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JORDAN DERIVATIONS ON BLOCK UPPER TRIANGULAR MATRIX ALGEBRAS

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Abstract. We provide that any Jordan derivation from the block upper triangular matrix algebra $\mathscr{T}=\mathscr{T}(n_1,n_2,\cdots,n_k)\subseteq M_n(\mathscr{C})$ into a 2-torsion free unital $\mathscr{T}$-bimodule is the sum of a derivation and an antiderivation.

1. Introduction

Throughout this paper $\mathscr{C}$ will denote a commutative ring with unity. Let $\mathscr{A}$ be an algebra over $\mathscr{C}$. Recall that a $\mathscr{C}$-linear map D from $\mathscr{A}$ into an $\mathscr{A}$-bimodule $\mathscr{M}$ is said to be a *Jordan derivation* if $D(ab+ba)=D(a)b+aD(b)+D(b)a+bD(a)$ for all $a,b\in\mathscr{A}$. It is called a *derivation* if $D(ab)=D(a)b+aD(b)$ for all $a,b\in\mathscr{A}$. If D is only additive, we will call D is an *additive (Jordan) derivation*. For an element $m\in\mathscr{M}$, the mapping $I_m:\mathscr{A}\to\mathscr{M}$, given by $I_m(a)=am-ma$, is a derivation which will be called an *inner derivation*. Also D is called an *antiderivation* if $D(ab)=D(b)a+bD(a)$ for all $a,b\in\mathscr{A}$. Clearly, each derivation or antiderivation is a Jordan derivation. The converse is, in general, not true (see [1]).

The question under what conditions that a map becomes a derivation attracted much attention of mathematicians and hence it is natural and interesting to find some conditions under which a Jordan derivation is a derivation. Herstein [4] proved that every additive Jordan derivation from a 2-torsion free prime ring into itself is an additive derivation. Brešar [2] proved that Herstein's result is true for 2-torsion free semiprime rings. Sinclair [8] proved that every continuous Jordan derivation on semisimple Banach algebras is a derivation. Johnson showed in [5] that a continuous Jordan derivation from a C^*-algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule is a derivation. Zhang in [9] proved that every Jordan derivation on nest algebras is an inner derivation. Li and Lu [7] showed that every additive Jordan derivation on reflexive algebras is an additive derivation which generalized the result in [9]. By a classical result of Jacobson and Rickart [6] every additive Jordan derivation on a full matrix ring over a 2-torsion free unital ring is an additive derivation. In [3], the author proved that any additive Jordan derivation from a full matrix ring over a unital ring into any of its 2-torsion free bimodule (not necessarily unital) is an additive derivation which generalized the result in [6].

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Benković [1] determined Jordan derivations on triangular matrices over commutative rings and proved that every Jordan derivation from the algebra of all upper triangular matrices into its arbitrary unital bimodule is the sum of a derivation and an antiderivation. Zhang and Yu [10] showed that every Jordan derivation of triangular algebras is a derivation.

In this note we prove that any Jordan derivation from the block upper triangular matrix algebra $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k) \subseteq M_n(\mathscr{C})$ into a 2-torsion free unital $\mathscr{T}$-bimodule is the sum of a derivation and an antiderivation, where $\mathscr{C}$ is a commutative ring with unity. This result generalizes the main result of [1].

2. Preliminaries

Throughout this paper, by $M_n(\mathscr{C})$, $n \geqslant 1$, we denote the algebra of all $n \times n$ matrices over $\mathscr{C}$, by $T_n(\mathscr{C})$ its subalgebra of all upper triangular matrices, and by $D_n(\mathscr{C})$ its subalgebra of all diagonal matrices. We shall denote the identity matrix by I. Also, E_{ij} is the matrix unit and $x_{i,j}$ is the (ij)th entry of $X \in M_n(\mathscr{C})$ for $1 \leqslant i, j \leqslant n$. Hence we have $E_{ii} X E_{jj} = x_{i,j} E_{ij}$ for $X \in M_n(\mathscr{C})$ and $1 \leqslant i, j \leqslant n$.

