Operators
and
Matrices
Volume 9, Number 1 (2015), 31–56
doi:10.7153/oam-09-02

ON AN UPPER HEAT KERNEL BOUND FOR SECOND ORDER ELLIPTIC OPERATORS ON BOUNDED REGIONS IN $\mathbb{R}^N$

MICHAEL M. H. PANG

(*Communicated by F. Gesztesy*)

Abstract. We revisit an upper heat kernel bound for second order uniformly elliptic operators H defined on bounded regions Ω in $\mathbb{R}^N$. This bound is of the type

$$K_H(t,x,y) \leqslant c_1 \max\{t^{-(\frac{N}{2}+a)},1\}e^{-E_1 t}\exp\left\{\frac{-|x-y|^2}{8\Lambda t}\right\}\phi_1(x)\phi_1(y)$$

where E_1 and ϕ_1 are, respectively, the ground state eigenvalue and the normalized ground state eigenfunction of H, Λ is the upper ellipticity constant of H, $a>0$ is a constant related to a lower bound of ϕ_1 near the boundary $\partial\Omega$, and $c_1>0$ is a constant which depends on Ω, E_1, the ellipticity constants of H, and a lower bound of ϕ_1 near $\partial\Omega$. In particular, this bound provides a corrected version of a bound originally studied in [2] for large time $t>0$.

Let H be a second order uniformly elliptic operator of divergence form defined on a bounded region Ω in $\mathbb{R}^{\mathbb{N}}$. Under certain assumptions on H and Ω, Davies and Pang gave an upper bound for the heat kernel $K_H(t,x,y)$ corresponding to e^{-Ht} in [2, Theorem 18]. In this paper we revisit this bound in [2], in particular, we provide a detailed corrected version of this bound for large time $t>0$. Our main result is stated in Theorem 16. It gives a bound of the form

$$K_H(t,x,y) \leqslant c_1 \max\{t^{-(\frac{N}{2}+a)},1\}e^{-E_1 t}\exp\left\{\frac{-|x-y|^2}{8\Lambda t}\right\}\phi_1(x)\phi_1(y)$$

where E_1 and ϕ_1 are, respectively, the ground state eigenvalue and the normalized ground state eigenfunction of H, Λ is the upper ellipticity constant of H, $a>0$ is a constant related to a lower bound of ϕ_1 near the boundary $\partial\Omega$ of Ω, and $c_1>0$ is a constant which depends on Ω, E_1, the ellipticity constants of H, and a lower bound of ϕ_1 near $\partial\Omega$ (see below for precise definitions of these constants and assumptions on them). As a corollary we obtain an upper bound for the quantity

$$C(t) = \inf\left\{\phi_1(x)^{-1}K_H(t,x,y)\phi_1(y)^{-1} : x,y\in\Omega\right\}$$

in Corollary 17. This quantity $C(t)$ is related to the fundamental eigenvalue gap of the elliptic operator H (see [2, Proposition 1]).

Mathematics subject classification (2010): 35K08.
Keywords and phrases: Upper heat kernel bound.

Paper OaM-09-02

Let Ω be a bounded region in $\mathbb{R}^N$, $N \geqslant 3$. Let $H \geqslant 0$ be the self-adjoint operator with domain $\mathrm{Dom}(H) \subseteq L^2(\Omega)$ associated to the quadratic form

$$Q(f) = \int_\Omega \sum_{i,j=1}^N a_{i,j}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}\, dx$$

with form domain $\mathrm{Quad}(H) = W_0^{1,2}(\Omega)$, where $a_{i,j}$ are measurable and satisfy

$$0 < \lambda \leqslant \{a_{i,j}(x)\} \leqslant \Lambda < \infty \qquad (x \in \Omega).$$

Let $\{E_i\}_{i=1}^\infty$ be the eigenvalues of H and let ϕ_i be an eigenfunction of E_i, $i = 1,2,3,\cdots$, and we can assume that $\phi_1 > 0$ and that $\{\phi_i\}_{i=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$. We shall also assume that Ω is strongly regular in the sense that there exists $c_1 = c_1(\Omega) \geqslant 1$ such that

$$c_1^{-1} \int_\Omega |f(x)|^2 d(x)^{-2}\, dx \leqslant \int_\Omega |\nabla f|^2\, dx \qquad (f \in C_c^\infty(\Omega))$$

where

$$d(x) = \min\{|x-y| : y \notin \Omega\} \qquad (x \in \Omega).$$

We shall also assume that there exist $a \geqslant 1$ and $b > 0$ such that

$$bd(x)^a \leqslant \phi_1(x) \qquad (x \in \Omega).$$

Let $\mathscr{U} : L^2(\Omega, \phi_1(x)^2 dx) \longrightarrow L^2(\Omega)$ be the unitary operator defined by

$$\mathscr{U} f = \phi_1 f \qquad (f \in L^2(\Omega, \phi_1(x)^2 dx)),$$

let $\widetilde{H} = \mathscr{U}^{-1} H \mathscr{U}$ be the self-adjoint operator with $\mathrm{Dom}(\widetilde{H}) \subseteq L^2(\Omega, \phi_1(x)^2 dx)$ unitarily equivalent to H under $\mathscr{U}$, and let $\widetilde{Q}$ be the quadratic form associated to $\widetilde{H}$. Then, by [2, Lemma 6], we have

PROPOSITION 1. *We have*

$$\int_\Omega f^2 (\log f) \phi_1^2\, dx \leqslant \varepsilon \widetilde{Q}(f) + \widetilde{\beta}(\varepsilon) \|f\|^2_{L^2(\Omega,\phi_1(x)^2 dx)} + \|f\|^2_{L^2(\Omega,\phi_1(x)^2 dx)} \log \|f\|_{L^2(\Omega,\phi_1(x)^2 dx)}$$

for all $0 \leqslant f \in L^1(\Omega, \phi_1(x)^2 dx) \cap L^\infty(\Omega, \phi_1(x)^2 dx) \cap Quad(\widetilde{H})$ *and all* $0 < \varepsilon < \infty$, *where*

$$\widetilde{\beta}(\varepsilon) = c_\Omega a \log a - \left(\frac{N}{4} + \frac{a}{2}\right)(\log \lambda + \log \varepsilon) - \log b + c_\Omega$$

for some $c_\Omega \geqslant 1$ *which depends only on* Ω.

LEMMA 2. *We have*

$$\widetilde{Q}(f) = \int_\Omega \Big(\sum_{i,j=1}^N a_{i,j}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \Big) \phi_1(x)^2\, dx + \int_\Omega E_1 f(x)^2 \phi_1(x)^2\, dx$$

for all $f \in C_c^\infty(\Omega)$.

Proof. Without loss of generality we may assume that $a_{i,j}$ are differentiable. Since $\widetilde{Q}$ is unitarily equivalent to Q under $\mathscr{U}$, we have, for all $f \in C_c^\infty(\Omega)$,

$$\begin{aligned}
&\widetilde{Q}(f) \qquad (1)\\
&= Q(\mathscr{U} f)\\
&= \int_\Omega \sum_{i,j=1}^N a_{i,j}(x) \frac{\partial}{\partial x_i}(\phi_1 f) \frac{\partial}{\partial x_j}(\phi_1 f)\, dx\\
&= \int_\Omega \sum_{i,j=1}^N a_{i,j}(x) \left[\frac{\partial \phi_1}{\partial x_i} f + \phi_1 \frac{\partial f}{\partial x_i}\right] \left[\frac{\partial \phi_1}{\partial x_j} f + \phi_1 \frac{\partial f}{\partial x_j}\right] dx\\
&= \int_\Omega \sum_{i,j=1}^N a_{i,j}(x) \left\{ \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_j} f^2 + \phi_1 \frac{\partial \phi_1}{\partial x_i} f \frac{\partial f}{\partial x_j} + \phi_1 \frac{\partial \phi_1}{\partial x_j} f \frac{\partial f}{\partial x_i} + \phi_1^2 \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \right\} dx\\
&= \int_\Omega \left(\sum_{i,j=1}^N a_{i,j}(x) \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_j} \right) f^2\, dx + 2\int_\Omega \sum_{i,j=1}^N a_{i,j}(x)\phi_1 \frac{\partial \phi_1}{\partial x_i} \frac{1}{2} \frac{\partial}{\partial x_j}(f^2)\, dx\\
&\quad + \int_\Omega \left(\sum_{i,j=1}^N a_{i,j}(x) \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \right) \phi_1(x)^2\, dx.
\end{aligned}$$

But

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^N a_{i,j}\phi_1 \frac{\partial \phi_1}{\partial x_i} \frac{\partial}{\partial x_j}(f^2)\, dx \qquad (2)\\
&= -\int_\Omega \sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{i,j}\phi_1 \frac{\partial \phi_1}{\partial x_i}\right) f^2\, dx\\
&= -\int_\Omega \left\{ \sum_{i,j=1}^N \left[\frac{\partial}{\partial x_j}\left(a_{i,j} \frac{\partial \phi_1}{\partial x_i}\right)\right] \phi_1 + \sum_{i,j=1}^N a_{i,j} \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_j} \right\} f^2\, dx\\
&= \int_\Omega E_1 \phi_1^2 f^2\, dx - \int_\Omega \left[\sum_{i,j=1}^N a_{i,j} \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_j} \right] f^2\, dx.
\end{aligned}$$

The lemma now follows from (1) and (2). □

We shall let $\psi : \Omega \to \mathbb{R}$ be a bounded C^∞ function satisfying

$$\sum_{i,j=1}^N a_{i,j}(x) \frac{\partial \psi}{\partial x_i}\frac{\partial \psi}{\partial x_j} \leqslant 1$$

on Ω. Let $\alpha \in \mathbb{R}$ and let $\Phi = e^{\alpha\psi}$. We shall let $\check{H} = \widetilde{H} - E_1$ and let $\check{Q}$ be the quadratic form associated to $\check{H} \geqslant 0$. Then, by Lemma 2, we have

$$\check{Q}(f) = \int_\Omega \left(\sum_{i,j=1}^N a_{i,j}(x) \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \right) \phi_1(x)^2 dx \qquad (3)$$

for all $f \in C_c^\infty(\Omega)$.

LEMMA 3. *We have*

$$(1-\mu)\check{Q}(f,f^{p-1}) \leqslant \check{Q}(\Phi f,\Phi^{-1}f^{p-1}) + \alpha^2\left\{1+\frac{(p-2)^2}{4(p-1)\mu}\right\}\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}$$

for all $p>2, 0<\mu<1,$ *and* $0\leqslant f\in C_c^\infty(\Omega)$.

