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A GENERALIZATION OF THE BROWN–PEARCY THEOREM

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Abstract. Let $\mathscr{A}$ be a unital separable simple exact C*-algebra. Suppose that either

1. $\mathscr{A}$ is purely infinite, or
2. $\mathscr{A}\otimes\mathscr{K}$ has strict comparison of positive elements and stable rank one, and $\mathscr{A}$ has unique tracial state.

Then for all $X\in\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$, X is a commutator if and only if X does not have the form $\alpha 1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})}+x$, for some $\alpha\in\mathbb{C}-\{0\}$ and for some x belonging to a proper ideal of $\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$.

1. Introduction

A *commutator* in a C*-algebra $\mathscr{C}$ is an element of the form $[x,y]=_{df} xy-yx$ for some $x,y\in\mathscr{C}$. The study of commutators in the context of operator theory has a long history, starting with Quantum Mechanics where the Heisenberg Uncertainty Principle is implied by a commutator relation. This is one of the original motivations for the development of "noncommutative mathematics" (operator algebras, "noncommutative topology", "noncommutative measure theory" etc.) which is a part of today's functional analysis.

Another early result is Shoda's 1940 result that for a field $\mathbb{F}$ with characteristic zero and $n\geqslant 1$, an element $[x_{j,k}]\in\mathbb{M}_n(\mathbb{F})$ is a commutator if and only if $Tr([x_{j,k}])=_{df}\sum_{j=1}^{n}x_{j,j}=0$ ([31]).

The questions of when an element of a C*-algebra (or more general ring) is a sum or limit (of sums) of commutators have been studied by many authors (e.g., [2], [3], [4], [5], [6], [7], [8], [9], [14], [22], [23], [25], [27], [28], [31], [32] etc.) with far-reaching connections and implications (e.g., equivalence relations on C*-algebras ([5]), noncommutative dimension theory in C*-algebras ([28]), operator decomposition questions ([22], [14]), and determinant theory and the uniqueness theorems of classification theory ([11], [32], [18]) etc.).

Perhaps one of the most definitive early results is the theorem of Brown and Pearcy, which showed that for a separable infinite dimensional Hilbert space $\mathscr{H}$ and for an operator $T\in\mathbb{B}(\mathscr{H})$, T is a commutator if and only if T is either a compact or nonthin operator, i.e., does not have the form $\alpha 1+S$ where $\alpha\in\mathbb{C}-\{0\}$ and $S\in\mathscr{K}(\mathscr{H})$ (the compact operators on $\mathscr{H}$). (See [3].)

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Since then, there have been many attempts to generalize the Brown–Pearcy Theorem to C*-algebras (or even Banach algebras) other than $\mathbb{B}(\mathscr{H})$. Among the most definitive generalizations are those for type III and type II_∞ factors, where the (corresponding) compact or nonthin operators are exactly the single commutators. (See [4] and [10]. See also [6].) There are also interesting generalizations where the analogous operators can be expressed as a sum of at least two commutators. (Examples include UHF-algebras and type II_1 factors ([22]). In fact, the following is an open question of Fack and de la Harpe (Marcoux): If $\mathscr{C}$ is a type II_1 factor (resp. UHF-algebra) with tracial state τ, then is it true that for all $x \in \mathscr{C}$, x is a single commutator if and only if $\tau(x) = 0$? The best result is two commutators, which follows from Marcoux's Commutator Reduction Argument ([22]). For similar results, see [8], [9], [14], [22], [23], [27], [28], [32] and the references therein.)

In this paper, we generalize the Brown–Pearcy Theorem to the context of multiplier algebras. Recall that for a C*-algebra $\mathscr{B}$, the multiplier algebra $\mathscr{M}(\mathscr{B})$, of $\mathscr{B}$, is the largest unital C*-algebra containing $\mathscr{B}$ as an essential ideal. For $\mathscr{K} = \mathscr{K}(\mathscr{H})$ the compact operators on a Hilbert space $\mathscr{H}$, $\mathscr{M}(\mathscr{K}) = \mathbb{B}(\mathscr{H})$. Moreover, for a C*-algebra $\mathscr{B}$, $\mathscr{M}(\mathscr{B})$ encodes the extension theory of $\mathscr{B}$, and multiplier algebras give the context for attempts to generalize BDF–Theory. (In fact, an essentially normal operator $T \in \mathbb{B}(\mathscr{H})$ is one where the self-commutator $[T, T^*]$ is compact.) Hence, multiplier algebras are natural objects to which to generalize the Brown–Pearcy Theorem.

In this paper, we prove the following result:

THEOREM 1.1. *Let $\mathscr{A}$ be a unital separable simple C*-algebra such that either*

1. *$\mathscr{A}$ is purely infinite, or*
2. *$\mathscr{A} \otimes \mathscr{K}$ has strict comparison of positive elements, $\mathscr{A}$ has stable rank one and unique tracial state, and every quasitrace on $\mathscr{A}$ is a trace.*

Let $X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$.

Then X is a single commutator if and only if X does not have the form $\alpha 1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})} + x$, where $\alpha \in \mathbb{C} - \{0\}$ and x is an element of a proper ideal of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$.

We note that, by [27], every element of the multiplier algebras in Theorem 1.1 is a sum of two commutators.

We also note that by Theorem 1.1 and [21] Theorem 5.3 (see also the remarks after [22] Theorem 5.2), if $\mathscr{A}$ is a unital separable simple C*-algebra satisfying the hypotheses of Theorem 1.1 then for all $X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ such that X does not have the form $\alpha 1 + x$ for some $\alpha \in \mathbb{C} - \{0\}$ and x in a proper ideal of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, X is a sum of 14 nilpotents of order two, and X is a linear combination of 56 projections.

Finally, we note that for the multiplier algebras in Theorem 1.1, the presence of a unital embedding of the Cuntz algebra O_2 seems to be a key ingredient of the proofs of Theorem 1.1. Hence, the next question seems natural:

QUESTION 1. Consider the Cuntz algebra O_2. Is it true that for all $x \in O_2$, x is a single commutator if and only if x does not have the form $\alpha 1_{O_2}$ for some $\alpha \in \mathbb{C} - \{0\}$?

Again, it is also known, by [27], that every element of O_2 is a sum of two commutators.

In fact, we do not know the answers to the following questions:

QUESTION 2.

1. Does there exist a unital separable simple nonelementary C*-algebra $\mathscr{A}$ such that for all $x \in \mathscr{A}$, x is a single commutator if and only if $\tau(x) = 0$ for all tracial states τ on $\mathscr{A}$?

2. Does there exist a unital separable simple C*-algebra $\mathscr{A}$ such that for all $x \in \mathscr{A}$, x is a single commutator if and only if x does not have the form $\alpha 1_{\mathscr{A}}$ for some $\alpha \in \mathbb{C} - \{0\}$?

2. Elements in the canonical ideal

For most of this paper, for a C*-algebra $\mathscr{B}$, we let lower case letters denote elements of $\mathscr{B}$. If $\mathscr{B}$ is nonunital, we let capital letters denote general elements (especially full elements) of $\mathscr{M}(\mathscr{B})$ and lower case letters for elements that we know are in a proper ideal of $\mathscr{M}(\mathscr{B})$ (e.g., $\mathscr{B}$). We occasionally vary from these conventions.

Also, for most of this paper, we use extensively the ideas developed in [2], [3], [4], and [25].

Finally, a good reference for multiplier algebras and the strict topology is the book [33].

The first lemma is a straightforward computation.

LEMMA 2.1. *Let $\mathscr{C}$ be a Banach space and let $\{x_{m,n}\}_{m,n\geqslant 1}$ be a biinfinite sequence in $\mathscr{C}$ such that*

$$\sum_{m,n\geqslant 1} \|x_{m,n}\| < \infty.$$

Then $\sum_{m,n\geqslant 1} x_{m,n}$ converges in norm to an element of $\mathscr{C}$.

LEMMA 2.2. *Let $\mathscr{B}$ be a separable C*-algebra, and suppose that $\{P_n\}_{n=1}^{\infty}$ is a sequence of pairwise orthogonal projections in $\mathscr{M}(\mathscr{B})$ and $\{T_{m,n}\}_{m,n\geqslant 1}$ is a biinfinite sequence in $\mathscr{M}(\mathscr{B})$ such that*

(a) $P_m \sim P_n$ in $\mathscr{M}(\mathscr{B})$ for all m,n,

(b) the sum $\sum_{n=1}^{\infty} P_n$ converges in the strict topology on $\mathscr{M}(\mathscr{B})$,

(c) $P_m T_{m,n} = T_{m,n} P_n = T_{m,n}$ for all m,n, and

(d) $\sum_{m,n\geqslant 1} \|T_{m,n}\| < \infty$.

Then $\sum_{m,n\geqslant 1} T_{m,n}$ is a commutator in $\mathscr{M}(\mathscr{B})$.

Proof. Firstly, by Lemma 2.1, $T = \sum_{m,n\geqslant 1} T_{m,n}$ converges in norm to an element in $\mathscr{M}(\mathscr{B})$.

Let $P =_{df} \sum_{n=1}^{\infty} P_n \in \mathscr{M}(\mathscr{B})$. Replacing $\mathscr{B}$ with $P\mathscr{B}P$ if necessary, we may assume that $P = 1_{\mathscr{M}(\mathscr{B})}$.

We may assume that $\mathscr{B}$ acts faithfully and nondegenerately on a separable infinite dimensional Hilbert space $\mathscr{H}$. We may then identify $\mathscr{M}(\mathscr{B})$ with the idealizer of $\mathscr{B}$ in $\mathbb{B}(\mathscr{H})$; i.e.,

$$\mathscr{M}(\mathscr{B}) = \{S \in \mathbb{B}(\mathscr{H}) : S\mathscr{B}, \mathscr{B}S \subseteq \mathscr{B}\}.$$

Let $\{E_{m,n}\}_{1\leqslant m,n<\infty}$ be a system of matrix units for a copy of $\mathscr{K}$ (the C*-algebra of compact operators) in $\mathscr{M}(\mathscr{B})$ such that for all $n \geqslant 1$, $E_{n,n} = P_n$.

By [2] Theorem 4, T is a commutator in $\mathbb{B}(\mathscr{H})$. More precisely, by inspection of the proof of [2] Theorem 4, we see that

$$T = [R,W] = RW - WR$$

where $R, W \in \mathbb{B}(\mathscr{H})$ are given by the following:

i.
$$R =_{df} \sum_{n=1}^{\infty} E_{n,n+1}.$$

ii. For all $m,n \geqslant 1$, let $W_{m,n} =_{df} P_m W P_n$. Then

$$W_{m,n} = \begin{cases} 0 & m = 1 \\ \sum_{k=0}^{m-2} E_{m,m-k-1} T_{m-k-1,n-k} E_{n-k,n} & 2 \leqslant m \leqslant n \\ \sum_{k=0}^{n-1} E_{m,n-k-1} T_{m-k-1,n-k} E_{n-k,n} & m > n. \end{cases}$$

Clearly, the sum for R converges strictly in $\mathscr{M}(\mathscr{B})$; i.e., $R \in \mathscr{M}(\mathscr{B})$.

To complete the proof, it suffice to show that $W = \sum_{m,n\geqslant 1} W_{m,n}$ converges strictly in $\mathscr{M}(\mathscr{B})$ (and hence, $W \in \mathscr{M}(\mathscr{B})$).

Firstly, note that since $W \in \mathbb{B}(\mathscr{H})$, $\|W\| < \infty$.

For each $N \geqslant 1$, let $Q_N =_{df} \sum_{n=1}^{N} P_n$.

Claim: For all $M_1 \geqslant 1$, $Q_{M_1} W(1 - Q_N) \to 0$ as $N \to \infty$.

Proof of Claim: We have that $Q_{M_1} W = \sum_{1\leqslant m\leqslant M_1, 1\leqslant n<\infty} W_{m,n}$.

Let $\gamma > 0$ be given. Since $\sum_{m,n\geqslant 1} \|T_{m,n}\| < \infty$, choose $N_1 \geqslant 1$ so that

$$\sum_{1\leqslant m<\infty, n\geqslant N_1} \|T_{m,n}\| < \gamma/(M_1+10)!.$$

Choose $N_2 \geqslant 1$ so that

$$N_2 - M_1 - 10 > N_1.$$

It follows, from the definition of W, that

$$\sum_{1\leqslant m\leqslant M_1, n\geqslant N_2} \|W_{m,n}\| < \gamma.$$

Hence, for all $N \geqslant N_2$,

$$\|Q_{M_1}W(1-Q_N)\| < \gamma.$$

Since γ was arbitrary, we have proven the claim.
End of the proof of the Claim.

