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SIMULTANEOUS EXTENSIONS OF A FAMILY OF LINEAR OPERATORS

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Abstract. In this paper we study when an arbitrary family of (positive) linear operators, defined on vector subspaces in an ordered vector space E with values in a Dedekind complete ordered vector space F, has a (positive) linear simultaneous extension. Some previous results of the first author concerning the existence of a (positive) linear common extension for two (positive) linear operators are generalized in the line of an appropriate theorem of D. Maharam. The results obtained are related to the classical Mazur-Orlicz theorem and its famous consequence, the Hahn-Banach theorem. Some applications of the main results pertaining to convex analysis, functional analysis and vector measure theory are given.

1. Introduction

In the theory of ordered vector spaces, *the existence of a positive linear extension* of a positive linear operator is not a trivial problem. Generally, such a problem is very different from the similar problem of extending an operator without preserving the positivity, because it is possible for a linear extension to fail to be positive.

The *basic problem* in the literature of the extension of positive linear operators can be stated as follows: If E is an ordered vector space, G is a vector subspace of E and T is a positive linear operator between G and a Dedekind complete ordered vector space F, is it possible to extend T to the whole E as a positive (and linear) operator if, for example, T is dominated on G by a monotone sublinear operator S acting between E and F?

A more difficult problem is that of the existence of a *common positive linear extension of two* positive linear operators.

An even more difficult problem arises when instead of two positive linear operators we have an *arbitrary family* of positive linear operators.

A way to solve this last problem is *to use an inequality* between two arbitrary linear combinations of the values of two families of maps $(f_\delta)_{\delta\in\Delta}$ and $(g_\delta)_{\delta\in\Delta}$. At the end, we will put this inequality in its simplest form.

The proofs of the obtained results use the technique of the auxiliary sublinear operator and apply the existence form of the Hahn-Banach theorem. ("For every sublinear operator $S : X \to F$ there exists a linear operator $L : X \to F$ such that $L \leqslant S$ on X".)

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What does the technique of the auxiliary sublinear operator consists of?

If in the statement of the Hahn-Banach existence theorem instead of X we consider an ordered vector space E, and, in addition, we assume that the sublinear operator $S : E \to F$ is monotone, then the linear operator $L : E \to F$ obtained by the Hahn-Banach theorem is positive.

The technique of the auxiliary sublinear operator is a method to prove the existence of a linear operator by using the Hahn-Banach existence theorem. This method has two steps:

1) construct a sublinear operator $S : E \to F$;

2) then apply the Hahn-Banach existence theorem, obtaining the linear operator $L : X \to F$ dominated by S on X.

According to the fact that if S is *monotone* then L is *positive*, the technique of the auxiliary sublinear operator can be used to extend positive linear operators by preserving the positivity.

This technique, due to V. Ptàk (1956), was used by him to give a much simpler proof to the Mazur-Orlicz theorem; see [10]. (We will later reproduce the statement of this theorem and in order to solve our problem we will use the idea of Ptàk's proof.)

Actually, the idea of Ptàk comes, perhaps, from the work of F. Riesz who in 1928 (see [11]) defined, in a particular case, a monotone sublinear operator $\overline{T}$ associated with a positive linear operator $T : G \to F$, where G is a majorizing vector subspace of E. Recall that G is a *majorizing subspace* of E if for all $x \in E$ there exists $v \in G$ such that $x \leqslant v$, and that:

$$\overline{T}(x) = \inf\{T(v) \,|\, v \in G, v \geqslant x\},\ x \in E.$$

The operator $\overline{T}$ has the following properties:

a) $\overline{T}$ is sublinear;

b) $\overline{T}$ is monotone;

c) $\overline{T} = T$ on G;

d) $L \leqslant \overline{T}$ on E, for any $L : E \to F$ which is a positive linear extension of T.

In this paper, we will use the technique of the auxiliary sublinear operator to obtain some positive linear operators, including a positive linear extension for an arbitrary family of positive linear operators.

What do we need in our proofs?

We need to give a convenient description of the linear space generated by a family of subsets. In what follows, we will denote by $\mathbb{N}^*$ the set of all natural numbers different from zero.

Firstly, recall that if X is a vector space and A, B are two nonempty subsets, then

$$span(A) = \left\{ \sum_{i=1}^{n} \lambda_i a_i \,|\, n \in \mathbb{N}^*,\, \lambda_i \in \mathbb{R},\, a_i \in A,\, i = 1,...,n \right\}$$

and therefore

$$span(A \cup B) = \left\{ \sum_{i=1}^{n} \lambda_i c_i \,|\, n \in \mathbb{N}^*,\, \lambda_i \in \mathbb{R},\, c_i \in A \cup B,\, i = 1,...,n \right\}.$$

Obviously, we can write:

$$span\,(A \cup B) =$$
$$\left\{ \sum_{i=1}^{n} \lambda_i a_i + \sum_{j=1}^{m} \mu_j b_j \,\middle|\, n,\, m \in \mathbb{N}^*,\, \lambda_i \in \mathbb{R},\, a_i \in A, i = 1,\dots,n,\, \mu_j \in \mathbb{R},\, b_j \in B,\, j = 1,\dots,m \right\}.$$

Now we will extend this description to the case in which A and B are replaced by a family $(A_\delta)_{\delta\in\Delta}$ of nonempty subsets in a vector space X.

We have:

$$span\left(\bigcup_{\delta\in\Delta} A_\delta\right) = \left\{ \sum_{i=1}^{n} \lambda_i c_i \,\middle|\, n \in \mathbb{N}^*,\, \lambda_i \in \mathbb{R},\, c_i \in \bigcup_{\delta\in\Delta} A_\delta,\, i = 1,\dots,n \right\}. \tag{1}$$

But we need to prove that $span\left(\bigcup\limits_{\delta\in\Delta} A_\delta\right)$ is the following set:

$$\left\{ \sum_{i=1}^{n} \left(\sum_{\delta\in\Delta} \lambda_{\delta i} a_{\delta i} \right) \middle|\, n \in \mathbb{N}^*,\, \forall i = 1,\dots,n,\, \lambda_{\delta i} \in \mathbb{R},\, (a_{\delta i})_{\delta\in\Delta} \in \Phi\left((A_\delta)_{\delta\in\Delta}\right) \right\}, \tag{2}$$

where $\Phi\left((A_\delta)_{\delta\in\Delta}\right)$ is the collection of all families $\{v_\delta \in A_\delta \,|\, \delta \in \Delta\}$ such that $v_\delta \neq 0$ for at most finitely many $\delta \in \Delta$.

Now let us prove (2).

In order to do this choose in (1) $x = \sum\limits_{i=1}^{n} \lambda_i c_i$, where $c_i \in \bigcup\limits_{\delta\in\Delta} A_\delta$ for all $i = 1,\dots,n$.

It follows that, for all $i = 1,\dots,n,$ there exists $\delta_i \in \Delta$ such that $c_i \in A_{\delta_i}$. Hence, we have $c_1 \in A_{\delta_1},\dots,c_n \in A_{\delta_n}$.

Of course, it is possible that some or all selected indices $\delta_1,\dots,\delta_n$ are equal.

And it is also possible, for example, to have another element from the set $\{c_1,\dots,c_2\}$, different from c_1, in the set A_{δ_1}. Let us denote by N_1 the set of indices of all these elements. Hence:

$$N_1 = \left\{ j \in \{1,\dots,n\} \,\middle|\, c_j \in A_{\delta_1} \right\}.$$

(Obviously, as we have already mentioned, it is possible that $N_1 = \{1,\dots,n\}$.) Let us denote by $a_{\delta_1 j}$ the elements $c_j \in A_{\delta_1}$ having the index j in N_1.

