infty]$. By Stechkin's inequality (see, e.g., [5, Theorem 17.1]), every function $a\in C(\overline{\mathbb{R}})$ of finite total variation belongs to $M_{p,w}$. We denote by $C_{p,w}(\overline{\mathbb{R}})$ the closure in $M_{p,w}$ of the set of all functions $a\in C(\overline{\mathbb{R}})$ with finite total variation. Let SAP be the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by $C(\overline{\mathbb{R}})$ and AP, and let $SAP_{p,w}$ stand for the smallest closed subalgebra of $M_{p,w}$ that contains $C_{p,w}(\overline{\mathbb{R}})$ and $AP_{p,w}$. Then $SAP_{p,w}\subset SAP$.

Let $C_b(\mathbb{R})$ be the C^*-algebra of all bounded continuous functions $a\colon\mathbb{R}\to\mathbb{C}$. Following [19] we denote by SO the C^*-algebra of *slowly oscillating at* ∞ functions,

$$SO:=\Big\{f\in C_b(\mathbb{R}):\lim_{x\to+\infty}\sup_{t,s\in[-2x,-x]\cup[x,2x]}|f(t)-f(s)|=0\Big\}. \tag{8.2}$$

Consider the commutative Banach algebra

$$SO^3:=\Big\{a\in SO\cap C^3(\mathbb{R}):\lim_{|x|\to\infty}(D^\gamma a)(x)=0,\ \gamma=1,2,3\Big\}$$

equipped with the norm $\|a\|_{SO^3}:=\max_{\gamma=0,1,2,3}\|D^\gamma a\|_{L^\infty(\mathbb{R})}$ where $(Da)(x)=xa'(x)$ for $x\in\mathbb{R}$. By [11, Corollary 2.10], $SO^3\subset M_{p,w}$. For $1<p<\infty$ and $w\in A_p(\mathbb{R})$, let $SO_{p,w}$ denote the closure of SO^3 in $M_{p,w}$. Clearly, $SO_{p,w}$ is a commutative Banach subalgebra of $M_{p,w}$.

Finally, let $[SO_{p,w},SAP_{p,w}]$ be the Banach subalgebra of $M_{p,w}$ generated by all functions in $SO_{p,w}$ and $SAP_{p,w}$. Clearly, $[SO_{p,w},SAP_{p,w}]$ is contained in the C^*-algebra $[SO,SAP]$ generated by SO and SAP.

We identify the points $t\in\dot{\mathbb{R}}$ with the evaluation functionals δ_t on $\dot{\mathbb{R}}$, $\delta_t(f)=f(t)$. If $\mathscr{A}$ is a C^*-subalgebra of $L^\infty(\mathbb{R})$ that contains $C(\dot{\mathbb{R}})$, then the fiber over ∞ of the maximal ideal space $\mathscr{M}(\mathscr{A})$ of $\mathscr{A}$ is defined by

$$\mathscr{M}_\infty(\mathscr{A}):=\{\xi\in\mathscr{M}(\mathscr{A}):\xi|_{C(\dot{\mathbb{R}})}=\delta_\infty\}.$$

Since $C(\dot{\mathbb{R}}) \subset SO$, we see that the maximal ideal space of SO can be represented as $\mathscr{M}(SO) = \mathbb{R} \cup \mathscr{M}_\infty(SO)$. By [2, Proposition 5], $\mathscr{M}_\infty(SO) = (\mathrm{clos}_{SO^*}\mathbb{R}) \setminus \mathbb{R}$, where $\mathrm{clos}_{SO^*}\mathbb{R}$ is the weak-star closure of $\mathbb{R}$ in SO^*, the dual space of SO.

LEMMA 8.1. [11, Lemma 3.4] *If $1 < p < \infty$ and $w \in A_p(\mathbb{R})$, then the maximal ideal spaces of $SO_{p,w}$ and SO coincide as sets, that is, $\mathscr{M}(SO_{p,w}) = \mathscr{M}(SO)$.*

Analogously to Lemma 8.1, one can prove that

$$\mathscr{M}([SO_{p,w}, SAP_{p,w}]) = \mathscr{M}([SO, SAP]) = \mathbb{R} \cup \mathscr{M}_\infty([SO, SAP]). \tag{8.3}$$

Equalities (8.3) and the Gelfand theory immediately yield the following assertion.