Proof. We have

$$\begin{aligned}
\check{Q}(\Phi f,\Phi^{-1}f^{p-1}) &= \int_\Omega\left(\sum_{i,j=1}^N a_{i,j}\frac{\partial}{\partial x_i}(\Phi f)\frac{\partial}{\partial x_j}(\Phi^{-1}f^{p-1})\right)\phi_1(x)^2\,dx \qquad (4)\\
&= \int_\Omega\left\{\sum_{i,j=1}^N a_{i,j}\left[\frac{\partial\Phi}{\partial x_i}f+\Phi\frac{\partial f}{\partial x_i}\right]\right.\\
&\quad\left.\times\left[-\Phi^{-2}\frac{\partial\Phi}{\partial x_j}f^{p-1}+\Phi^{-1}(p-1)f^{p-2}\frac{\partial f}{\partial x_j}\right]\right\}\phi_1(x)^2\,dx\\
&= \int_\Omega\left\{\sum_{i,j=1}^N a_{i,j}\left[\alpha\Phi\frac{\partial\psi}{\partial x_i}f+\Phi\frac{\partial f}{\partial x_i}\right]\right.\\
&\quad\left.\times\left[-\Phi^{-2}\alpha\Phi\frac{\partial\psi}{\partial x_j}f^{p-1}+\Phi^{-1}(p-1)f^{p-2}\frac{\partial f}{\partial x_j}\right]\right\}\phi_1(x)^2\,dx\\
&= \int_\Omega\left\{\sum_{i,j=1}^N a_{i,j}\left[-\alpha^2\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}f^p+\alpha(p-1)f^{p-1}\frac{\partial\psi}{\partial x_i}\frac{\partial f}{\partial x_j}\right.\right.\\
&\quad\left.\left.-\alpha f^{p-1}\frac{\partial f}{\partial x_i}\frac{\partial\psi}{\partial x_j}+(p-1)f^{p-2}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right]\right\}\phi_1(x)^2\,dx\\
&\geqslant \int_\Omega -\alpha^2 f^p\phi_1(x)^2\,dx\\
&\quad+\alpha(p-2)\int_\Omega\left[\sum_{i,j=1}^N a_{i,j}\frac{\partial\psi}{\partial x_i}\frac{\partial f}{\partial x_j}\right]f^{p-1}\phi_1(x)^2\,dx\\
&\quad+\int_\Omega\left[\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}(f^{p-1})\right]\phi_1(x)^2\,dx\\
&\geqslant -\alpha^2\int_\Omega f^p\phi_1(x)^2\,dx\\
&\quad-|\alpha|(p-2)\int_\Omega\left[\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right]^{1/2}f^{p-1}\phi_1(x)^2\,dx\\
&\quad+\int_\Omega\left[\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}(f^{p-1})\right]\phi_1(x)^2\,dx.
\end{aligned}$$

If $s>0$, then

$$\begin{aligned}
&\int_\Omega \left[\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right]^{1/2} f^{p-1}\phi_1(x)^2\,dx \qquad (5)\\
=&\frac{2}{p}\int_\Omega f^{p/2}\left[\sum_{i,j=1}^N a_{i,j}\frac{\partial}{\partial x_i}(f^{p/2})\frac{\partial}{\partial x_j}(f^{p/2})\right]^{1/2}\phi_1(x)^2\,dx\\
\leqslant&\frac{s}{p}\int_\Omega \left[\sum_{i,j=1}^N a_{i,j}\frac{\partial}{\partial x_i}(f^{p/2})\frac{\partial}{\partial x_j}(f^{p/2})\right]\phi_1(x)^2\,dx+\frac{1}{sp}\int_\Omega f^p\phi_1(x)^2\,dx\\
=&\frac{sp}{4(p-1)}\int_\Omega \left[\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}(f^{p-1})\right]\phi_1(x)^2\,dx+\frac{1}{sp}\int_\Omega f^p\phi_1(x)^2\,dx\\
=&\frac{sp}{4(p-1)}\check{Q}(f,f^{p-1})+\frac{1}{sp}\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}.
\end{aligned}$$

From (4) and (5) we have

$$\begin{aligned}
&\check{Q}(\Phi f,\Phi^{-1}f^{p-1}) \qquad (6)\\
\geqslant&-\alpha^2\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}\\
&-|\alpha|(p-2)\left\{\frac{sp}{4(p-1)}\check{Q}(f,f^{p-1})+\frac{1}{sp}\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}\right\}+\check{Q}(f,f^{p-1})\\
=&\check{Q}(f,f^{p-1})\left\{1-\frac{s|\alpha|p(p-2)}{4(p-1)}\right\}-\left\{\alpha^2+\frac{|\alpha|(p-2)}{sp}\right\}\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}.
\end{aligned}$$

Putting

$$s=\frac{4(p-1)\mu}{|\alpha|p(p-2)}$$

in (6) gives

$$\check{Q}(\Phi f,\Phi^{-1}f^{p-1})\geqslant(1-\mu)\check{Q}(f,f^{p-1})-\alpha^2\left[1+\frac{(p-2)^2}{4(p-1)\mu}\right]\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}$$

which is the lemma. □

LEMMA 4. *Suppose* $\beta:(0,\infty)\to\mathbb{R}$ *is a function satisfying*

$$\begin{aligned}
\int_\Omega f^2(\log f)\phi_1(x)^2\,dx\leqslant&\varepsilon\check{Q}(f)+\beta(\varepsilon)\|f\|^2_{L^2(\Omega,\phi_1(x)^2dx)} \qquad (7)\\
&+\|f\|^2_{L^2(\Omega,\phi_1(x)^2dx)}\log\|f\|_{L^2(\Omega,\phi_1(x)^2dx)}
\end{aligned}$$

for all $\varepsilon>0$ *and all* $0\leqslant f\in\mathrm{Dom}(\check{Q})\cap L^1(\Omega,\phi_1(x)^2dx)\cap L^\infty(\Omega,\phi_1(x)^2dx)$. *Then we* *have*

$$\int_\Omega f^p(\log f)\phi_1(x)^2\,dx \tag{8}$$
$$\leqslant \varepsilon\check{Q}(\Phi f,\Phi^{-1}f^{p-1})+\left\{\frac{2}{p}\beta((1-\mu)\varepsilon)+\varepsilon\alpha^2\left[1+\frac{(p-2)^2}{4(p-1)\mu}\right]\right\}\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}$$
$$+\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}\log\|f\|_{L^p(\Omega,\phi_1(x)^2dx)}$$

for all $\varepsilon>0$, $p>2, 0<\mu<1$ *and* $0\leqslant f\in C_c^\infty(\Omega)$.

Proof. Replacing f by $f^{p/2}$ in (7) we get

$$\frac{p}{2}\int_\Omega f^p(\log f)\phi_1(x)^2\,dx\leqslant\left(\frac{p^2}{4(p-1)}\right)\varepsilon\check{Q}(f,f^{p-1})+\beta(\varepsilon)\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}$$
$$+\frac{p}{2}\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}\log\|f\|_{L^p(\Omega,\phi_1(x)^2dx)}.$$

So

$$\int_\Omega f^p(\log f)\phi_1(x)^2\,dx\leqslant\varepsilon\check{Q}(f,f^{p-1})+\frac{2}{p}\beta(\varepsilon)\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)} \tag{9}$$
$$+\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}\log\|f\|_{L^p(\Omega,\phi_1(x)^2dx)}.$$

By (9) and Lemma 3 we have

$$\begin{aligned}\int_\Omega f^p(\log f)\phi_1(x)^2\,dx &\leqslant \varepsilon(1-\mu)\check{Q}(f,f^{p-1})+\frac{2}{p}\beta((1-\mu)\varepsilon)\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}\\ &\quad+\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}\log\|f\|_{L^p(\Omega,\phi_1(x)^2dx)}\\ &\leqslant\varepsilon\check{Q}(\Phi f,\Phi^{-1}f^{p-1})\\ &\quad+\left[\frac{2}{p}\beta((1-\mu)\varepsilon)+\varepsilon\alpha^2\left\{1+\frac{(p-2)^2}{4(p-1)\mu}\right\}\right]\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}\\ &\quad+\|f\|^p_{L^p(\Omega,\phi_1(x)^2dx)}\log\|f\|_{L^p(\Omega,\phi_1(x)^2dx)}.\quad\square\end{aligned}$$

More notations. If S is a vector space of functions defined on Ω, then we will write

$$S_+=\{f\in S: f(x)\geqslant 0\text{ for almost all }x\in\Omega\}.$$

Let

$$\check{\mathscr{D}}_1=\bigcup_{t>0}e^{-\check{H}t}\left(L^1(\Omega,\phi_1(x)^2dx)\cap L^\infty(\Omega,\phi_1(x)^2dx)\right)_+,$$

and let

$$\check{\mathscr{D}}=\Phi^{-1}\check{\mathscr{D}}_1.$$

Then $\check{\mathscr{D}}_1$ and $\check{\mathscr{D}}$ are both dense in $L^p(\Omega, \phi_1(x)^2dx)_+$ for $1 \leqslant p < \infty$. Also let

$$\check{K} = \Phi^{-1}\check{H}\Phi$$

with

$$\mathrm{Dom}(\check{K}) = \left\{f \in L^2(\Omega, \phi_1(x)^2dx) \;:\; \Phi f \in \mathrm{Dom}(\check{H})\right\}.$$

Then it is easy to check that $-\check{K}$ is the generator of the c_0-semigroup

$$e^{-\check{K}t} = \Phi^{-1}e^{-\check{H}t}\Phi$$

on $L^2(\Omega, \phi_1(x)^2dx)$. Note that $\check{\mathscr{D}}$ is invariant under the semigroup $e^{-\check{K}t}$. Since $e^{-\check{H}t}$ is holomorphic on $L^p(\Omega, \phi_1(x)^2dx)$ for all $1 < p < \infty$ (as $e^{-\check{H}t}$ is a symmetric Markov semigroup), the same is true for $e^{-\check{K}t}$. In particular $\check{\mathscr{D}}$ lies in the domain of $(-\check{K}_p)^n$ for all $n = 1, 2, 3, \cdots$, where $-\check{K}_p$ is the generator of $e^{-\check{K}t}$ on $L^p(\Omega, \phi_1(x)^2dx)$ for $1 < p < \infty$.