We now show that $W \in \mathcal{M}(\mathcal{B})$ and thus complete the proof.

Let $b \in \mathcal{B}$ be given. We want to prove that $bW \in \mathcal{B}$. We may assume that $\|b\| \leqslant 1$.

Since $\sum_{n=1}^{\infty} P_n$ converges strictly in $\mathcal{M}(\mathcal{B})$, choose $M_2 \geqslant 1$ so that $\|bQ_{M_2} - b\| < \varepsilon/(10(\|W\|+1))$ Hence, $\|bQ_{M_2}W - bW\| < \varepsilon/10$. By the Claim, we can find $N_3 \geqslant 1$ so that $\|bQ_{M_2}WQ_{N_3} - bQ_{M_2}W\| < \varepsilon/10$. Hence, $\|bW - bQ_{M_2}WQ_{N_3}\| < \varepsilon$. Since $bQ_{M_2}WQ_{N_3} \in \mathcal{B}$, bW is within ε of an element of $\mathcal{B}$. Since ε was arbitrary, $bW \in \mathcal{B}$ as we wish.

By a similar argument, $Wb \in \mathcal{B}$.

Since $b \in \mathcal{B}$ was arbitrary, $W \in \mathcal{M}(\mathcal{B})$, and this completes the proof. □

COROLLARY 2.3. *Let $\mathcal{B}$ be a separable stable C*-algebra. Then every element of $\mathcal{B}$ is a commutator in $\mathcal{M}(\mathcal{B})$.*

Proof. Since $\mathcal{B} \cong \mathcal{B} \otimes \mathcal{K}$, we may work with $\mathcal{B} \otimes \mathcal{K}$ (and $\mathcal{M}(\mathcal{B} \otimes \mathcal{K})$).

Note that $\mathcal{M}(\mathcal{B}) \otimes \mathbb{B}(\mathcal{H}) \cong \mathcal{M}(\mathcal{B}) \otimes \mathcal{M}(\mathcal{K})$ can be (naturally) realized as a unital *-subalgebra of $\mathcal{M}(\mathcal{B} \otimes \mathcal{K})$.

Let $\{e_{j,k}\}_{1 \leqslant j,k < \infty}$ be a system of matrix units for $\mathcal{K}$. Note that $\{\sum_{j=1}^{n} 1_{\mathcal{M}(\mathcal{B})} \otimes e_{j,j}\}_{n=1}^{\infty}$ is an approximate identity for $\mathcal{M}(\mathcal{B}) \otimes \mathcal{K}$, consisting of an increasing sequence of projections.

Let $x \in \mathcal{B} \otimes \mathcal{K}$ be arbitrary. We want to show that x is a commutator of $\mathcal{M}(\mathcal{B} \otimes \mathcal{K})$. We may assume that $\|x\| \leqslant 1$.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.

For each subset $\mathcal{F} \subseteq \mathbb{Z}_+$ ($\mathbb{Z}_+$ is the set of positive integers), let $P_{\mathcal{F}} =_{df} \sum_{j \in \mathcal{F}} 1_{\mathcal{M}(\mathcal{B})} \otimes e_{j,j}$. Note that the sum converges strictly in $\mathcal{M}(\mathcal{B} \otimes \mathcal{K})$ and $P_{\mathcal{F}}$ is a projection in $\mathcal{M}(\mathcal{B} \otimes \mathcal{K})$. Moreover, if $\mathcal{F}$ is infinite then $P_{\mathcal{F}} \sim 1_{\mathcal{M}(\mathcal{B} \otimes \mathcal{K})}$ in $\mathcal{M}(\mathcal{B} \otimes \mathcal{K})$.

We construct a sequence $\{\mathcal{E}_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{Z}_+$ and an increasing sequence of positive integers $\{N_n\}_{n=1}^{\infty}$ such that

i. $\mathcal{E}_n \subset \mathcal{E}_{n+1}$ for all $n \geqslant 1$,

ii. $\{1,2,3,\ldots,N_n\} \subseteq \mathcal{E}_n$ for all $n \geqslant 1$,

iii. $\mathcal{F}_n =_{df} \mathcal{E}_n - \mathcal{E}_{n-1}$ is an infinite set for all $n \geqslant 1$ (here we take $\mathcal{E}_0 =_{df} \emptyset$),

iv. $\|(1 - \sum_{j=1}^{N_n} 1_{\mathcal{M}(\mathcal{B})} \otimes e_{j,j})x\|, \|x(1 - \sum_{j=1}^{N_n} 1_{\mathcal{M}(\mathcal{B})} \otimes e_{j,j})\| < \varepsilon_{n+1}/(2(n+1))$ for all $n \geqslant 1$,

v. $\|P_{\mathcal{F}_m} x P_{\mathcal{F}_n}\|, \|P_{\mathcal{F}_n} x P_{\mathcal{F}_m}\| < \varepsilon_n/(2n)$ for all $2 \leqslant m \leqslant n$,

vi. $\bigcup_{n=1}^{\infty}\mathscr{E}_n = \mathbb{Z}_+$, and hence, $\sum_{n=1}^{\infty} P_{\mathscr{F}_n} = 1_{\mathscr{M}(\mathscr{B}\otimes\mathscr{K})}$ where the sum converges in the strict topology on $\mathscr{M}(\mathscr{B}\otimes\mathscr{K})$.

(Note that (v.) follows from (ii.) and (iv.).)

We denote the above statements by "$(*)$".

The construction is by induction on n.

Basis step $n=1$. Since $\{\sum_{j=1}^{m} 1_{\mathscr{M}(\mathscr{B})}\otimes e_{j,j}\}_{m=1}^{\infty}$ is an approximate identity for $\mathscr{M}(\mathscr{B})\otimes\mathscr{K}$, choose $N_1 \geqslant 1$ so that $\|(1-\sum_{j=1}^{N_1} 1_{\mathscr{M}(\mathscr{B})}\otimes e_{j,j})x\|, \|x(1-\sum_{j=1}^{N_1} 1_{\mathscr{M}(\mathscr{B})}\otimes e_{j,j})\| < \varepsilon_2/4$.

Let $\mathscr{E}_1 \subset \mathbb{Z}_+$ be such that

(a) $\{1,2,...,N_1+1\} \subset \mathscr{E}_1$, and

(b) $\mathscr{E}_1$ and $\mathbb{Z}_+ - \mathscr{E}_1$ are both infinite sets.

Induction step. Suppose that $\mathscr{E}_n$ has been constructed. We now construct $\mathscr{E}_{n+1}$.

Since $\{\sum_{j=1}^{m} 1_{\mathscr{M}(\mathscr{B})}\otimes e_{j,j}\}_{m=1}^{\infty}$ is an approximate identity for $\mathscr{M}(\mathscr{B})\otimes\mathscr{K}$, choose $N_{n+1} \geqslant N_n + 10$ so that $\|(1-\sum_{j=1}^{N_{n+1}} 1_{\mathscr{M}(\mathscr{B})}\otimes e_{j,j})x\|, \|x(1-\sum_{j=1}^{N_{n+1}} 1_{\mathscr{M}(\mathscr{B})}\otimes e_{j,j})\| < \varepsilon_{n+2}/(2(n+2))$.

Let $\mathscr{E}_{n+1} \subseteq \mathbb{Z}_+$ be such that $\mathscr{E}_n \cup \{1,2,...,N_{n+1}\} \subseteq \mathscr{E}_{n+1}$, $\mathscr{E}_{n+1} - \mathscr{E}_n$ is an infinite set, and $\mathbb{Z}_+ - \mathscr{E}_{n+1}$ is an infinite set.

This completes the inductive construction.

From $(*)$, we have that $\{P_{\mathscr{F}_n}\}_{n=1}^{\infty}$ (as defined in $(*)$) is a sequence of pairwise orthogonal projections in $\mathscr{M}(\mathscr{B})$ such that $P_{\mathscr{F}_n} \sim 1_{\mathscr{M}(\mathscr{B}\otimes\mathscr{K})}$ for all n, $\sum_{n=1}^{\infty} P_{\mathscr{F}_n} = 1_{\mathscr{M}(\mathscr{B}\otimes\mathscr{K})}$ where the sum converges strictly in $\mathscr{M}(\mathscr{B}\otimes\mathscr{K})$, and $\sum_{m,n\geqslant 1}\|P_{\mathscr{F}_m} x P_{\mathscr{F}_n}\| \leqslant \|P_{\mathscr{F}_1} x P_{\mathscr{F}_1}\| + \sum_{n=2}^{\infty}\varepsilon_n < \infty$.

Hence, by Lemma 2.2, x is a commutator in $\mathscr{M}(\mathscr{B})$. □

3. Some technical lemmas

Here and in the rest of the paper, we will say that a unital separable simple C*-algebra $\mathscr{A}$ is in the class $\mathfrak{R}$ if *either* (i.) $\mathscr{A}$ is purely infinite *or* (ii.) $\mathscr{A}$ is stably finite and all quasitraces extend to traces, and $\mathscr{A}\otimes\mathscr{K}$ has strict comparison of positive elements.

Firstly, multiplier elements with "large null space" are multiplier commutators.

LEMMA 3.1. *Let $\mathscr{B}$ be a separable stable C*-algebra, and let $X \in \mathscr{M}(\mathscr{B})$. Suppose that $P \in \mathscr{M}(\mathscr{B})$ is a projection such that $P \sim 1_{\mathscr{M}(\mathscr{B})}$ and $XP = 0$.*

Then X is a commutator of $\mathscr{M}(\mathscr{B})$.

Sketch of proof. This is essentially the argument of [25].

We sketch the short argument for the convenience of the reader.

Since $P \sim 1_{\mathscr{M}(\mathscr{B})}$, there exist a sequence $\{P_n\}_{n=1}^{\infty}$ of pairwise orthogonal projections in $\mathscr{M}(\mathscr{B})$ such that

(a) $P = \sum_{n=1}^{\infty} P_n$ where the sum converges strictly in $\mathscr{M}(\mathscr{B})$, and

(b) $P_n \sim 1_{\mathscr{M}(\mathscr{B})}$ for all $n \geqslant 1$.

Replacing $1-P$ with $(1-P)+P_1$ if necessary, we may assume that $1-P \sim 1_{\mathscr{M}(\mathscr{B})}$.

To simplify notation, denote $P_0 =_{df} 1-P$. Let $\{E_{m,n}\}_{0 \leqslant m,n < \infty}$ be a system of matrix units for a copy of $\mathscr{K}$ in $\mathscr{M}(\mathscr{B})$ so that $E_{m,m} = P_m$ for all $m \geqslant 0$.

Let $S =_{df} (-\sum_{n=0}^{\infty} E_{n,n+1} X) + (\sum_{n=0}^{\infty} E_{n,0} X E_{0,n+1})$, and let $R =_{df} \sum_{n=0}^{\infty} E_{n+1,n}$.

It is clear that the above sums converge strictly, and hence, $S, R \in \mathscr{M}(\mathscr{B})$.

Moreover, $X = [S,R]$. (See the proof of [25] Theorem 2.) □

LEMMA 3.2. *Let $\mathscr{C}$ be a unital C*-algebra, and suppose that $c \in \mathscr{C}$ is a commutator of $\mathscr{C}$.*

Then for any $x \in \mathbb{M}_2(\mathscr{C})$, if x has the form

$$x = \begin{bmatrix} c & * \\ * & 0 \end{bmatrix}$$

then x is a commutator of $\mathbb{M}_2(\mathscr{C})$.

Proof. This follows immediately from [22] Lemma 2.4. An elementary proof can be found in [3] Lemma 4.1. □

Following Brown and Pearcy, given a unital C*-algebra $\mathscr{C}$, and $x, y \in \mathscr{C}$, a *generalized sum* of x and y is an element (of $\mathscr{C}$) which has the form $s^{-1}xs + t^{-1}yt$ where s, t are invertible elements of $\mathscr{C}$.

LEMMA 3.3. *Let $\mathscr{C}$ be a unital C*-algebra. Suppose that $y, z, x_0 \in \mathscr{C}$ is such that some generalized sum of y and z is a commutator of $\mathscr{C}$, and x_0 is invertible in $\mathscr{C}$.*

Then for any $x \in \mathbb{M}_2(\mathscr{C})$, if x has the form

$$x = \begin{bmatrix} y & x_0 \\ * & z \end{bmatrix}$$

then x is a commutator in $\mathbb{M}_2(\mathscr{C})$.