For $n \geqslant 1$ and a finite sequence of positive integers $n_1, n_2, \cdots, n_k$ $(k \geqslant 1)$, satisfying $n_1 + n_2 + \cdots + n_k = n$, let $\mathscr{T}(n_1, n_2, \cdots, n_k)$ be the subalgebra of $M_n(\mathscr{C})$ of all matrices of the form

$$X = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ 0 & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{kk} \end{pmatrix},$$

where X_{ij} is an $n_i \times n_j$ matrix. We call such an algebra a *block upper triangular matrix algebra*. Also we call k is the number of *summands of* $\mathscr{T}(n_1, n_2, \cdots, n_k)$. Note that $M_n(\mathscr{C})$ is a special case of block upper triangular matrix algebras. In particular, if $k = 1$ with $n_1 = n$, then $\mathscr{T}(n_1, n_2, \cdots, n_k) = M_n(\mathscr{C})$. Also, when $k = n$ and $n_i = 1$ for every $1 \leqslant i \leqslant k$, we have $\mathscr{T}(n_1, n_2, \cdots, n_k) = T_n(\mathscr{C})$.

Let $F_1 = \sum_{i=1}^{n_1} E_i$ and $F_j = \sum_{i=1}^{n_j} E_{i+n_1+\cdots+n_{j-1}}$ for $2 \leqslant j \leqslant k$, where $E_l = E_{ll}$. Then $\{F_1, \ldots, F_k\}$ is a set of non-trivial idempotents of $\mathscr{T}(n_1, n_2, \cdots, n_k)$ such that $F_1 + \cdots + F_k = I$ and $F_i F_j = F_j F_i = 0$ for $1 \leqslant i, j \leqslant k$ with $i \neq j$. Moreover, we have $F_j \mathscr{T}(n_1, n_2, \cdots, n_k) F_j \cong M_{n_j}(\mathscr{C})$ for any $1 \leqslant j \leqslant k$. We use $\mathscr{D}(n_1, n_2, \cdots, n_k)$ for a subalgebra of $\mathscr{T}(n_1, n_2, \cdots, n_k)$ defined by

$$\mathscr{D}(n_1, n_2, \cdots, n_k) = F_1 \mathscr{T}(n_1, n_2, \cdots, n_k) F_1 + \cdots + F_k \mathscr{T}(n_1, n_2, \cdots, n_k) F_k.$$

Note that, if $\mathscr{T}(n_1, n_2, \cdots, n_k) = T_n(\mathscr{C})$, then $\mathscr{D}(n_1, n_2, \cdots, n_k) = D_n(\mathscr{C})$.

By $[X, Y] = XY - YX$ we denote the commutator or the Lie product of elements $X, Y \in M_n(\mathscr{C})$.

3. Main result

From [3, Theorem 3.2] and the fact that every Jordan derivation from $\mathscr{C}$ into its bimoduls is zero, we have the following lemma which will be needed in the proofs of our results.

LEMMA 3.1. *Every Jordan derivation from $M_n(\mathscr{C})$, for $n \geqslant 1$, into any of its bimodules is a derivation.*

In this note, our main result is the following theorem.

THEOREM 3.2. *Let $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k)$ be a block upper triangular algebras in $M_n(\mathscr{C})$ $(n \geqslant 1)$ and $\mathscr{M}$ be a 2-torsion free unital $\mathscr{T}$-bimodule. Suppose that $D : \mathscr{T} \to \mathscr{M}$ is a Jordan derivation. Then there exist a derivation $d : \mathscr{T} \to \mathscr{M}$ and an antiderivation $\alpha : \mathscr{T} \to \mathscr{M}$ such that $D = d + \alpha$ and $\alpha(\mathscr{D}(n_1, n_2, \cdots, n_k)) = \{0\}$. Moreover, d and α are uniquely determined.*

Proof. The proof is by induction on k, the number of summands of $\mathscr{T}$. If $k = 1$, then $\mathscr{T} = M_n(\mathscr{C})$ and $\mathscr{D}(n_1) = M_n(\mathscr{C})$. So by Lemma 3.1, D is derivation and $\alpha = 0$ is the only antiderivation such that $\alpha(\mathscr{D}(n_1)) = 0$. Hence the result is obvious in this case.