Now consider the set $\{1,\dots,n\} \setminus N_1$ and denote by N_2 its following subset:

$$N_2 = \left\{ j \in \{1,\dots,n\} \setminus N_1 \,\middle|\, c_j \in A_{\delta_2} \right\}.$$

We also denote by $a_{\delta_2 j}$ the elements c_j of A_{δ_2} having the index j in N_2 and so on. For any $p \leqslant n$, let N_p be the following set

$$N_p = \left\{ j \in \{1,\dots,n\} \setminus (N_1 \cup \dots \cup N_{p-1}) \,\middle|\, c_j \in A_{\delta_p} \right\},$$

and denote c_j (with $j \in N_p$) by $a_{\delta_p j}$.

We will stop when $N_1 \cup \dots \cup N_p = \{1,\dots,n\}$.

Therefore:

$$x = \sum_{j=1}^{n} \lambda_j c_j \Rightarrow x = \sum_{j\in N_1} \lambda_j a_{\delta_1 j} + \dots + \sum_{j\in N_p} \lambda_j a_{\delta_p j}.$$

In each of the sums appearing in the right side of the last equality, we denote λ_j by $\lambda_{\delta_1 j}$ (if $j \in N_1$), $\lambda_{\delta_2 j}$ (if $j \in N_2$),..., $\lambda_{\delta_p j}$ (if $j \in N_p$), respectively.

It follows that:

$$x = \sum_{j \in N_1} \lambda_{\delta_1 j} a_{\delta_1 j} + \cdots + \sum_{j \in N_p} \lambda_{\delta_p j} a_{\delta_p j} = \sum_{i=1}^{m} \lambda_{\delta_1 i} a_{\delta_1 i} + \cdots + \sum_{i=1}^{m} \lambda_{\delta_p i} a_{\delta_p i},$$

where m is the greatest of the cardinal numbers of the sets $N_1, \ldots, N_p$, and eventually, some coefficients in the summation of the right-hand side of this equality are equal to zero. So, we have:

$$x = \sum_{i=1}^{m} \left(\sum_{\delta \in \Delta} \lambda_{\delta i} a_{\delta i} \right),$$

where for each $i = 1, \ldots, m$, $(a_{\delta i})_{\delta \in \Delta} \in \Phi\left((A_\delta)_{\delta \in \Delta}\right)$. □

The main aim of this paper is to generalize the following result (also obtained as a generalization of the Mazur-Orlicz theorem - see Theorem 5 below).

PROPOSITION 1. ([5, Theorem 2.1]) *Let X be a vector space, F a Dedekind complete ordered vector space, A_1 and A_2 two nonempty arbitrary sets, $S : X \to F$ a sublinear operator, and $g_j : A_j \to X$ and $f_j : A_j \to F$, $j = 1, 2$, four maps. Then the following are equivalent:*

i) There exists $L : X \to F$ a linear operator such that:

a) $L \leqslant S$ on X, and

b) $f_1 \leqslant L \circ g_1$ on A_1 and $f_2 \leqslant L \circ g_2$ on A_2.

ii) The inequality

$$\sum_{i=1}^{n} \lambda_i f_1(a_{1i}) + \sum_{j=1}^{m} \mu_j f_2(a_{2j}) \leqslant S\left(\sum_{i=1}^{n} \lambda_i g_1(a_{1i}) + \sum_{j=1}^{m} \mu_j g_2(a_{2j}) \right) \tag{3}$$

holds for all $n, m \in \mathbb{N}^*$, $\{a_{11}, \ldots, a_{1n}\} \subset A_1$, $\lambda_1 \geqslant 0, \ldots, \lambda_n \geqslant 0$, $\{a_{21}, \ldots, a_{2m}\} \subset A_2$, $\mu_1 \geqslant 0, \ldots, \mu_m \geqslant 0$.

We shall achieve our main purpose by establishing *a general result* in the line of a theorem of Maharam (see Theorem 3 below). This general result formulated below as Theorem 7 will concern the simultaneous extension of an arbitrary family of linear operators. In this result we will denote by (4) the inequality corresponding to (3). The next goal will be to simplify the form of both sides of inequality (4).

This idea comes from the remark that the (classical) Hahn-Banach theorem can be seen as a consequence of the Mazur-Orlicz theorem.

2. Preliminaries

In this paper, the terminology, the notation and some mentioned results are generally considered to be classical in the theory of ordered vector spaces and linear operators (see, for example [1], [2] and [8]).

X and Y will be *vector spaces*, E and E_0 will be *ordered vector spaces*, and F will be a *Dedekind complete ordered vector space* (which means that, every nonempty order bounded set in F has a supremum or, equivalently, an infimum in F). The order relation between operators will be the *pointwise relation*, that is, if T and S are two operators defined on a set A with values in F, then $T \leqslant S$ on A means that $T(a) \leqslant S(a)$ for all $a \in A$. We will say that T is *dominated* (or, equivalently, *majorized*) *by* S *on* A. Also, a linear operator $T : E \to F$ will be called *positive*, if $T(x) \geqslant 0$ in F for all $x \geqslant 0$ in E.

An operator $S : X \to F$ is called *sublinear* if it is *subadditive*, that is, $S(x_1 + x_2) \leqslant S(x_1) + S(x_2)$ for all $x_1, x_2 \in X$, and *positively homogeneous*, that is, $S(\lambda x) = \lambda S(x)$, for all $x \in X$ and $\lambda \geqslant 0$.

If $G \subseteq X$ is a vector subspace of X and $T : G \to F$ is a linear operator, a linear operator $L : X \to F$ is called an *extension* of T, if $L(v) = T(v)$ for all v in G. Equivalently, we say that L *extends* T *to the whole space* X.

Also, if $(G_\delta)_{\delta \in \Delta}$ is an arbitrary family of vector subspaces of X, and for all $\delta \in \Delta$, $T_\delta : G_\delta \to F$ is a linear operator, we say that a linear operator $L : X \to F$ *simultaneously extends* all operators $(T_\delta)_{\delta \in \Delta}$, if $L(v_\delta) = T_\delta(v_\delta)$ for all v_δ in G_δ and all $\delta \in \Delta$. We say, equivalently, that L is *a simultaneous extension* of the family $(T_\delta)_{\delta \in \Delta}$, or that L is a *common extension* of the operators T_δ, $\delta \in \Delta$. Obviously, a necessary condition for the existence of a simultaneous extension of the family $(T_\delta)_{\delta \in \Delta}$ is that, for all $\delta \neq \delta'$ in Δ, $T_\delta = T_{\delta'}$ on $G_\delta \cap G_{\delta'}$. (According to the terminology employed in [7], the operators T_δ and $T_{\delta'}$ are then *consistent*.)

The following famous theorem is called the *Hahn-Banach theorem* (see, for example, [8, p. 44]).

THEOREM 2. *For every sublinear operator $S : X \to F$ there exists a linear operator $L : X \to F$ such that $L \leqslant S$ on X.*

This theorem is a very important tool in functional analysis. Theorem 2 will be very important for us because we will use it as the second part of the *technique of the auxiliary sublinear operator* applied to demonstrate the existence of some linear operators. (The first part of this technique consists of the construction of a sublinear operator associated with the family of operators which should be extended.)

Below we will state two results that gave us the ideas in this paper. The first of these results (Theorem 3) is the Maharam theorem (1972).