Proof. The argument is exactly the same as that of [3] Lemma 4.2. One notes that [12] Corollary 3.2 works in general Banach algebras. (See also [20] Theorem 10.) □

LEMMA 3.4. *Let $\mathscr{C}$ be a unital C*-algebra, and suppose that there exists an open subset $\mathfrak{O} \subseteq \mathscr{C}$ such that*

i. for all $z_1, z_2 \in \mathfrak{O}$, some generalized sum of z_1 and z_2 is a commutator of $\mathscr{C}$, and

ii. $\mathfrak{O}$ is closed under multiplication by nonzero scalars.

Suppose that $y_1, y_2, y_3, y_4 \in \mathscr{C}$ *and* z *is an invertible operator in* $\mathfrak{O}$.
Then for sufficiently large $\lambda > 0$, *the element* $x \in \mathbb{M}_2(\mathscr{C})$, *which is given by*

$$x = \begin{bmatrix} y_1 & y_2 + \lambda z \\ y_3 & y_4 \end{bmatrix},$$

is a commutator of $\mathbb{M}_2(\mathscr{C})$.

Proof. The proof is exactly the same as that of [4] Lemma 4.5, except that [4] Corollary 4.4 is replaced with (this paper) Lemma 3.3. □

LEMMA 3.5. *Let* $\mathscr{C}$ *be a unital C*-algebra such that there exists a unital *-embedding of* O_2 *into* $\mathscr{C}$. *Suppose that there exists an open subset* $\mathfrak{O} \subseteq \mathscr{C}$ *such that*

i. *for all* $z_1, z_2 \in \mathfrak{O}$, *some generalized sum of* z_1 *and* z_2 *is a commutator of* $\mathscr{C}$,

ii. $\mathfrak{O}$ *is closed under multiplication by nonzero scalars, and*

iii. $\mathfrak{O}$ *contains all elements of the form* $p + 2q$, *where* $p, q \in \mathscr{C}$ *are projections such that* $p + q = 1$, $p \perp q$ *and* $p \sim q \sim 1$.

Then for all $a \in \mathscr{C}$ *and all* $v \in \mathscr{C}$ *such that* v *is an isometry and* $1 - vv^* \sim 1$, *there exists an* $x \in \mathscr{C}$ *such that* $xv = 0$ *and* $a + vx$ *is a commutator in* $\mathscr{C}$.

Proof. Let $e =_{df} vv^*$. So $e \sim 1 - e \sim 1$. There exists a *-isomorphism $\Phi : \mathscr{C} \to \mathbb{M}_2(\mathscr{C})$ such that $\Phi(e) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $\Phi(1-e) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. The rest of the argument is exactly the same as that of [4] Lemma 4.6, except that we replace [4] Lemma 4.5 with (this paper) Lemma 3.4. □

DEFINITION 3.6. Let $\mathscr{C}$ be a unital C*-algebra such that there exists a unital *-embedding of O_2 into $\mathscr{C}$. Let $x \in \mathscr{C}$.

Then we say that x has *property (* π_0 *)* if there exists a *-isomorphism $\Phi : \mathscr{C} \to \mathbb{M}_2(\mathscr{C})$ such that

1. every minimal projection in $\mathbb{M}_2 \otimes 1_{\mathscr{C}}$ is Murray–von Neumann equivalent to $1_{\mathbb{M}_2(\mathscr{C})}$ in $\mathbb{M}_2(\mathscr{C})$, and
2. $\Phi(x)$ has the form

$$\Phi(x) = \begin{bmatrix} * & v \\ * & 0 \end{bmatrix},$$

where $v \in \mathscr{C}$ is an isometry such that $1_{\mathscr{C}} - vv^* \sim 1_{\mathscr{C}}$.

PROPOSITION 3.7. *Let* $\mathscr{C}$ *be a unital C*-algebra such that there exists a unital *-embedding of* O_2 *into* $\mathscr{C}$. *Suppose that there exists an open subset* $\mathfrak{O} \subseteq \mathscr{C}$ *such that*

i. *for all* $z_1, z_2 \in \mathfrak{O}$, *some generalized sum of* z_1 *and* z_2 *is a commutator of* $\mathscr{C}$, *ii.* $\mathfrak{O}$ *is closed under multiplication by nonzero scalars, and*

iii. $\mathfrak{O}$ *contains all elements of the form* $p+2q$*, where* $p,q\in\mathscr{C}$ *are projections such that* $p+q=1$*,* $p\perp q$ *and* $p\sim q\sim 1$*.*

If $x\in\mathscr{C}$ *has property* (π_0) *then* x *is a commutator in* $\mathscr{C}$*.*

Proof. The proof is exactly the same as that of [4] Theorem 1, except that [4] Lemmas 4.6 and 4.2 are replaced with (this paper) Lemmas 3.5, and 3.2 respectively. □

DEFINITION 3.8. Let $\mathscr{C}$ be a unital C*-algebra, and let $x\in\mathscr{C}$.

Then we say that x has *property* (π) if there exists a *-isomorphism $\Phi:\mathscr{C}\to\mathbb{M}_3(\mathscr{C})$ such that $\Phi(x)$ has the form

$$\Phi(x)=\begin{bmatrix}0 & * & *\\ 1 & * & *\\ 0 & * & *\end{bmatrix}.$$

LEMMA 3.9. *Let* $\mathscr{A}$ *be a unital separable simple C*-algebra in the class* $\mathfrak{R}$*. If* $A\in\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ *has property* (π) *then* A *has property* (π_0)*.*

Proof. Since $\mathscr{A}\otimes\mathscr{K}$ is stable, there is a unital *-embedding of O_2 into $\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$.

Since A has property (π), let $\Phi_1:\mathscr{M}(\mathscr{A}\otimes\mathscr{K})\to\mathbb{M}_3\otimes\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ be a *-isomorphism such that $\Phi_1(A)$ has the form

$$\Phi_1(A)=\begin{bmatrix}* & * & 0\\ * & * & 1\\ * & * & 0\end{bmatrix}.$$

Let $f_1,f_2\in\mathbb{M}_3\otimes\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ be given by $f_1=_{df} diag(1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})},1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})},0)$ and $f_2=_{df} diag(0,0,1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})})$.

If $\mathscr{A}$ is purely infinite then $f_1\sim f_2$ (since both are full projections in $\mathbb{M}_3\otimes\mathscr{M}(\mathscr{A}\otimes\mathscr{K})\sim\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ and since $\mathscr{A}\otimes\mathscr{K}$ has the corona factorization property ([15] Theorem 5.3; also [24] Proposition 2.5)).

If $\mathscr{A}$ is not purely infinite then (since $\mathscr{A}$ is in class $\mathfrak{R}$), $\tau(f_1)=\tau(f_2)=\infty$ for all $\tau\in T(\mathscr{A})$; and hence, $f_1\sim f_2$ in $\mathbb{M}_3\otimes\mathscr{M}(\mathscr{A}\otimes\mathscr{K})\cong\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ (again since both are full projections and since $\mathscr{A}\otimes\mathscr{K}$ has the corona factorization property ([24] Proposition 2.5)).

With respect to the decomposition $f_1+f_2=1$, A will have the form

$$A=\begin{bmatrix}* & V'\\ * & 0\end{bmatrix}$$

where V' is an element of $f_1(\mathbb{M}_3\otimes\mathscr{M}(\mathscr{A}\otimes\mathscr{K}))f_2$ such that $V'^*V'=f_{2,2}$ and $f_1-V'V'^*\sim f_1$.

Let $W_{1,2} \in \mathbb{M}_3 \otimes \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ be a partial isometry with initial projection f_2 and range projection f_1.

Let $\{e_{j,k}\}_{1 \leqslant j,k \leqslant 2}$ be system of matrix units for $\mathbb{M}_2$.

Let $\Phi_2 : \mathbb{M}_3 \otimes \mathscr{M}(\mathscr{A} \otimes \mathscr{K}) \to \mathbb{M}_2 \otimes f_{1,1}(\mathbb{M}_3 \otimes \mathscr{M}(\mathscr{A} \otimes \mathscr{K}))f_{1,1}$ be the *-isomorphism given by $\Phi_2(X) =_{df} (e_{1,1} \otimes f_1 X f_1) + (e_{1,2} \otimes f_1 X W_{1,2}^*) + (e_{2,1} \otimes W_{1,2} X f_1) + (e_{2,2} \otimes W_{1,2} X W_{1,2}^*)$, for all $X \in \mathbb{M}_3 \otimes \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$.

Noting that $f_{1,1}(\mathbb{M}_3 \otimes \mathscr{M}(\mathscr{A} \otimes \mathscr{K}))f_{1,1} \cong \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, we can take $\Phi =_{df} \Phi_2 \circ \Phi_1$, and Φ is the map that witnesses that A has property (π_0). □

LEMMA 3.10. *Let $\mathscr{A}$ be a unital separable simple C*-algebra in the class $\mathfrak{R}$. If $A, B \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ both have property (π) then some generalized sum of A and B is a commutator.*

Proof. There are two *-isomorphism $\Phi, \Psi : \mathscr{M}(\mathscr{A} \otimes \mathscr{K}) \to \mathbb{M}_3 \otimes \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ such that $\Phi(A)$ and $\Phi(B)$ have the forms stated in Definition 3.8.

Let $\{e_{j,k}\}_{1 \leqslant j,k \leqslant 3}$ be a system of matrix units for $\mathbb{M}_3$.

If $\mathscr{A}$ is purely infinite, then for all j, $e_{j,j} \otimes 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$ is Murray–von Neumann equivalent to $1_{\mathbb{M}_3} \otimes 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$ in $\mathbb{M}_3 \otimes \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ ([15] Theorem 5.3).

Suppose that $\mathscr{A}$ is not purely infinite (but still in $\mathfrak{R}$). Then for all j,k, and for all $\tau \in T(\mathscr{A})$, $\tau(e_{j,j} \otimes 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}) = \infty$.

Hence, $e_{j,j} \otimes 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})} \sim 1_{\mathbb{M}_3} \otimes 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$ in $\mathbb{M}_3 \otimes \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ for all j ([24] Proposition 2.5).

The rest of the argument is exactly the same as that of [4] Lemma 4.1, except that [25] Theorem 4 is replaced with (this paper) Lemma 3.1. □

4. The purely infinite case

For the rest of this paper, for a C*-algebra $\mathscr{B}$, let $\Gamma : \mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{B})/\mathscr{B}$ be the natural quotient map.

LEMMA 4.1. *Let $\mathscr{A}$ be a unital separable simple purely infinite C*-algebra. Suppose that $X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ is such that $\Gamma(X)$ is not a scalar multiple of the identity.*

Then there exists an $\alpha > 0$ with $\alpha < \|X\|^2$, where for every $\varepsilon > 0$, for every finite subset $\mathscr{F} \subset \mathscr{A} \otimes \mathscr{K}$, there exist projections $p, q \in \mathscr{A} \otimes \mathscr{K}$ such that

1. *$p \perp q$,*
2. *pa, ap, qa and ap are all within ε of 0, for all $a \in \mathscr{F}$, and*
3. *if $x =_{df} qXp$ then there exists $\beta \geqslant \alpha$ with $\beta \leqslant \|X\|^2$ and $\|x^*x - \beta p\|, \|xx^* - \beta q\| < \varepsilon$.*

Proof. Let $\varepsilon > 0$ and a finite subset $\mathscr{F} \subset \mathscr{A} \otimes \mathscr{K}$ be given. Contracting ε if necessary, we may assume that all the elements of $\mathscr{F}$ have norm less than or equal to one.

Since $\mathscr{A}$ is unital, there exists a projection $e \in \mathscr{A} \otimes \mathscr{K}$ such that ea, ae and eae are all within $\varepsilon/100$ of a for all $a \in \mathscr{F}$.

Since $\mathscr{A}$ is simple purely infinite, $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})/(\mathscr{A} \otimes \mathscr{K})$ is simple purely infinite and hence has real rank zero ([16]; see also [35] Theorem 3.3 and [29]). Hence, since $\Gamma(X)$ is not a scalar multiple of the identity, there exist nonzero orthogonal projections $R, S \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})/(\mathscr{A} \otimes \mathscr{K})$ such that $R\Gamma(X)S \neq 0$.

Let $\alpha =_{df} (1/2)\|R\Gamma(X)S\|^2 > 0$.