Assume inductively that $k \geqslant 1$ and the result holds for each block upper triangular algebra $\mathscr{T}(n_1, n_2, \cdots, n_k)$ with k summands.

Let $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_{k+1}) \subseteq M_n(\mathscr{C})$ be a block upper triangular algebra with $n_1 + n_2 + \cdots + n_{k+1} = n$.

Set $P = F_1$ and $Q = I - P = F_2 + \cdots + F_{k+1}$. Then P and Q are nontrivial idempotents of $\mathscr{T}$ such that $PQ = QP = 0$. Also $Q\mathscr{T}P = \{0\}$, $P\mathscr{T}P$ and $Q\mathscr{T}Q$ are subalgebras of $\mathscr{T}$ with unity P and Q, respectively, and $\mathscr{T} = P\mathscr{T}P \dotplus P\mathscr{T}Q \dotplus Q\mathscr{T}Q$ as sum of $\mathscr{C}$-linear spaces. Moreover, $P\mathscr{T}P \cong M_{n_1}(\mathscr{C})$ and $Q\mathscr{T}Q \cong \mathscr{T}(n_2, n_3, \cdots, n_{k+1}) \subseteq M_{n-n_1}(\mathscr{C})$ ($\mathscr{C}$-algebra isomorphisms) is a block upper triangular algebra with k summands $n_2, \cdots, n_{k+1}$, where $\mathscr{D}(n_2, \cdots, n_{k+1}) \cong F_2\mathscr{T}F_2 + \cdots + F_{k+1}\mathscr{T}F_{k+1}$.

Suppose $\mathscr{M}$ is a 2-torsion free unital $\mathscr{T}$-bimodule and $D : \mathscr{T} \to \mathscr{M}$ is a Jordan derivation. Define $\Delta : \mathscr{T} \to \mathscr{M}$ by $\Delta(X) = D(X) - I_B(X)$, where $B = PD(P)Q - QD(P)P$. Then Δ is a Jordan derivation such that $P\Delta(P)Q = Q\Delta(P)P = 0$. We will show that Δ is the sum of a derivation and an antiderivation.

We complete the proof by checking some steps.

Step 1. $\Delta(X) = P\Delta(PXP)P + P\Delta(PXQ)Q + Q\Delta(PXQ)P + Q\Delta(QXQ)Q$ for all $X \in \mathscr{T}$.

Let $X \in \mathscr{T}$. Since $P(QXQ) + (QXQ)P = 0$, we have

$$P\Delta(QXQ) + \Delta(P)QXQ + QXQ\Delta(P) + \Delta(QXQ)P = 0. \tag{3.1}$$

Multiplying this identity by P both on the left and on the right we arrive at $2P\Delta(QXQ)P = 0$ so $P\Delta(QXQ)P = 0$. Now, multiplying the Equation(3.1) from the left by P, from the right by Q and by the fact that $P\Delta(P)Q = 0$, we find $P\Delta(QXQ)Q = 0$. Similarly, from Equation(3.1) and the fact that $Q\Delta(P)P = 0$, we see that $Q\Delta(QXQ)P = 0$. Therefore, from above equations we get

$$\Delta(QXQ) = Q\Delta(QXQ)Q$$

Applying Δ to $(PXP)Q + Q(PXP) = 0$, we see that

$$PXP\Delta(Q) + \Delta(PXP)Q + Q\Delta(PXP) + \Delta(Q)PXP = 0. \tag{3.2}$$

By $\Delta(QXQ) = Q\Delta(QXQ)Q$, Equation(3.2) and using similar methods as above we obtain

$$\Delta(PXP) = P\Delta(PXP)P.$$

Since $P(PXQ) + (PXQ)P = PXQ$, we have

$$P\Delta(PXQ) + \Delta(P)PXQ + PXQ\Delta(P) + \Delta(PXQ)P = \Delta(PXQ). \tag{3.3}$$