PROPOSITION 5. *Let $\mathscr{H}$ be a Hilbert space and let $B \geqslant 0$ be a closed quadratic form with $\mathrm{Dom}(B) \subseteq \mathscr{H}$. Then it is well known that the inner product $\langle\cdot,\cdot\rangle_B$ defined by*

$$\langle f, g\rangle_B = \langle f, g\rangle_{\mathscr{H}} + B(f, g)$$

on $\mathrm{Dom}(B) \times \mathrm{Dom}(B)$ makes $(\mathrm{Dom}(B), \langle\cdot,\cdot\rangle_B)$ a Hilbert space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathrm{Dom}(B)$ and let $f \in \mathrm{Dom}(B)$. Suppose that

$$\|f_n - f\|_{\mathscr{H}} \longrightarrow 0 \qquad \textit{as} \quad n \longrightarrow \infty$$

and that

$$\sup\{\langle f_n, f_n\rangle_B \;:\; n = 1, 2, 3, \cdots\} < \infty.$$

Then $f_n \to f$ weakly in $(\mathrm{Dom}(B), \langle\cdot,\cdot\rangle_B)$ as $n \to \infty$, i.e., for all $g \in \mathrm{Dom}(B)$ we have

$$\langle f_n, g\rangle_B \longrightarrow \langle f, g\rangle_B \qquad \textit{as} \quad n \longrightarrow \infty.$$

REMARKS. Proposition 5 can be proved using the spectral theorem.

LEMMA 6. *Let $p > 2$. If $f \in \mathrm{Quad}(\check{H})$, then $\Phi^{-p}f \in \mathrm{Quad}(\check{H})$.*

Proof. Since $C_c^\infty(\Omega)$ is a form core of $\check{Q}$, it suffices to show that there exists $c \geqslant 1$ such that

$$c^{-1}\,|||f||| \leqslant \left|\left|\left|\Phi^{-p}f\right|\right|\right| \leqslant c\,|||f||| \qquad (f \in C_c^\infty(\Omega)), \tag{10}$$

where

$$|||f|||^2 = \|f\|^2_{L^2(\Omega, \phi_1(x)^2dx)} + \check{Q}(f) \qquad (f \in \mathrm{Quad}(\check{H})).$$

Let $f \in C_c^\infty(\Omega)$. Then

$$\begin{aligned}
\check{Q}(\Phi^{-p}f) &= \int_\Omega \left[\sum_{i,j=1}^N a_{i,j}\frac{\partial}{\partial x_i}(\Phi^{-p}f)\frac{\partial}{\partial x_j}(\Phi^{-p}f)\right]\phi_1(x)^2\,dx \qquad (11)\\
&= \int_\Omega \left[\sum_{i,j=1}^N a_{i,j}\left(-p\Phi^{-p-1}\alpha\Phi\frac{\partial\psi}{\partial x_i}f + \Phi^{-p}\frac{\partial f}{\partial x_i}\right)\right.\\
&\quad \left.\times\left(-p\Phi^{-p-1}\alpha\Phi\frac{\partial\psi}{\partial x_j}f + \Phi^{-p}\frac{\partial f}{\partial x_j}\right)\right]\phi_1(x)^2\,dx\\
&= \int_\Omega \left[\sum_{i,j=1}^N a_{i,j}\left(p^2\Phi^{-2p}\alpha^2 f^2\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j} - p\Phi^{-2p}\alpha f\frac{\partial\psi}{\partial x_i}\frac{\partial f}{\partial x_j}\right.\right.\\
&\quad \left.\left. - p\Phi^{-2p}\alpha f\frac{\partial f}{\partial x_i}\frac{\partial\psi}{\partial x_j} + \Phi^{-2p}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\right]\phi_1(x)^2\,dx.
\end{aligned}$$

We deal with each term in the last line in (11). First we have

$$\begin{aligned}
0 &\leqslant \int_\Omega \left[\sum_{i,j=1}^N a_{i,j}p^2\Phi^{-2p}\alpha^2 f^2\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\right]\phi_1(x)^2\,dx \qquad (12)\\
&= \int_\Omega p^2\Phi^{-2p}\alpha^2\left[\sum_{i,j=1}^N a_{i,j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\right]f^2\phi_1(x)^2\,dx\\
&\leqslant c(p,\alpha,\psi)\,\|f\|^2_{L^2(\Omega,\phi_1(x)^2dx)}
\end{aligned}$$

for some $c(p,\alpha,\psi) \geqslant 1$. Next we have

$$\begin{aligned}
&\left|\int_\Omega\left(\sum_{i,j=1}^N a_{i,j}(-p)\Phi^{-2p}\alpha f\frac{\partial\psi}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\phi_1(x)^2\,dx\right| \qquad (13)\\
&\leqslant p\,|\alpha|\int_\Omega\left|\sum_{i,j=1}^N a_{i,j}\frac{\partial\psi}{\partial x_i}\frac{\partial f}{\partial x_j}\right|\Phi^{-2p}\,|f|\,\phi_1(x)^2\,dx\\
&\leqslant p\,|\alpha|\int_\Omega\left\{\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right\}^{1/2}\Phi^{-2p}\,|f|\,\phi_1(x)^2\,dx\\
&\leqslant c(p,\alpha,\psi)\int_\Omega\left\{\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right\}^{1/2}|f|\,\phi_1(x)^2\,dx\\
&\leqslant \frac{1}{2}c(p,\alpha,\psi)\left\{\int_\Omega\left(\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\phi_1(x)^2\,dx\right\}^{1/2}\\
&\quad\times\left\{\int_\Omega f^2\phi_1(x)^2\,dx\right\}^{1/2}\\
&= \frac{1}{2}c(p,\alpha,\psi)\check{Q}(f)^{1/2}\,\|f\|_{L^2(\Omega,\phi_1(x)^2dx)}
\end{aligned}$$

$$\leqslant \frac{1}{4}c(p,\alpha,\psi)\left\{\check{Q}(f)+\|f\|^2_{L^2(\Omega,\phi_1(x)^2dx)}\right\}$$
$$=\frac{1}{4}c(p,\alpha,\psi)\,|||f|||^2.$$

Lastly

$$\int_\Omega \left[\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right]\Phi^{-2p}\phi_1(x)^2\,dx \tag{14}$$
$$\leqslant c(p,\alpha,\psi)\int_\Omega \left[\sum_{i,j=1}^N a_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right]\phi_1(x)^2\,dx$$
$$=c(p,\alpha,\psi)\check{Q}(f).$$

So, by (11), (12), (13), (14),

$$\check{Q}(\Phi^{-p}f)\leqslant c(p,\alpha,\psi)\,|||f|||^2<\infty, \tag{15}$$

therefore

$$\left|\left|\left|\Phi^{-p}f\right|\right|\right|^2\leqslant c(p,\alpha,\psi)\,|||f|||^2 \tag{16}$$

for some $c(p,\alpha,\psi)\geqslant 1$. Replacing α by $-\alpha$ we have, by (15),

$$\check{Q}(f)=\check{Q}(\Phi^p(\Phi^{-p}f))\leqslant c(p,-\alpha,\psi)\left|\left|\left|\Phi^{-p}f\right|\right|\right|^2$$

and thus

$$|||f|||^2\leqslant c(p,-\alpha,\psi)\left|\left|\left|\Phi^{-p}f\right|\right|\right|^2 \tag{17}$$

for some $c(p,-\alpha,\psi)\geqslant 1$. (10) now follows from (16) and (17). □

PROPOSITION 7. *Let $\mathscr{H}$ be a Hilbert space and let $f,f_n\in\mathscr{H}$ for $n=1,2,3,\cdots$. Suppose that*

$$\langle f,g\rangle_{\mathscr{H}}=\lim_{n\to\infty}\langle f_n,g\rangle_{\mathscr{H}} \qquad (g\in\mathscr{H}).$$

Then

$$\lim_{n\to\infty}\|f_n-f\|_{\mathscr{H}}=0$$

if and only if

$$\lim_{n\to\infty}\|f_n\|_{\mathscr{H}}=\|f\|_{\mathscr{H}}.$$

COROLLARY 8. *Let $\mathscr{H}$ be a Hilbert space and let $A\geqslant 0$ be a self-adjoint operator with $\operatorname{Dom}(A)\subseteq\mathscr{H}$. Let $g,g_n\in\operatorname{Quad}(A)$, $n=1,2,3,\cdots$. Suppose that*

$$\lim_{n\to\infty}\|g_n-g\|_{\mathscr{H}}=0 \tag{18}$$

and that

$$g_n\xrightarrow{w}g \quad \text{as} \quad n\to\infty \quad \text{in} \quad \operatorname{Quad}(A),$$

i.e.,

$$\langle g_n, f\rangle_{\mathscr{H}} + \left\langle A^{1/2} g_n, A^{1/2} f\right\rangle_{\mathscr{H}} \longrightarrow \langle g, f\rangle_{\mathscr{H}} + \left\langle A^{1/2} g, A^{1/2} f\right\rangle_{\mathscr{H}}$$

as $n \to \infty$ *for all* $f \in \mathrm{Quad}(A)$. *Suppose also that*

$$\limsup_{n\to\infty} |||g_n||| \leqslant |||g||| \tag{19}$$

where

$$|||f|||^2 = \|f\|^2_{\mathscr{H}} + \left\langle A^{1/2} f, A^{1/2} f\right\rangle_{\mathscr{H}} \qquad (f \in \mathrm{Quad}(A)).$$

Then

$$|||g_n - g||| \longrightarrow 0 \qquad as \quad n \longrightarrow \infty.$$

Proof. By the spectral theorem we represent A as multiplication by a positive function $a : M \longrightarrow [0,\infty)$, where (M, m) is a measure space, and

$$\mathrm{Dom}(A) = \left\{ f \in L^2(M) \;:\; af \in L^2(M)\right\}.$$

Take a subsequence $\{g_{n_i}\}_{i=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ such that

$$\lim_{i\to\infty} g_{n_i}(x) = g(x) \qquad (\text{a.e.} \quad x \in M).$$

Then, by Fatou's lemma, (18) and (19) ,

$$\begin{aligned}\int_M a g^2\, dm &\leqslant \liminf_{i\to\infty} \int_M a g_{n_i}^2\, dm \\ &\leqslant \limsup_{i\to\infty} \int_M a g_{n_i}^2\, dm \\ &= \limsup_{i\to\infty} \left\langle A^{1/2} g_{n_i}, A^{1/2} g_{n_i}\right\rangle_{\mathscr{H}} \\ &\leqslant \left\langle A^{1/2} g, A^{1/2} g\right\rangle_{\mathscr{H}} \\ &= \int_M a g^2\, dm.\end{aligned}$$

Thus

$$\lim_{i\to\infty} \int_M a g_{n_i}^2\, dm = \int_M a g^2\, dm,$$

i.e.

$$\lim_{i\to\infty} \left\langle A^{1/2} g_{n_i}, A^{1/2} g_{n_i}\right\rangle_{\mathscr{H}} = \left\langle A^{1/2} g, A^{1/2} g\right\rangle_{\mathscr{H}},$$

thus

$$\lim_{i\to\infty} |||g_{n_i}||| = |||g|||.$$

So for any subsequence $\{g_{n_i}\}_{i=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$, there exists a subsequence $\left\{g_{n_{i_j}}\right\}_{j=1}^{\infty}$ of $\{g_{n_i}\}_{i=1}^{\infty}$ such that

$$\lim_{j\to\infty} \left|\left|\left| g_{n_{i_j}} \right|\right|\right| = |||g|||.$$

Hence

$$\lim_{n\to\infty} |||g_n||| = |||g|||.$$

Since $g_n \xrightarrow{w} g$ as $n \to \infty$ in $\mathrm{Quad}(A)$, Proposition 7 implies that

$$\lim_{n\to\infty} |||g_n - g||| = 0. \quad \square$$

More notations. We let

$$\check{\mathscr{D}}_2 = \left(\mathrm{Quad}(\check{H}) \cap L^1(\Omega, \phi_1(x)^2 dx) \cap L^\infty(\Omega, \phi_1(x)^2 dx)\right)_+.$$

Note that $\check{\mathscr{D}}_1 \subseteq \check{\mathscr{D}}_2$.