Before formulating this theorem, we recall that the ordered vector space E, with the positive cone E_+, has an order unit $e \in E_+$, $e \neq 0$, if $E = \bigcup\limits_{\lambda \geqslant 0} [-\lambda e, \lambda e]$.

THEOREM 3. ([7]; see also [12, Theorem 6.3]) *Let E be a vector lattice with an order unit $e \in E_+$, $e \neq 0$, and $(G_\delta)_{\delta \in \Delta}$ a family of vector subspaces of E such that $e \in span\left(\bigcup\limits_{\delta \in \Delta} G_\delta\right)$. Let F be a Dedekind complete ordered vector space and let $\{T_\delta : G_\delta \to F \,|\, \delta \in \Delta\}$ be a family of positive linear operators. The following conditions are equivalent:*

i) There exists $T : E \to F$ a positive linear extension of the family $(T_\delta)_{\delta \in \Delta}$.

ii) The inequality $0 \leqslant T_\delta(v_\delta)$ *holds for every family* $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$ *satisfying* $0 \leqslant \sum\limits_{\delta\in\Delta} v_\delta$*, where* $\Phi\left((G_\delta)_{\delta\in\Delta}\right)$ *is the collection of all families* $\{v_\delta \in G_\delta \,|\, \delta \in \Delta\}$ *such that* $v_\delta \neq 0$ *for at most finitely many* $\delta \in \Delta$.

The following theorem is an easy generalization of the Maharam theorem. We recall that a vector subspace G of the ordered vector space E is called a *majorizing* subspace, if for every $x \in E$ there exists $v \in G$ such that $x \leqslant v$; obviously, if G is a vector subspace of E, and E has an ordered unit e, and, moreover, $e \in G$, then G is a majorizing subspace of E.

THEOREM 4. ([6, Theorem 5.4]) *Let* E *be an ordered vector space, and let* $(G_\delta)_{\delta\in\Delta}$ *be a family of vector subspaces of* E *such that there exists at least one which is majorizing, say* G_{δ_0}. *Let* F *be a Dedekind complete ordered vector space and let* $\{T_\delta\}_{\delta\in\Delta}$ *be a family of positive linear operators, where* $T_\delta : G_\delta \to F$. *Then the following are equivalent:*

i) The family $\{T_\delta : G_\delta \to F \,|\, \delta \in \Delta\}$ *has a positive common linear extension* $T : E \to F$.

ii) The implication $\sum\limits_{\delta\in\Delta} v_\delta \geqslant 0 \Rightarrow \sum\limits_{\delta\in\Delta} T_\delta(v_\delta) \geqslant 0$ *holds for every family* $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$.

3. Main results

One framework in which the problem of a simultaneous extension of a family of linear operators can be considered is as follows: Given an arbitrary ordered vector space E_0, a family of vector subspaces (or sets) $(G_\delta)_{\delta\in\Delta}$ in E_0, a Dedekind complete ordered vector space F, and a family $(T_\delta)_{\delta\in\Delta}$ of (positive) linear operators with $T_\delta : G_\delta \to F$ for all $\delta \in \Delta$, let $E = span(G_\delta)_{\delta\in\Delta}$ and find (necessary and) sufficient conditions under which there exists a (positive) linear operator $L : E \to F$ which simultaneously extends all operators $(T_\delta)_{\delta\in\Delta}$.

Another framework for considering the simultaneous extension problem is the following: Given an arbitrary real vector space X, a family $(A_\delta)_{\delta\in\Delta}$ of arbitrary subsets of X, a Dedekind complete ordered vector space F, and two families of arbitrary maps, namely $(f_\delta)_{\delta\in\Delta}$ and $(g_\delta)_{\delta\in\Delta}$ with $f_\delta : A_\delta \to F$ and $g_\delta : A_\delta \to X$ for all $\delta \in \Delta$, and, in addition, a sublinear operator $S : X \to F$ such that $f_\delta \leqslant S \circ g_\delta$ for all $\delta \in \Delta$, find the necessary and sufficient conditions under which these inequalities can be simultaneously extended by using a linear operator $L : X \to F$ instead of S. That is, find necessary and sufficient conditions under which there exists a linear operator $L : X \to F$ such that $f_\delta \leqslant L \circ g_\delta$ for all $\delta \in \Delta$.

We will start with the second question, which immediately reminds us of the Mazur-Orlicz theorem. This theorem is a very useful result for establishing the existence of a linear operator dominated by a sublinear operator. It was proved in 1953 by S. Mazur and W. Orlicz; see [9].

The following statement is the vectorial form of the original theorem of Mazur and Orlicz as it appears in a paper by W. Chojnacki and its erratum; see [3] and [4].

THEOREM 5. (Mazur-Orlicz theorem) *Let X be a vector space, F a Dedekind complete ordered vector space and $S : X \to F$ a sublinear operator. Let A be an arbitrary nonempty set, and let f and g be two maps, $f : A \to F$ and $g : A \to X$. The following conditions are equivalent:*

i) There exists a linear operator $L : X \to F$ with the properties

a) $L \leqslant S$ on X,

b) $f \leqslant L \circ g$ on A.

ii) The inequality

$$\sum_{i=1}^{n} \lambda_i f(a_i) \leqslant S\left(\sum_{i=1}^{n} \lambda_i g(a_i)\right)$$

holds for all finite subsets $\{a_1,...,a_n\} \subset A$ and $\{\lambda_1,...,\lambda_n\} \subset \mathbb{R}_+$.

It is immediate (see [6]) that the Hahn-Banach extension theorem is a consequence of the Mazur-Orlicz theorem.

COROLLARY 6. (Hahn-Banach theorem) *Let X be a vector space, F a Dedekind complete ordered vector space, and $S : X \to F$ a sublinear operator. Let G be a vector subspace of X and $T : G \to F$ a linear operator. Then the following conditions are equivalent:*

i) There exists a linear operator $L : X \to F$ with the properties

a) $L \leqslant S$ on X, and

b) $L = T$ on G.

ii) $T \leqslant S$ on G.

Proof. In *Theorem 5*, take $A = G$, $f = T$ and $g = i$ (the inclusion of G in X). $\square$

We remark that Proposition 1 extends the Mazur-Orlicz theorem for two sets A_1, A_2 and four maps f_1, f_2, g_1, g_2 instead of A, f and g, respectively.

Now we will generalize this result for an arbitrary family $(A_\delta)_{\delta\in\Delta}$ of sets and two arbitrary families of maps $(f_\delta)_{\delta\in\Delta}$ and $(g_\delta)_{\delta\in\Delta}$.

THEOREM 7. (Simultaneous extension theorem in the line of the Mazur-Orlicz theorem) *Let X be a vector space, F a Dedekind complete ordered vector space and $(A_\delta)_{\delta\in\Delta}$ a family of nonempty sets. Let $S : X \to F$ be a sublinear operator, and $(f_\delta)_{\delta\in\Delta}, (g_\delta)_{\delta\in\Delta}$ two families of maps, $g_\delta : A_\delta \to X$ and $f_\delta : A_\delta \to F$, $\delta \in \Delta$, such that for all $\delta \in \Delta$ with $0 \in A_\delta$ it follows that*

$$f_\delta(0) = g_\delta(0) = 0. \tag{$*$}$$

Then the following are equivalent:

i) There exists a linear operator $L : X \to F$ such that

a) $L \leqslant S$ on X, and

b) $f_\delta \leqslant L \circ g_\delta$ on A_δ for all $\delta \in \Delta$.

ii) The following inequality holds:

$$\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta} \lambda_{\delta i} f_\delta(a_{\delta i})\right) \leqslant S\left(\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta} \lambda_{\delta i} f_\delta(a_{\delta i})\right)\right), \tag{4}$$

where $n \in \mathbb{N}^*$ *and for each* $i \in \{1,...,n\}$, $(\lambda_{\delta i})_{\delta\in\Delta} \subset \mathbb{R}_+$ *and* $(a_{\delta i})_{\delta\in\Delta} \in \Phi\left((A_\delta)_{\delta\in\Delta}\right)$, *with* $\Phi\left((A_\delta)_{\delta\in\Delta}\right)$ *the collection of all* $(y_\delta)_{\delta\in\Delta} \in (A_\delta)_{\delta\in\Delta}$ *having* $y_\delta \neq 0$ *for at most finitely* $\delta \in \Delta$.