Multiplying Equation(3.3) by P both on the left and on the right and by the fact that $Q\Delta(P)P = 0$, we get $P\Delta(PXQ)P = 0$. Now multiplying Equation(3.3) by Q both on the left and on the right and by the fact that $Q\Delta(P)P = 0$, we have $Q\Delta(PXQ)Q = 0$. Hence from these equations we find

$$\Delta(PXQ) = P\Delta(PXQ)Q + Q\Delta(PXQ)P.$$

Now from above results we have

$$\begin{aligned}\Delta(X) &= \Delta(PXP) + \Delta(PXQ) + \Delta(QXQ)\\ &= P\Delta(PXP)P + P\Delta(PXQ)Q + Q\Delta(PXQ)P + Q\Delta(QXQ)Q.\end{aligned}$$

Step 2. $P\Delta(PXPYP)P = PXP\Delta(PYP)P + P\Delta(PXP)PYP$ for all $X,Y \in \mathscr{T}$.

$P\mathscr{M}P$ is a 2-torsion free unital $P\mathscr{T}P$-bimodule. Define $J : P\mathscr{T}P \to P\mathscr{M}P$ by $J(PXP) = P\Delta(PXP)P$. Clearly J is a well defined linear map. Since Δ is a Jordan derivation, it follows that J is a Jordan derivation. By Lemma 3.1 and the fact that $P\mathscr{T}P \cong M_{n_1}(\mathscr{C})$, we see that J is a derivation. So we obtain the result of this step.

Step 3. $P\Delta(PXPYQ)Q = PXP\Delta(PYQ)Q + P\Delta(PXP)PYQ$ and $P\Delta(PXQYQ)Q = PXQ\Delta(QYQ)Q + P\Delta(PXQ)QYQ$ for all $X,Y \in \mathscr{T}$.

Let $X,Y \in \mathscr{T}$. Applying Δ to the equations: $PXPYQ = (PXP)(PYQ) + (PYQ)(PXP)$ and $PXQYQ = (PXQ)(QYQ) + (QYQ)(PXQ)$, we get

$$\begin{aligned}&\Delta(PXPYQ) = PXP\Delta(PYQ) + \Delta(PXP)PYQ + PYQ\Delta(PXP) + \Delta(PYQ)PXP\\ &\text{and}\\ &\Delta(PXQYQ) = PXQ\Delta(QYQ) + \Delta(PXQ)QYQ + QYQ\Delta(PXQ) + \Delta(QYQ)PXQ.\end{aligned} \tag{3.4}$$

Multiplying these identities by P on the left and by Q on the right, from Step 1 we yield the result.

Step 4. There exists a derivation $g : Q\mathscr{T}Q \to Q\mathscr{M}Q$ and an antiderivation $\gamma : Q\mathscr{T}Q \to Q\mathscr{M}Q$ such that $Q\Delta(QXQ)Q = g(QXQ) + \gamma(QXQ)$ for all $X \in \mathscr{T}$. Moreover, $\gamma(F_2\mathscr{T}F_2 + \cdots + F_{k+1}\mathscr{T}F_{k+1}) = \{0\}$ and $PXQ\gamma(QYQ) = 0$ for all $X,Y \in \mathscr{T}$.

$Q\mathscr{M}Q$ is a 2-torsion free unital $Q\mathscr{T}Q$-bimodule. Define $G : Q\mathscr{T}Q \to Q\mathscr{M}Q$ by $G(QXQ) = Q\Delta(QXQ)Q$. Clearly G is a well defined linear map. Since Δ is a Jordan derivation, we see that G is a Jordan derivation. In view of the isomorphisms $Q\mathscr{T}Q \cong \mathscr{T}(n_2,n_3,\cdots,n_{k+1}) \subseteq M_{n-n_1}(\mathscr{C})$, $\mathscr{D}(n_2,\cdots,n_{k+1}) \cong F_2\mathscr{T}F_2 + \cdots + F_{k+1}\mathscr{T}F_{k+1}$ and induction hypothesis, there exists a derivation $g : Q\mathscr{T}Q \to Q\mathscr{M}Q$ and an antiderivation $\gamma : Q\mathscr{T}Q \to Q\mathscr{M}Q$ such that $Q\Delta(QXQ)Q = G(QXQ) = g(QXQ) + \gamma(QXQ)$ for all

$X \in \mathscr{T}$. Also, $\gamma(F_2\mathscr{T}F_2+\cdots+F_{k+1}\mathscr{T}F_{k+1})=\{0\}$. We will show that $PXQ\gamma(QYQ)=0$ for all $X,Y \in \mathscr{T}$.

