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ALGEBRAIC PROPERTIES OF THE SET OF OPERATORS WITH 0 IN THE CLOSURE OF THE NUMERICAL RANGE

CRISTINA DIOGO

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Abstract. Sets of operators which have zero in the closure of the numerical range are studied. For some particular sets $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$, we characterize the set of all operators $A \in \mathscr{B}(\mathscr{H})$ such that $0 \in \overline{W(TA)}$ for every $T \in \mathscr{T}$.

1. Introduction and preliminaries

Let $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators on a separable complex Hilbert space $\mathscr{H}$ and $\mathscr{S}_{\mathscr{H}} = \{x \in \mathscr{H};\ \|x\| = 1\}$ be the unit sphere of $\mathscr{H}$. The numerical range of $A \in \mathscr{B}(\mathscr{H})$ is defined by

$$W(A) = \{\langle Ax, x\rangle;\ x \in \mathscr{S}_{\mathscr{H}}\}.$$

It is well known that $W(A)$ is a convex subset of the complex plane $\mathbb{C}$ (Toeplitz-Hausdorff Theorem) which contains in its closure the convex hull of the spectrum $\sigma(A)$, i.e., $\operatorname{conv}(\sigma(A)) \subseteq \overline{W(A)}$. If A is normal, then $\operatorname{conv}(\sigma(A)) = \overline{W(A)}$. For an arbitrary operator A, $\operatorname{conv}(\sigma(A))$ is the intersection of the closures of numerical ranges of all operators which are similar to A (Hildebrandt's Theorem). This and other properties of the numerical range can be found, for instance, in [5, 6, 8]. To determine the numerical range of an arbitrary operator is a difficult task. However, there are some classes of operators for which a complete description of $W(A)$ is known (see [7] and references cited therein). For instance, if $\mathscr{H}$ is a two-dimensional space, then each operator A can be represented by a matrix of the form $\begin{bmatrix} \lambda & \omega \\ 0 & \mu \end{bmatrix}$ with respect to a suitable orthonormal basis. By the Elliptic Range Theorem (see [5]) we have that $W(A)$ is the elliptical disc with foci at the eigenvalues λ, μ and with semiaxes $\frac{1}{2}|\omega|$ and $\frac{1}{2}\sqrt{|\omega|^2 + |\lambda - \mu|^2}$. A similar result holds for quadratic operators on any Hilbert space (see [9]). One among the important problems related to the numerical ranges is to find necessary and sufficient conditions on an operator A such that $0 \in \overline{W(A)}$. This problem has been addressed by many authors (see, for instance, [1, 4]) and in this paper we are

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also concerned with it. We study the set of all operators which have 0 in the closure of the numerical range, i.e.,

$$\mathscr{W}_{\{0\}} = \{A \in \mathscr{B}(\mathscr{H});\ 0 \in \overline{W(A)}\}.$$

It is obvious that this is a proper non-empty subset of $\mathscr{B}(\mathscr{H})$. We will use the following notation: for $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$, let $\mathscr{A}^* = \{A^*;\ A \in \mathscr{A}\}$. It is easy to see that $\mathscr{W}_{\{0\}}$ is selfadjoint in the sense that $\mathscr{W}_{\{0\}}^* = \mathscr{W}_{\{0\}}$. Moreover, from [3, Theorem 3.6] it follows easily that if $T \in \mathscr{B}(\mathscr{H})$ is an invertible operator, then $T \in \mathscr{W}_{\{0\}}$ if and only if $T^{-1} \in \mathscr{W}_{\{0\}}$.

In [2], it was shown that $\mathscr{W}_{\{0\}}$ is not closed under addition and multiplication. Let $\mathscr{B}_L \subseteq \mathscr{B}(\mathscr{H})$ be the set of all operators which are not left invertible and $\mathscr{B}_R \subseteq \mathscr{B}(\mathscr{H})$ be the set of all operators which are not right invertible. If $A \in \mathscr{B}_L$, then $TA \in \mathscr{B}_L$ for any $T \in \mathscr{B}(\mathscr{H})$, which gives $\mathscr{B}(\mathscr{H})\mathscr{B}_L \subseteq \mathscr{W}_{\{0\}}$. Similarly, $\mathscr{B}_R\mathscr{B}(\mathscr{H}) \subseteq \mathscr{W}_{\{0\}}$. Taking this into account, it is natural to consider an algebraic structure in $\mathscr{W}_{\{0\}}$ which can be described in the following way. Let $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ be a non-empty set of operators. It is easily seen that

$$\mathfrak{Q}_{\mathscr{T}} = \{A \in \mathscr{B}(\mathscr{H});\quad 0 \in \overline{W(TA)} \text{ for every } T \in \mathscr{T}\}$$

is the largest set of operators such that $\mathscr{T}\mathfrak{Q}_{\mathscr{T}} \subseteq \mathscr{W}_{\{0\}}$. Analogously,

$$\mathfrak{R}_{\mathscr{T}} = \{A \in \mathscr{B}(\mathscr{H});\quad 0 \in \overline{W(AT)} \text{ for every } T \in \mathscr{T}\}$$

is the largest set of operators such that $\mathfrak{R}_{\mathscr{T}}\mathscr{T} \subseteq \mathscr{W}_{\{0\}}$. Let $\mathscr{B}_0 = \mathscr{B}_L \cup \mathscr{B}_R$ be the set of all non-invertible operators. For a non-empty set $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$, we define $\mathscr{Q}_{\mathscr{T}} = \mathfrak{Q}_{\mathscr{T}} \backslash \mathscr{B}_0$ and, similarly, $\mathscr{R}_{\mathscr{T}} = \mathfrak{R}_{\mathscr{T}} \backslash \mathscr{B}_0$. The next proposition follows easily from [2, Proposition 2.6].

