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# A TRANSMISSION PROBLEM FOR THE HELMHOLTZ EQUATION WITH HIGHER ORDER BOUNDARY CONDITIONS

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*Abstract.* The present paper deals with some properties for certain classes of Wiener-Hopf operators associated with a wave diffraction problem. This diffraction problem is mathematically modeled by the Helmholtz equation and higher order boundary conditions on an infinite strip. Different types of operator relations are exhibited for different kinds of operators acting between Lebesgue and Bessel potential spaces on a finite interval and on the positive half-line. In particular, the operators under study are analyzed in detail in what concerns their Fredholm property. At the end, an operator normalization procedure is applied to the critical orders of the spaces where the problem is not normally solvable.

## 1. Introduction

By using methods from operator theory, this paper deals with a boundary value problem for the Helmholtz equation arising in wave diffraction theory. Namely, the boundary value problem is derived from the problem of diffraction of an electromagnetic wave by an infinite strip where certain higher order boundary conditions arise.

We will consider a time-harmonic electromagnetic plane wave incident on a strip in $\mathbb{R}^3$ defined by some inhomogeneous, isotropic, dielectric material and invariant in the z-direction and where we consider independence of electro-magnetic fields on z. So, if we consider the Cartesian coordinate system $Oxyz$ with the y-axis vertically upwards, perpendicular to the strip, the problem can be considered as to be two-dimensional and the strip will be therefore represented by

$$\Sigma := ]0,a[, \qquad \text{for } 0 < a < \infty.$$

This kind of mathematical problems and, in particular, the canonical boundary value problems for time harmonic waves governed by the Helmholtz equation was first studied by A. Sommerfeld [38]. Since then, a great number of different approaches have been presented and used in the applied mathematics and operators literature for

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studying various kinds of wave diffraction problems. For related works and physical relevance of this kind of problems we refer to [3], [4], [5], [8]–[10], [12]–[17], [20]–[22], [25], [28]–[29], [32] and [34].

The electromagnetic wave propagation is governed by the time-harmonic Maxwell equations

$$\nabla \times E - iw\mu H = 0 \qquad \text{and} \qquad \nabla \times H + (iw\eta - \lambda)E = 0,$$

with time dependence $exp(-w\tau)$, frequency $w > 0$ and where $\eta$ denotes the electric permittivity of the medium, $\mu$ is the magnetic permeability, $\lambda$ is the electric conductivity, and $E$ and $H$ represent the electric and magnetic fields, respectively. Since the electric and magnetic fields satisfy the two-dimensional Helmholtz equation, with the same wave number, we consider the electric and the magnetic fields denoted by $u$ and, in addition, certain higher order boundary conditions are posed on the strip. It should be mentioned that the boundary conditions envolving normal derivatives of the nth-order was considered in several works, see, for instance, [34, 35, 36, 37, 23, 40] and the references cited therein.

As mentioned we want to understand better what are the operators behind such a problem. Especially, we will deal with Wiener-Hopf operators and convolution type operators on finite intervals with semi-almost periodic Fourier symbol matrices. Thus, one of the main goals of the present work is the use of an operator theoretical machinery and different kinds of operator relations that will translate the problem into the study of properties of certain known types of operators. In particular the well-posedness of the problem and the Fredholm properties are analyzed. Another goal is to describe when the operators associated with the problem enjoy the Fredholm property. As we shall see, this will depend on the initial space order parameters.

Another goal of this work is to improve the results presented in [13]. We will determine the corresponding Fredholm index to our operators related with the problem.

At the end of the paper, an operator normalization is applied to the case of critical orders of the spaces where the problem does not enjoy the Fredholm property.

## 2. Formulation of the problem

In view of presenting the mathematical formulation of the problem, we need first to introduce some notation for the spaces we will be using.

Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz space, the space of all rapidly decreasing functions and $\mathscr{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$.