COROLLARY 8.2. *If $1 < p < \infty$ and $w \in A_p^0(\mathbb{R})$, then the Banach algebra $[SO_{p,w}, SAP_{p,w}]$ is inverse closed in the C^*-algebras $[SO, SAP]$ and $L^\infty(\mathbb{R})$, that is, if $a \in [SO_{p,w}, SAP_{p,w}]$ is invertible in $L^\infty(\mathbb{R})$, then $a^{-1} \in [SO_{p,w}, SAP_{p,w}]$ as well.*

By [20], any function $a \in SAP$ is uniquely represented in the form

$$a = a_+u_+ + a_-u_- + a_0 \tag{8.4}$$

where $a_\pm \in AP$, $a_0 \in C(\dot{\mathbb{R}})$, $a_0(\infty) = 0$, $u_\pm(x) = (1 \pm \tanh x)/2$, and the mappings $\nu_\pm : a \mapsto a_\pm$ are C^*-algebra homomorphisms of SAP onto AP.

According to [19, Section 3], the C^*-algebras SO and SAP are asymptotically independent in the following sense:

PROPOSITION 8.3. *The fiber $\mathscr{M}_\infty([SO, SAP])$ is naturally homeomorphic to the set $\mathscr{M}_\infty(SO) \times \mathscr{M}_\infty(SAP)$, that is, for every character $\mu \in \mathscr{M}_\infty([SO, SAP])$ there are characters $\xi \in \mathscr{M}_\infty(SO)$ and $\nu \in \mathscr{M}_\infty(SAP)$ such that $\mu|_{SO} = \xi$ and $\mu|_{SAP} = \nu$.*

By Proposition 8.3, we can identify characters $\mu \in \mathscr{M}_\infty([SO, SAP])$ with pairs $(\xi, \nu) \in \mathscr{M}_\infty(SO) \times \mathscr{M}_\infty(SAP)$, and hence for every $\xi \in \mathscr{M}_\infty(SO)$ we obtain a homomorphism

$$\beta_\xi : [SO, SAP] \to SAP|_{\mathscr{M}_\infty(SAP)}, \quad (\beta_\xi\, \varphi)(\nu) = (\xi, \nu)\varphi \quad \text{for } \nu \in \mathscr{M}_\infty(SAP).$$

Thus, for every $\varphi \in [SO, SAP]$ there exists a non-unique function $\varphi_\xi \in SAP$ with uniquely determined almost periodic representatives $\varphi_{\xi,\pm}$ at $\pm\infty$ such that $\beta_\xi\, \varphi = \varphi_\xi\big|_{\mathscr{M}_\infty(SAP)}$. Since the fiber $\mathscr{M}_\infty(AP)$ is homeomorphic to $\mathscr{M}(AP)$, identifying $\mathscr{M}_\infty(SAP)$ and $\mathscr{M}_\infty(AP) \times \mathscr{M}_\infty(AP)$, we conclude that the maps

$$\gamma_\pm : \varphi_\xi\,|_{\mathscr{M}_\infty(SAP)} \mapsto \varphi_{\xi,\pm}|_{\mathscr{M}_\infty(AP)} \mapsto \varphi_{\xi,\pm}$$

are C^*-algebra homomorphisms of $SAP|_{\mathscr{M}_\infty(SAP)}$ onto AP. Thus the maps

$$\nu_{\xi,\pm} = \gamma_\pm \circ \beta_\xi : [SO, SAP] \to AP, \quad \nu_{\xi,\pm}\, \varphi = \varphi_{\xi,\pm} \tag{8.5}$$

are well-defined C^*-algebra homomorphisms for every $\xi \in \mathscr{M}_\infty(SO)$.