REMARK. Note that the assumption $(*)$ implies that $(f_\delta(a_\delta))_{\delta\in\Delta} \in \Phi\left(f_\delta(A_\delta)_{\delta\in\Delta}\right)$ and $(g_\delta(a_\delta))_{\delta\in\Delta} \in \Phi\left(g_\delta(A_\delta)_{\delta\in\Delta}\right)$ if $(a_\delta)_{\delta\in\Delta} \in \Phi\left((A_\delta)_{\delta\in\Delta}\right)$. So, for example, in the sum $\sum\limits_{\delta\in\Delta} f_\delta(a_\delta)$, only a finite number of terms are nonzero.

Proof. Obviously, i) implies ii), for

$$\begin{aligned}\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta}\lambda_{\delta i}f_\delta(a_{\delta i})\right) &\overset{i)b)}{\leqslant} \sum_{i=1}^{n}\left(\sum_{\delta\in\Delta}\lambda_{\delta i}(L\circ g_\delta)(a_{\delta i})\right)\\ &\overset{L\ linear}{=} L\left(\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta}\lambda_{\delta i}g_\delta(a_{\delta i})\right)\right)\\ &\overset{i)a)}{\leqslant} S\left(\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta}\lambda_{\delta i}g_\delta(a_{\delta i})\right)\right).\end{aligned}$$

To obtain the converse, we will apply the technique of the auxiliary sublinear operator. For every $x \in X$, define $S_1(x)$ as the infimum of the set

$$\left\{S\left(x+\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta}\lambda_{\delta i}g_\delta(a_{\delta i})\right)\right)-\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta}\lambda_{\delta i}f_\delta(a_{\delta i})\right)\right\},$$

where the infimum is taken over all the $(a_{\delta i})_{\delta\in\Delta} \in \Phi\left((A_\delta)_{\delta\in\Delta}\right)$ and $(\lambda_{\delta i})_{\delta\in\Delta} \subset \mathbb{R}_+$, with $n \in \mathbb{N}^*$ and $i = 1,...,n$.

First, note that $S_1(x)$ exists, because using ii) and the sublinearity of S, we have:

$$\begin{aligned}\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta}\lambda_{\delta i}f_\delta(a_{\delta i})\right) &\leqslant S\left(\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta}\lambda_{\delta i}g_\delta(a_{\delta i})\right)\right)\\ &\leqslant S\left(x+\sum_{\delta\in\Delta}\lambda_{\delta i}g_\delta(a_{\delta i})\right)+S(-x)\end{aligned}$$

and further

$$-S(-x) \leqslant S\left(x+\sum_{\delta\in\Delta}\lambda_{\delta i}g_\delta(a_{\delta i})\right)-\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta}\lambda_{\delta i}f_\delta(a_{\delta i})\right).$$

(Note that this inequality holds in the Dedekind complete ordered vector space F.) Also, from the definition of $S_1(x)$ it follows that

$$-S(-x) \leqslant S_1(x) \leqslant S(x), \quad x \in X. \tag{5}$$

It is not difficult to show that S_1 is a sublinear operator.

Then by using the existence form of the Hahn-Banach theorem (see Theorem 2), it follows that there exists a linear operator $L : X \to F$ such that

$$L(x) \leqslant S_1(x), \quad x \in X.$$

Therefore, by using (5) it follows that

$$L(x) \leqslant S(x), \quad x \in X$$

that is, i a) holds.

To prove i b), note that, for each $\delta \in \Delta$,

$$\begin{aligned} L(-g_\delta(a_\delta)) &\leqslant S_1(-g_\delta(a_\delta)) \\ &\leqslant S(-g_\delta(a_\delta) + g_\delta(a_\delta)) - f_\delta(a_\delta) = -f_\delta(a_\delta). \end{aligned}$$

Hence, because L is linear, we obtain $f_\delta \leqslant L \circ g_\delta$. □

In what follows we will try to simplify the form of both sides of the inequality (4).

4. An application of the simultaneous extension result of the Mazur-Orlicz type theorem in convex analysis

Now we shall formulate a version of Theorem 7 for the *ordered vector spaces* as domain spaces. In this setting we shall assume that the operator $S : E \to F$ is not only sublinear but it is also *monotone*, that is, if $x \leqslant y$ in E, then $S(x) \leqslant S(y)$ in F. Under these circumstances, the appropriate extension L will not only be linear, but will also be positive.

Note that, any linear operator L dominated by a monotone sublinear operator S will automatically be positive (see for example [6, Remark 2.3]).

We chose the setting of ordered vector spaces because then inequality (4) becomes simpler and, consequently, easier to be checked in applications (see (6) below).

Also, the family $(A_\delta)_{\delta \in \Delta}$ of arbitrary sets becomes a family of *convex* sets and the two families $(g_\delta)_{\delta \in \Delta}$ and $(f_\delta)_{\delta \in \Delta}$ of arbitrary maps become two families of *convex* and *concave* operators, respectively.

THEOREM 8. (Simultaneous extension theorem in the line of Mazur-Orlicz theorem for ordered vector spaces) *Let E be an ordered vector space, F a Dedekind complete ordered vector space and $S : E \to F$ a monotone sublinear operator. Let $(K_\delta)_{\delta \in \Delta}$ be an arbitrary family of nonempty convex sets, $(P_\delta)_{\delta \in \Delta}$ with $P_\delta : K_\delta \to E$ $(\delta \in \Delta)$ a family of convex operators and $(Q_\delta)_{\delta \in \Delta}$ with $Q_\delta : K_\delta \to F$ $(\delta \in \Delta)$ a family of concave operators, such that, for all $\delta \in \Delta$ with $0 \in K_\delta$ it follows that $P_\delta(0) = Q_\delta(0) = 0$.*

The following conditions are equivalent:

i) There exists a positive linear operator $L : E \to F$ with the properties:

a) $L \leqslant S$ on E, and

b) $Q_\delta \leqslant L \circ P_\delta$ on K_δ, for all $\delta \in \Delta$.

ii) The following inequality holds

$$\sum_{\delta \in \Delta} \lambda_\delta Q_\delta(a_\delta) \leqslant S\left(\sum_{\delta \in \Delta} \lambda_\delta P_\delta(a_\delta)\right) \tag{6}$$

for $(\lambda_\delta)_{\delta\in\Delta} \subset \mathbb{R}_+$ *and* $(a_\delta)_{\delta\in\Delta} \in \Phi\left((K_\delta)_{\delta\in\Delta}\right)$, *where* $\Phi\left((K_\delta)_{\delta\in\Delta}\right)$ *denotes the collection of all* $(y_\delta)_{\delta\in\Delta} \in (K_\delta)_{\delta\in\Delta}$ *with* $y_\delta \neq 0$ *for at most finitely many* $\delta \in \Delta$.