PROPOSITION 1.1. *Let $\mathscr{T}$, $\mathscr{T}_1$, and $\mathscr{T}_2$ be arbitrary non-empty subsets of $\mathscr{B}(\mathscr{H})$. Then*

(i) $(\mathscr{Q}_{\mathscr{T}})^* = \mathscr{R}_{\mathscr{T}^*}$;

(ii) if $I \in \mathscr{T}$, then $\mathscr{Q}_{\mathscr{T}} \subseteq \mathscr{W}_{\{0\}}$;

(iii) if $\mathscr{T}_1 \subseteq \mathscr{T}_2$, then $\mathscr{Q}_{\mathscr{T}_1} \supseteq \mathscr{Q}_{\mathscr{T}_2}$.

According to this result, it is enough to consider sets $\mathscr{Q}_{\mathscr{T}}$ because the properties of $\mathscr{R}_{\mathscr{T}}$ are similar. The algebraic properties of $\mathscr{Q}_{\mathscr{T}}$, for an arbitrary $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$, are studied in Section 2. In Section 3, we characterize $\mathscr{Q}_{\mathscr{T}}$ for some particular sets $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$. Namely, when $\mathscr{T} = \mathscr{W}_{\{0\}}$, some properties of $\mathscr{Q}_{\mathscr{W}_{\{0\}}}$ are studied and it is also shown that if $\mathscr{H}$ is finite dimensional, then $\mathscr{Q}_{\mathscr{W}_{\{0\}}}$ contains only non-zero scalar multiples of the identity matrix. In the end of the section, we are concerned with $\mathscr{Q}_{\mathscr{S}}$, where $\mathscr{S}$ is the set of all selfadjoint operators.

2. Properties of $\mathscr{Q}_{\mathscr{T}}$

Let $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ be a non-empty set. Denote $\mathbb{C}\mathscr{T} = \{\lambda T; \lambda \in \mathbb{C}, T \in \mathscr{T}\}$. It is easily seen that $\mathscr{Q}_{\mathscr{T}} = \mathscr{Q}_{\mathbb{C}\mathscr{T}}$ and also that $\mathscr{Q}_{\mathscr{T}} = \mathbb{C}\mathscr{Q}_{\mathscr{T}} \setminus \{0\}$.

PROPOSITION 2.1. *If $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ is an arbitrary non-empty subset, then $\mathscr{Q}_{\mathscr{T}} = \mathscr{Q}_{\overline{\mathscr{T}}}$.*

Proof. It is obvious that $\mathscr{Q}_{\overline{\mathscr{T}}} \subseteq \mathscr{Q}_{\mathscr{T}}$, so we are left to prove the opposite inclusion. Let $A \in \mathscr{Q}_{\mathscr{T}}$ and $T \in \overline{\mathscr{T}}$. Let $(T_n)_{n=1}^{\infty} \subseteq \mathscr{T}$ be a sequence whose limit is T. Then, for $\varepsilon > 0$, there exists n_ε such that $\|T_n - T\| < \varepsilon$ for every $n \geqslant n_\varepsilon$. Since $A \in \mathscr{Q}_{\mathscr{T}}$, we have $0 \in \overline{W(T_nA)}$ for each index n. On the other hand,

$$\begin{aligned}\overline{W(T_nA)} = \overline{W((T_n-T)A + TA)} &\subseteq \overline{W((T_n-T)A)} + \overline{W(TA)} \\ &\subseteq \overline{\mathbb{D}(0, \|(T_n-T)A\|)} + \overline{W(TA)} \subseteq \overline{\mathbb{D}(0, \varepsilon\|A\|)} + \overline{W(TA)},\end{aligned}$$

which means that $\overline{W(T_nA)}$ is in the $\varepsilon\|A\|$-hull of $\overline{W(TA)}$ if $n \geqslant n_\varepsilon$. Since ε is arbitrarily small, we conclude that $0 \in \overline{W(TA)}$, i.e., $A \in \mathscr{Q}_{\overline{\mathscr{T}}}$. □

By a similar reasoning it can be shown that $\mathfrak{Q}_{\mathscr{T}}$ is a closed subset of $\mathscr{B}(\mathscr{H})$.

PROPOSITION 2.2. *Let $\{\mathscr{T}_i;\ i \in \mathbb{I}\}$ be an arbitrary family of subsets of $\mathscr{B}(\mathscr{H})$. Then*

(i) $\bigcap_{i\in\mathbb{I}} \mathscr{Q}_{\mathscr{T}_i} = \mathscr{Q}_{\cup_i \mathscr{T}_i}$ *and*

(ii) $\bigcup_{i\in\mathbb{I}} \mathscr{Q}_{\mathscr{T}_i} \subseteq \mathscr{Q}_{\cap_i \mathscr{T}_i}$.

Proof. (i) Let $A \in \bigcap_{i\in\mathbb{I}} \mathscr{Q}_{\mathscr{T}_i}$. If $T \in \mathscr{T}_i$, for some $i \in \mathbb{I}$, then $0 \in \overline{W(TA)}$. Hence, $0 \in \overline{W(TA)}$ for every $T \in \cup_i \mathscr{T}_i$ and therefore $A \in \mathscr{Q}_{\cup_i \mathscr{T}_i}$. Now, for the opposite inclusion, since $\mathscr{T}_i \subseteq \cup_i \mathscr{T}_i$ for any $i \in \mathbb{I}$, we have $\mathscr{Q}_{\cup_i \mathscr{T}_i} \subseteq \mathscr{Q}_{\mathscr{T}_i}$ and therefore $\mathscr{Q}_{\cup_i \mathscr{T}_i} \subseteq \bigcap_{i\in\mathbb{I}} \mathscr{Q}_{\mathscr{T}_i}$.

(ii) Since $\cap_i \mathscr{T}_i \subseteq \mathscr{T}_i$ for any $i \in \mathbb{I}$, one has $\mathscr{Q}_{\mathscr{T}_i} \subseteq \mathscr{Q}_{\cap_i \mathscr{T}_i}$. Hence, $\bigcup_{i\in\mathbb{I}} \mathscr{Q}_{\mathscr{T}_i} \subseteq \mathscr{Q}_{\cap_i \mathscr{T}_i}$. □

It can be shown by an example that the inclusion in (ii) is strict.