Lift R, S to orthogonal positive elements $A, B \in (1-e)\mathscr{M}(\mathscr{A} \otimes \mathscr{K})(1-e)$ with norm one (i.e., $e \perp A \perp B \perp e$, $\Gamma(A) = R$, $\Gamma(B) = S$ and $\|A\| = \|B\| = 1$).

Choose a number $\delta_1 > 0$ so that for all $Z \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ and for every projection $R \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, if ZZ^* is within δ_1 of R then Z^*RZ is within $\varepsilon/(100(2\alpha+1))$ of Z^*Z. Contracting $\delta_1 > 0$ if necessary, we may assume that $\delta_1 < \varepsilon/100$.

Choose $\delta_2 > 0$ so that for all $Z \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, if Z^*Z within δ_2 of a projection then ZZ^* is within $\delta_1/(100(2\alpha+1))$ of a projection in $\mathrm{Her}(ZZ^*)$. Contracting $\delta_2 > 0$ if necessary, we may assume that $\delta_2 < \varepsilon/100$.

Since $\mathscr{A} \otimes \mathscr{K}$ has real rank zero ([34]), we can choose projections $q' \in \overline{A(\mathscr{A} \otimes \mathscr{K})A}$ and $p' \in \overline{B(\mathscr{A} \otimes \mathscr{K})B}$ so that $\beta =_{df} \|q'Xp'\|^2 \geqslant (100/51)\alpha > 0$.

Note that $p' \perp q'$.

Let $h_1 : [0, \beta] \to [0,1]$ be the unique continuous function satisfying:

$$h_1(s) \begin{cases} = 1 & s \in [\beta - \frac{\delta_2 \beta}{1000(\beta+1)(\|X\|+1)}, \beta + 1] \\ = 0 & s \in [0, \beta - \frac{\delta_2 \beta}{100(\beta+1)(\|X\|+1)}] \\ \text{linear} & \text{on } [\beta - \frac{\delta_2 \beta}{100(\beta+1)(\|X\|+1)}, \beta - \frac{\delta_2 \beta}{1000(\beta+1)(\|X\|+1)}]. \end{cases}$$

Hence, $h_1(p'X^*q'Xp') \neq 0$ (indeed, $\|h_1(p'X^*q'Xp')\| = 1$).

Since $\mathscr{A} \otimes \mathscr{K}$ has real rank zero, let $p \in \mathrm{Her}(h_1(p'X^*q'Xp'))$ be a nonzero projection.

Hence, $p \leqslant p'$ and $pX^*q'Xp = pp'X^*q'Xp'p$ is within $\frac{\delta_2 \beta}{100(\beta+1)}$ of βp. Hence, $\beta^{-1}pX^*q'Xp$ is within $\frac{\delta_2}{100(\beta+1)}$ of the projection p.

Hence, by our choice of δ_2, $\beta^{-1}q'XpX^*q'$ is within $\delta_1/(100(2\alpha+1))$ of a projection, say $q \in \mathrm{Her}(q')$. Hence, $\beta^{-1}qXpX^*q$ is within $\delta_1/(100(2\alpha+1))$ of q. Hence, we have that $qXpX^*q$ is within $\varepsilon/100$ of βq.

Also, by our choice of δ_1, $\beta^{-1}pX^*qXp$ is within $\varepsilon/(100(2\alpha+1))$ of $\beta^{-1}pX^*q'Xp$. Hence, $\beta^{-1}pX^*qXp$ is within $\frac{\varepsilon}{50(\beta+1)}$ of p. Hence, pX^*qXp is wthin ε of βp. $\square$

LEMMA 4.2. *Let $\mathscr{C}$ be a C^*-algebra, and let $x \in \mathscr{C}$. Suppose that there exist projections $p, q \in \mathscr{C} - \{0\}$ such that*

i. $p \perp q$,

ii. px^*qxp *is invertible in* $p\mathscr{C}p$, *and*

iii. $qxpx^*q$ *is invertible in* $q\mathscr{C}q$.

Then

(a) *either xpx^* is invertible or 0 is an isolated point of the spectrum of xpx^* (and hence the support projection of xpx^* is in $\mathscr{C}$),*

(b) *if $r \in \mathscr{C}$ is the support projection of xpx^* then $\|pr\| < 1$, and*

(c) *if $q' \in \mathscr{C}$ is a projection with $q' \perp p$ and $r \sim q'$, and $v \in \mathscr{C}$ is a partial isometry with $v^*v = q'$ and $vv^* = r$, then $(p+v)^*(p+v)$ is an invertible element of $(p+q')\mathscr{C}(p+q')$.*

Proof. Since px^*qxp is invertible in $p\mathscr{C}p$ and $0 \leqslant px^*qxp \leqslant px^*xp$, px^*xp is invertible in $p\mathscr{C}p$.

Hence, (in $\mathscr{C}$) 0 is an isolated point in the spectrum of px^*xp. Hence, either xpx^* is invertible or 0 is an isolated point in the spectrum of xpx^*.

Hence, let $r \in \mathscr{C}$ be the support projection of xpx^*.

Suppose that $\mathscr{C}$ acts faithfully and nondegenerately on a Hilbert space $\mathscr{H}$. Hence, $xp|_{p\mathscr{H}}$ is a (continuous) bijective linear map in $\mathbb{B}(p\mathscr{H}, r\mathscr{H})$. Hence, by the Open Mapping Theorem, there exists $T \in \mathbb{B}(r\mathscr{H}, p\mathscr{H})$ such that $T \circ (xp) = p$. Hence, there exists $\alpha > 0$ such that $\|xp(h)\| \geqslant \alpha \|h\|$ for all $h \in \mathscr{H}$.

Also, $qxp|_{p\mathscr{H}}$ is an invertible bijective linear map in $\mathbb{B}(p\mathscr{H}, q\mathscr{H})$.

Hence, let $\beta > 0$ be such that $\|qxp(h)\| \geqslant \beta \|h\|$ for all $h \in p\mathscr{H}$.

Choose $\delta > 0$ so that $\delta < \min\{1/100, \alpha/100, \beta/100\}$.

Suppose, to the contrary, that $\|pr\| = 1$.

Since r is the range projection of xp (and xp is surjective onto $r\mathscr{H}$) and since we have assumed that $\|pr\| = 1$, choose $h \in p\mathscr{H}$ with $\|h\| = 1$ so that if $k =_{df} xh = xph$ then $\|pk\|^2 \geqslant \|k\|^2 - \delta^2$. (Note that since $\|h\| = 1$, $\|k\| = \|xph\| \geqslant \alpha \|h\| = \alpha > \delta$. Hence, $\|k\|^2 - \delta^2 \geqslant 0$.)

Hence, $\|k\|^2 = \|pk\|^2 + \|(1-p)k\|^2 \geqslant \|k\|^2 - \delta^2 + \|(1-p)k\|^2$.

Hence, $\|(1-p)k\|^2 \leqslant \delta^2$. I.e., $\|(1-p)k\| \leqslant \delta \leqslant \beta/100$.

But $\|(1-p)k\| \geqslant \|qk\| = \|qxph\| \geqslant \beta \|h\| = \beta$. This is a contradiction.

Hence, we must have that $\|pr\| < 1$.

Hence, by [4] Lemma 2.1, $p\mathscr{H} \cap r\mathscr{H} = \{0\}$ and $p\mathscr{H} + r\mathscr{H}$ is a closed linear subspace of $\mathscr{H}$.

Suppose that $q' \in \mathscr{C}$ is a projection with $q' \perp p$ and $r \sim q'$, and suppose that $v \in \mathscr{C}$ is a partial isometry with $v^*v = q'$ and $vv^* = r$. Then $(p+v)|_{p\mathscr{H}+q'\mathscr{H}}$ is a bijective linear map in $\mathbb{B}(p\mathscr{H} + q'\mathscr{H}, p\mathscr{H} + r\mathscr{H})$. Hence, by the Open Mapping Theorem, there exists $S \in \mathbb{B}(p\mathscr{H} + r\mathscr{H}, p\mathscr{H} + q'\mathscr{H})$ such that $S \circ (p+v) = p + q'$. Hence, $(p+v)^*(p+v)$ is an invertible element of $(p+q')\mathscr{C}(p+q')$. □

LEMMA 4.3. *Let $\mathscr{A}$ be a unital separable simple purely infinite C*-algebra. Suppose that $X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ is such that $\Gamma(X)$ is not a scalar multiple of the identity.*

Then for every $\varepsilon > 0$, there exist projections $P, Q, R, S \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ such that

1. $P \sim Q \sim S \sim 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$,

2. $S \perp P \perp Q \perp S$,

3. *PX^*QXP is invertible in $P\mathscr{M}(\mathscr{A}\otimes\mathscr{K})P$*
4. *$QXPX^*Q$ is invertible in $Q\mathscr{M}(\mathscr{A}\otimes\mathscr{K})Q$,*
5. *0 is an isolated point of the spectrum of XPX^*,*
6. *R is the left support projection of XP,*
7. *$\|PR\| < 1$,*
8. *$\|SR\| < \varepsilon$, and*
9. *$\Gamma(S)\Gamma(R) = 0$.*
10. *if $Q' \in \mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ is a projection such that $Q' \sim 1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})}$ and $Q' \perp P$, and if $V \in \mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ is a partial isometry such that $V^*V = Q'$ and $VV^* = R$ then $(P+V)^*(P+V)$ is an invertible element of $(P+Q')\mathscr{M}(\mathscr{A}\otimes\mathscr{K})(P+Q')$.*

Proof. Apply Lemma 4.1 to X to get a number $\|X\|^2 > \alpha > 0$.

Since $\mathscr{A}\otimes\mathscr{K}$ is separable, let $b \in (\mathscr{A}\otimes\mathscr{K})_+$ be a strictly postive element with $\|b\| = 1$.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in $(0, 1/100)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < 1/100$.

We now construct two sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ of projections in $\mathscr{A}\otimes\mathscr{K}$ and a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of numbers in $(0,\infty)$ such that the following statements hold:

1. $p_m \perp p_n$ and $q_m \perp q_n$ for all $m \neq n$.
2. $p_m \perp q_n$ for all m, n.
3. $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ both converge in the strict topology on $\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$. In particular, for all $n \geqslant 2$, $p_n b$ and $q_n b$ are both within ε_n of 0.
4. $\|X\|^2 \geqslant \alpha_n \geqslant \alpha/2$ for all n.
5. $\|p_n X^* q_n X p_n - \alpha_n p_n\| < \varepsilon_n$ for all n.
6. $\|q_n X p_n X^* q_n - \alpha_n q_n\| < \varepsilon_n$ for all n.
7. $\|q_m X p_n\| < 1/1000^{m+n}$ for all $m \neq n$.

The construction is by induction on n.

Basis step $n = 1$. Apply Lemma 4.1 to get projections $p_1, q_1 \in \mathscr{A}\otimes\mathscr{K}$ and $\alpha_1 \geqslant \alpha$ so that

(a) $p_1 \perp p_2$, and

(b) $\|p_1 X^* q_1 X p_1 - \alpha_1 p_1\|, \|q_1 X p_1 X^* q_1 - \alpha_1 q_1\| < \varepsilon_1$.

We denote the above statements by "$(*)$".

Induction step. Suppose that $p_1, p_2, ..., p_n, q_1, q_2, ..., q_n, \alpha_1, \alpha_2, ..., \alpha_n$ have been constructed. We now construct $p_{n+1}, q_{n+1}, \alpha_{n+1}$.

Choose $\delta_1 > 0$ so that for any C*-algebra $\mathscr{C}$, if $r_1, r_2, r_3 \in \mathscr{C}$ are projections such that $r_1 \perp r_2$ and $\|r_1 r_3\|, \|r_2 r_3\| < \delta_1$, then there exist projections $r_1', r_2' \in \mathscr{C}$ such that $r_j' \sim r_j$ $(j = 1,2)$, $r_3 \perp r_1' \perp r_2' \perp r_3$, and

$$\|r_j' - r_j\| < \min\left\{\frac{\varepsilon_{n+1}}{1000(\|X\| + \|X\|^2 + 1)}, \frac{1}{1000^{2n+3}(\|X\| + 1)}\right\} \quad (j = 1,2).$$

Contracting δ_1 if necessary, we may assume that $\delta_1 < \varepsilon_{n+1}/1000$.