Proof. Firstly, we remark that inequality (6) is equivalent with inequality (4) from Theorem 7. Indeed, it is obviously that (4) implies (6) if we put $A_\delta = K_\delta,\ g_\delta = P_\delta$ and $f_\delta = Q_\delta$, for all $\delta \in \Delta$.

To prove the converse, consider $a_{\delta i} \in K_\delta$ and $\lambda_{\delta i} \geqslant 0$ for each $i = 1,...,n$ $(n \in \mathbb{N}^*)$ and $\delta \in \Delta$. Suppose that, for each $\delta \in \Delta$, we have $\lambda_\delta = \sum_{i=1}^{n} \lambda_{\delta i} > 0$ and denote $\alpha_{\delta i} = \frac{\lambda_{\delta i}}{\lambda_\delta}$, for each $i = 1,...,n$. It follows that $\sum_{i=1}^{n} \alpha_{\delta i} = 1$ and hence, using the fact that the operator P_δ is *convex* and the operator Q_δ is *concave,* we have:

$$P_\delta\left(\sum_{i=1}^{n} \alpha_{\delta i} a_{\delta i}\right) \leqslant \sum_{i=1}^{n} \alpha_{\delta i} P_\delta(a_{\delta i}) \text{ and } Q_\delta\left(\sum_{i=1}^{n} \alpha_{\delta i} a_{\delta i}\right) \geqslant \sum_{i=1}^{n} \alpha_{\delta i} Q_\delta(a_{\delta i})$$

for all $\delta \in \Delta$.

Next, using (6) and the condition that S is monotone, we obtain:

$$\begin{aligned}
\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta} \lambda_{\delta i} Q_\delta(a_{\delta i})\right) &= \sum_{\delta\in\Delta} \lambda_\delta \left(\sum_{i=1}^{n} \frac{\lambda_{\delta i}}{\lambda_\delta} Q_\delta(a_{\delta i})\right) = \sum_{\delta\in\Delta} \lambda_\delta \left(\sum_{i=1}^{n} \alpha_{\delta i} Q_\delta(a_{\delta i})\right) \\
&\leqslant \sum_{\delta\in\Delta} \lambda_\delta Q_\delta\left(\sum_{i=1}^{n} \alpha_{\delta i} a_{\delta i}\right) \overset{(6)}{\leqslant} S\left(\sum_{\delta\in\Delta} \lambda_\delta P_\delta\left(\sum_{i=1}^{n} \alpha_{\delta i} a_{\delta i}\right)\right) \\
&\overset{S\ monotone}{\leqslant} S\left(\sum_{\delta\in\Delta} \lambda_\delta \left(\sum_{i=1}^{n} \alpha_{\delta i} P_\delta(a_{\delta i})\right)\right) \\
&= S\left(\sum_{i=1}^{n}\left(\sum_{\delta\in\Delta} \lambda_{\delta i} P_\delta(a_{\delta i})\right)\right),
\end{aligned}$$

where $n \in \mathbb{N}^*$ and, for each $i = 1,...,n$, $(\lambda_{\delta i})_{\delta\in\Delta} \subset \mathbb{R}_+$ and $(a_{\delta i})_{\delta\in\Delta} \in \Phi\left((K_\delta)_{\delta\in\Delta}\right)$.

To complete the proof that ii) implies i), let us observe, as we already mentioned, that L is a positive operator because $L \leqslant S$ and S is monotone. □

We remark that Theorem 8 generalizes Theorem 2.4 in [5] and also Theorem 2.4 in [6].

5. Some applications of the simultaneous extension theorem in the line of the Mazur-Orlicz theorem in functional analysis

In the following result (a Hahn-Banach type theorem) which generalizes [5, Theorem 2.8], we will apply Theorem 7 to obtain a necessary and sufficient condition for a family of linear operators on subspaces of a vector space to be extendable to the whole space.

THEOREM 9. (Simultaneous extension of a family of linear operators) *Let* X *be a vector space,* $(G_\delta)_{\delta\in\Delta}$ *a family of vector subspaces of* X *and* F *a Dedekind complete ordered vector space. Let* $S : X \to F$ *be a sublinear operator and, for each* $\delta \in \Delta$, *let* $T_\delta : G_\delta \to F$ *be a linear operator. The following conditions are equivalent:*

i) There exists $L: X \to F$ *a linear operator such that:*

a) $L \leqslant S$ *on* X*, and*

b) $L = T_\delta$ *on* G_δ *for all* $\delta \in \Delta$.

ii) The following inequality holds:

$$\sum_{\delta\in\Delta} T_\delta(v_\delta) \leqslant S\left(\sum_{\delta\in\Delta} v_\delta\right), \tag{7}$$

for all $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$, *where* $\Phi\left((G_\delta)_{\delta\in\Delta}\right)$ *denotes the collection of all* $(v_\delta)_{\delta\in\Delta} \in (G_\delta)_{\delta\in\Delta}$ *such that* $v_\delta \neq 0$ *for at most finitely many* $\delta \in \Delta$.

Proof. Obviously, i) implies ii). To prove the converse, apply Theorem 7 with $A_\delta = G_\delta$, $f_\delta = T_\delta$, and g_δ chosen to be the inclusion of G_δ in X, for each $\delta \in \Delta$. We thus ensure the existence of a linear operator $L: X \to F$ such that $L \leqslant S$ on X and $T_\delta \leqslant L$ on G_δ for all $\delta \in \Delta$. Actually, because T_δ and L are linear, we even have $T_\delta = L$ on G_δ, that is, L is an extension of T_δ, for all $\delta \in \Delta$. $\square$

Note that inequality (7) implies that:

1) $T_\delta \leqslant S$ on G_δ for each $\delta \in \Delta$;

2) $T_{\delta'} = T_{\delta''}$ on $G_{\delta'} \cap G_{\delta''}$ (that is, $T_{\delta'}$ and $T_{\delta''}$ are *consistent*) for all $\delta',\ \delta'' \in \Delta$, $\delta' \neq \delta''$.

(Indeed, to prove 2), let $v \in G_{\delta'} \cap G_{\delta''}$ and put in (7) $v_{\delta'} = v$, $v_{\delta''} = -v$ and $v_\delta = 0$ for all $\delta \in \Delta \backslash \{\delta', \delta''\}$. Then $T_{\delta'}(v_{\delta'}) + T_{\delta''}(v_{\delta''}) \leqslant S(0) = 0$ and hence $T_{\delta'}(v) \leqslant -T_{\delta''}(-v) = T_{\delta''}(v)$. Similarly, we have the converse inequality $T_{\delta''}(v) \leqslant T_{\delta'}(v)$ and hence $T_{\delta'}(v) = T_{\delta''}(v)$).

The next result (again a Hahn-Banach type theorem) is a version of Theorem 9 in the ordered vector spaces setting.

THEOREM 10. (Simultaneous extension of a family of positive linear operators) *Let* E *be an ordered vector space,* $(G_\delta)_{\delta\in\Delta}$ *a family of vector subspaces of* E *and* F *a Dedekind complete ordered vector space. For each* $\delta \in \Delta$*, let* $T_\delta: G_\delta \to F$ *be a positive linear operator. The following conditions are equivalent:*

i) There exists $L: E \to F$ *which is a positive linear extension of* T_δ *for all* $\delta \in \Delta$.

ii) There exists a monotone sublinear operator $S: E \to F$ *such that*

$$\sum_{\delta\in\Delta} T_\delta(v_\delta) \leqslant S\left(\sum_{\delta\in\Delta} v_\delta\right)$$

for all $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$.