EXAMPLE 2.3. Let $\mathscr{T}_1 = \{I, N_1\}$ and $\mathscr{T}_2 = \{I, N_2\}$, where $N_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right]$ and $N_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -i \end{smallmatrix}\right]$. Since $\mathscr{T}_1 \cap \mathscr{T}_2 = \{I\}$, we have $\mathscr{Q}_{\mathscr{T}_1 \cap \mathscr{T}_2} = \mathscr{W}_{\{0\}} \backslash \mathscr{B}_0$. Taking $D = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]$ we have, by the Elliptic Range Theorem, that $W(D) = [-i, i]$ and therefore $0 \in W(D)$. Hence, $D \in \mathscr{Q}_{\mathscr{T}_1 \cap \mathscr{T}_2}$. On the other hand, $W(N_1 D) = [i, 1]$ and $W(N_2 D) = [i, -1]$ which means that $D \notin \mathscr{Q}_{\mathscr{T}_1} \cup \mathscr{Q}_{\mathscr{T}_2}$.

Let $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ be an arbitrary non-empty set. Denote by $\tau = \{\mathscr{T}_i;\ i \in \mathbb{I}\}$ the family of all subsets $\mathscr{T}_i \subseteq \mathscr{B}(\mathscr{H})$ such that $\mathscr{Q}_{\mathscr{T}} \subseteq \mathscr{Q}_{\mathscr{T}_i}$. It is easy to see that

$$\widehat{\mathscr{T}} := \overline{\cup_{i\in\mathbb{I}} \mathscr{T}_i} \tag{2.1}$$

is the largest set in τ. Namely, since $\mathscr{Q}_{\mathscr{T}} \subseteq \mathscr{Q}_{\mathscr{T}_i}$, we have, by Proposition 2.2, that $\mathscr{Q}_{\mathscr{T}} \subseteq \bigcap_{i\in\mathbb{I}} \mathscr{Q}_{\mathscr{T}_i} = \mathscr{Q}_{\cup_i \mathscr{T}_i}$. Hence, $\mathscr{Q}_{\mathscr{T}} \subseteq \mathscr{Q}_{\widehat{\mathscr{T}}}$. Because of $\mathscr{T} \subseteq \widehat{\mathscr{T}}$ we also have the other inclusion and we may conclude that for each $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ there exists the largest subset $\widehat{\mathscr{T}} \subseteq \mathscr{B}(\mathscr{H})$, which is given by (2.1), such that $\mathscr{Q}_{\mathscr{T}} = \mathscr{Q}_{\widehat{\mathscr{T}}}$.

For $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$, let $\mathscr{T}_0 = \mathscr{T} \cap \mathscr{B}_0$ and $\mathscr{T}_{inv} = \mathscr{T} \setminus \mathscr{T}_0 = \{T \in \mathscr{T};\ T \text{ is invertible}\}$. Since $\mathscr{T} = \mathscr{T}_0 \cup \mathscr{T}_{inv}$, it follows, by Proposition 2.2, that $\mathscr{Q}_{\mathscr{T}} = \mathscr{Q}_{\mathscr{T}_0} \cap \mathscr{Q}_{\mathscr{T}_{inv}}$. Therefore, it is enough to consider only $\mathscr{Q}_{\mathscr{T}_{inv}}$ as $\mathscr{Q}_{\mathscr{T}_0}$ consists of all invertible operators in $\mathscr{B}(\mathscr{H})$.

Let $\mathscr{T}$ be a non-empty set of invertible operators and let $\mathscr{T}^{-1} = \{T^{-1};\ T \in \mathscr{T}\}$. Let us now establish the relation between $\mathscr{Q}_{\mathscr{T}^{-1}}$ and $\mathscr{Q}_{\mathscr{T}^*}$.

PROPOSITION 2.4. *Let $\mathscr{T}$ be an arbitrary non-empty set of invertible operators in $\mathscr{B}(\mathscr{H})$. Then $(\mathscr{Q}_{\mathscr{T}^{-1}})^* = (\mathscr{Q}_{\mathscr{T}^*})^{-1}$.*

Proof. If $A \notin \mathscr{Q}_{\mathscr{T}}$, then there exists $T \in \mathscr{T}$ such that $TA \notin \mathscr{W}_{\{0\}}$. It follows that $A^{-1}T^{-1} \notin \mathscr{W}_{\{0\}}$. Hence we have that $A^{-1} \notin \mathscr{R}_{\mathscr{T}^{-1}}$, which is equivalent to $A^{-1} \notin \left(\mathscr{Q}_{(\mathscr{T}^{-1})^*}\right)^*$ by Proposition 1.1. We conclude that $A \in \mathscr{Q}_{\mathscr{T}}$ if $A^{-1} \in \left(\mathscr{Q}_{(\mathscr{T}^{-1})^*}\right)^*$. Equivalently, if $A^* \in \mathscr{Q}_{(\mathscr{T}^{-1})^*}$, then $A^{-1} \in \mathscr{Q}_{\mathscr{T}}$. After interchanging $\mathscr{T}$ and $\mathscr{T}^*$, it follows

$$(\mathscr{Q}_{\mathscr{T}^{-1}})^* \subseteq (\mathscr{Q}_{\mathscr{T}^*})^{-1}. \tag{2.2}$$

Now let $\mathscr{S} = (\mathscr{T}^{-1})^*$. Since (2.2) holds for every set of invertible operators, we have $(\mathscr{Q}_{\mathscr{S}^{-1}})^* \subseteq (\mathscr{Q}_{\mathscr{S}^*})^{-1}$ or, equivalently, $(\mathscr{Q}_{\mathscr{T}^*})^* \subseteq (\mathscr{Q}_{\mathscr{T}^{-1}})^{-1}$, which gives the desired equality. □

Using Proposition 1.1 we can write the last result in the following form.