By Step 3 and above results for all $X,Y,Z \in \mathscr{T}$, we have

$$\begin{aligned} P\Delta(PXQYQZQ)Q &= PXQ\Delta(QYQZQ)Q + P\Delta(PXQ)QYQZQ \\ &= PXQg(QYQZQ) + PXQ\gamma(QYQZQ) + P\Delta(PXQ)QYQZQ. \end{aligned}$$

On the other hand,

$$\begin{aligned} P\Delta(PXQYQZQ)Q &= PXQYQ\Delta(QZQ)Q + P\Delta(PXQYQ)QZQ \\ &= PXQYQ\Delta(QZQ)Q + PXQ\Delta(QYQ)QZQ + P\Delta(PXQ)QYQZQ \\ &= PXQYQg(QZQ) + PXQYQ\gamma(QZQ) + PXQg(QYQ)QZQ \\ &\quad + PXQ\gamma(QYQ)QZQ + P\Delta(PXQ)QYQZQ \\ &= PXQg(QYQZQ) + PXQ\gamma(QZQYQ) + P\Delta(PXQ)QYQZQ, \end{aligned}$$

since g is a derivation and γ is an antiderivation. By comparing the two expressions for $P\Delta(PXQYQZQ)Q$, we arrive at

$$PXQ\gamma([QYQ,QZQ]) = 0, \tag{3.5}$$

for all $X,Y,Z \in \mathscr{T}$. Now from the fact that $Q = F_2+\cdots+F_{k+1}$ and $F_jQ = QF_j = F_j$ for all $2 \leqslant j \leqslant k+1$, we have

$$\begin{aligned} QXQ - \sum_{j=2}^{k+1} F_jXF_j &= (\sum_{j=2}^{k+1} F_j)QXQ - \sum_{j=2}^{k+1} F_jXF_j = \sum_{j=2}^{k+1}(F_jXQ - F_jXF_j) \\ &= \sum_{j=2}^{k+1} F_jX(Q-F_j) = \sum_{j=2}^{k+1}[F_j, F_jX(Q-F_j)], \end{aligned} \tag{3.6}$$

for all $X \in \mathscr{T}$. Note that $F_j, F_jX(Q-F_j) \in Q\mathscr{T}Q$. By Equation (3.5), (3.6) and $\gamma(F_2\mathscr{T}F_2+\cdots+F_{k+1}\mathscr{T}F_{k+1})=\{0\}$, we conclude that

$$\begin{aligned} PXQ\gamma(QYQ) &= PXQ\gamma(QYQ - \sum_{j=2}^{k+1} F_jYF_j + \sum_{j=2}^{k+1} F_jYF_j) \\ &= PXQ\gamma(QYQ - \sum_{j=2}^{k+1} F_jYF_j) \\ &= PXQ\gamma([F_j, F_jY(Q-F_j)]) = 0, \end{aligned}$$

for all $X,Y \in \mathscr{T}$.

Step 5. $PXQ\Delta(PYQ)P = 0$ and $Q\Delta(PXQ)PYQ = 0$ for all $X,Y \in \mathscr{T}$.

Multiplying Equations (3.4) by Q on the left and by P on the right, we have

$$Q\Delta(PXPYQ)P = Q\Delta(PYQ)PXP$$

and (3.7)

$$Q\Delta(PXQYQ)P = QYQ\Delta(PXQ)P,$$

for all $X,Y \in \mathscr{T}$. Now applying Δ to $(PXQ)(PYQ)+(PYQ)(PXQ)=0$ for any $X,Y \in \mathscr{T}$, we see that

$$PXQ\Delta(PYQ)+\Delta(PXQ)PYQ+PYQ\Delta(PXQ)+\Delta(PYQ)PXQ=0.$$

From this identity we get the following equations.