LEMMA 9. *Let $\beta : (0,\infty) \longrightarrow \mathbb{R}$ be a function such that*

$$\begin{aligned}
&\int_\Omega \left(\Phi^{-1} g\right)^p \log(\Phi^{-1} g) \phi_1(x)^2 \, dx \\
&\leqslant \varepsilon \check{Q}(g, \Phi^{-p} g^{p-1}) \\
&\quad + \left\{ \frac{2}{p} \beta((1-\mu)\varepsilon) + \varepsilon \alpha^2 \left[1 + \frac{(p-2)^2}{4(p-1)\mu} \right] \right\} \left\|\Phi^{-1} g\right\|^p_{L^p(\Omega, \phi_1(x)^2 dx)} \\
&\quad + \left\|\Phi^{-1} g\right\|^p_{L^p(\Omega, \phi_1(x)^2 dx)} \log \left\|\Phi^{-1} g\right\|_{L^p(\Omega, \phi_1(x)^2 dx)}
\end{aligned} \tag{20}$$

for all $\varepsilon > 0, p > 2, 0 < \mu < 1$ and $0 \leqslant g \in C_c^\infty(\Omega)$. Then the inequality (20) also holds for all $g \in \check{\mathscr{D}}_2$.

Proof. Let $g \in \check{\mathscr{D}}_2$. Since $g \in L^1(\Omega, \phi_1(x)^2 dx) \cap L^\infty(\Omega, \phi_1(x)^2 dx)$ and Φ^{-1} is bounded, $\Phi^{-1} g \in L^1(\Omega, \phi_1(x)^2 dx) \cap L^\infty(\Omega, \phi_1(x)^2 dx)$. Hence $\Phi^{-1} g \in L^q(\Omega, \phi_1(x)^2 dx)$ for all $1 \leqslant q \leqslant \infty$. Since $x \longmapsto x \log x$ is a bounded function on any bounded interval, we have

$$\int_\Omega \left(\Phi^{-1} g\right)^p \log\left(\Phi^{-1} g\right) \phi_1(x)^2 \, dx < \infty, \qquad (p > 2)$$

and thus the left side of (20) makes sense.

Since $g \in L^\infty$, say $\|g\|_\infty \leqslant k$, and $p > 2$, there exists a function $F : [0,\infty) \longrightarrow \mathbb{R}$ such that

$$F(x) = x^{p-1} \qquad (0 \leqslant x \leqslant 2k+1),$$

that F has bounded derivative on $[0,\infty)$, and that

$$g^{p-1} = F(g).$$

So by the Beurling-Deny criteria (see [1, Theorem 1.3.3.]), there exists $c \geqslant 1$ such that

$$\check{Q}(g^{p-1}) \leqslant c \check{Q}(g) < \infty.$$

Clearly $g^{p-1} \in L^1(\Omega, \phi_1(x)^2 dx)$, we deduce that $g \in \check{\mathscr{D}}_2$ implies that $g^{p-1} \in \check{\mathscr{D}}_2$, hence, by Lemma 6, $\Phi^{-p} g^{p-1} \in \mathrm{Quad}(\check{H})$, and thus $\Phi^{-p} g^{p-1} \in \check{\mathscr{D}}_2$. Next, given any $g \in \check{\mathscr{D}}_2$, suppose

$$\|g\|_\infty \leqslant k. \tag{21}$$

Then there exists a sequence $\{h_n\}_{n=1}^\infty$ in $C_c^\infty(\Omega)$ such that

$$\begin{aligned} |||h_n - g|||^2 &= \left\langle \check{H}^{1/2}(h_n - g), \check{H}^{1/2}(h_n - g) \right\rangle_{L^2(\Omega, \phi_1(x)^2 dx)} \\ &\quad + \|h_n - g\|_{L^2(\Omega, \phi_1(x)^2 dx)} \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

For each $n = 1, 2, 3, \cdots$ there exists a function $F_n : (-\infty, \infty) \longrightarrow [0, \infty)$ such that F_n is C^∞, that

$$\begin{aligned} 0 \leqslant F_n(x) &\leqslant 2k \qquad (x \in \mathbb{R}), \\ 0 \leqslant F_n'(x) &\leqslant 1 \qquad (x \in \mathbb{R}), \\ F(0) &= 0, \\ F(x) &\leqslant x \qquad (x > 0), \end{aligned}$$

and that if we put

$$g_n = F_n(h_n), \qquad (n = 1, 2, 3, \cdots),$$

then g_n satisfies the following:

$$\begin{aligned} &0 \leqslant g_n \in C_c^\infty(\Omega), \\ &0 \leqslant g_n \leqslant 2k \qquad (n = 1, 2, 3, \cdots), \end{aligned} \tag{22}$$

$$\|g_n - g\|_{L^2(\Omega, \phi_1(x)^2 dx)} \to 0 \qquad \text{as} \quad n \to \infty, \tag{23}$$

and

$$\begin{aligned} \check{Q}(g_n) &= \int_\Omega F_n'(h_n)^2 \left(\sum_{i,j=1}^N a_{i,j} \frac{\partial h_n}{\partial x_i} \frac{\partial h_n}{\partial x_j} \right) \phi_1(x)^2 \, dx \\ &\leqslant \int_\Omega \left(\sum_{i,j=1}^N a_{i,j} \frac{\partial h_n}{\partial x_i} \frac{\partial h_n}{\partial x_j} \right) \phi_1(x)^2 \, dx \\ &= \check{Q}(h_n). \end{aligned}$$

Thus we have

$$\limsup_{n\to\infty} |||g_n||| \leqslant |||g||| \,.$$

Hence, by Proposition 5,

$$g_n \longrightarrow g \quad \text{weakly in } \mathrm{Quad}(\check{H}) \text{ as } \quad n \longrightarrow \infty.$$

So, by Corollary 8,

$$|||g_n - g||| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty. \tag{24}$$

Let $F:[0,\infty)\longrightarrow\mathbb{R}$ be the function described earlier in the proof of this lemma. Then the same argument used for g^{p-1} before shows that there exists $c'\geqslant 1$ such that

$$\check{Q}(g_n^{p-1})\leqslant c\,\check{Q}(g_n)\leqslant c'\check{Q}(g)\qquad (n=1,2,3,\cdots).$$

Since multiplication by Φ^{-p} is a bounded operator on $\mathrm{Quad}(\check{H})$, by the proof of Lemma 6, there exists $c''\geqslant 1$ such that

$$\left|\left|\left|\Phi^{-p}g_n^{p-1}\right|\right|\right|\leqslant c''\qquad (n=1,2,3,\cdots). \tag{25}$$

By (21), (22), (23), we have

$$\left\|g_n^{p-1}-g^{p-1}\right\|_{L^2(\Omega,\phi_1(x)^2dx)}\longrightarrow 0\qquad \text{as}\quad n\longrightarrow\infty,$$

hence

$$\left\|\Phi^{-p}g_n^{p-1}-\Phi^{-p}g^{p-1}\right\|_{L^2(\Omega,\phi_1(x)^2dx)}\longrightarrow 0\qquad \text{as}\quad n\longrightarrow\infty.$$

So Proposition 5 implies that

$$\Phi^{-p}g_n^{p-1}\longrightarrow\Phi^{-p}g^{p-1}\qquad \text{weakly in } \mathrm{Quad}(\check{H}) \text{ as } n\longrightarrow\infty. \tag{26}$$

Thus we have, by (24), (25), (26),

$$\begin{aligned}&\left|\check{Q}(g_n,\Phi^{-p}g_n^{p-1})-\check{Q}(g,\Phi^{-p}g^{p-1})\right|\\ &\leqslant\left|\check{Q}(g_n-g,\Phi^{-p}g_n^{p-1})\right|+\left|\check{Q}(g,\Phi^{-p}g_n^{p-1})-\check{Q}(g,\Phi^{-p}g^{p-1})\right|\\ &\to 0\qquad \text{as}\quad n\longrightarrow\infty.\end{aligned} \tag{27}$$

By (21), (22), (23), we have

$$\|g_n-g\|_{L^p(\Omega,\phi_1(x)^2dx)}\longrightarrow 0\qquad \text{as}\quad n\longrightarrow\infty,$$

and hence

$$\left\|\Phi^{-1}g_n-\Phi^{-1}g\right\|_{L^p(\Omega,\phi_1(x)^2dx)}\longrightarrow 0\qquad \text{as}\quad n\longrightarrow\infty. \tag{28}$$

Taking a subsequence of $\{g_n\}_{n=1}^{\infty}$ if necessary, we can, by (23), assume that

$$g(x)=\lim_{n\longrightarrow\infty}g_n(x)\qquad (\text{a.e. } x\in\Omega).$$

So by Fatou's lemma we have

$$\begin{aligned}&\int_\Omega\left(\Phi^{-1}g\right)^p\log_+(\Phi^{-1}g)\phi_1(x)^2\,dx\\ &\leqslant\liminf_{n\to\infty}\int_\Omega\left(\Phi^{-1}g_n\right)^p\log_+(\Phi^{-1}g_n)\phi_1(x)^2\,dx.\end{aligned} \tag{29}$$