We will develop our study in a framework of Bessel potential spaces denoted by $\mathscr{H}^s$ defined by

$$\mathscr{H}^s(\mathbb{R}^n) = \left\{ \varphi \in \mathscr{S}'(\mathbb{R}^n) : \|\varphi\|_{\mathscr{H}^s(\mathbb{R}^n)} := \left\| \mathscr{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathscr{F}\varphi \right\|_{L^2(\mathbb{R}^n)} < +\infty \right\},$$

where $\mathscr{F} = \mathscr{F}_{x \mapsto \xi}$ is the Fourier transformation in $\mathbb{R}^n$ defined by

$$(\mathscr{F}\phi)(\xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} \phi(x) dx, \qquad \xi \in \mathbb{R}^n.$$

For a given domain $\mathscr{D}$, on $\mathbb{R}^n$, we denote by $\widetilde{\mathscr{H}^s}(\mathscr{D})$ the closed subspace of $\mathscr{H}^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathscr{D}}$, and $\mathscr{H}^s(\mathscr{D})$ denotes the space of generalized functions on $\mathscr{D}$ which have extensions into $\mathbb{R}^n$ that belong to $\mathscr{H}^s(\mathbb{R}^n)$. The space $\widetilde{\mathscr{H}^s}(\mathscr{D})$ is endowed with the subspace topology, and on $\mathscr{H}^s(\mathscr{D})$ we introduce the norm of the quotient space $\mathscr{H}^s(\mathbb{R}^n)/\widetilde{\mathscr{H}^s}(\mathbb{R}^n\backslash\overline{\mathscr{D}})$. Throughout the paper we will use the notation $\mathbb{R}^n_\pm := \{x = (x_1,\ldots,x_{n-1},x_n) \in \mathbb{R}^n : \pm x_n > 0\}$.

We are now in position to formulate our boundary-transmission problem. Given a positive integer $m$ and let $\varepsilon \geqslant 0$, we are interested in studying if the solution of the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) u = 0 \qquad \text{in } \Omega := \mathbb{R}^2 \setminus \overline{\Sigma},$$

satisfying the boundary conditions

$$\begin{cases} u_m^+ = h \\ u_m^- = h \end{cases} \qquad \text{on } \Sigma, \tag{2.1}$$

with $u \in L^2\left(\mathbb{R}^2\right)$ and

$$u_{|\Omega} \in \mathscr{H}^{1+\varepsilon}\left(\Omega\right),$$

enjoy the Fredholm property, where $k$ is a given wave number, $u_m^\pm := \left(\frac{\partial^m u}{\partial y^m}\right)_{|y=\pm 0}$ denote the traces of the normal derivatives of $u$ and $h \in \mathscr{H}^{\frac{1}{2}-m+\varepsilon}\left(\Sigma\right)$ is an arbitrarily given element. The derivatives are understood for smooth functions with compact support in the classical sense and for (weak) solutions $u \in \mathscr{H}^1\left(\Omega\right)$ by help of the representation formula

$$u(x,y) = \mathscr{F}^{-1} exp\left[-t(\xi)y\right] \cdot \mathscr{F} u_0^+(x)\chi_+(y) + \mathscr{F}^{-1} exp\left[t(\xi)y\right] \cdot \mathscr{F} u_0^-(x)\chi_-(y), \tag{2.2}$$

as continuous mappings into those spaces, continuously extended from the dense subspace of smooth functions with compact support. In the representation formula (2.2), $\chi_\pm$ is the Heaviside unit-step function of $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$ and $t(\xi) = (\xi^2 - k^2)^{\frac{1}{2}} = t_+(\xi)t_-(\xi)$, with the squareroot functions $t_\pm$ defined by

$$t_\pm(\xi) = (\xi \pm k)^{\frac{1}{2}} = |\xi \pm k|^{\frac{1}{2}}\, exp\left[\frac{1}{2} i \arg(\xi \pm k)\right], \qquad \xi \in \mathbb{R},$$

with branch cuts $\Gamma_\mp = \{\pm k \pm i\rho, \rho \geqslant 0\}$ respectively, $\arg(\xi - k) \in \left]-\frac{3\pi}{2}, \frac{\pi}{2}\right[$ and $\arg(\xi + k) \in \left]-\frac{\pi}{2}, \frac{3\pi}{2}\right[$.

Furthermore, we consider the wave number $k$ as a complex number with $Re\, k > 0$ and $Im\, k > 0$ due to a dissipative medium.

## 3. First results

In this section, after reducing our problem to a system of convolution type equations, we shall present certain extension methods in view to obtain corresponding operator relations between the operators associated to the problem and new Wiener-Hopf operators. Such extension methods and relations derived from [13] and [16] will be therefore used in the next section to study the Fredholm property of the operators associated with the problem. For more details about operator relations in the study of boundary value problems see [39].

It is known that the function $u \in L^2(\mathbb{R}^2)$ with $u_{|\mathbb{R}^2_\pm} \in \mathscr{H}^{1+\varepsilon}(\mathbb{R}^2_\pm)$ satisfies the Helmholtz equation in $\mathbb{R}^2_\pm$ in the weak $\mathscr{H}^1$ sense if and only if it is represented by the formula (2.2).