The C^*-algebra $[SO, SAP]$ consists of all functions of the form

$$c = \lim_{k\to\infty} \sum\nolimits_{i=1}^{m_k} b_{i,k}\, a_{i,k} \tag{8.6}$$

where $b_{i,k} \in SO$, $a_{i,k} \in SAP$, and limit is taken in the norm $\|\cdot\|_{L^\infty(\mathbb{R})}$. Therefore, for every $\xi \in \mathscr{M}_\infty(SO)$, the maps $\nu_{\xi,\pm} : [SO, SAP] \to AP$ act by the rule

$$\nu_{\xi,\pm} c = \lim_{k\to\infty} \sum\nolimits_{i=1}^{m_k} \xi(b_{i,k})\nu_\pm(a_{i,k}). \tag{8.7}$$

On the other hand, by [2, Section 4], for every $c \in [SO, SAP]$ and every $\xi \in \mathscr{M}_\infty(SO)$ there exist sequences $g_\pm = \{g_n^\pm\} \to \pm\infty$ such that for $x \in \mathbb{R}$,

$$\lim_{n\to\infty} b_{i,k}(x + g_n^\pm) = \xi(b_{i,k}), \quad \lim_{n\to\infty} a_{i,k}(x + g_n^\pm) = (\nu_\pm a_{i,k})(x),$$

where for all i,k the convergence is uniform on $\mathbb{R}$ for $a_{i,k}$ and is uniform on compact subsets of $\mathbb{R}$ for $b_{i,k}$. Consequently, for the function (8.6) we obtain

$$(\nu_{\xi,\pm} c)(x) = \lim_{n\to\infty} c(x + g_n^\pm), \quad \text{for } x \in \mathbb{R}. \tag{8.8}$$

Applying (8.8), we see that

$$wW^0(\nu_{\xi,\pm} c)w^{-1} I = \operatorname*{s-lim}_{n\to\infty} \left(e_{g_n^\mp}\, wW^0(c)w^{-1}\, e_{g_n^\pm} I\right),$$

which implies that the maps $\nu_{\xi,\pm} : [SO, SAP] \to AP$ restricted to the Banach algebra $[SO_{p,w}, SAP_{p,w}]$ are Banach algebra homomorphisms

$$\nu_{\xi,\pm} : [SO_{p,w}, SAP_{p,w}] \to AP_{p,w} \ \text{ for all } \ \xi \in \mathscr{M}_\infty(SO).$$

9. Fredholmness of convolution type operators

As usual, χ_γ will stand for the multiplication operator by the characteristic function of a set $\gamma \subset \mathbb{R}$. Also, let $J = \bigcup_{m=1}^n J_m$, where

$$J_m = [a_{m-1}, a_m] \ \ (m = 1,2,\dots,n-1), \ \ 0 = a_0 < a_1 < a_2 < \dots < a_{n-1} < \infty. \tag{9.1}$$

Applying the results of Sections 2 and 3, we establish here Fredholm criteria for the convolution type operator

$$W := \chi_J \sum_{m=1}^{n-1} \mathscr{F}^{-1} K_m \mathscr{F} \chi_m : L^p(J,w) \to L^p(J,w), \tag{9.2}$$

where $K_m \in [SO_{p,w}, SAP_{p,w}]$, $\chi_m = \chi_{J_m}$, and functions in $L^p(J,w)$ are extended by zero to $\mathbb{R} \setminus J$.

Following [5, p. 22], we say that two bounded linear operators A and B are strongly Φ-equivalent if either both operators are not normally solvable or both A and B are normally solvable and

$$\dim \operatorname{Ker} A = \dim \operatorname{Ker} B, \quad \dim \operatorname{Coker} A = \dim \operatorname{Coker} B.$$

Given a measurable set $\gamma \subset \mathbb{R}$, let $L_n^p(\gamma, w)$ be the Banach space of vector functions $f = [f_k]_{k=1}^n$ with entries $f_k \in L^p(\gamma, w)$ and the norm

$$\|f\|_{L_n^p(\gamma,w)} = \left(\sum_{k=1}^{n} \|f_k\|_{L^p(\gamma,w)}^p \right)^{1/p}.$$

By analogy with [21], [1, Lemma 2.3] and [13, Lemma 2], we obtain the following result for weighted Lebesgue spaces. The proof is provided in order to make the exposition self-contained.