Proof. To show that i) implies ii) let $S = L$. For the converse implication, we recall again that if ii) holds and $L \leqslant S$, then L is positive because S is monotone. $\square$

The following statement is a consequence of the previous result; the condition that the sublinear operator S is monotone will be dropped here.

PROPOSITION 11. *Let E be an ordered vector space, $(G_\delta)_{\delta\in\Delta}$ a family of vector subspaces of E and F a Dedekind complete ordered vector space. For all $\delta \in \Delta$, let $T_\delta : G_\delta \to F$ be a positive linear operator. Then the following are equivalent:*

i) There exists a positive linear operator $L : E \to F$ such that $L = T_\delta$ on G_δ for all $\delta \in \Delta$.

ii) There exists $S : E \to F$ a sublinear operator such that:

$$\sum_{\delta\in\Delta} v_\delta \leqslant v \text{ in } E \Rightarrow \sum_{\delta\in\Delta} T_\delta(v_\delta) \leqslant S(v) \text{ in } F, \tag{8}$$

where $v \in E$ and $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$.

Proof. To prove that i) implies ii), put $S = L$ and use that L is a positive linear extension for all T_δ, $\delta \in \Delta$. We have:

$$\sum_{\delta\in\Delta} v_\delta \leqslant v \Rightarrow \sum_{\delta\in\Delta} L(v_\delta) \leqslant L(v) \Rightarrow \sum_{\delta\in\Delta} T_\delta(v_\delta) \leqslant S(v).$$

To establish the converse implication ii)$\Rightarrow$i), let $S : E \to F$ be a sublinear operator satisfying (8). Apply *the technique of the auxiliary sublinear operator*, defining $S_1 : E \to F$ by the following formula

$$S_1(v) = \inf\{S(w)\,|w \in E, w \geqslant v\},$$

for each $v \in E$. This infimum exists in F, because the set $\{S(w)\,|w \in E, w \geqslant v\}$ is minorized in F by $-S(-v)$. Indeed, with $v_\delta = 0$ $(\delta \in \Delta)$ and $u \geqslant 0$, we have

$$0 = \sum_{\delta\in\Delta} T_\delta(v_\delta) \leqslant S(u) = S(v+u-v) \leqslant S(v+u) + S(-v).$$

Hence, $-S(-v) \leqslant S(v+u)$, for all $u \geqslant 0$, or, equivalently, $-S(-v) \leqslant S(w)$ for all $w \in E$ with $w \geqslant u$. Obviously, $S_1 \leqslant S$ on E. In addition, S_1 has the following properties:

1) S_1 is a sublinear operator;

2) S_1 is monotone;

3) $\sum_{\delta\in\Delta} T_\delta(v_\delta) \leqslant S_1\left(\sum_{\delta\in\Delta} v_\delta\right)$ for all $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$.

Now, using the implication ii)$\Rightarrow$i) of Theorem 10 with S replaced by S_1 we obtain a positive linear extension of T_δ, for all $\delta \in \Delta$. □

Now remark that Theorem 4 and Theorem 3 (Maharam theorem) are both consequences of our Theorem 10. Indeed, suppose that $(G_\delta)_{\delta\in\Delta}$ is a family of vector subspaces of E of which one, say G_{δ_0}, is a majorizing subspace. Then it follows that ii) of Theorem 4 implies ii) of Theorem 10.

Indeed, suppose that

$$\sum_{\delta\in\Delta} v_\delta \leqslant 0 \Rightarrow \sum_{\delta\in\Delta} T_\delta(v_\delta) \leqslant 0 \tag{9}$$

for all $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$. Define:

$$T : span\left(\bigcup_{\delta\in\Delta} G_\delta\right) \to F$$

by

$$T\left(\sum_{\delta\in\Delta} v_\delta\right) = \sum_{\delta\in\Delta} T_\delta\left(v_\delta\right),$$

for all $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$.

The operator T has the following properties:

1) T is well-defined, according to (9);
2) T is linear;
3) T is positive.

Because G_{δ_0} is majorizing, it follows that $G = span\left(\bigcup_{\delta\in\Delta} G_\delta\right)$ is majorizing, too. Define $S : E \to F$ by $S(x) = \overline{T}(x)$, for all $x \in E$, that is,

$$S(x) = \inf\{T(z) \,|\, z \in G,\, z \geqslant x\}.$$

It is known that S is a *monotone sublinear* operator and that $T \leqslant S$ on E. We have for all $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$,

$$\sum_{\delta\in\Delta} T_\delta\left(v_\delta\right) = T\left(\sum_{\delta\in\Delta} v_\delta\right) \leqslant S\left(\sum_{\delta\in\Delta} v_\delta\right),$$

that is, the condition ii) of Theorem 10 is satisfied.

6. An application of our simultaneous extension result of the Hahn-Banach type theorem in vector measure theory

Let Ω be a nonempty *set*. For a set $A \subseteq \Omega$, denote by $\chi_A : \Omega \to \{0,1\}$ its *indicator function*.

Let $\mathscr{A}$ be an *algebra of subsets* of Ω and let

$$D(\mathscr{A}) = span\{\chi_A \,|\, A \in \mathscr{A}\}.$$

(By an *algebra of subsets* of a set Ω we understand a nonempty family of subsets of Ω that contains the null set $\emptyset$, the complement of each its member, and the union of any two of its members.)

Endowed with the *pointwise* operations and order relation, $D(\mathscr{A})$ is a (real) *vector lattice.*

Let Y be a vector space. Recall that a map (a set function) $\varphi : \mathscr{A} \to Y$ is called a *vector measure* if

$$\varphi(A \cup B) = \varphi(A) + \varphi(B)$$

for all disjoint sets $A,\, B \in \mathscr{A}$.

As a consequence, a vector measure on $\mathscr{A}$ is a *finitely additive set function*, that is,

$$\varphi\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \varphi(A_i)$$

for every finite collection $(A_i)_{i=1}^{n}$ of disjoint sets of $\mathscr{A}$.

It is known that every vector measure $\varphi : \mathscr{A} \to Y$ induces a *linear operator* $T : D(\mathscr{A}) \to Y$ given by

$$T\left(\sum_{i=1}^{n} \alpha_i \chi_{A_i}\right) = \sum_{i=1}^{n} \alpha_i \varphi(A_i).$$

T is called the *representing linear operator* of φ or, equivalently, the *elementary integral* with respect to φ.

Conversely, every linear operator $T : D(\mathscr{A}) \to Y$ induces a vector measure $\varphi : \mathscr{A} \to Y$ given by

$$\varphi(A) = T(\chi_A)$$

for all $A \in \mathscr{A}$.

So, there exists a one-to-one correspondence between vector measures $\varphi : \mathscr{A} \to Y$ and linear operators $T : D(\mathscr{A}) \to Y$. This correspondence can be used to obtain some results on vector measures from the corresponding results concerning linear operators.

Given an *ordered vector space* F, a vector measure $\varphi : \mathscr{A} \to F$ is called *positive* if $\varphi(A) \geqslant 0$ for all $A \in \mathscr{A}$. It is immediate that a vector measure is positive if and only if its representing linear operator is positive.

In what follows, Ω will be a nonempty set, $E = 2^{\Omega}$, and F a Dedekind complete ordered vector space.

Let $(\mathscr{A}_\delta)_{\delta\in\Delta}$ be a family of algebras of subsets of Ω and $(\varphi_\delta)_{\delta\in\Delta}$ a family of *vector measures*, $\varphi_\delta : \mathscr{A}_\delta \to F$, for each $\delta \in \Delta$.