Since $p>2$, $x\longmapsto x^{p-2}\log x$ is a bounded function on any bounded interval of $(0,\infty)$. By

$$\begin{aligned}&\int_\Omega\left(\Phi^{-1}g\right)^p\log_-(\Phi^{-1}g)\phi_1(x)^2\,dx-\int_\Omega\left(\Phi^{-1}g_n\right)^p\log_-(\Phi^{-1}g_n)\phi_1(x)^2\,dx\\ &=\int_\Omega(\Phi^{-1}g)^2\left[(\Phi^{-1}g)^{p-2}\log_-(\Phi^{-1}g)-(\Phi^{-1}g_n)^{p-2}\log_-(\Phi^{-1}g_n)\right]\phi_1(x)^2\,dx\\ &\quad+\int_\Omega\left[(\Phi^{-1}g)^2-(\Phi^{-1}g_n)^2\right](\Phi^{-1}g_n)^{p-2}\log_-(\Phi^{-1}g_n)\phi_1(x)^2\,dx,\end{aligned} \tag{30}$$

(23) and the dominated convergence theorem, we see that

$$\begin{aligned}&\int_\Omega (\Phi^{-1}g)^p \log_-(\Phi^{-1}g)\phi_1(x)^2\,dx \\ &= \lim_{n\to\infty}\int_\Omega (\Phi^{-1}g_n)^p \log_-(\Phi^{-1}g_n)\phi_1(x)^2\,dx.\end{aligned} \tag{31}$$

Finally (29) and (31) give

$$\begin{aligned}&\int_\Omega (\Phi^{-1}g)^p \log(\Phi^{-1}g)\phi_1(x)^2\,dx \\ &\leqslant \limsup_{n\to\infty}\int_\Omega (\Phi^{-1}g_n)^p \log(\Phi^{-1}g_n)\phi_1(x)^2\,dx.\end{aligned} \tag{32}$$

Since (20) holds for $g_n,\ n=1,2,3,\ldots,$ as $0\leqslant g_n\in C_c^\infty(\Omega),$ the lemma now follows from (32), (27), and (28). □

THEOREM 10. *Let $\varepsilon(p)>0$ be a continuous function defined for $p>2$ and let $0<\mu<1$. Let $\beta:(0,\infty)\longrightarrow\mathbb{R}$ be a function satisfying the hypothesis in Lemma 9. Put*

$$\Gamma(p)=\frac{2}{p}\beta((1-\mu)\varepsilon(p))+\varepsilon(p)\alpha^2\left[1+\frac{(p-2)^2}{4(p-1)\mu}\right]\qquad(p>2).$$

If

$$t=\int_2^\infty\frac{\varepsilon(p)}{p}dp,\quad M=\int_2^\infty\frac{\Gamma(p)}{p}dp$$

are both finite, then

$$\left\|e^{-\check{K}t}\right\|_{L^2(\Omega,\phi_1(x)^2dx)\to L^\infty(\Omega,\phi_1(x)^2dx)}\leqslant e^M.$$

Proof. Define the function $S:[2,\infty)\longrightarrow[0,t)$ by

$$S(p)=\int_2^p\frac{\varepsilon(\tau)}{\tau}d\tau\qquad(p\geqslant 2).$$

Then $S(2)=0,\ S$ is strictly increasing, and

$$S'(p)=\frac{\varepsilon(p)}{p}\qquad(p\geqslant 2).$$

Let $P:[0,t)\longrightarrow[2,\infty)$ be the inverse function of S. Then $P(0)=2$ and

$$P'(\sigma)=\frac{P(\sigma)}{\varepsilon(P(\sigma))}\qquad(0\leqslant\sigma<t).$$

We also define the function $N_1:[2,\infty)\longrightarrow[0,M)$ by

$$N_1(p)=\int_2^p\frac{\Gamma(\tau)}{\tau}d\tau\qquad(p\geqslant 2)$$

and we let

$$N(\sigma) = N_1(P(\sigma)) \qquad (0 \leqslant \sigma < t).$$

Then $N(0) = 0$ and

$$N'(\sigma) = \frac{\Gamma(P(\sigma))}{\varepsilon(P(\sigma))} \qquad (0 \leqslant \sigma < t).$$

Let $f \in \check{D}$ and put

$$f_\sigma = e^{-\check{K}\sigma} f \qquad (0 \leqslant \sigma < t).$$

Then

$$\begin{aligned}
&\frac{d}{d\sigma}\left\{\log\left[e^{-N(\sigma)}\, \|f_\sigma\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}\right]\right\} \\
=&\frac{d}{d\sigma}\left\{-N(\sigma) + P(\sigma)^{-1}\log\|f_\sigma\|^{P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}\right\} \\
=&-\frac{\Gamma(P(\sigma))}{\varepsilon(P(\sigma))} - P(\sigma)^{-1}\varepsilon(P(\sigma))^{-1}\log\|f_\sigma\|^{P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} \\
&+P(\sigma)^{-1}\,\|f_\sigma\|^{-P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}\,\frac{d}{d\sigma}\left(\|f_\sigma\|^{P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}\right).
\end{aligned} \tag{33}$$

Consider the last term on the right side of (33):

$$\begin{aligned}
&\frac{d}{d\sigma}\left(\|f_\sigma\|^{P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}\right) \\
=&\lim_{h\to 0} h^{-1}\int_\Omega\left\{\left(e^{-\check{K}(\sigma+h)}f\right)^{P(\sigma+h)} - \left(e^{-\check{K}\sigma}f\right)^{P(\sigma)}\right\}\phi_1(x)^2\,dx \\
=&\lim_{h\to 0} h^{-1}\int_\Omega\left\{\left(e^{-\check{K}(\sigma+h)}f\right)^{P(\sigma+h)} - \left(e^{-\check{K}\sigma}f\right)^{P(\sigma+h)}\right\}\phi_1(x)^2\,dx \\
&+h^{-1}\int_\Omega\left\{\left(e^{-\check{K}\sigma}f\right)^{P(\sigma+h)} - \left(e^{-\check{K}\sigma}f\right)^{P(\sigma)}\right\}\phi_1(x)^2\,dx \\
=&\lim_{h\to 0}\int_\Omega P(\sigma+h)\left(e^{-\check{K}(\sigma+h)}f(x)\longleftrightarrow e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1} \\
&\times h^{-1}\left[e^{-\check{K}(\sigma+h)}f - e^{-\check{K}\sigma}f\right]\phi_1(x)^2\,dx \\
&+\int_\Omega (e^{-\check{K}\sigma}f)\log(e^{-\check{K}\sigma}f)(e^{-\check{K}\sigma}f)^{P(\sigma+k)-1} \\
&\times h^{-1}\left[P(\sigma+h)-P(\sigma)\right]\phi_1(x)^2\,dx
\end{aligned} \tag{34}$$

where we have used

$$e^{-\check{K}(\sigma+h)}f(x)\longleftrightarrow e^{-\check{K}\sigma}f(x)$$

to denote a number between $e^{-\check{K}(\sigma+h)}f(x)$ and $e^{-\check{K}\sigma}f(x)$, and $k = k(h,x)$ satisfies

$$0 \leqslant |k| \leqslant |h|.$$

Now we have

$$\begin{aligned}
&\int_\Omega P(\sigma+h)\left(e^{-\check{K}(\sigma+h)}f(x)\longleftrightarrow e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1} \\
&\times h^{-1}\left[e^{-\check{K}(\sigma+h)}f-e^{-\check{K}\sigma}f\right]\phi_1(x)^2\,dx \\
=&P(\sigma+h)\left\{\int_\Omega\left(e^{-\check{K}(\sigma+h)}f(x)\longleftrightarrow e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1}\right. \\
&\times\left(h^{-1}\left[e^{-\check{K}(\sigma+h)}f-e^{-\check{K}\sigma}f\right]+\check{K}e^{-\check{K}\sigma}f\right)\phi_1(x)^2\,dx \\
&\left.-\int_\Omega\left(e^{-\check{K}(\sigma+h)}f(x)\longleftrightarrow e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1}(\check{K}e^{-\check{K}\sigma}f)\phi_1(x)^2\,dx\right\}.
\end{aligned} \tag{35}$$

Note that

$$e^{-\check{K}(\sigma+h)}f+e^{-\check{K}\sigma}f\in L^1(\Omega,\phi_1(x)^2dx)\cap L^\infty(\Omega,\phi_1(x)^2dx) \tag{36}$$

and there exists $M\geqslant 1,$ depending on $\sigma,$ such that

$$\left\|\left(e^{-\check{K}(\sigma+h)}f+e^{-\check{K}\sigma}f\right)^{P(\sigma+h)-1}\right\|_{L^2(\Omega,\phi_1(x)^2dx)}\leqslant M \tag{37}$$

for all $|h|\leqslant M^{-1}.$ So we have

$$\begin{aligned}
&\left|\int_\Omega\left(e^{-\check{K}(\sigma+h)}f(x)\longleftrightarrow e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1}\right. \\
&\left.\times\left(h^{-1}\left[e^{-\check{K}(\sigma+h)}f-e^{-\check{K}\sigma}f\right]+\check{K}e^{-\check{K}\sigma}f\right)\phi_1(x)^2\,dx\right| \\
\leqslant&\int_\Omega\left(e^{-\check{K}(\sigma+h)}f+e^{-\check{K}\sigma}f\right)^{P(\sigma+h)-1} \\
&\times\left|\left(h^{-1}\left[e^{-\check{K}(\sigma+h)}f-e^{-\check{K}\sigma}f\right]+\check{K}e^{-\check{K}\sigma}f\right)\right|\phi_1(x)^2\,dx \\
\leqslant&\left\|\left(e^{-\check{K}(\sigma+h)}f+e^{-\check{K}\sigma}f\right)^{P(\sigma+h)-1}\right\|_{L^2(\Omega,\phi_1(x)^2dx)} \\
&\times\left\|h^{-1}\left[e^{-\check{K}(\sigma+h)}f-e^{-\check{K}\sigma}f\right]+\check{K}e^{-\check{K}\sigma}f\right\|_{L^2(\Omega,\phi_1(x)^2dx)} \\
\to&\, 0\qquad\text{as}\quad h\to 0.
\end{aligned} \tag{38}$$

Next note that

$$\begin{aligned}
&\left|\left(e^{-\check{K}(\sigma+h)}f(x)\longleftrightarrow e^{-\check{K}\sigma}f(x)\right)-e^{-\check{K}\sigma}f(x)\right| \\
\leqslant&\left|e^{-\check{K}(\sigma+h)}f(x)-e^{-\check{K}\sigma}f(x)\right|\qquad (x\in\Omega).
\end{aligned} \tag{39}$$