LEMMA 9.1. *The convolution type operator $W : L^p(J,w) \to L^p(J,w)$ given by (9.2) is strongly Φ-equivalent to the Wiener-Hopf operator*

$$W_G := \chi_+ \mathscr{F}^{-1} G \mathscr{F} : L_n^p(\mathbb{R}_+, w) \to L_n^p(\mathbb{R}_+, w) \tag{9.3}$$

where $\chi_+ = \chi_{\mathbb{R}_+}$,

$$G = \begin{bmatrix} e_{\lambda_1} & 0 & \dots & 0 & 0 \\ 0 & e_{\lambda_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e_{\lambda_{n-1}} & 0 \\ K_1 & K_2 e_{\varepsilon_1} & \dots & K_{n-1} e_{\varepsilon_{n-2}} & e_{\lambda_n} \end{bmatrix} \in [SO_{p,w}, SAP_{p,w}]_{n\times n}, \tag{9.4}$$

$$\lambda_m = a_m - a_{m-1}, \quad \lambda_n = \varepsilon_{n-1}, \quad \varepsilon_m = -(\lambda_1 + \ldots + \lambda_m) \quad (m = 1,2,\ldots,n-1), \tag{9.5}$$

and a_m are given by (9.1).

Proof. Setting for $m = 1,2,\ldots,n-1$,

$$W_m := \chi_+ \mathscr{F}^{-1} K_m \mathscr{F} : L^p(\mathbb{R}_+, w) \to L^p(\mathbb{R}_+, w),$$
$$V_m^{\pm 1} := \chi_+ \mathscr{F}^{-1} e_{\pm\lambda_m} \mathscr{F} : L^p(\mathbb{R}_+, w) \to L^p(\mathbb{R}_+, w),$$

where λ_m are given by (9.5) and the operators V_m are right invertible, with right inverses V_m^{-1}, we conclude that the operator $W : L^p(J,w) \to L^p(J,w)$ is strongly Φ-equivalent to the operator

$$W_0 := \sum_{m=1}^{n-1} W_m \chi_m + \left(\chi_+ - \sum_{m=1}^{n-1} \chi_m \right) : L^p(\mathbb{R}_+, w) \to L^p(\mathbb{R}_+, w), \tag{9.6}$$

where the projections χ_m can be represented in the form

$$\begin{aligned} &\chi_1 = I - V_1^{-1} V_1, \quad \text{with } I = \chi_+, \\ &\chi_m = V_1^{-1} V_2^{-1} \cdots V_{m-1}^{-1} (I - V_m^{-1} V_m) V_{m-1} \cdots V_2 V_1 \quad (m = 2,3,\ldots,n-1). \end{aligned} \tag{9.7}$$

Taking $\widehat{W} = W_G$, where W_G is given by (9.3), and applying the equalities

$$\widehat{W} := \begin{bmatrix} V_1 & 0 & \dots & 0 & 0 \\ 0 & V_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & V_{n-1} & 0 \\ W_1 & W_2V_1^{-1} & \dots & W_{n-1}V_1^{-1}V_2^{-1}\cdots V_{n-2}^{-1} & V_1^{-1}V_2^{-1}\cdots V_{n-1}^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \\ W_1V_1^{-1} & W_2V_1^{-1}V_2^{-1} & \dots & W_{n-1}V_1^{-1}V_2^{-1}\cdots V_{n-1}^{-1} & W_0 \end{bmatrix} Y,$$

where W_0 is given by (9.6) and

$$Y = \begin{bmatrix} V_1 & 0 & \dots & 0 & 0 \\ 0 & V_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & V_{n-1} & 0 \\ \chi_1 & \chi_2V_1^{-1} & \dots & \chi_{n-1}V_1^{-1}V_2^{-1}\cdots V_{n-2}^{-1} & V_1^{-1}V_2^{-1}\cdots V_{n-1}^{-1} \end{bmatrix},$$
$$Y^{-1} = \begin{bmatrix} V_1^{-1} & 0 & \dots & 0 & \chi_1 \\ 0 & V_2^{-1} & \dots & 0 & V_1\chi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & V_{n-1}^{-1} & V_{n-2}\cdots V_2V_1\chi_{n-1} \\ 0 & 0 & \dots & 0 & V_{n-1}\cdots V_2V_1 \end{bmatrix},$$

we immediately infer the strong Φ-equivalence of the operators W_G and W_0, which completes the proof. □

Since $APP \subset AP_{p,w}$, [12, Theorem 7.2] immediately implies the following result. Note that the switch from right (as in [12]) to left factorization is caused by the accepted in this paper definition (8.1) of the Fourier transform.