We say that the family $(\varphi_\delta)_{\delta\in\Delta}$ *can be simultaneously extended* to the whole E, if there exists a vector measure $\varphi : E \to F$ such that $\varphi(A) = \varphi_\delta(A)$ for all $\delta \in \Delta$ and $A \in \mathscr{A}_\delta$.

The following result [12, Theorem 6.4] gives a *necessary* and *sufficient* condition for a family $(\varphi_\delta)_{\delta\in\Delta}$ to be simultaneously extendable under the assumption that φ_δ is positive for all $\delta \in \Delta$ (and consequently any extension will be positive, too).

Using our Theorem 10, we can give another proof of this result.

THEOREM 12. *Let Ω be a nonempty set $E = 2^{\Omega}$ and F a Dedekind complete ordered vector space. Let $(\mathscr{A}_\delta)_{\delta\in\Delta}$ be a family of algebras of subsets of Ω and $(\varphi_\delta)_{\delta\in\Delta}$ a family of positive vector measures, with $\varphi_\delta : \mathscr{A}_\delta \to F$ for all $\delta \in \Delta$.*

Then the following are equivalent:

i) The family $(\varphi_\delta)_{\delta\in\Delta}$ can be simultaneously extended to a positive vector measure $\varphi : E \to F$.

ii) $\sum_{i=1}^{q} \chi_{A_i} \leqslant \sum_{i=q+1}^{q+r} \chi_{A_i} \Rightarrow \sum_{i=1}^{q} \varphi_{\delta_i}(A_i) \leqslant \sum_{i=q+1}^{q+r} \varphi_{\delta_i}(A_i)$, for all $q,\ r \in \mathbb{N}^*$, $A_1, \ldots, A_{q+r} \in \bigcup_{\delta\in\Delta} \mathscr{A}_\delta$ and for all $\delta_1, \ldots, \delta_{q+r} \in \Delta$ such that $A_i \in \mathscr{A}_{\delta_i}$ for all $i = 1, \ldots, q+r$.

Proof. i)$\Rightarrow$ii) is obvious. Conversely, suppose that ii) is valid.

First, we will prove that ii)$\Leftrightarrow$ii$'$) where ii$'$) is the following statement:

$$\text{ii}'\text{)}\ \sum_{i=1}^{q}\mu_i\chi_{A_i}\leqslant\sum_{i=q+1}^{q+r}\mu_i\chi_{A_i}\Rightarrow\sum_{i=1}^{q}\mu_i\varphi_{\delta_i}(A_i)\leqslant\sum_{i=q+1}^{q+r}\mu_i\varphi_{\delta_i}(A_i),$$

for all $q,\ r\in\mathbb{N}^*$, $\mu_1,...,\mu_{q+r}\in\mathbb{R}_+$, $A_i\in\mathscr{A}_{\delta_i}$, with $\delta_i\in\Delta$ and $i=1,...,q+r$.

Obviously ii$'$)$\Rightarrow$ii). Conversely, suppose that ii) holds and that

$$\sum_{i=1}^{q}\mu_i\chi_{A_i}\leqslant\sum_{i=q+1}^{q+r}\mu_i\chi_{A_i}\tag{10}$$

as in the hypothesis of ii$'$).

We consider three cases.

Case 1. Suppose that $\mu_1,...,\mu_{q+r}\in\mathbb{N}^*$. Define the sets $(B_i)_{i=1}^{\mu_1+\cdots+\mu_{q+r}}$ as follows:

$$\begin{aligned}&B_1=B_2=...=B_{\mu_1}=A_1\\&B_{\mu_1+1}=B_{\mu_1+2}=...=B_{\mu_1+\mu_2}=A_2\\&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\&B_{\mu_1+\mu_2+\cdots+\mu_{q+r-1}+1}=B_{\mu_1+\mu_2+\cdots+\mu_{q+r}}=A_{q+r}.\end{aligned}$$

Denoting $m=\mu_1+\cdots+\mu_q$ and $n=\mu_{q+1}+\cdots+\mu_{q+r}$ it follows that $m\geqslant q$ and $n\geqslant r$ because for each $i=1,...,q+r,\ \mu_i\geqslant 1$. Then inequality (10) becomes

$$\sum_{i=1}^{m}\chi_{B_i}\leqslant\sum_{i=m+1}^{m+n}\chi_{B_i}.$$

Therefore, from ii) it follows that

$$\sum_{i=1}^{q}\mu_i\varphi_{\delta_i}(A_i)=\sum_{i=1}^{m}\varphi_{\delta_i}(B_i)\leqslant\sum_{i=m+1}^{m+n}\varphi_{\delta_i}(B_i)=\sum_{i=q+1}^{q+r}\mu_i\varphi_{\delta_i}(A_i).$$

Case 2. Assume that $\mu_i\in\mathbb{Q}_+$ for all $i=1,...,q+r$. Let us suppose that $\mu_i=\frac{s_i}{t_i}$, where $s_i\in\mathbb{N}$ and $t_i\in\mathbb{N}^*$ for all $i=1,...,q+r$. Denote by t the least common multiple of t_i for all $i=1,...,q+r$. It follows that for each $i=1,...,q+r$ there exists $k_i\in\mathbb{N}^*$ such that $t=k_it_i$.

If we assume that $\sum_{i=1}^{q}\mu_i\chi_{A_i}\leqslant\sum_{i=q+1}^{q+r}\mu_i\chi_{A_i}$, then

$$\sum_{i=1}^{q}\frac{s_i}{t_i}\chi_{A_i}\leqslant\sum_{i=q+1}^{q+r}\frac{s_i}{t_i}\chi_{A_i}\Rightarrow\sum_{i=1}^{q}\frac{s_ik_i}{t}\chi_{A_i}\leqslant\sum_{i=q+1}^{q+r}\frac{s_ik_i}{t}\chi_{A_i}.$$

By using *Case 1*, we obtain:

$$\begin{aligned}&\sum_{i=1}^{q}s_ik_i\varphi_{\delta_i}(A_i)\leqslant\sum_{i=q+1}^{q+r}s_ik_i\varphi_{\delta_i}(A_i)\Rightarrow\\&\sum_{i=1}^{q}\frac{s_i}{t_i}\varphi_{\delta_i}(A_i)\leqslant\sum_{i=q+1}^{q+r}\frac{s_i}{t_i}\varphi_{\delta_i}(A_i)\Rightarrow\\&\sum_{i=1}^{q}\mu_i\varphi_{\delta_i}(A_i)\leqslant\sum_{i=q+1}^{q+r}\mu_i\varphi_{\delta_i}(A_i).\end{aligned}$$

Case 3. Suppose that $\mu_i \in \mathbb{R}_+$ for all $i = 1,...,q+r$. We will use *Case 2* and that F is Archimedean. More precisely for all $i = 1,...,q+r$ there exist two sequences of positive rational numbers, $(\lambda_{i,n})_{n\in\mathbb{N}^*}$ *increasing* and $(\rho_{i,n})_{n\in\mathbb{N}^*}$ *decreasing,* such that $\lambda_{i,n} \to \mu_i$ and $\rho_{i,n} \to \mu_i$ as $n \to \infty$. We know that $\sum\limits_{i=1}^{q} \mu_i \chi_{A_i} \leqslant \sum\limits_{i=q+1}^{q+r} \mu_i \chi_{A_i}$. Then because $0 \leqslant \lambda_{i,n} \leqslant \mu_i \leqslant \rho_{i,n}$ for each $i = 1,...,q+r$ and each $n \in \mathbb{N}^*$, it follows that $\sum\limits_{i=1}^{q} \lambda_{i,n}\chi_{A_i} \leqslant \sum\limits_{i=q+1}^{q+r} \rho_{i,n}\chi_{A_i}$. According to Case 2, it follows that $\sum\limits_{i=1}^{q} \lambda_{i,n}\varphi_{\delta_i}(A_i) \leqslant \sum\limits_{i=q+1}^{q+r} \rho_{i,n}\varphi_{\delta_i}(A_i)$ for all $n \in \mathbb{N}^*$. Because F is Archimedean according to [2, Proposition 2, p.67], we may pass to the limit with n in both inequalities, obtaining $\sum\limits_{i=1}^{q} \mu_i\varphi_{\delta_i}(A_i) \leqslant \sum\limits_{i=q+1}^{q+r} \mu_i\varphi_{\delta_i}(A_i)$.