$$PXQ\Delta(PYQ)P+PYQ\Delta(PXQ)P=0 \tag{3.8}$$

and

$$Q\Delta(PXQ)PYQ+Q\Delta(PYQ)PXQ=0 \tag{3.9}$$

for all $X,Y \in \mathscr{T}$. Let $1 \leqslant i,k \leqslant n_1$ and $n_1 < j,l \leqslant n$ be arbitrary. By Equations(3.7) and Equation(3.8) we have

$$\begin{aligned}
E_{ij}\Delta(E_{kl})P &= E_{ij}\Delta(E_{ki}E_{il})P = E_{ij}\Delta(E_{il})E_{ki}\\
&= E_{ij}\Delta(E_{ij}E_{jl})E_{ki} = E_{ij}E_{jl}\Delta(E_{ij})E_{ki} = E_{il}\Delta(E_{ij})E_{ki}\\
&= -E_{ij}\Delta(E_{il})E_{ki} = -E_{ij}\Delta(E_{ki}E_{il})P = -E_{ij}\Delta(E_{kl})P,
\end{aligned}$$

since $E_{ki} \in P\mathscr{T}P$, $E_{ij},E_{il},E_{kl} \in P\mathscr{T}Q$, $E_{jl} \in Q\mathscr{T}Q$. So $E_{ij}\Delta(E_{kl})P=0$. Also by Equations(3.7) and Equation(3.9) we find

$$\begin{aligned}
Q\Delta(E_{ij})E_{kl} &= Q\Delta(E_{il}E_{lj})E_{kl} = E_{lj}\Delta(E_{il})E_{kl}\\
&= E_{lj}\Delta(E_{ik}E_{kl})E_{kl} = E_{lj}\Delta(E_{kl})E_{ik}E_{kl} = E_{lj}\Delta(E_{kl})E_{il}\\
&= -E_{lj}\Delta(E_{il})E_{kl} = -Q\Delta(E_{il}E_{lj})E_{kl} = -Q\Delta(E_{ij})E_{kl},
\end{aligned}$$

since $E_{ik} \in P\mathscr{T}P$, $E_{ij},E_{il},E_{kl} \in P\mathscr{T}Q$, $E_{lj} \in Q\mathscr{T}Q$. Hence $Q\Delta(E_{ij})E_{kl}=0$. For any $X,Y \in \mathscr{T}$, let $PXQ=\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} x_{i,j}E_{ij}$ and $PYQ=\sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n} y_{k,l}E_{kl}$. Therefore, by identities $E_{ij}\Delta(E_{kl})P=0$, $Q\Delta(E_{ij})E_{kl}=0$ and linearity of Δ it follows that

$$\begin{aligned}
PXQ\Delta(PYQ)P &= \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} x_{i,j}E_{ij}\Delta(\sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n} y_{k,l}E_{kl})P\\
&= \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}\sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n} x_{i,j}y_{k,l}E_{ij}\Delta(E_{kl})P=0,
\end{aligned}$$

and

$$\begin{aligned}
Q\Delta(PXQ)PYQ &= \sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n} Q\Delta(\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} x_{i,j}E_{ij})y_{k,l}E_{kl}\\
&= \sum_{k=1}^{n_1}\sum_{l=n_1+1}^{n}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} x_{i,j}y_{k,l}Q\Delta(E_{ij})E_{kl}=0.
\end{aligned}$$

Step 6. The mapping $\delta : \mathscr{T} \to \mathscr{M}$, given by

$$\delta(X)=P\Delta(PXP)P+P\Delta(PXQ)Q+g(QXQ)$$

is a derivation and the mapping $\alpha : \mathscr{T} \to \mathscr{M}$, given by

$$\alpha(X) = Q\Delta(PXQ)P + \gamma(QXQ)$$

is an antiderivation such that $\alpha(\mathscr{D}(n_1,n_2,\cdots,n_{k+1})) = \{0\}$. Moreover, $\Delta = \delta + \alpha$.

Clearly, δ is a linear map. By Steps 2, 3, 4 and the fact that $Q\mathscr{T}P = \{0\}$ one can check directly that δ is a derivation.