THEOREM 9.2. *Let $1 < p < \infty$, $w \in A_p^0(\mathbb{R})$, and let $a \in [SO_{p,w}, SAP_{p,w}]_{n\times n}$. If for every $\xi \in \mathscr{M}_\infty(SO)$ the matrix functions $a_{\xi,\pm} = v_{\xi,\pm}a$ admit left APP factorizations, then the Wiener-Hopf operator $W(a) = \chi_+\mathscr{F}^{-1}a\mathscr{F}$ is Fredholm on the space $L_n^p(\mathbb{R}_+, w)$ if and only if the following three conditions are satisfied:*

(i) $\det a(x) \neq 0$ *for all $x \in \mathbb{R}$;*

(ii) for every $\xi \in \mathscr{M}_\infty(SO)$, the left APP factorizations of the matrix functions $a_{\xi,\pm}$ are canonical;

(iii) for every $\xi \in \mathscr{M}_\infty(SO)$ and all $j = 1,2,\dots,N$, the eigenvalues $\eta_{\xi,j}$ of the matrix $\mathbf{d}^{-1}(a_{\xi,-})\mathbf{d}(a_{\xi,+})$ satisfy the condition

$$\frac{1}{p} + \frac{1}{2\pi}\arg\eta_{\xi,j} \notin \mathbb{Z}. \tag{9.8}$$

From Lemma 9.1 and Theorem 9.2 we obtain

THEOREM 9.3. *Let $1 < p < \infty$, $w \in A_p^0(\mathbb{R})$, $K_m \in [SO_{p,w}, SAP_{p,w}(\overline{\mathbb{R}})]$, and let $G \in [SO_{p,w}, SAP_{p,w}]_{n\times n}$ be given by (9.4)–(9.5). If for every $\xi \in \mathscr{M}_\infty(SO)$ the matrix functions $G_{\xi,\pm} = v_{\xi,\pm}G$ are in $APP_{n\times n}$ and admit left APP factorizations, then the convolution type operator (9.2) is Fredholm on the space $L^p(J,w)$ if and only if the following two conditions are satisfied:*

(i) for every $\xi \in \mathscr{M}_\infty(SO)$, the left APP factorizations of the matrix functions $G_{\xi,\pm}$ are canonical;

(ii) for every $\xi \in \mathscr{M}_\infty(SO)$ and all $j = 1,2,\dots,n$, the eigenvalues $\eta_{\xi,j}$ of the matrix $\mathbf{d}^{-1}(G_{\xi,-})\mathbf{d}(G_{\xi,+})$ satisfy condition (9.8).

Assume now that for every $m = 1,2,\dots,n-1$,

$$K_m = b_{m,-}u_- + b_{m,+}u_+ \in [SO_{p,w}, C_{p,w}(\overline{\mathbb{R}})] \tag{9.9}$$

where $b_{m,\pm} \in SO_{p,w}$ and the functions $u_\pm \in C_{p,w}(\overline{\mathbb{R}})$ are given by $u_\pm(x) = (1 \pm \tanh x)/2$ for all $x \in \overline{\mathbb{R}}$. Then Theorems 9.3, 2.1, 3.2 and 3.3 imply the following Fredholm criterion for the convolution type operator (9.2).