By Lemma 4.1, let $p_{n+1}', q_{n+1}' \in \mathscr{A} \otimes \mathscr{K}$ be orthogonal projections and $\alpha_{n+1} \geqslant \alpha$ such that the following statements are true:

i. $p_{n+1}' \perp q_{n+1}'$.

ii. $\|p_{n+1}' X^* q_{n+1}' X p_{n+1}' - \alpha_{n+1} p_{n+1}'\|, \|p_{n+1}' X^* q_{n+1}' X p_{n+1}' - \alpha_{n+1} p_{n+1}'\| < \varepsilon_{n+1}/1000$.

iv. $p_{n+1}' \left(\sum_{m=1}^n (p_m + q_m)\right)$, $q_{n+1}' \left(\sum_{m=1}^n (p_m + q_m)\right)$, $p_{n+1}' b$ and $q_{n+1}' b$ are all within δ_1 of 0.

iv. $\|q_m X p_{n+1}'\|, \|q_{n+1}' X p_m\| < 1/1000^{2n+3}$ for all $m \leqslant n$.

We denote the above statements by "$(*)$".

By our choice of δ_1 and by statements $(*)$, we have that there exists projections $p_{n+1}, q_{n+1} \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ such that $p_{n+1} \sim p_{n+1}'$, $q_{n+1} \sim q_{n+1}'$, $q_{n+1} \perp \left(\sum_{m=1}^n (p_m + q_m)\right) \perp p_{n+1} \perp q_{n+1}$, and

$$\|p_{n+1} - p_{n+1}'\|, \ \|q_{n+1} - q_{n+1}'\| < \min\left\{\frac{\varepsilon_{n+1}}{1000(\|X\| + \|X\|^2 + 1)}, \frac{1}{1000^{2n+3}(\|X\| + 1)}\right\}.$$

From this and statements $(*)$, we have that

1. $\|p_{n+1} X^* q_{n+1} X p_{n+1} - \alpha_{n+1} p_{n+1}\|, \|q_{n+1} X p_{n+1} X^* q_{n+1} - \alpha_{n+1} q_{n+1}\| < \varepsilon_{n+1}$,

2. $p_{n+1} b$ and $q_{n+1} b$ are within ε_{n+1} of 0, and

3. $\|q_m X p_{n+1}\|, \|q_{n+1} X p_m\| < 1/1000^{m+n+1}$ for all $m \leqslant n$.

This completes the inductive construction.

Choose $\delta > 0$ so that if $A, E, E' \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ where (a) A is positive, (b) $\|A\| \leqslant \|X\|^2$, (c) E and E' are projections, (d) $EAE = A$,(e) $A \geqslant (\alpha/10)E$, and (f) $\|E'A\| < \delta$, then $\|E'E\| < \varepsilon$.

Choose $N \geqslant 1$ so that for all $n \geqslant N$, $\varepsilon_n < \alpha/100$.

Let $\{n(1,k)\}_{k=1}^\infty$ and $\{n(2,k)\}_{k=1}^\infty$ be two disjoint subsequences of the positive integers greater than or equal to N, $\{N, N+1, ...\}$ (so, as sets, $\{n(1,k)\}_{k=1}^\infty \cup \{n(2,k)\}_{k=1}^\infty \subseteq \{N, N+1, N+2, ...\}$ and $\{n(1,k)\}_{k=1}^\infty \cap \{n(2,k)\}_{k=1}^\infty = \emptyset$), such that

$$\sum_{k \neq l} \|q_{n(1,k)} X p_{n(1,l)}\| < \frac{\min\{\sqrt{(\alpha/100)}, \alpha/100\}}{(100(\|X\| + 1))} \tag{4.1}$$

and

$$\sum_{1\leqslant k,l<\infty} \|q_{n(2,k)}Xp_{n(1,l)}\| < \delta/(100(1+\|X\|)). \tag{4.2}$$

Let $P,Q,S \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be projections given by $P =_{df} \sum_{k=1}^{\infty} p_{n(1,k)}$, $Q =_{df} \sum_{k=1}^{\infty} q_{n(1,k)}$, and $S =_{df} \sum_{k=1}^{\infty} q_{n(2,k)}$ where the sums converge strictly on $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.

From statements $(*)$ and the definitions of P,Q,S, it follows immediately that $Q\perp P\perp S\perp Q$, $P\sim Q\sim S\sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$, PX^*QXP is invertible in $P\mathcal{M}(\mathcal{A}\otimes\mathcal{K})P$, and $QXPX^*Q$ is invertible in $Q\mathcal{M}(\mathcal{A}\otimes\mathcal{K})Q$.

Hence, by Lemma 4.2, (i.) either XPX^* is invertible (in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$) or 0 is an isolated point in the spectrum of XPX^*, (ii.) the support projection of XPX^*, say R, is an element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$, (iii.) $\|PR\|<1$, and (iv.) if $Q'\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ is a projection with $Q'\sim 1$ and $Q'\perp P$, and if $V\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ is a partial isometry with initial projection Q' and range projection R, then $(P+V)^*(P+V)$ is an invertible element of $(P+Q')\mathcal{M}(\mathcal{A}\otimes\mathcal{K})(P+Q')$.

Moreover, by the definitions of P,Q and by equation (4.1), $PX^*QXP \geqslant (\alpha/10)P$. Hence, since $PX^*XP \geqslant PX^*QXP$, we have that $PX^*XP \geqslant (\alpha/10)P$. Hence, it follows that $XPX^* \geqslant (\alpha/10)R$. (Recall that $R\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ is the support projection of XPX^*.) But also, by equation (4.2) and by the definitions of P and S, $\|SXPX^*\|<\delta$. Hence, by the definition of δ, we must have that $\|SR\|<\varepsilon$. And this implies that $R\neq 1$; so 0 is an isolated point in the spectrum of XPX^*.

Finally, since $\sum_{k\geqslant K,1\leqslant l<\infty}\|q_{n(2,k)}Xp_{n(1,l)}X^*\|\to 0$ as $K\to\infty$ (and since XPX^* is invertible in $R\mathcal{M}(\mathcal{A}\otimes\mathcal{K})R$), it follows that $\Gamma(S)\Gamma(R)=0$. □

LEMMA 4.4. *Let $\mathcal{B}$ be a separable stable C*-algebra, and let $\Gamma:\mathcal{M}(\mathcal{B})\to\mathcal{M}(\mathcal{B})/\mathcal{B}$ be the natural quotient map.*

Let $X\in\mathcal{M}(\mathcal{B})$ be an operator. Suppose that $P,Q,R,S\in\mathcal{M}(\mathcal{B})$ are projections such that

1. $P\sim Q\sim S\sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$,
2. $S\perp P\perp Q\perp S$,
3. PX^*QXP *is invertible in* $P\mathcal{M}(\mathcal{A}\otimes\mathcal{K})P$
4. $QXPX^*Q$ *is invertible in* $Q\mathcal{M}(\mathcal{A}\otimes\mathcal{K})Q$,
5. 0 *is an isolated point of the spectrum of* XPX^*,
6. R *is the left support projection of* XP,
7. $\|PR\|<1$,
8. $\|SR\|<1/10$,
9. $\Gamma(S)\Gamma(R)=0$, *and*
10. *if* $Q'\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ *is a projection such that* $Q'\sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$ *and* $Q'\perp P$, *and if* $V\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ *is a partial isometry such that* $V^*V=Q'$ *and* $VV^*=R$ *then* $(P+V)^*(P+V)$ *is an invertible element of* $(P+Q')\mathcal{M}(\mathcal{A}\otimes\mathcal{K})(P+Q')$.

Then X is similar to an operator with property (π) in $\mathcal{M}(\mathcal{B})$.

Proof. Now since PX^*QXP is invertible in $P\mathcal{M}(\mathcal{B})P$ and $0 \leqslant PX^*QXP \leqslant PX^*XP$, PX^*XP is invertible in $P\mathcal{M}(\mathcal{B})P$. Hence, $R \sim P$ $(\sim 1_{\mathcal{M}(\mathcal{B})})$ in $\mathcal{M}(\mathcal{B})$.

Since $Q \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$, let $Q', Q'' \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be orthogonal projections such that $Q' \sim Q'' \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$ and $Q = Q' + Q''$.

Let $V \in \mathcal{M}(\mathcal{B})$ be a partial isometry such that $V^*V = Q'$ and $VV^* = R$. By hypothesis, $(P+V)^*(P+V)$ is an invertible element of $(P+Q')\mathcal{M}(\mathcal{B})(P+Q')$. Hence, either $(P+V)(P+V)^*$ is invertible or 0 is an isolated point in the spectrum of $(P+V)(P+V)^*$. Hence, let $T \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be the support projection of $(P+V)(P+V)^*$. Hence, $(P+V)(P+V)^*$ is an invertible element of $T\mathcal{M}(\mathcal{A}\otimes\mathcal{K})T$.

Since $S \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$, $S \perp P$ and $\Gamma(S)\Gamma(R) = 0$, $\Gamma(S) \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})}$ and $\Gamma(S) \perp \Gamma(T)$. Hence, $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})} \preceq 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})} - \Gamma(T)$. Hence, since $\mathcal{A}\otimes\mathcal{K}$ is stable, $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - T \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. Also, since $\mathcal{A}\otimes\mathcal{K}$ is stable and since $Q'' \leqslant 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (P+Q')$, $Q''' =_{df} 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (P+Q') \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. Hence, let $W \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be a partial isometry with $W^*W = Q'''$ and $WW^* = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - T$.

Let $Y \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be the invertible element that is given by $Y =_{df} P+V+W$.

Therefore, $Y^{-1}XYP = Y^{-1}XP = Y^{-1}RXP = V^*RXP = Q'V^*RXP$. Thus, with respect to the decomposition $P+Q'+Q''' = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$, $Y^{-1}XY$ has the form

$$Y^{-1}XY = \begin{bmatrix} 0 & * & * \\ Z & * & * \\ 0 & * & * \end{bmatrix},$$

where $Z =_{df} Q'V^*RXP$.

Moreover, Z^*Z and ZZ^* are invertible elements of $P\mathcal{M}(\mathcal{A}\otimes\mathcal{K})P$ and $Q'\mathcal{M}(\mathcal{A}\otimes\mathcal{K})Q'$ respectively. Let $Z = U|Z|$ be the Polar Decomposition of Z. Then $|Z|$ is an invertible element of $P\mathcal{M}(\mathcal{A}\otimes\mathcal{K})P$, and the partial isometry U is an element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Hence, let $Z_1 \in P\mathcal{M}(\mathcal{A}\otimes\mathcal{K})P$ be the inverse of $|Z|$ in $P\mathcal{M}(\mathcal{A}\otimes\mathcal{K})P$. Let $Y_1 \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be the invertible element given by $Y_1 = Z_1 + (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - P)$.

Then, with respect to the decomposition $P+Q'+Q''' = 1$, $Y_1^{-1}Y^{-1}XYY_1$ has the form

$$Y_1^{-1}Y^{-1}XYY_1 = \begin{bmatrix} 0 & * & * \\ U & * & * \\ 0 & * & * \end{bmatrix}.$$

Note that $U^*U = P$ and $UU^* = Q'$.

From this and the fact that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K}) \cong P\mathcal{M}(\mathcal{A}\otimes\mathcal{K})P$, we can construct a *-isomorphism $\Phi : \mathcal{M}(\mathcal{A}\otimes\mathcal{K}) \to \mathbb{M}_3 \otimes \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ which witnesses that $Y_1^{-1}Y^{-1}XYY_1$ is an operator with property (π). (E.g., see the argument in Lemma 3.9.) □

THEOREM 4.5. *Let $\mathcal{A}$ be a unital separable simple purely infinite C*-algebra. Let $X \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Then X is a commutator if and only if X is either compact or nonthin, i.e., X does not have the form $\alpha 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + x$ where $\alpha \in \mathbb{C} - \{0\}$ and $x \in \mathcal{A}\otimes\mathcal{K}$.*

Proof. Note that since $\mathscr{A}$ is simple purely infinite, the only proper nontrivial ideal of $\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ is $\mathscr{A}\otimes\mathscr{K}$. (E.g., see [35] Theorem 3.3 or [29].)

The "only if" direction is straightforward. (If $\alpha\in\mathbb{C}-\{0\}$ and $x\in\mathscr{A}\otimes\mathscr{K}$ then $\Gamma(\alpha 1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})}+x)=\alpha 1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})/(\mathscr{A}\otimes\mathscr{K})}$; and no nonzero scalar multiple of the unit, of a unital C*-algebra, can be a commutator.)

We now prove the "if" direction.

If $X\in\mathscr{A}\otimes\mathscr{K}$, then, by Corollary 2.3, X is a commutator.