Writing

$$m(f,\sigma,h,x)=\left(e^{-\check{K}(\sigma+h)}f(x)\longleftrightarrow e^{-\check{K}\sigma}f(x)\right)-e^{-\check{K}\sigma}f(x) \tag{40}$$

we have

$$
\begin{aligned}
&\left(e^{-\check{K}(\sigma+h)}f(x) \longleftrightarrow e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1} \qquad (41)\\
=&\left(e^{-\check{K}\sigma}f(x)+m(f,\sigma,h,x)\right)^{P(\sigma+h)-1}\\
=&\left(e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1}\\
&+(P(\sigma+h)-1)\left(e^{-\check{K}\sigma}f(x)+jm(f,\sigma,h,x)\right)^{P(\sigma+h)-2}m(f,\sigma,h,x)
\end{aligned}
$$

for some $0 \leqslant j \leqslant 1$ depending on f,σ,h and x. Let $W : \Omega \longrightarrow \mathbb{R}$ be defined by

$$
W(x)=\begin{cases}\left\|e^{-\check{K}\sigma}f\right\|^{P(\sigma+1)-1}_{L^\infty(\Omega,\phi_1(x)^2dx)} & (e^{-\check{K}\sigma}f(x)>1)\\ e^{-\check{K}\sigma}f(x) & (0\leqslant e^{-\check{K}\sigma}f(x)\leqslant 1).\end{cases}
$$

Then $W \in L^2(\Omega,\phi_1(x)^2dx)$ and

$$
\left(e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1} \leqslant W(x) \qquad (x\in\Omega,\ |h|<1). \tag{42}
$$

So

$$
\int_\Omega W(x)\left|\check{K}e^{-\check{K}\sigma}f\right|\phi_1(x)^2\,dx<\infty
$$

and hence the dominated convergence theorem implies that

$$
\begin{aligned}
&\int_\Omega\left(e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1}(\check{K}e^{-\check{K}\sigma}f)\phi_1(x)^2\,dx \qquad (43)\\
\longrightarrow&\int_\Omega\left(e^{-\check{K}\sigma}f(x)\right)^{P(\sigma)-1}(\check{K}e^{-\check{K}\sigma}f)\phi_1(x)^2\,dx \quad \text{as} \quad h\to 0.
\end{aligned}
$$

Also, by (39), for $|h|<1$

$$
\begin{aligned}
&\left|\int_\Omega(P(\sigma+h)-1)\left(e^{-\check{K}\sigma}f(x)+jm(f,\sigma,h,x)\right)^{P(\sigma+h)-2}\right. \qquad (44)\\
&\left.\times m(f,\sigma,h,x)(\check{K}e^{-\check{K}\sigma}f)\phi_1(x)^2\,dx\right|\\
\leqslant&(P(\sigma+h)-1)\left(\left\|e^{-\check{K}\sigma}f\right\|_{L^\infty(\Omega,\phi_1(x)^2dx)}\right.\\
&\left.+2\sup\left\{\left\|e^{-\check{K}(\sigma+l)}f\right\|_{L^\infty(\Omega,\phi_1(x)^2dx)}\ :\ |l|<1\right\}+1\right)^{P(\sigma+1)-2}\\
&\times\int_\Omega\left|e^{-\check{K}(\sigma+h)}f-e^{-\check{K}\sigma}f\right|\left|\check{K}e^{-\check{K}\sigma}f\right|\phi_1(x)^2\,dx\\
\longrightarrow&\,0 \qquad \text{as} \quad h\to 0.
\end{aligned}
$$

By (41), (43), (44), we have

$$\int_\Omega \left(e^{-\check{K}(\sigma+h)}f(x) \longleftrightarrow e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1}(\check{K}e^{-\check{K}\sigma}f)\phi_1(x)^2\,dx \tag{45}$$
$$\longrightarrow \int_\Omega \left(e^{-\check{K}\sigma}f\right)^{P(\sigma)-1}(\check{K}e^{-\check{K}\sigma}f)\phi_1(x)^2\,dx \quad \text{as} \quad h\to 0.$$

Thus, by (35), (38), (45), we have

$$\int_\Omega P(\sigma+h)\left(e^{-\check{K}(\sigma+h)}f(x) \longleftrightarrow e^{-\check{K}\sigma}f(x)\right)^{P(\sigma+h)-1} \tag{46}$$
$$\times h^{-1}\left[e^{-\check{K}(\sigma+h)}f - e^{-\check{K}\sigma}f\right]\phi_1(x)^2\,dx$$
$$\longrightarrow -P(\sigma)\int_\Omega \left(e^{-\check{K}\sigma}f\right)^{P(\sigma)-1}(\check{K}e^{-\check{K}\sigma}f)\phi_1(x)^2\,dx \quad \text{as } h\to 0.$$

Next note that $W \in L^1(\Omega,\phi_1(x)^2dx)$. Also since $e^{-\check{K}\sigma}f \in L^\infty(\Omega,\phi_1(x)^2dx)$, we have

$$(e^{-\check{K}\sigma}f)\log(e^{-\check{K}\sigma}f) \in L^\infty(\Omega,\phi_1(x)^2dx).$$

Thus, by dominated convergence and (42),

$$\int_\Omega (e^{-\check{K}\sigma}f)\log(e^{-\check{K}\sigma}f)\left(e^{-\check{K}\sigma}f\right)^{P(\sigma+k)-1} \tag{47}$$
$$\times h^{-1}\left[P(\sigma+h)-P(\sigma)\right]\phi_1(x)^2\,dx$$
$$\to P'(\sigma)\int_\Omega \left(e^{-\check{K}\sigma}f\right)^{P(\sigma)}\log(e^{-\check{K}\sigma}f)\,\phi_1(x)^2\,dx$$
$$= \frac{P(\sigma)}{\varepsilon(P(\sigma))}\int_\Omega \left(e^{-\check{K}\sigma}f\right)^{P(\sigma)}\log(e^{-\check{K}\sigma}f)\,\phi_1(x)^2\,dx \quad \text{as } h\to 0.$$

So, by (33), (34), (35), (38), (45), (46), (47) and Lemma 9, we have

$$\frac{d}{d\sigma}\left\{\log\left[e^{-N(\sigma)}\|f_\sigma\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}\right]\right\} \tag{48}$$
$$= -\frac{\Gamma(P(\sigma))}{\varepsilon(P(\sigma))} - P(\sigma)^{-1}\varepsilon(P(\sigma))^{-1}\log\|f_\sigma\|^{P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}$$
$$+P(\sigma)^{-1}\|f_\sigma\|^{-P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}$$
$$\times\left\{-P(\sigma)\left\langle \check{K}f_\sigma, f_\sigma^{P(\sigma)-1}\right\rangle_{L^2(\Omega,\phi_1(x)^2dx)} + \frac{P(\sigma)}{\varepsilon(P(\sigma))}\int_\Omega f_\sigma^{P(\sigma)}(\log f_\sigma)\phi_1(x)^2\,dx\right\}$$
$$= \varepsilon(P(\sigma))^{-1}\|f_\sigma\|^{-P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}\left\{\int_\Omega f_\sigma^{P(\sigma)}(\log f_\sigma)\phi_1(x)^2\,dx\right.$$
$$-\varepsilon(P(\sigma))\left\langle \check{K}f_\sigma, f_\sigma^{P(\sigma)-1}\right\rangle_{L^2(\Omega,\phi_1(x)^2dx)} - \Gamma(P(\sigma))\|f_\sigma\|^{P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}$$
$$\left.-\|f_\sigma\|^{P(\sigma)}_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}\log\|f_\sigma\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}\right\}$$
$$\leqslant 0.$$

Thus for all $0 \leqslant \sigma < t$ we have

$$e^{-N(\sigma)} \|f_\sigma\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} \leqslant \|f\|_{L^2(\Omega,\phi_1(x)^2dx)}. \tag{49}$$

We next show that

$$\left\|e^{-\check{K}t} f\right\|_{L^\infty(\Omega,\phi_1(x)^2dx)} = \lim_{\sigma\uparrow t} \left\|e^{-\check{K}\sigma} f\right\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)}. \tag{50}$$

To do this we first show that there exists $M > 0$ such that

$$\left\|e^{-\check{K}\sigma_1} f - e^{-\check{K}\sigma_2} f\right\|_{L^p(\Omega,\phi_1(x)^2dx)} \leqslant M\,|\sigma_1 - \sigma_2| \tag{51}$$

for all $p \geqslant 2$ and $\sigma_1, \sigma_2 \in [t/2, t]$. Since $f \in \check{\mathscr{D}}$,

$$f = \Phi^{-1} e^{-\check{H}s} g \tag{52}$$

for some $s > 0$ and $0 \leqslant g \in L^1(\Omega, \phi_1(x)^2dx) \cap L^\infty(\Omega, \phi_1(x)^2dx)$. Since $\check{\mathscr{D}} \subseteq \mathrm{Dom}(-\check{K}_p) \subseteq L^p(\Omega, \phi_1(x)^2dx)$ for $1 < p < \infty$, where $-\check{K}_p$ is the generator of $e^{-\check{K}\sigma}$ in $L^p(\Omega, \phi_1(x)^2dx)$, we have

$$\check{K}f = \Phi^{-1}\check{H}\Phi\Phi^{-1}e^{-\check{H}s}g = \Phi^{-1}e^{-\check{H}s/2}\check{H}e^{-\check{H}s/2}g. \tag{53}$$

By Proposition 1 and [1, Corollary 2.2.8.], there exists $M_1 > 0$ such that

$$\left\|e^{-\check{H}s/2}\right\|_{L^2(\Omega,\phi_1(x)^2dx)\longrightarrow L^\infty(\Omega,\phi_1(x)^2dx)} \leqslant M_1. \tag{54}$$

So, by (53) and (54),

$$\|\check{K}f\|_{L^\infty(\Omega,\phi_1(x)^2dx)} \leqslant \|\Phi^{-1}\|_{L^\infty(\Omega)} M_1 \left\|\check{H}e^{-\check{H}s/2}g\right\|_{L^2(\Omega,\phi_1(x)^2dx)}. \tag{55}$$