THEOREM 9.4. *Let $1 < p < \infty$, $w \in A_p^0(\mathbb{R})$, (9.1) hold, and let $K_m \in [SO_{p,w}, C_{p,w}(\overline{\mathbb{R}})]$ and $G \in [SO_{p,w}, SAP_{p,w}]_{n\times n}$ be given by (9.9) and (9.4)–(9.5), respectively. Then the convolution type operator*

$$W = \chi_J \sum_{m=1}^{n-1} \mathscr{F}^{-1} K_m \mathscr{F} \chi_{J_m} : L^p(J,w) \to L^p(J,w)$$

is Fredholm on the space $L^p(J,w)$ if and only if

(i) $b_{m,\pm}(\xi) \neq 0$ for every $\xi \in \mathscr{M}_\infty(SO)$ and every $m = 1,2,\dots,n-1$;

(ii) for every $\xi \in \mathscr{M}_\infty(SO)$ and all $j = 1,2,\dots,n$, the numbers $\eta_{\xi,j}$, where

$$\begin{aligned}
&\eta_{\xi,1} = b_{1,-}^{-1}(\xi)b_{1,+}(\xi),\\
&\eta_{\xi,k} = b_{k,-}^{-1}(\xi)b_{k,+}(\xi)b_{k-1,-}(\xi)b_{k-1,+}^{-1}(\xi) \quad (k = 2,3,\dots,n-1),\\
&\eta_{\xi,n} = b_{n-1,-}(\xi)b_{n-1,+}^{-1}(\xi),
\end{aligned} \tag{9.10}$$

satisfy condition (9.8).

Proof. We deduce from (9.4) and (9.9) that

$$G_{\xi,\pm} = \begin{bmatrix} e_{\lambda_1} & 0 & \dots & 0 & 0 \\ 0 & e_{\lambda_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e_{\lambda_{n-1}} & 0 \\ b_{1,\pm}(\xi) & b_{2,\pm}(\xi)e_{\varepsilon_1} & \dots & b_{n-1,\pm}(\xi)e_{\varepsilon_{n-2}} & e_{\lambda_n} \end{bmatrix}$$

where, in view of (9.1) and (9.5), $\lambda_n = -(\lambda_1 + \ldots + \lambda_{n-1}) < 0$ and

$$\lambda_m = a_m - a_{m-1} > 0, \;\; \varepsilon_m = -(\lambda_1 + \ldots + \lambda_m) < 0 \;\text{ for all }\; m = 1,2,\ldots,n-1.$$

By Theorem 2.1, for all $\xi \in \mathscr{M}_\infty(SO)$ the matrix functions $G_{\xi,\pm}$ admit left *APP* factorizations. Then according to Theorem 9.3 the operator W is Fredholm on the space $L^p(J,w)$ if and only if these *APP* factorizations of $G_{\xi,\pm}$ are canonical, and for every $\xi \in \mathscr{M}_\infty(SO)$ and all $j = 1,2,\ldots,n$, the eigenvalues $\eta_{\xi,j}$ of the matrix $\mathbf{d}^{-1}(G_{\xi,-})\mathbf{d}(G_{\xi,+})$ satisfy (9.8). Changing the order of rows and columns of the matrices $G_{\xi,\pm}$ according to the permutation $(n-1,n-2,\ldots,2,1)(n)$ we see that the matrix functions

$$\widetilde{G}_{\xi,\pm} = \begin{bmatrix} e_{\lambda_{n-1}} & & & & \\ & e_{\lambda_{n-2}} & & & \\ & & \ddots & & \\ & & & e_{\lambda_1} & \\ b_{n-1,\pm}(\xi)e_{\varepsilon_{n-2}} & b_{n-2,\pm}(\xi)e_{\varepsilon_{n-3}} & \dots & b_{1,\pm}(\xi) & e_{\lambda_n} \end{bmatrix}, \qquad (9.11)$$

where

$$\varepsilon_{n-k} = -(\lambda_1 + \ldots + \lambda_{n-k}) = (\lambda_{n-1} + \ldots + \lambda_{n-k+1}) - \lambda_n \;\; (k = 2,3,\ldots,n),$$

satisfy the conditions of Theorem 3.2. In particular, by Theorem 3.2, the existence of canonical left *APP* factorizations of $G_{\xi,\pm}$ implies that $b_{m,\pm}(\xi) \neq 0$ for all $m = 1,2,\ldots,n-1$ and all $\xi \in \mathscr{M}_\infty(SO)$. By Theorem 3.3, the matrix functions $\widetilde{G}_{\xi,\pm}$ admit left canonical *APP* factorizations $\widetilde{G}_{\xi,\pm} = \widetilde{G}^+_{\xi,\pm}\widetilde{G}^-_{\xi,\pm}$ with