COROLLARY 2.5. *Let $\mathscr{T}$ be an arbitrary non-empty set of invertible operators in $\mathscr{B}(\mathscr{H})$. Then $(\mathscr{Q}_{\mathscr{T}^{-1}})^{-1} = \mathscr{R}_{\mathscr{T}}$.*

In general, $\mathscr{Q}_{\mathscr{T}} \neq \mathscr{R}_{\mathscr{T}}$. For instance, let $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathscr{T} = \{T\}$. If $A = \begin{bmatrix} a & -a \\ 0 & b \end{bmatrix}$, where $0 < a \leqslant \frac{\sqrt{2}-1}{\sqrt{2}+1} b$, we have $W(AT) = [a,b]$, which means that $0 \notin W(AT)$, i.e. $A \notin \mathscr{R}_{\mathscr{T}}$. On the other hand, by the Elliptical Range Theorem, $W(TA)$ is the elliptical disc with foci at a and b and the major semiaxis $\frac{\sqrt{2}}{2}(b-a)$. Hence, it is easy to check that 0 is inside this elliptical disc so $0 \in W(TA)$, i.e., $A \in \mathscr{Q}_{\mathscr{T}}$. However, as the following proposition shows, $\mathscr{Q}_{\mathscr{T}}$ and $\mathscr{R}_{\mathscr{T}}$ contain the same set of unitary operators.

PROPOSITION 2.6. *Let $\mathscr{T}$ be an arbitrary non-empty set of operators in $\mathscr{B}(\mathscr{H})$. If U is unitary, then $U \in \mathscr{Q}_{\mathscr{T}}$ if and only if $U \in \mathscr{R}_{\mathscr{T}}$.*

Proof. If $U \in \mathscr{Q}_{\mathscr{T}}$, then $0 \in \overline{W(TU)}$ for any $T \in \mathscr{T}$. Since the numerical range is unitarily invariant, one has $W(TU) = W(U^*UTU) = W(UT)$. Therefore $0 \in \overline{W(UT)}$ for any $T \in \mathscr{T}$, which means that $U \in \mathscr{R}_{\mathscr{T}}$. The opposite implication is proved similarly. □

Let $\mathscr{T}$ be a set of invertible operators and $T \in \mathscr{T}$. For every $A \in \mathscr{R}_{\mathscr{T}}$ we have $AT \in \mathscr{W}_{\{0\}}$, which means that $T \in \mathscr{Q}_{\mathscr{R}_{\mathscr{T}}}$. Therefore, we conclude that $\mathscr{T} \subseteq \mathscr{Q}_{\mathscr{R}_{\mathscr{T}}}$. The inclusion $\mathscr{T} \subseteq \mathscr{R}_{\mathscr{Q}_{\mathscr{T}}}$ is obvious, as well. Taking this into account we have the following result.

PROPOSITION 2.7. *Let $\mathscr{T}$ be an arbitrary non-empty subset of invertible operators in $\mathscr{B}(\mathscr{H})$. Then $\mathscr{Q}_{\mathscr{R}_{\mathscr{Q}_{\mathscr{T}}}} = \mathscr{Q}_{\mathscr{T}}$ and $\mathscr{R}_{\mathscr{Q}_{\mathscr{R}_{\mathscr{T}}}} = \mathscr{R}_{\mathscr{T}}$.*

Proof. We will prove only the first equality since the proof of the second one is similar. Since $\mathscr{T} \subseteq \mathscr{Q}_{\mathscr{R}_{\mathscr{T}}}$ for every non-empty set $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ of invertible operators, we have, in particular, that $\mathscr{Q}_{\mathscr{T}} \subseteq \mathscr{Q}_{\mathscr{R}_{\mathscr{Q}_{\mathscr{T}}}}$. On the other hand, taking into account that $\mathscr{T} \subseteq \mathscr{R}_{\mathscr{Q}_{\mathscr{T}}}$ and Proposition 1.1, we have the opposite inclusion. □

COROLLARY 2.8. *Let $\mathscr{T}$ be a non-empty subset of invertible operators in $\mathscr{B}(\mathscr{H})$. Then $\mathscr{T} = \mathscr{Q}_{\mathscr{R}_{\mathscr{T}}}$ if and only if there exists $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ such that $\mathscr{T} = \mathscr{Q}_{\mathscr{S}}$. Similarly, $\mathscr{T} = \mathscr{R}_{\mathscr{Q}_{\mathscr{T}}}$ if and only if there exists $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ such that $\mathscr{T} = \mathscr{R}_{\mathscr{S}}$.*

Proof. If $\mathscr{T} = \mathscr{Q}_{\mathscr{R}_{\mathscr{T}}}$, then $\mathscr{S} = \mathscr{R}_{\mathscr{T}}$. On the other hand, if there exists $\mathscr{S}$ such that $\mathscr{T} = \mathscr{Q}_{\mathscr{S}}$, then, by Proposition 2.7, we have that $\mathscr{T} = \mathscr{Q}_{\mathscr{S}} = \mathscr{Q}_{\mathscr{R}_{\mathscr{Q}_{\mathscr{S}}}} = \mathscr{Q}_{\mathscr{R}_{\mathscr{T}}}$. The second statement can be proved analogously. □

This result raise a question which sets of invertible operators $\mathscr{T}$ can be realized as $\mathscr{Q}_{\mathscr{S}}$ for some $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$. We are concerned with this problem in the following section.

3. $\mathscr{Q}_{\mathscr{T}}$ of some sets $\mathscr{T}$

In this section we obtain descriptions of $\mathscr{Q}_{\mathscr{T}}$ and $\mathfrak{Q}_{\mathscr{T}}$ for some particular sets $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$. It is easily seen that $\mathfrak{Q}_{\mathscr{B}(\mathscr{H})} = \mathscr{B}_L$ (and $\mathfrak{R}_{\mathscr{B}(\mathscr{H})} = \mathscr{B}_R$). Let $\mathscr{P} = \{P \in \mathscr{B}(\mathscr{H});\, P^2 = P = P^*\}$ be the set of all orthogonal projections on $\mathscr{H}$. It is clear that the only invertible element in $\mathscr{P}$ is the identity operator, so $\mathfrak{Q}_{\mathscr{P}} = \mathscr{W}_{\{0\}}$. But, of course, usually the characterization of $\mathfrak{Q}_{\mathscr{T}}$, and consequently of $\mathscr{Q}_{\mathscr{T}}$, is not trivial.