THEOREM 3.1. *Let $m$ be an odd number. The initial problem is equivalently rewritten as the finite interval convolution type equation*

$$W_{t^m,\Sigma}\varphi = -2h, \tag{3.1}$$

*where*

$$W_{t^m,\Sigma} = r_\Sigma \mathscr{F}^{-1} t^m \cdot \mathscr{F} : \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\Sigma) \to \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\Sigma). \tag{3.2}$$

*Proof.* Let the densities $\vartheta$ and $\varphi$ defined by

$$\begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} u_1^+ - u_1^- \\ u_0^+ - u_0^- \end{bmatrix} \in \widetilde{\mathscr{H}}^{-\frac{1}{2}+\varepsilon}(\Sigma) \times \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\Sigma).$$

For any integer $j$, by the Fourier transform properties, follows that

$$u_j^+ = (-1)^j \mathscr{F}^{-1} t^j \cdot \mathscr{F} u_0^+ \quad \text{and} \quad u_j^- = \mathscr{F}^{-1} t^j \cdot \mathscr{F} u_0^-.$$

Using these formulas, it is possible to write the Dirichlet data (on $y = 0$) in the form

$$\begin{bmatrix} u_0^+ \\ u_0^- \end{bmatrix} = \mathscr{B}_{\Phi_\mathscr{B}} \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} \tag{3.3}$$

where $\mathscr{B}_{\Phi_\mathscr{B}} = \mathscr{F}^{-1}\Phi_\mathscr{B} \cdot \mathscr{F} : \widetilde{\mathscr{H}}^{-\frac{1}{2}+\varepsilon}(\Sigma) \times \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\Sigma) \to \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}) \times \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R})$ is a convolution operator with Fourier symbol

$$\Phi_\mathscr{B} = -\frac{1}{2} \begin{bmatrix} t^{-1} & -1 \\ t^{-1} & 1 \end{bmatrix}. \tag{3.4}$$

Now, by the use of (3.3), it is possible to rewrite the boundary condition (2.1) as

$$r_\Sigma \mathscr{C}_{\Phi_\mathscr{C}} \begin{bmatrix} u_0^+ \\ u_0^- \end{bmatrix} = \begin{bmatrix} h \\ h \end{bmatrix}$$

where $\mathscr{C}_{\Phi_\mathscr{C}} = \mathscr{F}^{-1}\Phi_\mathscr{C} \cdot \mathscr{F} : \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}) \times \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}) \to \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}) \times \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R})$ is an invertible convolution operator with Fourier symbol

$$\Phi_\mathscr{C} = \begin{bmatrix} (-1)^m t^m & 0 \\ 0 & t^m \end{bmatrix}. \tag{3.5}$$

From (3.4) and (3.5) we obtain

$$-\frac{1}{2} r_\Sigma \mathscr{F}^{-1} \begin{bmatrix} (-1)^m t^{m-1} & (-1)^{m+1} t^m \\ t^{m-1} & t^m \end{bmatrix} \cdot \mathscr{F} \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} h \\ h \end{bmatrix}.$$

In the last identity, if we multiply the second equation by $(-1)^m$ and then adding and subtracting the two equations, we obtain a new convolution type equation

$$\widetilde{\mathscr{W}}_{\Phi_m,\Sigma} \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} h+(-1)^m h \\ h-(-1)^m h \end{bmatrix} \tag{3.6}$$

where the convolution type operator $\widetilde{\mathscr{W}}_{\Phi_m,\Sigma}$ is defined by

$$\widetilde{\mathscr{W}}_{\Phi_m,\Sigma} = r_\Sigma \mathscr{F}^{-1} \Phi_m \cdot \mathscr{F} : \widetilde{\mathscr{H}}^{-\frac{1}{2}+\varepsilon}(\Sigma) \times \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\Sigma) \to \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\Sigma) \times \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\Sigma)$$

with

$$\Phi_m = \begin{bmatrix} (-1)^{m+1} t^{m-1} & 0 \\ 0 & (-1)^m t^m \end{bmatrix}.$$

If $m$ is odd, then $h+(-1)^m h = 0$ and therefore (3.6) turns out to have the form

$$\widetilde{\mathscr{W}}_{\Phi_m,\Sigma} \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ h-(-1)^m h \end{bmatrix}.$$

In this way, from the above identities, the initial problem is equivalently rewritten as the equation

$$\mathscr{W}_{t^m,\Sigma} \varphi = -2h,$$

where $\mathscr{W}_{t^m,\Sigma}$ is defined like in (3.2). $\square$

For the even case we have a corresponding result.