$$\widetilde{G}^+_{\xi,\pm} = \begin{bmatrix} b^{-1}_{n-1,\pm}(\xi) & -b^{-1}_{n-1,\pm}(\xi)e_{\lambda_{n-1}} & & & \\ & \ddots & \ddots & & \\ & & b^{-1}_{2,\pm}(\xi) & -b^{-1}_{2,\pm}(\xi)e_{\lambda_2} & \\ & & & b^{-1}_{1,\pm}(\xi) & b^{-1}_{1,\pm}(\xi)e_{\lambda_1} \\ & & & & 1 \end{bmatrix},$$

$$\widetilde{G}^-_{\xi,\pm} = \begin{bmatrix} b^{-1}_{n-1,\pm}(\xi)e_{-\lambda_{n-1}} & -b^{-1}_{n-1,\pm}(\xi) & & & \\ & \ddots & \ddots & & \\ & & b^{-1}_{2,\pm}(\xi)e_{-\lambda_2} & -b^{-1}_{2,\pm}(\xi) & \\ & & & b^{-1}_{1,\pm}(\xi)e_{-\lambda_1} & b^{-1}_{1,\pm}(\xi) \\ -1 & & & & 0 \end{bmatrix}^{-1}.$$

Hence

$$\mathbf{d}(\widetilde{G}_{\xi,+}) = \begin{bmatrix} 0 & & & & -b_{n-1,+}^{-1}(\xi) \\ -b_{n-2,+}^{-1}(\xi)b_{n-1,+}(\xi) & 0 & & & \\ & \ddots & \ddots & & \\ & & -b_{1,+}^{-1}(\xi)b_{2,+}(\xi) & 0 & \\ & & & b_{1,+}(\xi) & 0 \end{bmatrix},$$

$$\mathbf{d}^{-1}(\widetilde{G}_{\xi,-}) = \begin{bmatrix} 0 & -b_{n-2,-}(\xi)b_{n-1,-}^{-1}(\xi) & & & \\ & \ddots & \ddots & & \\ & & -b_{1,-}(\xi)b_{2,-}^{-1}(\xi) & 0 & \\ & & 0 & b_{1,-}^{-1}(\xi) & \\ -b_{n-1,-}(\xi) & & & & 0 \end{bmatrix},$$

which implies that

$$\mathbf{d}^{-1}(\widetilde{G}_{\xi,-})\mathbf{d}(\widetilde{G}_{\xi,+}) = \operatorname{diag}\left[\eta_{\xi,n-1},\ldots,\eta_{\xi,1},\eta_{\xi,n}\right],$$

where $\eta_{\xi,j}$ for $j = 1,2,\ldots,n$ are given by (9.10). It remains to invoke condition (ii) of Theorem 9.3. $\square$

REFERENCES

[1] M. A. BASTOS, YU. I. KARLOVICH, AND A. F DOS SANTOS, *The invertibility of convolution type operators on a union of intervals and the corona theorem*, Integral Equations and Operator Theory **42** (2002), 22–56.

[2] M. A. BASTOS, YU. I. KARLOVICH, AND B. SILBERMANN, *Toeplitz operators with symbols generated by slowly oscillating and semi-almost periodic matrix functions*, Proc. London Math. Soc. (3) **89**, 3 (2004), 697–737.

[3] A. S. BESICOVITCH, *Almost periodic functions*, Dover Publications Inc., New York, 1955.

[4] A. BÖTTCHER AND YU. I. KARLOVICH, *Carleson curves, Muckenhoupt weights, and Toeplitz operators*, Birkhäuser Verlag, Basel and Boston, 1997.

[5] A. BÖTTCHER, YU. I. KARLOVICH, AND I. M. SPITKOVSKY, *Convolution operators and factorization of almost periodic matrix functions*, Operator Theory: Advances and Applications, vol. 131, Birkhäuser Verlag, Basel and Boston, 2002.

[6] A. BRUDNYI, L. RODMAN, AND I. M. SPITKOVSKY, *Non-denseness of factorable matrix functions*, J. Functional Analysis **261** (2011), 1969–1991.