Hence, it suffices to prove that for all $X\in\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ such that $\Gamma(X)$ is not a scalar multiple of $1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})/(\mathscr{A}\otimes\mathscr{K})}$, X is a commutator. Let $\mathfrak{O}$ consist of all such elements X.

$\mathfrak{O}$ is an (norm topology) open subset of $\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$. (Since the set of all $Y\in\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ with $\Gamma(Y)$ being a scalar multiple of the unit is closed.)

Clearly, $\mathfrak{O}$ is closed under multiplication by nonzero scalars. It is also clear that for all projections $P,Q\in\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$ such that $P\perp Q$, $P\sim Q\sim 1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})}$ and $P+Q=1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})}$, $P+2Q\in\mathfrak{O}$. Moreover, note that since $\mathscr{A}\otimes\mathscr{K}$ is stable, there exists a unital *-embedding of the Cuntz algebra O_2 into $\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$.

By Lemma 4.3, Lemma 4.4 and Lemma 3.9, every element of $\mathfrak{O}$ is similar to an operator with both properties (π) and (π_0). Hence, by Lemma 3.10, for all $X_1,X_2\in\mathfrak{O}$, some generalized sum of X_1 and X_2 is a commutator.

Hence, by Proposition 3.7, for every $X\in\mathfrak{O}$, X is a commutator. □

5. The stably finite case: Part I

At some point, in the sections to follow, we will use the notion of Cuntz subequivalence. For a C*-algebra $\mathscr{C}$ and positive elements $a,b\in\mathscr{C}_+$, we say that a is *Cuntz subequivalent* to b ("$a\preceq b$") if there exists a sequence $\{x_n\}$ in $\mathscr{C}$ such that $x_nbx_n^*\to a$. If a and b are projections, then a is Cuntz subequivalent to b if and only if a is Murray–von Neumann subequivalent to b.

PROPOSITION 5.1. *Suppose that $\mathscr{A}$ is a unital separable simple C*-algebra with stable rank one and in class $\mathfrak{R}$.*

Suppose that $P,Q\in\mathscr{M}(\mathscr{A}\otimes\mathscr{K})-\mathscr{A}\otimes\mathscr{K}$ are two projections such that $\tau(P)=\tau(Q)$ for all $\tau\in T(A)$.

Then $P\sim Q$ in $\mathscr{M}(\mathscr{A}\otimes\mathscr{K})$.

Proof. This is [19] Proposition 4.2. We note that the proof works even without the finiteness assumption. □

PROPOSITION 5.2. *Suppose that $\mathscr{A}$ is a unital separable simple stably finite C*-algebra in class $\mathfrak{R}$. Suppose, in addition, that for every bounded strictly positive affine lower semicontinuous function $f:T(\mathscr{A})\to(0,\infty)$, there exists a nonzero $a\in(\mathscr{A}\otimes\mathscr{K})_+$ which is not Cuntz equivalent to a projection such that $d_\tau(a)=f(\tau)$ for all $\tau\in T(\mathscr{A})$.*
(E.g., $\mathscr{A}$ can be unital simple exact finite and $\mathscr{Z}$-stable.)

Then for every strictly positive, affine, lower semicontinuous function $f : T(A) \to (0, \infty]$, *there exists a projection* $P \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K}) - (\mathscr{A} \otimes \mathscr{K})$ *such that*

$$\tau(P) = f(\tau)$$

for all $\tau \in T(\mathscr{A})$

Proof. This is [19] Corollary 4.6. □

DEFINITION 5.3. Let $\mathscr{B}$ be a separable, nonunital, nonelementary simple C*-algebra, and let $\{e_n\}_{n=1}^{\infty}$ be an approximate unit for $\mathscr{B}$.

Let $\mathscr{I}_{min}$ be the closure of the set

$$\{X \in \mathscr{M}(\mathscr{B}) : \forall a \in \mathscr{B}_+ - \{0\}, \exists n_0 \text{ s.t. } (e_m - e_n)X^*X(e_m - e_n) \preceq a, \forall m > n \geqslant n_0\}.$$

By [17], $\mathscr{I}_{min}$ is independent of choice of approximate unit $\{e_n\}_{n=1}^{\infty}$, and also $\mathscr{I}_{min}$ is the unique smallest C*-ideal in $\mathscr{M}(\mathscr{B})$ which properly contains $\mathscr{B}$ (see [17] Lemma 2.1, Remark 2.2, Lemma 2.4 and Remark 2.9).

LEMMA 5.4. *Let* $\mathscr{A}$ *be a unital simple separable stably finite C*-algebra in class* $\mathfrak{R}$. *Let* $\mathscr{I}_{min} \subseteq \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ *be the C*-ideal defined in Definition in 5.3.*

Let $a, b \in (\mathscr{I}_{min})_+ - (A \otimes \mathscr{K})$ *such that* $\|b\| \leqslant 1$ *and* b *induces a continuous function on* $T(A)$. *Suppose that*

$$\inf\{\tau(b) - d_\tau(a) : \tau \in T(A)\} > 0.$$

Then $a \preceq b$.

Proof. This is [19] Lemma 4.1. □

DEFINITION 5.5. Let $\mathscr{C}$ be a C*-algebra. $\mathscr{C}$ is said to have the *Hjelmborg–Rordam Property* if for every $a \in \mathscr{C}_+$, and for every $\varepsilon > 0$, there exists $b \in \mathscr{C}_+$ with $\|(a - \varepsilon)_+ b\| < \varepsilon$ and $(a - \varepsilon)_+ \preceq b$.

If $\mathscr{C}$ is a separable C*-algebra, then $\mathscr{C}$ has the Hjelmborg–Rordam Property if and only if $\mathscr{C}$ is stable (see [13] and [30]).

LEMMA 5.6. *Let* $\mathscr{A}$ *be a unital separable simple C*-algebra with stable rank one and in class* $\mathfrak{R}$.

Let $A \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})_+$ *be a full positive element, and let* $\mathscr{I}_{min} \subseteq \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ *be the C*-ideal defined in Definition 5.3.*

Then $\overline{A \mathscr{I}_{min} A}$ *has the Hjelmborg–Rordam Property.*

Proof. We may assume that $\|A\| = 1$.

Let $a \in (\overline{A \mathscr{I}_{min} A})_+ - (\mathscr{A} \otimes \mathscr{K})$ and let $\varepsilon > 0$ be given. Contracting ε if necessary, we may assume that $\varepsilon < 1/10$ and $\|a\| \leqslant 1$.

Since $a \in \mathscr{I}_{min}$, $L =_{df} \sup_{\tau \in T(\mathscr{A})} d_\tau((a-\varepsilon/100)_+) < \infty$.

Choose $N \geqslant 1$ and $\delta > 0$ so that $(A-\delta)_+^{1/N}(a-\varepsilon/100)_+$, $(a-\varepsilon/100)_+(A-\delta)_+^{1/N}$ and $(A-\delta)_+^{1/N}(a-\varepsilon/100)_+(A-\delta)_+^{1/N}$ are all within $\varepsilon/100$ of $(a-\varepsilon/100)_+$. Contracting $\delta > 0$ if necessary, we may assume that $\delta < \min\{\varepsilon/100, 1/100\}$.

Further contracting $\delta > 0$ if necessary, we may assume that $(A-\delta)_+$ is a full element of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$.

Let $h_1 : [0,1] \to [0,1]$ be the unique continuous function which is given by

$$h_1(s) \begin{cases} = 1 & s \in [\varepsilon/100, 1] \\ = 0 & [0, \varepsilon/1000] \\ \text{linear} & \text{on } [\varepsilon/1000, \varepsilon/100]. \end{cases}$$

Let $a' =_{df} h_1((A-\delta)_+^{1/N}(a-\varepsilon/100)_+(A-\delta)_+^{1/N}) \in \mathscr{I}_{min}$.

Hence, $a'(A-\delta)_+^{1/N}(a-\varepsilon/100)_+(A-\delta)_+^{1/N}$ and $(A-\delta)_+^{1/N}(a-\varepsilon/100)_+(A-\delta)_+^{1/N}a'$ are both within $\varepsilon/100$ of $(A-\delta)_+^{1/N}(a-\varepsilon/100)_+(A-\delta)_+^{1/N}$. Hence, $a'(a-\varepsilon/100)_+$ and $(a-\varepsilon/100)_+a'$ are both within $3\varepsilon/100$ of $(a-\varepsilon/100)_+$.

Let $h_2 : [0,1] \to [0,1]$ be the unique continuous function such that

$$h_2(s) \begin{cases} = 1 & s \in [\delta/10, 1] \\ = 0 & s = 0 \\ \text{linear} & \text{on } [0, \delta/10]. \end{cases}$$

Then $h_2(A)a' = a'$.

Moreover, $h_2(A)$ is a full positive element of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, and $\overline{h_2(A)\mathscr{I}_{min}h_2(A)} = \overline{A\mathscr{I}_{min}A}$.

Since any ideal of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, that properly contains $\mathscr{A} \otimes \mathscr{K}$, must contain $\mathscr{I}_{min}$, $h_2(A) - a'$ is a full positive element of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ ([17] Remark 2.9). Hence, since $\mathscr{A} \otimes \mathscr{K}$ has the corona factorization property ([24] Proposition 2.5), there exists $X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ such that $X(h_2(A) - a')X^* = 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$. From this and Proposition 5.2, there exists a projection $p \in \mathscr{I}_{min} - (\mathscr{A} \otimes \mathscr{K})$ such that $p \in \mathrm{Her}(h_2(A) - a')$ and $\tau(p) \geqslant L + 1$ for all $\tau \in T(\mathscr{A})$.

Note that since $p \in \mathscr{I}_{min}$ and since p is a projection, p induces a continuous function on $T(\mathscr{A})$.

Hence, by Lemma 5.4, $(a-\varepsilon)_+ \preceq p$. Finally, since $pa' = 0$, $\|p(a-\varepsilon)_+\| < \varepsilon$.

Since a was arbitrary, $\overline{A\mathscr{I}_{min}A}$ has the Hjelmborg–Rordam Property. □

Note that though $\mathscr{I}_{min}$ (as above) has the Hjelmborg–Rordam Property, it need not be stable, since $\mathscr{I}_{min}$ is not separable.

LEMMA 5.7. *Let $\mathscr{B}$ be a separable stable C*-algebra and let $\tilde{\mathscr{B}}$ be the unitization of $\mathscr{B}$.*

Then $\mathscr{B} \subseteq \overline{GL(\tilde{\mathscr{B}})}$, where the closure is in the norm topology.

Proof. This is [1] Lemma 4.3.2. □

LEMMA 5.8. *Let $\mathscr{A}$ be a unital separable simple C*-algebra with stable rank one and in class $\mathfrak{R}$. Let $A \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})_+$ be a full positive element.*

For every $\varepsilon > 0$, for every $\alpha > 0$ and for every finite subset $\mathscr{F} \subset \overline{A\mathscr{I}_{min}A}$, there exists a projection $p \in \overline{A\mathscr{I}_{min}A} - (\mathscr{A} \otimes \mathscr{K})$ such that $\tau(p) \geqslant \alpha$ for all $\tau \in T(\mathscr{A})$, and $\|px - x\|, \|xp - x\| < \varepsilon$ for all $x \in \mathscr{F}$.

In particular, $\overline{A\mathscr{I}_{min}A}$ has an (netwise) approximate unit consisting of projections.

Proof. We may assume that $\mathscr{F}$ contains a nonzero element.

Let $a \in (\overline{A\mathscr{I}_{min}A})_+$ be given by $a =_{df} \sum_{x\in\mathscr{F}}(x^*x + xx^*)/\|\sum_{x\in\mathscr{F}}(x^*x + xx^*)\|$.

Find $\delta > 0$ so that if $R \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ is a projection with $\|R(a-\delta)_+ - (a-\delta)_+\| < \delta$ then $\|Rx - x\|, \|xR - x\| < \varepsilon$ for all $x \in \mathscr{F}$.

Contracting δ if necessary, we may assume that $\delta < \min\{1/100, \varepsilon/100\}$.

Since $a \in \mathscr{I}_{min}$, $L =_{df} \sup_{\tau\in T(\mathscr{A})} d_\tau((a - \delta/100)_+) < \infty$.

Since A is a full positive element of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ and since $\mathscr{A} \otimes \mathscr{K}$ has the corona factorization property ([24] Proposition 2.5), there exists $X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ such that $XAX^* = 1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})}$. From this and Proposition 5.2, let $q \in \overline{A\mathscr{I}_{min}A} - (\mathscr{A} \otimes \mathscr{K})$ be a projection such that $\tau(q) > \max\{L + 10, \alpha + 10\}$ for all $\tau \in T(\mathscr{A})$.