THEOREM 3.2. *Let $m$ be an even number. The initial problem is equivalently rewritten as the finite interval convolution type equation*

$$\mathscr{W}_{t^{m-1},\Sigma} \vartheta = -2h, \tag{3.7}$$

*where*

$$\mathscr{W}_{t^{m-1},\Sigma} = r_\Sigma \mathscr{F}^{-1} t^{m-1} \cdot \mathscr{F} : \widetilde{\mathscr{H}}^{-\frac{1}{2}+\varepsilon}(\Sigma) \to \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\Sigma). \tag{3.8}$$

*Proof.* By a similar procedure to the one presented for the odd case, if $m$ is even then $h-(-1)^m h = 0$ in (3.6). Thus, we obtain

$$\widetilde{\mathscr{W}}_{\Phi_m,\Sigma} \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} h+(-1)^m h \\ 0 \end{bmatrix}$$

and therefore the initial problem is equivalently rewritten as the single equation (3.7) with $\mathscr{W}_{t^{m-1},\Sigma}$ defined in (3.8). $\square$

In the last two theorems the problem is equivalently rewritten as the finite interval convolution type equations (3.1) and (3.7), for $m$ odd and $m$ even, respectively, in the sense that if $u$ is a solution of the problem with Dirichlet traces $u_0^{\pm}$ on $y = \pm 0$, then

$$\begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \mathscr{B}^{-1}_{\Phi_{\mathscr{B}}} \begin{bmatrix} u_0^+ \\ u_0^- \end{bmatrix}$$

provides solutions $\varphi$ and $\vartheta$ of the equations (3.1) and (3.7), for $m$ odd and $m$ even, respectively. On the other hand, if $\varphi$ and $\vartheta$ are solutions of the equations (3.1) and (3.7), depending if $m$ is odd or even, respectively, then $u$ given by the representation formula (2.2) with $(u_0^+, u_0^-)^T$, provided by the use of (3.3), is a solution of the problem.

We will now recall some extension methods in view of obtaining corresponding operator relations between the operators related to the problem and new Wiener-Hopf operators. But first, we introduce an important definition, see [1, 19, 26].

DEFINITION 3.1. [1] Let us consider two operators $A : X_1 \to Y_1$ and $B : X_2 \to Y_2$, acting between Banach spaces.

(i) The operators $A$ and $B$ are said to be algebraically equivalent after extension if there exist additional Banach spaces $Z_1$ and $Z_2$ and invertible linear operators

$$E : Y_2 \times Z_2 \to Y_1 \times Z_1 \qquad \text{and} \qquad F : X_1 \times Z_1 \to X_2 \times Z_2$$

such that

$$\begin{bmatrix} A & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} B & 0 \\ 0 & I_{Z_2} \end{bmatrix} F. \tag{3.9}$$

(ii) If, in addition to (i), the invertible and linear operators $E$ and $F$ in (3.9) are bounded, then we will say that $A$ and $B$ are topologically equivalent after extension operators or, to simply, we say that $A$ and $B$ are equivalent after extension operators [1].

(iii) $A$ and $B$ are said to be equivalent operators in the particular case when

$$A = EBF,$$

for some bounded invertible linear operators

$$E : Y_2 \to Y_1 \qquad \text{and} \qquad F : X_1 \to X_2.$$

Now, we will rewrite a theorem of [6] to our case. First, we present the results when $n$ is an odd number.

THEOREM 3.3. *Let $m$ be an odd number. The finite interval convolution type operator defined by*

$$W_{t^m,\Sigma} = r_\Sigma \mathscr{F}^{-1} t^m \cdot \mathscr{F} : \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\Sigma) \to \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\Sigma)$$

*is algebraically equivalent after extension to the Wiener-Hopf operator*

$$\begin{aligned} &\mathrm{W}_{\Phi_{od},\mathbb{R}_+} : \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \\ &\mathrm{W}_{\Phi_{od},\mathbb{R}_+} = r_+ \mathscr{F}^{-1} \Phi_{od} \cdot \mathscr{F} \end{aligned} \tag{3.10}$$

*where $\Phi_{od}$ is the Fourier symbol*

$$\Phi_{od}(\xi) = \begin{bmatrix} e^{-ia\xi} & 0 \\ t^m(\xi) & -e^{ia\xi} \end{bmatrix}.$$