Hence for $p \geqslant 2$

$$\begin{aligned}
\|\check{K}f\|^p_{L^p(\Omega,\phi_1(x)^2dx)} &= \int_\Omega \left|\check{K}f\right|^p \phi_1(x)^2\, dx \\
&= \left(\int_{\{x\in\Omega\,:\,|\check{K}f|\geqslant 1\}} + \int_{\{x\in\Omega\,:\,|\check{K}f|<1\}}\right) \left|\check{K}f\right|^p \phi_1(x)^2\, dx \\
&\leqslant \|\Phi^{-1}\|^p_{L^\infty(\Omega)} M_1^p \left\|\check{H}e^{-\check{H}s/2}g\right\|^p_{L^2(\Omega,\phi_1(x)^2dx)} + \|\check{K}f\|^2_{L^2(\Omega,\phi_1(x)^2dx)}
\end{aligned}$$

and so

$$\begin{aligned}
&\|\check{K}f\|_{L^p(\Omega,\phi_1(x)^2dx)} \\
&\leqslant 2^{1/p} \max\left[\|\Phi^{-1}\|_{L^\infty(\Omega)} M_1 \left\|\check{H}e^{-\check{H}s/2}g\right\|_{L^2(\Omega,\phi_1(x)^2dx)}, \|\check{K}f\|^{2/p}_{L^2(\Omega,\phi_1(x)^2dx)}\right] \\
&\leqslant M_2
\end{aligned} \tag{56}$$

for some $M_2 > 0$ independent of p. Therefore

$$\begin{aligned} &\left\| e^{-\check{K}\sigma_1} f - e^{-\check{K}\sigma_2} f \right\|_{L^p(\Omega,\phi_1(x)^2dx)} \\ &\leqslant \sup\left\{ \left\| \frac{d}{d\sigma}\left(e^{-\check{K}\sigma} f\right) \right\|_{L^p(\Omega,\phi_1(x)^2dx)} : \sigma \in [t/2,t] \right\} |\sigma_1 - \sigma_2| \\ &= \sup\left\{ \left\| e^{-\check{K}\sigma} \check{K} f \right\|_{L^p(\Omega,\phi_1(x)^2dx)} : \sigma \in [t/2,t] \right\} |\sigma_1 - \sigma_2|. \end{aligned} \tag{57}$$

Since $e^{-\check{H}s}$ is a symmetric Markov semigroup on $L^2(\Omega, \phi_1(x)^2dx)$,

$$\left\| e^{-\check{K}\sigma} \right\|_{L^p(\Omega,\phi_1(x)^2dx) \longrightarrow L^p(\Omega,\phi_1(x)^2dx)} \leqslant \left\| \Phi^{-1} \right\|_{L^\infty(\Omega)} \|\Phi\|_{L^\infty(\Omega)}. \tag{58}$$

Thus (56), (57), (58) imply that

$$\left\| e^{-\check{K}\sigma_1} f - e^{-\check{K}\sigma_2} f \right\|_{L^p(\Omega,\phi_1(x)^2dx)} \leqslant \left\| \Phi^{-1} \right\|_{L^\infty(\Omega)} \|\Phi\|_{L^\infty(\Omega)} M_2 |\sigma_1 - \sigma_2| \tag{59}$$

for all $p \geqslant 2$ and $\sigma_1, \sigma_2 \in [t/2,t]$. So (51) holds with

$$M = \left\| \Phi^{-1} \right\|_{L^\infty(\Omega)} \|\Phi\|_{L^\infty(\Omega)} M_2.$$

Since $f \in \check{\mathscr{D}}$, we have

$$e^{-\check{K}\sigma} f \in L^1(\Omega, \phi_1(x)^2dx) \cap L^\infty(\Omega, \phi_1(x)^2dx) \qquad (\sigma \geqslant 0),$$

so for all $\sigma \geqslant 0$

$$\left\| e^{-\check{K}\sigma} f \right\|_{L^\infty(\Omega,\phi_1(x)^2dx)} = \lim_{p\to\infty} \left\| e^{-\check{K}\sigma} f \right\|_{L^p(\Omega,\phi_1(x)^2dx)}. \tag{60}$$

Hence, by (51) and (60), for all $\sigma \in [t/2,t)$

$$\begin{aligned} &\left| \left\| e^{-\check{K}t} f \right\|_{L^\infty(\Omega,\phi_1(x)^2dx)} - \left\| e^{-\check{K}\sigma} f \right\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} \right| \\ &\leqslant \left| \left\| e^{-\check{K}t} f \right\|_{L^\infty(\Omega,\phi_1(x)^2dx)} - \left\| e^{-\check{K}t} f \right\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} \right| \\ &\quad + \left| \left\| e^{-\check{K}t} f \right\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} - \left\| e^{-\check{K}\sigma} f \right\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} \right| \\ &\leqslant \left| \left\| e^{-\check{K}t} f \right\|_{L^\infty(\Omega,\phi_1(x)^2dx)} - \left\| e^{-\check{K}t} f \right\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} \right| \\ &\quad + \left\| e^{-\check{K}t} f - e^{-\check{K}\sigma} f \right\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} \\ &\leqslant \left| \left\| e^{-\check{K}t} f \right\|_{L^\infty(\Omega,\phi_1(x)^2dx)} - \left\| e^{-\check{K}t} f \right\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} \right| + M|\sigma - t| \\ &\to 0 \qquad \text{as} \quad \sigma \uparrow t \end{aligned}$$

and (50) is proved. Thus by (49)

$$\begin{aligned}\left\|e^{-\check{K}t}f\right\|_\infty &= \lim_{\sigma\uparrow t}\left\|e^{-\check{K}\sigma}f\right\|_{L^{P(\sigma)}(\Omega,\phi_1(x)^2dx)} \\ &\leqslant \lim_{\sigma\uparrow t} e^{N(\sigma)}\|f\|_{L^2(\Omega,\phi_1(x)^2dx)} \\ &= e^M\|f\|_{L^2(\Omega,\phi_1(x)^2dx)}\,. \end{aligned} \tag{61}$$

If $0\leqslant f\in L^2(\Omega,\phi_1(x)^2dx)$, then there exists a sequence $\{u_n\}_{n=1}^\infty$ in $\check{\mathscr{D}}_1$ such that

$$\lim_{n\to\infty} u_n = \Phi f \qquad \text{in} \quad L^2(\Omega,\phi_1(x)^2dx).$$

Then $\left\{\Phi^{-1}u_n\right\}_{n=1}^\infty$ is a sequence in $\check{\mathscr{D}}$ such that

$$f_n = \Phi^{-1}u_n \longrightarrow f \qquad \text{in} \quad L^2(\Omega,\phi_1(x)^2dx) \quad \text{as} \quad n\longrightarrow\infty.$$

So, by what we have proved in (61),

$$\left\|e^{-\check{K}t}f_n\right\|_\infty \leqslant e^M\|f_n\|_{L^2(\Omega,\phi_1(x)^2dx)}\,.$$

Hence, using a subsequence of $\{f_n\}_{n=1}^\infty$ is necessary, we have

$$\left\|e^{-\check{K}t}f\right\|_\infty \leqslant e^M\|f\|_{L^2(\Omega,\phi_1(x)^2dx)}\,.$$

Finally, for a general $f\in L^2(\Omega,\phi_1(x)^2dx)$ we have, by the positivity of $e^{-\check{K}t}$,

$$\left|e^{-\check{K}t}f\right| \leqslant e^{-\check{K}t}|f|\,.$$

So

$$\begin{aligned}\left\|e^{-\check{K}t}f\right\|_\infty &\leqslant \left\|e^{-\check{K}t}|f|\right\|_\infty \\ &\leqslant e^M\||f|\|_{L^2(\Omega,\phi_1(x)^2dx)} \\ &= e^M\|f\|_{L^2(\Omega,\phi_1(x)^2dx)}\end{aligned}$$

which completes the proof of the theorem. □

THEOREM 11. *We have, for all* $t>0$,

$$\left\|e^{-\check{K}t}\right\|_{L^2(\Omega,\phi_1(x)^2dx)\to L^\infty(\Omega,\phi_1(x)^2dx)} \leqslant c_\Omega e^{c_\Omega a\log a}(\lambda t)^{-\left(\frac{N}{4}+\frac{a}{2}\right)}b^{-1}e^{\frac{1}{3}E_1t+2\alpha^2t}$$

for some $c_\Omega\geqslant 1$ *which depends only on* Ω.

Proof. In Theorem 10 put $\mu = \frac{1}{2}$,

$$\varepsilon(p) = 8tp^{-2} \qquad (2 < p < \infty)$$

(so that $\int_2^\infty \frac{\varepsilon(p)}{p} dp = t$) and, by Proposition 1, Lemmas 4 and 9, we can put

$$\beta(\varepsilon) = c'_\Omega a \log a - (\frac{N}{4} + \frac{a}{2})(\log\lambda + \log\varepsilon) + E_1\varepsilon - \log b + c'_\Omega$$

for some $c'_\Omega \geqslant 1$ which depends only on Ω. Then

$$\begin{aligned} M &= \int_2^\infty p^{-1}\left\{2p^{-1}\beta(\frac{1}{2}\varepsilon(p)) + \varepsilon(p)\alpha^2\left[1 + \frac{(p-2)^2}{2(p-1)}\right]\right\} dp \qquad (62) \\ &= \int_2^\infty 2p^{-2}\left[c'_\Omega a\log a - \left(\frac{N}{4} + \frac{a}{2}\right)(\log\lambda + \log(\frac{1}{2}\varepsilon(p)))\right. \\ &\quad \left. + \frac{1}{2}E_1\varepsilon(p) - \log b + c'_\Omega\right] + p^{-1}\varepsilon(p)\alpha^2\left[1 + \frac{(p-2)^2}{2(p-1)}\right] dp \\ &= \int_2^\infty 2p^{-2}\left[c'_\Omega a\log a - \left(\frac{N}{4} + \frac{a}{2}\right)(\log\lambda + \log(4t) - 2\log p)\right. \\ &\quad \left. + \frac{4E_1 t}{p^2} - \log b + c'_\Omega\right] + 8\alpha^2 t p^{-3}\left[1 + \frac{(p-2)^2}{2(p-1)}\right] dp. \end{aligned}$$

For $p > 2$ we have

$$1 + \frac{(p-2)^2}{2(p-1)} \leqslant \frac{p}{2}.$$

Therefore, from (62), we have

$$\begin{aligned} M \leqslant & c'_\Omega a\log a - \left(\frac{N}{4} + \frac{a}{2}\right)\log(4\lambda t) \\ & + \left(\frac{N}{2} + a\right)(\log 2 + 1) + \frac{1}{3}E_1 t - \log b + c'_\Omega + 2\alpha^2 t \\ \leqslant & c''_\Omega a\log a - \left(\frac{N}{4} + \frac{a}{2}\right)\log(\lambda t) + \frac{1}{3}E_1 t - \log b + c''_\Omega + 2\alpha^2 t \end{aligned}$$

for some $c''_\Omega \geqslant 1$ which depends only on Ω. □

THEOREM 12. *Let $K_{\check{H}}(t,x,y)$ be the heat kernel of $e^{-\check{H}t}$. Then there exists $c_\Omega \geqslant 1$, depending only on Ω, such that*

$$K_{\check{H}}(t,x,y) \leqslant c_\Omega e^{c_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)} b^{-2} e^{\frac{1}{3}E_1 t} \exp\left\{-\left(\frac{[\psi(x)-\psi(y)]^2}{8t}\right)\right\}$$

for all $x, y \in \Omega$ and all $t > 0$.