Hence, by Lemma 5.4, $(a - \delta/100)_+ \preceq q$.

Hence, there exists $y \in \overline{A\mathscr{I}_{min}A}$ such that $(a - \delta/10)_+ = y^*y$ and $yy^* \leqslant q$.

Let $y = v|y|$ be the Polar Decomposition of y. Hence, $(a - \delta/10)_+ = |y|^2$ and $v|y|^2v^* \leqslant q$.

Since $\overline{A\mathscr{I}_{min}A}$ has the Hjelmborg–Rordam Property, there exists a separable C*-subalgebra $\mathscr{D} \subset \overline{A\mathscr{I}_{min}A}$ such that $\{(a - \delta/10)_+, y, q\} \subset \mathscr{D}$ and $\mathscr{D}$ has the Hjelmborg–Rordam Property. Hence, since $\mathscr{D}$ is separable, $\mathscr{D}$ is a stable C*-algebra.

Hence, by Lemma 5.7, $y \in \overline{GL(\tilde{\mathscr{D}})}$, where $\tilde{\mathscr{D}}$ is the unitization of $\mathscr{D}$. Hence, by [26] Theorem 5, let $u \in \tilde{\mathscr{D}}$ ($\subset \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$) be a unitary such that $u(|y|^2 - \delta/10)_+u^* = v(|y|^2 - \delta/10)_+v^* \leqslant q$.

Hence, $(a - \delta/5)_+ = (|y|^2 - \delta/10)_+ \leqslant u^*qu$, and $p =_{df} u^*qu$ is a projection in $\overline{A\mathscr{I}_{min}A}$ such that $\tau(p) \geqslant \alpha$ for all $\tau \in T(\mathscr{A})$. In particular, $p(a - \delta/5)_+ = (a - \delta/5)_+$. So $p(a - \delta)_+ = (a - \delta)_+$. Hence, by our choice of δ, $\|px - x\|, \|xp - x\| < \varepsilon$ for all $x \in \mathscr{F}$. □

LEMMA 5.9. *Let $\mathscr{A}$ be a unital separable simple C*-algebra with stable rank one and in class $\mathfrak{R}$.*

Let $x \in \mathscr{I}_{min}$ be given.

Then there exists a sequence $\{p_n\}_{n=1}^\infty$ of pairwise orthogonal projections in $\mathscr{I}_{min} - (\mathscr{A} \otimes \mathscr{K})$ such that

1. *$\tau(p_n) \geqslant 10$ for all $n \geqslant 1$,*
2. *the sum $\sum_{n=1}^\infty p_n$ converges in the strict topology on $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, and*
3. *$\|(\sum_{n=1}^N p_n)x - x\|, \|x(\sum_{n=1}^N p_n) - x\|, \|(\sum_{n=1}^N p_n)x(\sum_{n=1}^N p_n) - x\| \to 0$ as $N \to \infty$.*

Sketch of proof. The proof is an easy induction argument, repeatedly using Lemma 5.8. (In particular, we will use Lemma 5.8 (many times) to find an appropriate increasing sequence $\{r_n\}$ of projections and then take $p_n =_{df} r_{n+1} - r_n$ for all n. To ensure strict convergence of $\sum p_n$, we will need the finite sets $\mathscr{F}$ (notation as in Lemma 5.8) to contain a fixed strictly positive element of $\mathscr{A} \otimes \mathscr{K}$. Also, at some point, we need to use the following perturbation result: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if e, e' are projections with $\|ee' - e'\| < \delta$ then there exists a projection $e'' \leqslant e$ with $\|e' - e''\| < \varepsilon$.) □

THEOREM 5.10. *Let $\mathscr{A}$ be a unital separable simple C*-algebra with stable rank one and in class $\mathfrak{R}$.*

If $x \in \mathscr{I}_{min}$ then x is a commutator in $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$.

Sketch of proof. By Lemma 5.9, let $\{r_n\}_{n=1}^{\infty}$ be a sequence of pairwise orthogonal projections in $\mathscr{I}_{min} - (\mathscr{A} \otimes \mathscr{K})$ such that

1. $\tau(r_n) \geqslant 10$ for all $n \geqslant 1$,
2. the sum $\sum_{n=1}^{\infty} r_n$ converges in the strict topology on $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, and
3. $\|(\sum_{n=1}^{N} r_n)x - x\|, \|x(\sum_{n=1}^{N} r_n) - x\|, \|(\sum_{n=1}^{N} r_n)x(\sum_{n=1}^{N} r_n) - x\| \to \infty$ as $N \to \infty$.

Claim: There exists a sequence $\{Q_n\}_{n=1}^{\infty}$ of pairwise orthogonal projections in $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ such that

(a) $Q_n \sim 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$ for all n,

(b) $\sum_{n=1}^{\infty} Q_n$ converges in the strict topology on $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$,

(c) $(\sum_{n=1}^{\infty} Q_n)x = x(\sum_{n=1}^{\infty} Q_n) = x$, and

(d) $\sum_{1 \leqslant m,n < \infty} \|Q_m x Q_n\| < \infty$.

Sketch of proof of Claim The proof is exactly the same as that of Corollary 2.3, except that for all $j \geqslant 1$, the projection $1_{\mathscr{M}(\mathscr{B})} \otimes e_{j,j}$ (notation as in the proof of Corollary 2.3) is replaced with r_j (notation as in this proof). Moreover, for all $n \geqslant 1$, $P_{\mathscr{F}_n}$ (notation as in the proof of Corollary 2.3) will be replaced with Q_m (notation as in this proof) for some $m \geqslant 1$. Note that this means that each Q_m will be a strict sum of infinitely many r_js.
End of proof of the Claim.

From the Claim and from Lemma 2.2, it follows that x is a commutator of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$. □

6. The stably finite case: Part II

In this section, we will assume that $\mathscr{A}$ is a unital separable simple C*-algebra with stable rank one, unique tracial state, and in class $\mathfrak{R}$. As a consequence, $\mathscr{I}_{min}$ (defined in the previous section) is the unique C*-ideal of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ that sits properly between $\mathscr{A} \otimes \mathscr{K}$ and $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$; i.e., $\mathscr{I}_{min}$ is the unique C*-ideal for which the inclusions $\mathscr{A} \otimes \mathscr{K} \subset \mathscr{I}_{min} \subset \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ are proper. Moreover, if τ is the unique tracial state of $\mathscr{A}$ then $\mathscr{I}_{min} = \overline{\{X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K}) : \tau(X^*X) < \infty\}}$. (E.g., see [29]; see also [36] Proposition 2.9 or [35] Proposition 3.6.)

For the rest of the paper, we let $\Gamma_{\mathscr{I}_{min}} : \mathscr{M}(\mathscr{A} \otimes \mathscr{K}) \to \mathscr{M}(\mathscr{A} \otimes \mathscr{K})/\mathscr{I}_{min}$ be the natural quotient map.

LEMMA 6.1. *Let $\mathscr{A}$ be a unital separable simple C*-algebra with stable rank one, unique tracial state τ, and in class $\mathfrak{R}$. Suppose that $X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ is such that $\Gamma_{min}(X)$ is not a scalar multiple of the identity.*

Then there exists an $\alpha > 0$ with $\alpha < \|X\|^2$, where for every $\varepsilon > 0$, for every finite subset $\mathscr{F} \subset \mathscr{I}_{min}$, there exist projections $p, q \in \mathscr{I}_{min} - (\mathscr{A} \otimes \mathscr{K})$ such that

1. $\tau(p), \tau(q) \geqslant 10$,
2. $p \perp q$,
3. *pa, ap, qa and ap are all within ε of 0, for all $a \in \mathscr{F}$, and*
4. *if $x =_{df} qXp$ then there exists $\beta \geqslant \alpha$ with $\beta \leqslant \|X\|^2$ and $\|x^*x - \beta p\|, \|xx^* - \beta q\| < \varepsilon$.*

Proof. The proof is similar to but more complicated than that of Lemma 4.1.

Note that $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})/\mathscr{I}_{min}$ is simple purely infinite and hence has real rank zero (see, for example, [16]). Hence, since $\Gamma_{min}(X)$ is not a scalar multiple of the identity, there exist nonzero orthogonal projections $R, S \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})/\mathscr{I}_{min}$ such that $R\Gamma_{min}(X)S \neq 0$.

Lift R, S to orthogonal positive elements $A, B \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ with norm one (i.e., $A \perp B$, $\Gamma_{min}(A) = R$, $\Gamma_{min}(B) = S$ and $\|A\| = \|B\| = 1$).

Let $\gamma =_{df} \|R\Gamma_{min}(X)S\|^2 > 0$.

Let $\delta_1 =_{df} \min\{1/100, \gamma/100\}$.

Let $h_0 : [0, 2\|X\|^2 + 10] \to [0, 1]$ be the unique continuous function that is given by

$$h_0(s) \begin{cases} = 1 & s = \gamma \\ = 0 & s \in [0, \gamma - \delta_1] \cup [\gamma + \delta_1, 2\|X\|^2 + 10] \\ \text{is linear} & \text{on } [\gamma - \delta_1, \gamma] \\ \text{is linear} & \text{on } [\gamma, \gamma + \delta_1]. \end{cases}$$

Note that from the definitions of γ and h_0, $h_0(AXB^2X^*A)$ is a full positive element of $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$. Hence, since $\mathscr{A} \otimes \mathscr{K}$ has the corona factorization property,

$1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \preceq h_0(AXB^2X^*A)$ (e.g., see [24] Proposition 2.5). Hence, there exists a projection $Q \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that $Q \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$ and $Q \in \mathrm{Her}(h_0(AXB^2X^*A))$. It follows, from the definition of h_0, that $QAXB^2X^*AQ$ is within a distance $\min\{1/100, \gamma/100\}$ of γQ. Hence, $QAXB^2X^*AQ$ is a full positive element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.

But since $Q \in \mathrm{Her}(A)$ and $\Gamma_{min}(A) = R$ is a projection, $\Gamma_{min}(Q) \leqslant R = \Gamma_{min}(A)$. Hence, $\Gamma_{min}(QXB^2X^*Q) = \Gamma_{min}(QAXB^2X^*AQ)$. Hence, QXB^2X^*Q is a full positive element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Hence, BX^*QXB is a full positive element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.

Let $\gamma' =_{df} \|\Gamma_{min}(BX^*QXB)\| > 0$. Let $\delta_2 =_{df} \min\{1/100, \gamma'/100\}$. Let $h_1 : [0, 2\|X\|^2 + 10] \to [0,1]$ be the unique continuous function such that

$$h_1(s) \begin{cases} = 1 & s = \gamma' \\ = 0 & s \in [0, \gamma' - \delta_2] \cup [\gamma + \delta_2, 2\|X\|^2 + 10] \\ \text{is linear} & \text{on } [\gamma - \delta_2, \gamma] \\ \text{is linear} & \text{on } [\gamma, \gamma + \delta_2]. \end{cases}$$

By the definitions of γ' and h_1, $h_1(BX^*QXB)$ is a full positive element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Hence, since $\mathcal{A}\otimes\mathcal{K}$ has the corona factorization property, $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \preceq h_1(BX^*QXB)$. Hence, there exists a projection $P \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that $P \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$ and $P \in \mathrm{Her}(h_1(BX^*QXB))$. It follows, from the definition of h_1, that PBX^*QXBP is within a distance $\min\{1/100, \gamma'/100\}$ of $\gamma' P$. Hence, PBX^*QXBP is a full positive element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Since $P \in \mathrm{Her}(B)$ and $\Gamma_{min}(B) = S$ is a projection, $\Gamma_{min}(P) \leqslant S = \Gamma_{min}(B)$. Hence, $\Gamma_{min}(PX^*QXP) = \Gamma_{min}(PBX^*QXBP)$. Hence, PX^*QXP is a full positive element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.

Hence, $P, Q \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ are full projections (each MvN equivalent to the unit) with $P \perp Q$ such that QXP is a full element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.

Let $\alpha =_{df} (1/2)\|\Gamma_{min}(QXP)\|^2 > 0$.

Let $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset \mathcal{I}_{min}$ be given. Contracting ε if necessary, we may assume that all elements of $\mathcal{F}$ have norm less than or equal to one.

By Lemma 5.8, there exists projections $e \in P\mathcal{I}_{min}P - (\mathcal{A}\otimes\mathcal{K})$ and $f \in Q\mathcal{I}_{min}Q - (\mathcal{A}\otimes\mathcal{K})$ such that ea, ae, eae, fa, af, faf are within $\varepsilon/100$ of Pa, aP, PaP, Qa, aQ, QaQ respectively, for all $a \in \mathcal{F}$.