3.1. Positive semidefinite operators

Let $\mathscr{B}_+$ be the set of all positive semidefinite operators on $\mathscr{H}$. In [2], we showed that

$$\mathfrak{Q}_{\mathscr{B}_+} = \{A \in \mathscr{B}(\mathscr{H});\ 0 \in \operatorname{conv}(\sigma(A))\}, \tag{3.1}$$

which gives

$$\mathscr{Q}_{\mathscr{B}_+} = \{A \in \mathscr{B}(\mathscr{H});\ 0 \in \operatorname{conv}(\sigma(A)) \setminus \sigma(A)\}. \tag{3.2}$$

We will use this result to characterize $\mathfrak{Q}_{[0,C]}$, where $[0,C] = \{T \in \mathscr{B}(\mathscr{H});\ 0 \leqslant T \leqslant C\}$, for a given $C \in \mathscr{B}_+$. If C is non-invertible, then each operator T in $[0,C]$ is non-invertible. Namely, if C is not invertible, then $\sqrt{C}$ is also not invertible. Hence 0 is in its approximate point spectrum. Let $(x_n)_{n=1}^{\infty} \subseteq \mathscr{S}(\mathscr{H})$ be a sequence of vectors such that $\|\sqrt{C}x_n\| \to 0$. From

$$\|\sqrt{T}x_n\|^2 = \langle Tx_n, x_n\rangle \leqslant \langle Cx_n, x_n\rangle = \|\sqrt{C}x_n\|^2 \to 0$$

we derive that $0 \in \sigma(T)$ and therefore T is not invertible. Since it is a normal operator, it is left and right non-invertible. Hence, for a non-invertible C, one has $\mathscr{Q}_{[0,C]} = \mathscr{B}(\mathscr{H})\backslash\mathscr{B}_0$. Assume now that C is invertible. Since $[0,C] \subseteq \mathscr{B}_+$, we have $\mathscr{Q}_{[0,C]} \supseteq \mathscr{Q}_{\mathscr{B}_+}$. In fact, these two sets are equal.

THEOREM 3.1. *Let $C \in \mathscr{B}_+$ be invertible. Then $\mathscr{Q}_{[0,C]} = \mathscr{Q}_{\mathscr{B}_+}$.*

Proof. Assume that there exists $A \in \mathscr{Q}_{[0,C]}$ such that $A \notin \mathscr{Q}_{\mathscr{B}_+}$. Therefore, there is a positive and invertible operator $P \in \mathscr{B}_+$ such that $PA \notin \mathscr{W}_{\{0\}}$. Since P and C are positive and invertible operators, we have that $\overline{W(P)} = [c(P), \|P\|]$ and $\overline{W(C)} = [c(C), \|C\|]$, where the Crawford numbers $c(P)$ and $c(C)$ are positive ([3, Theorem 3.6]). Hence, taking $E = \frac{c(C)P}{\|P\|}$ it is easy to see that $E \in [0,C]$ and $EA \notin \mathscr{W}_{\{0\}}$, which is a contradiction since $A \in \mathscr{Q}_{[0,C]}$. $\square$

Now we are able to show that, for a general set $\mathscr{T}$, there is not the smallest set $\check{\mathscr{T}}$ such that $\mathscr{Q}_{\check{\mathscr{T}}} = \mathscr{Q}_{\mathscr{T}}$.

EXAMPLE 3.2. Let $\mathscr{T} = \mathscr{B}_+$. First we show that

$$\mathscr{C} = \bigcap_{\substack{C \in \mathscr{B}_+ \\ C \text{ invertible}}} [0,C]$$

is the singleton containing 0. Assume that there is $A \in \mathscr{C}$ such that $A \neq 0$. Then there is $\lambda \in W(A) \subseteq\ ]0, \|A\|]$, which means that $\lambda = \langle Ax, x\rangle$ for some $x \in \mathscr{S}_{\mathscr{H}}$. Let $0 < \mu < \lambda$. Then $\langle \mu x, x\rangle < \langle Ax, x\rangle$ and therefore $\langle (A - \mu I)x, x\rangle > 0$. Hence $A \notin [0, \mu I]$. This is a contradiction because $A \in \mathscr{C}$.

Assume that $\check{\mathscr{B}}_+$, the smallest set such that $\mathscr{Q}_{\check{\mathscr{B}}_+} = \mathscr{Q}_{\mathscr{B}_+}$, exists. Then, by Theorem 3.1, we would have $\check{\mathscr{B}}_+ \subseteq [0,C]$, for every invertible positive definite C, which would imply that $\check{\mathscr{B}}_+ = \{0\}$. However, $\mathscr{Q}_{\{0\}} = \mathscr{B}(\mathscr{H})\backslash\mathscr{B}_0$. Thus, $\check{\mathscr{B}}_+$ does not exist.

3.2. Unitary and normal operators

Let $\mathscr{U} \subseteq \mathscr{B}(\mathscr{H})$ be the set of all unitary operators and $\mathscr{N} \subseteq \mathscr{B}(\mathscr{H})$ the set of all normal operators.

PROPOSITION 3.3. $\mathfrak{Q}_{\mathscr{U}} = \mathscr{B}_0 = \mathfrak{Q}_{\mathscr{N}}$.