[7] C. CORDUNEANU, *Almost periodic functions*, J. Wiley & Sons, 1968.

[8] J. B. GARNETT, *Bounded analytic functions*, Academic Press, New York, 1981.

[9] R. HUNT, B. MUCKENHOUPT, AND R. WHEEDEN, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251.

[10] YU. I. KARLOVICH AND J. LORETO HERNÁNDEZ, *Wiener-Hopf operators with semi-almost periodic matrix symbols on weighted Lebesgue spaces*, Integral Equations and Operator Theory **62**, 1 (2008), 85–128.

[11] YU. I. KARLOVICH AND J. LORETO HERNÁNDEZ, *Wiener-Hopf operators with slowly oscillating matrix symbols on weighted Lebesgue spaces*, Integral Equations and Operator Theory **64**, 2 (2009), 203–237.

[12] YU. I. KARLOVICH AND J. LORETO HERNÁNDEZ, *Wiener-Hopf operators with oscillating symbols on weighted Lebesgue spaces*, Recent trends in Toeplitz and pseudodifferential operators, Operator Theory: Advances and Applications, vol. 210, Birkhäuser Verlag, Basel, 2010, pp. 123–145.

[13] YU. I. KARLOVICH AND J. LORETO HERNÁNDEZ, *Convolution type operators with oscillating symbols on weighted Lebesgue spaces on a union of intervals*, Mathematica **54 (77)** (2012), no. Special Issue, 104–119.

[14] YU. I. KARLOVICH AND I. M. SPITKOVSKY, *Factorization of almost periodic matrix functions and (semi) Fredholmness of some convolution type equations*, No. 4421–85 dep., VINITI, Moscow, 1985, in Russian.

[15] YU. I. KARLOVICH AND I. M. SPITKOVSKY, *Factorization of almost periodic matrix-valued functions and the Noether theory for certain classes of equations of convolution type*, Mathematics of the USSR, Izvestiya **34** (1990), 281–316.

[16] B. M. LEVITAN AND V. V. ZHIKOV, *Almost periodic functions and differential equations*, Cambridge University Press, 1982.

[17] C. V. M. VAN DER MEE, L. RODMAN, AND I. M. SPITKOVSKY, *Factorization of block triangular matrix functions with off diagonal binomials*, Operator Theory: Advances and Applications **160** (2005), 423–437.

[18] C. V. M. VAN DER MEE, L. RODMAN, I. M. SPITKOVSKY, AND H. J. WOERDEMAN, *Factorization of block triangular matrix functions in Wiener algebras on ordered abelian groups*, Operator Theory: Advances and Applications **149** (2004), 441–465.

[19] S. C. POWER, *Fredholm Toeplitz operators and slow oscillation*, Canad. J. Math. **32**, 5 (1980), 1058–1071.

[20] D. SARASON, *Toeplitz operators with semi-almost periodic symbols*, Duke Math. J. **44**, 2 (1977), 357–364.

[21] I. M. SPITKOVSKY, *Factorization of several classes of semi-almost periodic matrix functions and applications to systems of convolution equations*, Izvestiya VUZ., Mat. (1983), no. 4, 88–94 (in Russian), English translation in *Soviet Math. – Iz. VUZ* **27** (1983), 383–388.

(Received November 15, 2013)

Yu. I. Karlovich
Facultad de Ciencias
Universidad Autónoma del Estado de Morelos
Av. Universidad 1001, Col. Chamilpa,
C.P. 62209 Cuernavaca, Morelos
México
e-mail: karlovich@uaem.mx

J. Loreto-Hernández
Facultad de Ciencias
Universidad Autónoma de San Luis Potosí
Lateral av. Salvador Nava s/n, Col. Lomas
C.P. 78290 San Luis Potosí (SLP)
México
e-mail: juan@fc.uaslp.mx

I. M. Spitkovsky
Department of Mathematics
College of William and Mary
Williamsburg, VA 23187-8795
USA
and
Division of Science and Math
New York University Abu Dhabi (NYUAD)
Saadiyat Island, P.O. Box 129188
Abu Dhabi, UAE
e-mail: ilya@math.wm.edu, imspitkovsky@gmail.com