It is clear that α is a linear map. For each $X,Y \in \mathscr{T}$, by Equations(3.7), Steps 4, 5 and the fact that $Q\mathscr{T}P = \{0\}$, we have

$$\begin{aligned}\alpha(XY) &= Q\Delta(PXPYQ)P + Q\Delta(PXQYQ)P + \gamma(QXQYQ)\\ &= Q\Delta(PYQ)PXP + QYQ\Delta(PXQ)P + QYQ\gamma(QXQ) + \gamma(QYQ)QXQ\\ &\quad + Q\Delta(PYQ)PXQ + PYQ\Delta(PXQ)P + PYQ\gamma(QXQ) + \gamma(QYQ)QXP\\ &= Y\alpha(X) + \alpha(Y)X.\end{aligned}$$

Let $F_1X_1F_1 + F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1}$ be an arbitrary element of $\mathscr{D}(n_1,n_2,\cdots,n_{k+1})$. Since $\gamma(F_2\mathscr{T}F_2 + \cdots + F_{k+1}\mathscr{T}F_{k+1}) = \{0\}$, $F_1Q = QF_1 = 0$, $PF_j = F_jP = 0$ and $F_jQ = QF_j = F_j$ for any $2 \leqslant j \leqslant k$, it follows that

$$\begin{aligned}&\alpha(F_1X_1F_1 + F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1})\\ &\quad = Q\Delta(P(F_1X_1F_1 + F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1})Q)P\\ &\qquad + \gamma(Q(F_1X_1F_1 + F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1})Q)\\ &\quad = \gamma(F_2X_2F_2 + \cdots + F_{k+1}X_{k+1}F_{k+1}) = 0.\end{aligned}$$

So $\alpha(\mathscr{D}(n_1,n_2,\cdots,n_{k+1})) = \{0\}$. By Steps 1, 4, it is obvious that $\Delta = \delta + \alpha$.

Now from the above results we have $D - I_B = \Delta = \delta + \alpha$, where $\delta : \mathscr{T} \to \mathscr{M}$ is a derivation, $\alpha : \mathscr{T} \to \mathscr{M}$ is an antiderivation and $\alpha(\mathscr{D}(n_1,n_2,\cdots,n_{k+1})) = \{0\}$. So the mapping $d : \mathscr{T} \to \mathscr{M}$ given by $d = \delta + I_B$ is a derivation and we find $D = d + \alpha$.

Finally, we will show that d and α are uniquely determined. Suppose that $D = d' + \alpha'$, where $d' : \mathscr{T} \to \mathscr{M}$ is a derivation, $\alpha' : \mathscr{T} \to \mathscr{M}$ is an antiderivation and $\alpha'(\mathscr{D}(n_1,n_2,\cdots,n_{k+1})) = \{0\}$. Hence $D_{|Q\mathscr{T}Q} : Q\mathscr{T}Q \to \mathscr{M}$, the restriction of D to $Q\mathscr{T}Q$, is a Jordan derivation. So $D_{|Q\mathscr{T}Q} = d_{|Q\mathscr{T}Q} + \alpha_{|Q\mathscr{T}Q} = d'_{|Q\mathscr{T}Q} + \alpha'_{|Q\mathscr{T}Q}$, where $d_{|Q\mathscr{T}Q}, d'_{|Q\mathscr{T}Q} : Q\mathscr{T}Q \to \mathscr{M}$ are derivations, $\alpha_{|Q\mathscr{T}Q}, \alpha'_{|Q\mathscr{T}Q} : Q\mathscr{T}Q \to \mathscr{M}$ are antiderivations and $\alpha_{|Q\mathscr{T}Q}, \alpha'_{|Q\mathscr{T}Q}(F_2\mathscr{T}F_2 + \cdots + F_{k+1}\mathscr{T}F_{k+1}) = \{0\}$. Since $Q\mathscr{T}Q \cong \mathscr{T}(n_2,n_3,\cdots,n_{k+1}) \subseteq M_{n-n_1}(\mathscr{C})$, $\mathscr{D}(n_2,\cdots,n_{k+1}) \cong F_2\mathscr{T}F_2 + \cdots + F_{k+1}\mathscr{T}F_{k+1}$, by the uniqueness in induction hypothesis it follows that $\alpha_{|Q\mathscr{T}Q} = \alpha'_{|Q\mathscr{T}Q}$ and $d_{|Q\mathscr{T}Q} = d'_{|Q\mathscr{T}Q}$. Define $\beta : \mathscr{T} \to \mathscr{M}$ by $\beta = \alpha - \alpha'$. Clearly β is a linear map and $\beta = \alpha - \alpha' = d' - d$. So β is a derivation and an antiderivation. Since $\alpha(\mathscr{D}(n_1,n_2,\cdots,n_{k+1})) = \alpha'(\mathscr{D}(n_1,n_2,\cdots,n_{k+1})) = \{0\}$, it follows that $\alpha(PXP) = \alpha'(PXP) = 0$ for all $X \in \mathscr{T}$, so $\beta(PXP) = 0$ for all $X \in \mathscr{T}$. Also from $\alpha_{|Q\mathscr{T}Q} = \alpha'_{|Q\mathscr{T}Q}$, we have $\beta(QXQ) = 0$ for all $X \in \mathscr{T}$. Now observe that $\beta(P) = \beta(Q) = 0$. Then, since β is a derivation and an antiderivation, we have