Proof. Since the numerical range is unitarily invariant, one has $\mathfrak{Q}_{\mathscr{U}} = \mathfrak{R}_{\mathscr{U}}$. It follows from $\mathscr{U} \subseteq \mathscr{B}(\mathscr{H})$ that $\mathfrak{Q}_{\mathscr{U}} \supseteq \mathfrak{Q}_{\mathscr{B}(\mathscr{H})} = \mathscr{B}_L$ and $\mathfrak{Q}_{\mathscr{U}} = \mathfrak{R}_{\mathscr{U}} \supseteq \mathfrak{R}_{\mathscr{B}(\mathscr{H})} = \mathscr{B}_R$, which gives $\mathfrak{Q}_{\mathscr{U}} \supseteq \mathscr{B}_0$. On the other hand, if $A \in \mathscr{B}(\mathscr{H})$ is invertible with polar decomposition $A = UP$, where $U \in \mathscr{U}$ and $P > 0$, then $0 \notin \overline{W(P)} = \overline{W(U^*A)}$, i.e., $A \notin \mathfrak{Q}_{\mathscr{U}}$, which proves the other inclusion.

To prove the second equality, let us suppose that there is a normal operator N such that $0 \notin \overline{W(NA)}$. Then 0 is not in $\sigma(NA)$. This means that NA is invertible, and hence N is right invertible. It follows that the normal N is invertible. Thus so is $A = N^{-1}(NA)$, which proves that $\mathfrak{Q}_{\mathscr{U}} \supseteq \mathscr{B}_0$. The reverse containment follows from $\mathfrak{Q}_{\mathscr{N}} \subseteq \mathfrak{Q}_{\mathscr{U}} = \mathscr{B}_0$. □

3.3. Operators with 0 in the closure of the numerical range

In order to characterize $\mathscr{Q}_{\mathscr{W}_{\{0\}}}$, we list some properties of this set of operators.

LEMMA 3.4. *Let $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$ and U be unitary. Then $U^*AU \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$.*

Proof. Let U be unitary. Since $W(T) = W(UTU^*)$ for any $T \in \mathscr{B}(\mathscr{H})$ we have $T \in \mathscr{W}_{\{0\}}$ if and only if $UTU^* \in \mathscr{W}_{\{0\}}$. Hence, if $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$, then $0 \in \overline{W(UTU^*A)}$ for every $T \in \mathscr{W}_{\{0\}}$. This means that $0 \in \overline{W(TU^*AU)}$ for every $T \in \mathscr{W}_{\{0\}}$, and therefore $U^*AU \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. □

PROPOSITION 3.5. *$\mathscr{Q}_{\mathscr{W}_{\{0\}}}$ is a semigroup which contains the identity operator I.*

Proof. It is obvious that $I \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. Suppose that $A, B \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. Let $T \in \mathscr{W}_{\{0\}}$ be arbitrary. Then $TA \in \mathscr{W}_{\{0\}}$. Since $B \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$, one has $0 \in \overline{W(TAB)}$, and we conclude that $AB \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. □

LEMMA 3.6. *If $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$, then $A \notin \mathscr{W}_{\{0\}}$.*

Proof. Let $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. If A were in $\mathscr{W}_{\{0\}}$, then $A^{-1} \in \mathscr{W}_{\{0\}}$ and one would have $0 \in \overline{W(A^{-1}A)} = \{1\}$, which is a contradiction. □

Taking into account that $\left(\mathscr{W}_{\{0\}} \backslash \mathscr{B}_0\right)^{-1} = \mathscr{W}_{\{0\}} \backslash \mathscr{B}_0 = \mathscr{W}^*_{\{0\}} \backslash \mathscr{B}_0$, it follows from Proposition 2.4 and Corollary 2.5 that

$$\left(\mathscr{Q}_{\mathscr{W}_{\{0\}}}\right)^* = \left(\mathscr{Q}_{\mathscr{W}_{\{0\}}}\right)^{-1} = \mathscr{R}_{\mathscr{W}_{\{0\}}}. \tag{3.3}$$

To prove that $\mathscr{Q}_{\mathscr{W}_{\{0\}}}$ is selfadjoint, i.e., $\left(\mathscr{Q}_{\mathscr{W}_{\{0\}}}\right)^* = \mathscr{Q}_{\mathscr{W}_{\{0\}}}$, we need the following lemma.

LEMMA 3.7. *Let $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$ and let $A = UP$ be its polar decomposition. Then*

(i) $P^{-1}U \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$;

(ii) $U^2 \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$ *and* $(U^*)^2 \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$.

Proof. (i) Let $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. By (3.3), $(A^*)^{-1} \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. Since U is unitary and P is positive definite, we have that $(A^*)^{-1} = UP^{-1} \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$ and it follows, by Lemma 3.4, that $P^{-1}U \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$.

(ii) By (i), $P^{-1}U \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. Since $\mathscr{Q}_{\mathscr{W}_{\{0\}}}$ is a semigroup, we have $A(P^{-1}U) = U^2 \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. By (3.3), one has $(U^2)^* \in \mathscr{R}_{\mathscr{W}_{\{0\}}}$ and consequently, by Proposition 2.6, $(U^2)^* \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. $\square$

PROPOSITION 3.8. $\left(\mathscr{Q}_{\mathscr{W}_{\{0\}}}\right)^{-1} = \mathscr{Q}_{\mathscr{W}_{\{0\}}} = \left(\mathscr{Q}_{\mathscr{W}_{\{0\}}}\right)^{*}.$

Proof. Let $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$ and let $A = UP$ be its polar decomposition. Taking into account Proposition 3.5 and Lemma 3.7, we have $A^{-1} = P^{-1}U^* = (P^{-1}U)(U^*)^2 \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. This proves the first equality and the second follows by (3.3). $\square$

By Propositions 3.5 and 3.8, we have that $\mathscr{Q}_{\mathscr{W}_{\{0\}}}$ is a group.

LEMMA 3.9. *Let $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. If $B \in \mathscr{B}(\mathscr{H})$ is such that $B \notin \mathscr{W}_{\{0\}}$, then $AB \notin \mathscr{W}_{\{0\}}$.*

Proof. Let $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$ and $B \in \mathscr{B}(\mathscr{H})$ be such that $0 \notin \overline{W(B)}$. If 0 were in $\overline{W(AB)}$, then one would have $0 \in \overline{W(A^{-1}(AB))} = \overline{W(B)}$ since $A^{-1} \in \mathscr{R}_{\mathscr{W}_{\{0\}}}$ by (3.3). This is a contradiction. $\square$

Now we will characterize $\mathscr{Q}_{\mathscr{W}_{\{0\}}}$ as the set of all non-zero scalar multiples of the identity operator if the underlying space is finite dimensional. We believe that the same result holds also in the infinite dimensional case. We start with a lemma, which holds in any separable complex Hilbert space.