Hence we proved that ii)$\Leftrightarrow$ii$'$). To prove ii)$\Rightarrow$i) we shall show that ii$'$)$\Rightarrow$i).

For all $\delta \in \Delta$, define $G_\delta = D(\mathscr{A}_\delta)\left(= span\left\{\chi_{A_\delta} \mid A_\delta \in \mathscr{A}_\delta\right\}\right)$ and consider $T_\delta : G_\delta \to F$ the *representing linear operator* of φ_δ, that is,

$$T_\delta\left(\sum_{i=1}^{n} \alpha_i\chi_{A_i}\right) = \sum_{i=1}^{n} \alpha_i\varphi_\delta(A_i).$$

As it turns out, for all $\delta \in \Delta$, G_δ is a *majorizing* subspace in the space $D(E)$, denoted by E_0. (Recall that $D(E) = D(2^\Omega) = span\{\chi_M | M \subseteq \Omega\}$.) To show this, we have to prove that if $\delta \in \Delta$ and $x \in E_0$, then there exists $v_\delta \in G_\delta$ such that $x \leqslant v_\delta$. Take $x = \sum\limits_{i=1}^{n} \alpha_i\chi_{M_i}$, with $n \in \mathbb{N}^*$, and $M_i \subseteq \Omega$, for $i = 1,...,n$. It follows that

$$x \leqslant \sum_{i=1}^{n} |\alpha_i|\,\chi_{M_i} \leqslant \sum_{i=1}^{n} |\alpha_i|\,\chi_\Omega.$$

Now, take $v_\delta = \sum\limits_{i=1}^{n} |\alpha_i|\,\chi_\Omega$ and note that $v_\delta \in G_\delta$, because $\mathscr{A}_\delta$ is an algebra of subsets of Ω and $G_\delta = span\left\{\chi_{A_\delta} | A_\delta \in \mathscr{A}_\delta\right\}$.

Since for all $\delta \in \Delta$ the vector measure φ_δ is *positive*, it follows that the operator T_δ is *positive*, too.

Define $T : span\left(\bigcup\limits_{\delta\in\Delta} G_\delta\right) \to F$ by

$$T\left(\sum_{\delta\in\Delta} f_\delta\right) = \sum_{\delta\in\Delta} T_\delta(f_\delta),$$

where $(f_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$.

(Recall that $\Phi\left((G_\delta)_{\delta\in\Delta}\right)$ is the family of all $\{f_\delta \in G_\delta | \delta \in \Delta\}$ such that $f_\delta \neq 0$ for at most finitely many $\delta \in \Delta$.)

Firstly, we will prove that T is *well-defined*. It will suffice to prove that

$$0 \leqslant \sum_{\delta\in\Delta} f_\delta \Rightarrow 0 \leqslant \sum_{\delta\in\Delta} T_\delta(f_\delta)$$

for all $(f_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$.

Let $\delta_1, ..., \delta_n$ be the indices of all nonzero elements in the set $\{f_\delta \,|\, \delta \in \Delta\}$. Then the inequality $0 \leqslant \sum\limits_{\delta\in\Delta} f_\delta$ reduces to the inequality

$$0 \leqslant f_{\delta_1} + \cdots + f_{\delta_n}, \tag{11}$$

with $f_{\delta_1} \in G_{\delta_1}, ..., f_{\delta_n} \in G_{\delta_n}$, that is,

$$f_{\delta_1} = \sum_{i=1}^{m_1} \lambda_{\delta_1 i} \chi_{A_{\delta_1 i}}, \text{ where } A_{\delta_1 i} \in \mathscr{A}_{\delta_1} \text{ and } \lambda_{\delta_1 i} \in \mathbb{R}, \text{ for all } i = 1, ..., m_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_{\delta_n} = \sum_{i=1}^{m_n} \lambda_{\delta_n i} \chi_{A_{\delta_n i}}, \text{ where } A_{\delta_n i} \in \mathscr{A}_{\delta_n} \text{ and } \lambda_{\delta_n i} \in \mathbb{R}, \text{ for all } i = 1, ..., m_n.$$

Then (11) becomes

$$0 \leqslant \sum_{i=1}^{m_1} \lambda_{\delta_1 i} \chi_{A_{\delta_1 i}} + \cdots + \sum_{i=1}^{m_n} \lambda_{\delta_n i} \chi_{A_{\delta_n i}}.$$

By renumbering the indices in the right side of this inequality, starting with the second sum, we obtain:

$$0 \leqslant \sum_{i=1}^{m_1} \lambda_{\delta_1 i} \chi_{A_{\delta_1 i}} + \sum_{i=m_1+1}^{m_1+m_2} \lambda_{\delta_2 i} \chi_{A_{\delta_2 i}} + \cdots + \sum_{i=m_1+\cdots+m_{n-1}+1}^{m_1+\cdots+m_n} \lambda_{\delta_n i} \chi_{A_{\delta_n i}}. \tag{12}$$

Note that the right side of inequality (12) is a sum having $p = m_1 + m_2 + \cdots + m_n$ terms. Move to the left side of the inequality all terms having negative coefficients changing the sign on the way, and apply ii$'$). Next, in the new inequality obtained, move all the terms from the left to the right side with a due change of sign. We obtain:

$$0 \leqslant \sum_{i=1}^{m_1} \lambda_{\delta_1 i} \varphi_{\delta_1 i}\left(A_{\delta_1 i}\right) + \cdots + \sum_{i=1}^{m_n} \lambda_{\delta_n i} \varphi_{\delta_n i}\left(A_{\delta_n i}\right),$$

that is,

$$0 \leqslant \sum_{\delta\in\Delta} T_\delta\left(f_\delta\right).$$

Therefore, the operator T is well-defined. Moreover, T is linear and positive.

Note that the vector space $G = span\left(\bigcup\limits_{\delta\in\Delta} G_\delta\right)$ is a *majorizing* subspace in $E_0 = D(E)$ because each G_δ is so.

Define $S : E_0 \to F$ by $S(f) = \inf\{T(g) \,|\, g \in G, g \geqslant f\}$, that is, $S(f) = \overline{T}(f)$. It is known that S is a monotone sublinear operator and that $T \leqslant S$ on E_0.

For all $(f_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$ we have:

$$\sum_{\delta\in\Delta} T_\delta\left(f_\delta\right) = T\left(\sum_{\delta\in\Delta} f_\delta\right) \leqslant S\left(\sum_{\delta\in\Delta} f_\delta\right).$$

Then, according to the implication ii)$\Rightarrow$i) of Theorem 10, it follows that the family $(T_\delta)_{\delta\in\Delta}$ can be simultaneously extended to a linear operator $L : E \to F$.

Define $\varphi : E \to F$ by $\varphi(A) = L(\chi_A)$ for all $A \subseteq \Omega$. Clearly, φ is a *positive* vector measure which extends all the vector measures φ_δ, $\delta \in \Delta$. $\square$

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