$$\beta(PXQ) = P\beta(XQ) + \beta(P)XQ = P\beta(XQ) = P(Q\beta(X) + \beta(Q)X) = 0.$$

So

$$\beta(X) = \beta(PXP) + \beta(PXQ) + \beta(QXQ) = 0$$

for all $X \in \mathscr{T}$. Therefore, $\alpha = \alpha'$ and hence $d = d'$. The proof of Theorem 3.2 is thus completed. $\square$

We have the following corollary, which was proved in [1].

COROLLARY 3.3. *Let $T_n(\mathscr{C})$ be an upper triangular matrix algebra and $\mathscr{M}$ be a 2-torsion free unital $T_n(\mathscr{C})$-bimodule. Suppose that $D : T_n(\mathscr{C}) \to \mathscr{M}$ is a Jordan derivation. Then there exist a derivation $d : T_n(\mathscr{C}) \to \mathscr{M}$ and an antiderivation $\alpha : T_n(\mathscr{C}) \to \mathscr{M}$ such that $D = d + \alpha$ and $\alpha(D_n(\mathscr{C})) = \{0\}$. Moreover, d and α are uniquely determined.*

REMARK 3.4. In the main theorem it is possible that the antiderivation α equals to zero i.e. $\alpha = 0$. But the theorem doesn't say when any Jordan derivation on block upper triangular matrix algebras is a derivation. So this question may be interested that under what conditions every Jordan derivation from the block upper triangular matrix algebra $\mathscr{T} = \mathscr{T}(n_1, n_2, \cdots, n_k) \subseteq M_n(\mathscr{C})$ into a 2-torsion free unital $\mathscr{T}$-bimodule is a derivation?

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REFERENCES

[1] D. BENKOVIĆ, *Jordan derivations and antiderivations on triangular matrices*, Linear Algebra Appl. **397** (2005), 235–244.
[2] M. BREŠAR, *Jordan derivation on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), 1003–1006.
[3] H. GHAHRAMANI, *Characterizing Jordan derivations of matrix rings through zero products*, Math. Slovaca, in press.
[4] I. N. HERSTEIN, *Jordan derivations on prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.
[5] B. E. JOHNSON, *Symmetric amenability and the nonexistence of Lie and Jordan derivations*, Math. Proc. Camb. Phil. Soc. **120** (1996), 455–473.
[6] N. JACOBSON, C. E. RICKART, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc. **69** (3) (1950), 479–502.
[7] J. LI AND F. Y. LU, *Additive Jordan derivations of reflexive algebras*, J. Math. Anal. Appl. **329** (2007), 102–111.
[8] A. M. SINCLAIR, *Jordan homomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209–214.
[9] J. H. ZHANG, *Jordan derivations on nest algebras*, Acta Math. Sinica, **41** (1998), 205–212.
[10] J. H. ZHANG AND W. Y. YU, *Jordan derivations of triangular algebras*, Linear Algebra Appl. **419** (2006), 251–255.

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