LEMMA 3.10. *If $U \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$ is unitary, then $U = \lambda I$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$.*

Proof. Let $U \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$ be unitary. Since the spectrum $\sigma(U)$ is a subset of the unit circle and U is normal, we have $\overline{W(U)} = \operatorname{conv}(\sigma(U)) \subseteq \overline{\mathbb{D}}$. Assume that there is a number $\mu \in \overline{W(U)}$ such that $|\mu| < 1$. By Lemma 3.6, $\mu \neq 0$. Hence, μ^{-1} exists and $|\mu^{-1}| > 1$. Since $\mu \in \overline{W(U)}$, we have $0 \in \overline{W(U - \mu I)}$, i.e., $U - \mu I \in \mathscr{W}_{\{0\}}$. By Proposition 3.8, $U^{-1} \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$ and therefore $(U - \mu I)U^{-1} \in \mathscr{W}_{\{0\}}$. Since $\mu \neq 0$, it follows $U^{-1} - \mu^{-1} I \in \mathscr{W}_{\{0\}}$, that is, $\mu^{-1} \in \overline{W(U^{-1})}$. However $U^{-1} = U^*$ is unitary and therefore $\overline{W(U^{-1})} \subseteq \overline{\mathbb{D}}$, which is a contradiction. We have proved that $\overline{W(U)}$ does not contain numbers of modulus strictly less than 1. Because of the convexity of $\overline{W(U)}$, we may conclude that $\overline{W(U)} = \{\lambda\}$ for some number λ of modulus 1. Hence $U = \lambda I$. $\square$

PROPOSITION 3.11. *If $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$, then $A = \lambda P$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and P is positive definite.*

Proof. Let $A = UP$ be the polar decomposition of $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. Since A is invertible and P is a positive definite operator, we have $0 \notin \overline{W(P)}$ and therefore $0 \notin \overline{W(P^{-1})}$. Hence, by Lemma 3.9, $0 \notin \overline{W(AP^{-1})} = \overline{W(U)}$.

On the other hand, by Lemma 3.7, $U^2 \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$. Since U^2 is unitary one has, by Lemma 3.10, that $U^2 = \mu I$ for some $\mu \in \mathbb{C}$, $|\mu| = 1$. Let $\lambda \in \mathbb{C}$, $|\lambda| = 1$, be such that $\mu = \lambda^2$. If $U \neq \pm\lambda I$, then λ and $-\lambda$ are in the spectrum $\sigma(U)$ and consequently $0 \in \overline{W(U)}$, which is a contradiction. Hence, either $U = \lambda I$ or $U = -\lambda I$, i.e., $A = \lambda P$ or $A = -\lambda P$. □

LEMMA 3.12. *Let $P = \operatorname{diag}\{1, p_1, \ldots, p_{n-1}\}$ be a non-scalar positive definite matrix with eigenvalues $0 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n = 1$ (which means that $p_1 < 1$). Let $B = \left[\begin{smallmatrix} 1 & \omega \\ 0 & 1 \end{smallmatrix}\right]$, where $2 \leqslant \omega < \frac{2}{\sqrt{p_1}}$. Then $A := B \oplus \operatorname{diag}\{1/p_2, \ldots, 1/p_{n-1}\} \in \mathscr{W}_{\{0\}}$ and $AP \notin \mathscr{W}_{\{0\}}$.*

Proof. Since $\omega \geqslant 2$, one has $0 \in W(B)$, which means that 0 is also in the numerical range of A and therefore $A \in \mathscr{W}_{\{0\}}$. Let $C = \begin{bmatrix} 1 & \omega p_1 \\ 0 & p_1 \end{bmatrix}$. Then $AP = C \oplus I_{n-2}$ and therefore $W(AP) = \operatorname{conv}(W(C) \cup W(I_{n-2}))$. By the Elliptical Range Theorem, $W(C)$ is an elliptical disc with foci at 1 and p_1 and the major axis $\sqrt{\omega^2 p_1^2 + (1-p_1)^2}$. It follows that the inequality describing $W(C)$ is $|z-1| + |z-p_1| \leqslant \sqrt{\omega^2 p_1^2 + (1-p_1)^2}$. It is obvious now that $1 \in W(C)$, i.e., $W(AP) = W(C)$. Since $\omega < \frac{2}{\sqrt{p_1}}$, one has $1 + p_1 > \sqrt{\omega^2 p_1^2 + (1-p_1)^2}$, which means that $0 \notin W(AP)$. □

PROPOSITION 3.13. *If $P \in \mathbb{M}_n$ is a non-scalar positive definite matrix, then $P \notin \mathscr{Q}_{\mathscr{W}_{\{0\}}}$.*

Proof. Let P be a non-scalar positive definite matrix with eigenvalues $0 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$ (which means that $p_1 < p_n$). Then $\frac{1}{p_n}P$ is positive definite with eigenvalues $\frac{p_1}{p_n} \leqslant \frac{p_2}{p_n} \leqslant \cdots \leqslant \frac{p_{n-1}}{p_n} \leqslant 1$. Let $U \in \mathbb{M}_n$ be a unitary matrix such that $U(\frac{1}{p_n}P)U^* = \operatorname{diag}\{1, p_1/p_n, \ldots, p_{n-1}/p_n\}$. By Lemma 3.12, there exists $A \in \mathscr{W}_{\{0\}}$ such that $0 \notin W(AU(\frac{1}{p_n}P)U^*) = \frac{1}{p_n}W(U^*AUP)$. Let $T = U^*AU$. Then $T \in \mathscr{W}_{\{0\}}$ and $0 \notin W(TP)$. □

THEOREM 3.14. *If $\dim(\mathscr{H}) < \infty$, then $\mathscr{Q}_{\mathscr{W}_{\{0\}}} = \{\lambda I;\ \lambda \in \mathbb{C}\backslash\{0\}\}$.*

Proof. If $A \in \mathscr{Q}_{\mathscr{W}_{\{0\}}}$, then $A = \lambda I$ for some $\lambda \neq 0$ by Propositions 3.11 and 3.13. □

We would like to point out the following equivalent formulation of Theorem 3.14. If $\dim(\mathscr{H}) < \infty$ and $A \in \mathscr{B}(\mathscr{H})$ is an invertible non-scalar operator, then there exists an operator $T \in \mathscr{B}(\mathscr{H})$ such that $0 \in \overline{W(T)}$ and $0 \notin W(TA)$.