Let $Q' \in Q\mathcal{M}(\mathcal{A}\otimes\mathcal{K})Q$ and $P' \in P\mathcal{M}(\mathcal{A}\otimes\mathcal{K})P$ be projections that are given by $Q' =_{df} Q - f$ and $P' =_{df} P - e$. Note that Q', P' are both full projections in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ and $Q'XP'$ is a full element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ with $\|\Gamma_{min}(Q'XP')\|^2 = 2\alpha > 0$. Also, $Q' \perp P'$.

Choose a number $\delta_3 > 0$ so that for any C*-algebra $\mathcal{C}$, for all $z \in \mathcal{C}$ and for every projection $r \in \mathcal{C}$, if zz^* is within δ_1 of r then z^*rz is within $\varepsilon/(100(2\alpha+1))$ of z^*z. Contracting $\delta_3 > 0$ if necessary, we may assume that $\delta_3 < \varepsilon/100$.

Choose $\delta_4 > 0$ so that for any C*-algebra $\mathcal{C}$, for all $z \in \mathcal{C}$, if z^*z is within δ_4 of a projection then zz^* is within $\delta_3/(100(2\alpha+1))$ of a projection in $\mathrm{Her}(zz^*)$. Contracting $\delta_4 > 0$ if necessary, we may assume that $\delta_4 < \varepsilon/100$.

Let $h_2 : [0, \|X\|^2+10] \to [0,1]$ be the unique continuous function satisfying:

$$h_2(s) \begin{cases} = 1 & s \in \left[2\alpha - \frac{\delta_4 \alpha}{1000(\alpha+1)(\|X\|^2+1)}, 2\alpha + \frac{\delta_4 \alpha}{1000(\alpha+1)(\|X\|^2+1)}\right] \\ = 0 & s \in \left[0, 2\alpha - \frac{\delta_4 \alpha}{100(\alpha+1)(\|X\|^2+1)}\right] \cup \left[2\alpha + \frac{\delta_4 \alpha}{100(\alpha+1)(\|X\|^2+1)}, \|X\|^2+10\right] \\ \text{linear} & \text{on } \left[2\alpha - \frac{\delta_4 \alpha}{100(\alpha+1)(\|X\|^2+1)}, 2\alpha - \frac{\delta_4 \alpha}{1000(\alpha+1)(\|X\|^2+1)}\right] \\ \text{linear} & \text{on } \left[2\alpha + \frac{\delta_4 \alpha}{1000(\alpha+1)(\|X\|^2+1)}, 2\alpha + \frac{\delta_4 \alpha}{100(\alpha+1)(\|X\|^2+1)}\right]. \end{cases}$$

Hence, by the definitions of h_2 and α, $\|\Gamma_{min}(h_2(P'X^*Q'XP'))\| = 1$. Hence, by Lemma 5.8, let $p \in \mathrm{Her}_{\mathscr{I}_{min}}(h_2(P'X^*Q'XP'))$ be a nonzero projection such that $\tau(p) \geqslant 15$ and $\tau(pX^*Q'Xp) = \tau(pP'X^*Q'XP'p) \geqslant 15$.

Hence, $p \leqslant P'$ and $pX^*Q'Xp$ is within $\frac{\delta_4 \alpha}{100(\alpha+1)}$ of $2\alpha p$.

Hence, $(1/(2\alpha))pX^*Q'Xp$ is within $\frac{\delta_4}{200(\alpha+1)}$ of the projection p.

Hence, by our choice of δ_4, $(1/(2\alpha))Q'XpX^*Q'$ is within $\frac{\delta_3}{100(2\alpha+1)}$ of a projection, say $q \in \mathrm{Her}_{\mathscr{I}_{min}}(Q') =_{df} Q'\mathscr{I}_{min}Q'$. So $qXpX^*q$ is within $\varepsilon/100$ of $2\alpha q$.

Also, by our choice of δ_3, $(1/(2\alpha))pX^*qXp$ is wthin $\varepsilon/(100(2\alpha+1))$ of $(1/(2\alpha))pX^*Q'Xp$. Hence, pX^*qXp is within ε of $2\alpha p$.

Taking $\beta =_{df} 2\alpha$, we are done. □

LEMMA 6.2. *Let $\mathscr{A}$ be a unital separable simple C*-algebra with stable rank one, unique tracial state, and in class $\mathfrak{R}$.*

Suppose that $X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ is such that $\Gamma_{min}(X)$ is not a scalar multiple of the identity.

Then for every $\varepsilon > 0$, there exist projections $P, Q, R, S \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ such that

1. *$P \sim Q \sim S \sim 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$,*
2. *$S \perp P \perp Q \perp S$,*
3. *PX^*QXP is invertible in $P\mathscr{M}(\mathscr{A} \otimes \mathscr{K})P$*
4. *$QXPX^*Q$ is invertible in $Q\mathscr{M}(\mathscr{A} \otimes \mathscr{K})Q$,*
5. *0 is an isolated point of the spectrum of XPX^*,*
6. *R is the left support projection of XP,*
7. *$\|PR\| < 1$,*
8. *$\|SR\| < \varepsilon$, and*
9. *$\Gamma(S)\Gamma(R) = 0$.*
10. *if $Q' \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ is a projection such that $Q' \sim 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$ and $Q' \perp P$, and if $V \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ is a partial isometry such that $V^*V = Q'$ and $VV^* = R$ then $(P+V)^*(P+V)$ is an invertible element of $(P+Q')\mathscr{M}(\mathscr{A} \otimes \mathscr{K})(P+Q')$.*

Proof. The proof is exactly the same as that of Lemma 4.3, except that we use Lemma 6.1 in place of Lemma 4.1. □

THEOREM 6.3. *Let $\mathscr{A}$ be a unital separable simple C*-algebra with stable rank one and unique tracial state, such that every quasitrace is a trace, and $\mathscr{A} \otimes \mathscr{K}$ has strict comparison of positive elements.*

Let $X \in \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$.

Then X is a commutator if and only if X does not have the form $\alpha 1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})} + x$ where $\alpha \in \mathbb{C} - \{0\}$ and $x \in \mathscr{I}_{min}$.

Proof. The proof is exactly the same as that of Theorem 4.5, except that Lemma 4.3 and Corollary 2.3 is replaced with Lemma 6.2 and Theorem 5.10 respectively. Also, the map Γ is replaced with Γ_{min}. □

REFERENCES

[1] B. BLACKADAR, L. ROBERT, A. TIKUISIS, A. S. TOMS AND W. WINTER, *An algebraic approach to the radius of comparison*, Trans. Amer. Math. Soc. **364**, 7 (2012), 3657–3674.

[2] A. BROWN, P. R. HALMOS AND C. PEARCY, *Commutators of operators on a Hilbert space*, Canad. J. Math. **17** (1965), 695–708.

[3] A. BROWN AND C. PEARCY, *Structure of commutators of operators*, Annals of Math. Second Series, **82**, 1 (1965), 112–127.

[4] A. BROWN AND C. PEARCY, *Commutators in factors of type III*, Canad. J. Math. **18** (1965), 1152–1160.

[5] J. CUNTZ AND G. K. PEDERSEN, *Equivalence and traces on C*-algebras*, J. Funct. Anal. **33**, 2 (1979), 135–164.

[6] D. DOSEV, W. B. JOHNSON AND G. SCHECHTMAN, *Commutators on L_p, $1 \leqslant p < \infty$*, J. Amer. Math. Soc. **26**, 1 (2013), 101–127.

[7] K. DYKEMA AND A. SKRIPKA, *On single commutators in II_1 factors*, Proc. Amer. Math. Soc. **140**, 3 (2012), 931–940.

[8] T. FACK, *Finite sums of commutators in C*-algebras*, Annales de l'Institut Fourier **32**, 1 (1982), 129–137.

[9] T. FACK AND P. DE LA HARPE, *Sommes de commutateurs dans les algebres de von Neumann finies continues*, Ann. Inst. Fourier (Grenoble) **30** (1980) 49–73.

[10] H. HALPERN, *Commutators in properly infinite von Neumann algebras*, Trans. Amer. Math. Soc. **139** (1969), 55–73.

[11] P. DE LA HARPE AND G. SKANDALIS, *Produits finis de commutateurs dans les C*-algebres*, Ann. Inst. Fourier (Grenoble) **34**, 4 (1984), 169–202.

[12] D. A. HERRERO, *Approximation of Hilbert space operators I.*, volume 224 of Pitman Research Notes in Math., Longman Scientific and Technical, Harlow, New York, second edition, 1989.

[13] J. HJELMBORG AND M. RORDAM, *On stability of C*-algebras*, J. Funct. Anal. **155**, 1 (1998), 153–171.

[14] V. KAFTAL, P. W. NG AND S. ZHANG, *Commutators and linear spans of projections in certain finite C*-algebras*, J. Funct. Anal. **266**, 4 (2014), 1883–1912.

[15] D. KUCEROVSKY AND P. W. NG, *The corona factorization property and approximate unitary equivalence*, Houston J. Math. **32**, 2 (2006), 531–550.

[16] D. KUCEROVSKY, P. W. NG AND F. PERERA, *Purely infinite corona algebras of simple C*-algebras*, Math. Ann. **346**, 1 (2010), 23–40.

[17] H. LIN, *Simple C*-algebras with continuous scales and simple corona algebras*, Proceedings of the American Mathematical Society, **112**, 3 (1991), 871–880.

[18] H. LIN, *Asymptotic unitary equivalence and classification of simple amenable C*-algebras*, Invent. Math. **183**, 2 (2011), 385–450. A copy is available at `http://arxiv.org/pdf/0806.0636`.

[19] H. LIN AND P. W. NG, *The corona algebra of stabilized Jiang–Su algebra*, Preprint. A copy can be found at `http://arxiv.org/pdf/1302.4135`.
[20] G. LUMER AND M. ROSENBLUM, *Linear operator equations*, Proc. Amer. Math. Soc. **10** (1959), 32–41.
[21] L. W. MARCOUX, *Linear spans of projections in certain C*-algebras*, Indiana Univ. Math. J. **51** (2002), 753–771.
[22] L. W. MARCOUX, *Sums of small numbers of commutators*, J. Operator Theory **56**, 1 (2006), 111–142.
[23] P. W. NG, *Commutators in* $C_r^*(\mathbb{F}_\infty)$, Houston J. Math, **40**, 2 (2014), 421–446.
[24] P. W. NG AND W. WINTER, *Nuclear dimension and the corona factorization property*, Int. Math. Res. Not. IMRN (2010), no. 2, 261–278.
[25] C. PEARCY, *On commutators of operators on Hilbert space*, Proc. Amer. Math. Soc. **10** (1965), 53–59.
[26] G. K. PEDERSEN, *Unitary extensions and polar decompositions in a C*-algebra*, J. Operator Theory **17** (1987), 357–364.
[27] C. POP, *Finite sums of commutators*, Proc. Amer. Math. Soc., **130**, 10 (2002), 3039–3041.
[28] L. ROBERT, *Nuclear dimension and sums of commutators*.
Indiana Univ. Math. J. (to appear).
[29] M. RORDAM, *Ideals in the multiplier algebra of a stable C*-algebra*, J. Operator Theory **25**, 2 (1991), 283–298.
[30] M. RORDAM, *Stable C*-algebras*, Advanced Studies in Pure Mathematics **38** "Operator Algebras and Applications". Edited by Hideki Kosaki. (2004), 177–199.
[31] K. SHODA, *Einige Satze uber Matrizen*, Japanese J. Math. **13** (1936), 361–365.
[32] K. THOMSEN, *Finite sums and products of commutators in inductive limit C*-algebras*, Ann. Inst. Fourier, Grenoble, **43**, 1 (1993), 225–249.
[33] N. E. WEGGE-OLSEN, *K-theory and C*-algebras. A friendly approach*, Oxford University Press, New York, 1993.
[34] S. ZHANG, *A property of purely infinite simple C*-algebras*, Proc. Amer. Math. Soc. **109**, 3 (1990), 717–720.
[35] S. ZHANG, *A riesz decomposition property and ideal structure of multiplier algebras*, J. Operator Theory **24**, 2 (1990), 209–225.
[36] S. ZHANG, *Certain C*-algebras with real rank zero and their corona a multiplier algebrs. Part I*, Pacific Journal of Mathematics **155**, 1 (1992), 169–197.

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