Proof. Replacing α by $-\alpha$ in Theorem 11 and taking adjoint, we have

$$\left\| e^{-\check{K}t} \right\|_{L^1(\Omega,\phi_1(x)^2dx)\to L^2(\Omega,\phi_1(x)^2dx)} \leqslant c'_\Omega e^{c'_\Omega a\log a}(\lambda t)^{-(\frac{N}{4}+\frac{a}{2})} b^{-1} e^{\frac{1}{3}E_1 t+2\alpha^2 t} \quad (t>0).$$

Hence,

$$\left\| e^{-\check{K}t} \right\|_{L^1(\Omega,\phi_1(x)^2dx)\to L^\infty(\Omega,\phi_1(x)^2dx)} \leqslant c''_\Omega e^{c''_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)} b^{-2} e^{\frac{1}{3}E_1 t+2\alpha^2 t} \quad (t>0).$$

Since

$$e^{-\check{K}t} = \Phi^{-1} e^{-\check{H}t}\Phi,$$

we have, for all $t>0$,

$$\begin{aligned} K_{\check{H}}(t,x,y) \leqslant& \Phi(x)\Phi(y)^{-1} c''_\Omega e^{c''_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)} b^{-2} e^{\frac{1}{3}E_1 t+2\alpha^2 t} \\ =& c''_\Omega e^{c''_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)} b^{-2} e^{\frac{1}{3}E_1 t} \\ &\times \exp\left\{\alpha\left[\psi(x)-\psi(y)\right]+2\alpha^2 t\right\}. \end{aligned} \tag{63}$$

So we choose $\alpha\in\mathbb{R}$ which minimizes

$$\alpha\left[\psi(x)-\psi(y)\right]+2\alpha^2 t,$$

i.e. we choose

$$\alpha = \frac{\psi(y)-\psi(x)}{4t}.$$

Thus (63) gives

$$K_{\check{H}}(t,x,y) \leqslant c_\Omega e^{c_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)} b^{-2} e^{\frac{1}{3}E_1 t} \exp\left\{-\left(\frac{\left[\psi(x)-\psi(y)\right]^2}{8t}\right)\right\}$$

for all $x,y\in\Omega$ and all $t>0$. $\square$

COROLLARY 13. *Let $d(x,y)$ be the metric on Ω defined by*

$$d(x,y) = \sup\left\{ |\psi(x)-\psi(y)| \ : \ \psi \text{ is a bounded } C^\infty \text{ function on } \Omega \text{ satisfying } \sum_{i,j=1}^{N} a_{i,j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j} \leqslant 1 \text{ on } \Omega\right\}.$$

Then, for all $x,y\in\Omega$ and all $t>0$, we have

$$K_{\check{H}}(t,x,y) \leqslant c_\Omega e^{c_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)} b^{-2} e^{\frac{1}{3}E_1 t} \exp\left\{-\frac{d(x,y)^2}{8t}\right\}$$

where $c_\Omega \geqslant 1$ is as in Theorem 12.

COROLLARY 14. *We have, for all $x,y\in\Omega$ and all $t>0$,*

$$K_{\check{H}}(t,x,y)\leqslant c_\Omega e^{c_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)}b^{-2}e^{\frac{1}{3}E_1 t}\exp\left\{-\frac{|x-y|^2}{8\Lambda t}\right\}$$

where $c_\Omega\geqslant 1$ is as in Theorem 12.

Proof. Let $d(x,y)$ be the metric in Corollary 13. Then, by our assumption on $\{a_{i,j}\}$, we have

$$\begin{aligned}d(x,y)\geqslant \sup\Big\{\,|\psi(x)-\psi(y)|\ :\ &\psi \text{ is a bounded}\\ &C^\infty \text{ function on } \Omega \text{ satisfying}\\ &\sum_{i,j=1}^{N}\Lambda\delta_{i,j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\leqslant 1 \text{ on } \Omega\Big\}\\ =\sup\Big\{\,|\psi(x)-\psi(y)|\ :\ &\psi \text{ is a bounded}\\ &C^\infty \text{ function on } \Omega \text{ satisfying}\\ &|\nabla\psi|^2\leqslant\Lambda^{-1}\quad\text{on}\quad\Omega\Big\}\\ =|x-y|\,\Lambda^{-\frac{1}{2}}.\end{aligned}\tag{64}$$

The result now follows from (64) and Corollary 13. □

LEMMA 15. *Let $K_H(t,x,y)$ be the heat kernel of e^{-Ht}. Then, for all $x,y\in\Omega$ and all $t>0$, we have*

$$K_H(t,x,y)\leqslant c_\Omega e^{c_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)}b^{-2}e^{-\frac{2}{3}E_1 t}\exp\left\{\frac{-|x-y|^2}{8\Lambda t}\right\}\phi_1(x)\phi_1(y)$$

where c_Ω is as in Theorem 12.

Proof. Let $K_{\widetilde{H}}(t,x,y)$ be the heat kernel of $e^{-\widetilde{H}t}$. Recall that $\check{H}=\widetilde{H}-E_1$. So Corollary 14 implies that, for all $x,y\in\Omega$ and all $t>0$,

$$K_{\widetilde{H}}(t,x,y)\leqslant c_\Omega e^{c_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)}b^{-2}e^{-\frac{2}{3}E_1 t}\exp\left\{-\frac{|x-y|^2}{8\Lambda t}\right\}.\tag{65}$$

The lemma now follows from (65) since

$$K_{\widetilde{H}}(t,x,y)=\phi_1(x)^{-1}K_H(t,x,y)\phi_1(y)^{-1}$$

for all $x,y\in\Omega$ and all $t>0$. □

THEOREM 16. *Let $K_H(t,x,y)$ be the heat kernel of e^{-Ht}. Then*

$$K_H(t,x,y) \leqslant c_1 \max\{t^{-(\frac{N}{2}+a)},1\}e^{-E_1 t}\exp\left\{\frac{-|x-y|^2}{8\Lambda t}\right\}\phi_1(x)\phi_1(y)$$

for all $x,y\in\Omega$ and all $t>0$, where, with $c_\Omega \geqslant 1$ as in Theorem 12,

$$\begin{aligned} c_1 &= c_1(\Omega,a,b,\lambda,\Lambda,E_1)\\ &= \max\Bigg\{c_\Omega e^{c_\Omega a\log a}\lambda^{-(\frac{N}{2}+a)}b^{-2}e^{E_1},\\ &\qquad (c_\Omega e^{c_\Omega a\log a}\lambda^{-(\frac{N}{2}+a)}2^{\frac{N}{2}+a}b^{-2})^2\exp\left\{E_1+\frac{\operatorname{diam}(\Omega)^2}{8\Lambda}\right\},1\Bigg\}\\ &\geqslant 1. \end{aligned}$$

Proof. By Lemma 15 we have

$$K_H(t,x,y) \leqslant c_\Omega e^{c_\Omega a\log a}(\lambda t)^{-(\frac{N}{2}+a)}b^{-2}\exp\left\{\frac{-|x-y|^2}{8\Lambda t}\right\}\phi_1(x)\phi_1(y) \tag{66}$$

for all $x,y\in\Omega$ and all $t>0$. In particular, we have

$$0<K_H(1/2,x,y)\leqslant c_\Omega e^{c_\Omega a\log a}\lambda^{-(\frac{N}{2}+a)}2^{\frac{N}{2}+a}b^{-2}\phi_1(x)\phi_1(y)$$

for all $x,y\in\Omega$. If $t>1$, then, since $|x-y|\leqslant \operatorname{diam}(\Omega)$, we have

$$\begin{aligned} K_H(t,x,y) &= \int_\Omega\int_\Omega K_H(1/2.x,u)K_H(t-1,u,v)K_H(1/2,v,y)\,du\,dv \\ &\leqslant \int_\Omega\int_\Omega c_2\phi_1(x)\phi_1(u)K_H(t-1,u,v)c_2\phi_1(v)\phi_1(y)\,du\,dv\\ &= c_2^2\phi_1(x)\phi_1(y)\int_\Omega\int_\Omega K_H(t-1,u,v)\phi_1(u)\phi_1(v)\,du\,dv\\ &= c_2^2 e^{-E_1(t-1)}\phi_1(x)\phi_1(y)\\ &= c_2^2\exp\left\{E_1+\frac{\operatorname{diam}(\Omega)^2}{8\Lambda}\right\}e^{-E_1 t}\exp\left\{\frac{-|x-y|^2}{8\Lambda t}\right\}\phi_1(x)\phi_1(y) \end{aligned} \tag{67}$$

where we have written

$$c_2 = c_\Omega e^{c_\Omega a\log a}\lambda^{-(\frac{N}{2}+a)}2^{\frac{N}{2}+a}b^{-2}. \tag{68}$$

The theorem now follows follows (66), (67) and (68). □

REMARKS. (i) Theorem 16 can also be proved by combining Lemma 15 with [1, Corollary 4.2.5].

(ii) If one wants to express c_1 in Theorem 16 in terms of constants depending only on $\Omega,a,b,\lambda,\Lambda$, one can easily replace E_1 in terms of constants depending only on Λ and the inradius of Ω.

COROLLARY 17. *For $t>0$ let*

$$C(t)=\inf\left\{\phi_1(x)^{-1}K_H(t,x,y)\phi_1(y)^{-1}:\ x,y\in\Omega\right\}.$$

Then

$$C(t)\leqslant c_1\max\{t^{-(\frac{N}{2}+a)},1\}e^{-E_1t}\exp\left\{-\frac{\operatorname{diam}(\Omega)^2}{8\Lambda t}\right\}\quad (t>0)$$

where $c_1\geqslant 1$ is as in Theorem 16.

Acknowledgement. We wish to thank E. B. Davies, M. van den Berg, D. Krejcirik, and Y. Pinchover for communications on this paper.

REFERENCES

[1] E. B. DAVIES, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, London/New York, 1989
[2] E. B. DAVIES, M. M. H. PANG, *The eigenvalue gap for second order elliptic operators with Dirichlet boundary conditions*, J. Diff. Equations **88** (1990), 46–70.

(Received December 3, 2013)

Michael M. H. Pang
Department of Mathematics
University of Missouri-Columbia
Columbia, MO 65211, USA
e-mail: pangm@missouri.edu