CONJECTURE 3.15. Let $\mathscr{H}$ be an arbitrary complex Hilbert space. If $A \in \mathscr{B}(\mathscr{H})$ is an invertible non-scalar operator, then there exists an operator $T \in \mathscr{B}(\mathscr{H})$ such that $0 \in \overline{W(T)}$ but $0 \notin \overline{W(TA)}$.

3.4. Selfadjoint operators

Let us denote by $\mathscr{S}$ the set of all selfadjoint operators in $\mathscr{B}(\mathscr{H})$. Since $\mathscr{B}_+ \subseteq \mathscr{S}$, we conclude that $\mathscr{Q}_{\mathscr{S}} \subseteq \mathscr{Q}_{\mathscr{B}_+}$. Let us show that $\mathscr{Q}_{\mathscr{S}}$ is a proper subset of $\mathscr{Q}_{\mathscr{B}_+}$. Namely, if $H \in \mathscr{S}$ is invertible such that its spectrum has positive and negative values, then $0 \notin \sigma(H)$ but $0 \in \operatorname{conv}(\sigma(H)) = \overline{W(H)}$. Therefore, by (3.2), we have that $H \in \mathscr{Q}_{\mathscr{B}_+}$. On the other hand, taking $S = H^{-1}$, which is also a selfadjoint operator, we conclude that $SH \notin \mathscr{W}_{\{0\}}$, that is, $H \notin \mathscr{Q}_{\mathscr{S}}$.

CONJECTURE 3.16. Let $\mathscr{H}$ be a finite-dimensional complex Hilbert space. If $A \in \mathscr{B}(\mathscr{H})$ is invertible, then there exists a selfadjoint operator $H \in \mathscr{B}(\mathscr{H})$ such that $0 \notin \overline{W(HA)}$.

The following result gives some evidence that this conjecture holds.

PROPOSITION 3.17. *Let $\mathscr{H}$ be a separable complex Hilbert space and $A \in \mathscr{B}(\mathscr{H})$ an invertible quadratic operator. Then there exists a selfadjoint operator $H \in \mathscr{B}(\mathscr{H})$ such that $0 \notin \overline{W(HA)}$.*

Proof. It is obvious that the proposition holds for non-zero scalar operators. Assume therefore that A is a non-scalar invertible quadratic operator with eigenvalues $\lambda,\ \mu \in \mathbb{C}\backslash\{0\}$. By [9, Theorem 2.1] and because of the unitary invariance of the numerical range we can assume that A has a block matrix representation $\begin{bmatrix} \lambda I & P & 0 \\ 0 & \mu I & 0 \\ 0 & 0 & \gamma I \end{bmatrix}$, where $\gamma \in \{\lambda, \mu\}$ and P positive semidefinite. If $\gamma = \mu$, then let $H = \begin{bmatrix} I & 0 & 0 \\ 0 & rI & 0 \\ 0 & 0 & rI \end{bmatrix}$, and if $\gamma = \lambda$, then let $H = \begin{bmatrix} I & 0 & 0 \\ 0 & rI & 0 \\ 0 & 0 & I \end{bmatrix}$, where $r = \varepsilon|r|$ $(\varepsilon \in \{1,-1\})$ is a real number such that

$$\varepsilon \operatorname{Re}(\lambda\overline{\mu}) \geqslant 0 \quad \text{and} \quad |r| > \frac{\|P\|^2}{2(|\lambda||\mu| + \varepsilon \operatorname{Re}(\lambda\overline{\mu}))}. \tag{3.4}$$

When $\gamma = \mu$, then $HA = \begin{bmatrix} \lambda I & P & 0 \\ 0 & r\mu I & 0 \\ 0 & 0 & r\mu I \end{bmatrix}$ and when $\gamma = \lambda$, then $HA = \begin{bmatrix} \lambda I & P & 0 \\ 0 & r\mu I & 0 \\ 0 & 0 & \lambda I \end{bmatrix}$. In both cases, HA is a quadratic operator. Hence, by [9, Theorem 2.1], the numerical range of HA is an elliptical disc with foci at λ, $r\mu$, and with the minor axis $\|P\|$. Therefore the major axis is $\sqrt{\|P\|^2 + |\lambda - r\mu|^2}$ and the inequality which describes this elliptical disc is

$$|z-\lambda| + |z - r\mu| \leqslant \sqrt{\|P\|^2 + |\lambda - r\mu|^2}. \tag{3.5}$$

It follows from (3.4) that

$$2|r||\lambda||\mu| + 2r\operatorname{Re}(\lambda\overline{\mu}) > \|P\|^2,$$

which gives

$$|\lambda| + |r\mu| > \sqrt{\|P\|^2 + |\lambda - r\mu|^2}.$$

This shows that 0 is not in the elliptical disc (3.5). We conclude that for a selfadjoint operator H, where r is chosen to satisfy (3.4), one has $0 \notin \overline{W(HA)}$. $\square$

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Cristina Diogo
Instituto Universitário de Lisboa
Departamento de Matemática
Av. das Forças Armadas
1649-026 Lisboa, Portugal
and
Center for Mathematical Analysis, Geometry, and Dynamical Systems
Mathematics Department
Instituto Superior Técnico, Universidade de Lisboa
Av. Rovisco Pais
1049-001 Lisboa, Portugal
e-mail: cristina.diogo@iscte.pt