*Proof.* The equivalence is consequence of Kuijper's extension methods and, for more details, we advise to see [6], [26] and [27]. In abridged form and without many details the equivalence after extension relation can be directly obtained by computing the following operator composition

$$\begin{bmatrix} \mathrm{W}_{t^m,\Sigma} & 0 & 0 \\ 0 & I_{\mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+)} & 0 \\ 0 & 0 & I_{\widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+)} \end{bmatrix} = ETF,$$

where

$$\begin{aligned} T &: KerA \times N^{\frac{1}{2}+\varepsilon} \times \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \longrightarrow \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times ImB \times M^{\frac{1}{2}-m+\varepsilon} \\ T &= \begin{bmatrix} 0 & A_{|N^{\frac{1}{2}+\varepsilon}} & 0 \\ C_1 & C_2 & B_{ImB} \\ C_3 & C_4 & 0 \end{bmatrix}, \end{aligned}$$

$E$ and $F$ are invertible operators defined by

$$E : \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times ImB \times M^{\frac{1}{2}-m+\varepsilon} \longrightarrow \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\Sigma) \times \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+)$$

$$E = \begin{bmatrix} -q_{M^{\frac{1}{2}-m+\varepsilon}} C_4 \left( A_{|N^{\frac{1}{2}+\varepsilon}} \right)^{-1} & 0 & q_{M^{\frac{1}{2}-m+\varepsilon}} \\ I_{\mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+)} & 0 & 0 \\ -[B_{ImB}]^{-1} C_2 \left( A_{|N^{\frac{1}{2}+\varepsilon}} \right)^{-1} & [B_{ImB}]^{-1} & 0 \end{bmatrix}$$

and

$$F : \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\Sigma) \times \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \longrightarrow KerA \times N^{\frac{1}{2}+\varepsilon} \times \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+)$$

$$F = \begin{bmatrix} R & 0 & 0 \\ 0 & \left( A_{|N^{\frac{1}{2}+\varepsilon}} \right)^{-1} & 0 \\ -(B_{ImB})^{-1} C_1 R & 0 & I_{\widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+)} \end{bmatrix},$$

respectively, for some algebraic decompositions (see [24])

$$\widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) = KerA \times N^{\frac{1}{2}+\varepsilon} \qquad \text{and} \qquad \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) = ImB \times M^{\frac{1}{2}-m+\varepsilon}$$

for convenient subspaces

$$N^{\frac{1}{2}+\varepsilon} \subset \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \qquad \text{and} \qquad M^{\frac{1}{2}-m+\varepsilon} \subset \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+),$$

$q_{M^{\frac{1}{2}-m+\varepsilon}}$ is defined between $M^{\frac{1}{2}-m+\varepsilon}$ and $M^{\frac{1}{2}-m+\varepsilon}/ImB$ by $q_{M^{\frac{1}{2}-m+\varepsilon}}(\cdot) = q(\cdot)$ where $q$ is the quotient map from $\mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+)$ to $\mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+)/ImB$,

$$A = r_+\mathscr{F}^{-1}e^{-i\xi a}\cdot\mathscr{F} : \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+),$$

$$B = r_+\mathscr{F}^{-1}(-e^{i\xi a})\cdot\mathscr{F} : \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \longrightarrow \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+),$$

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = r_+\mathscr{F}^{-1}t^m\cdot\mathscr{F} : KerA \times N^{\frac{1}{2}+\varepsilon} \longrightarrow ImB \times M^{\frac{1}{2}-m+\varepsilon},$$

$B_{ImB}$ is the isomorphism defined from $\widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+)$ to $ImB$ by $B_{ImB}(\cdot) = B(\cdot)$ and $R$ the identity operator defined between $\widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\Sigma)$ and $KerA$. $\square$

In the next result we obtain an operator relation with a new operator acting between Lebesgue spaces by the use of the lifting procedure and choosing convenient auxiliary bounded invertible operators. We will use the notation $L^2_+(\mathbb{R}) := \widetilde{\mathscr{H}}^0(\mathbb{R}_+)$. For the proof see [13].

THEOREM 3.4. *Let $m$ be an odd number. The Wiener-Hopf operator* $\mathrm{W}_{\Phi_{od},\mathbb{R}_+}$ *in (3.10) is equivalent to the Wiener-Hopf operator*

$$\widehat{\mathrm{W}}_{\widehat{\Phi}_{od},\mathbb{R}_+} = r_+\mathscr{F}^{-1}\widehat{\Phi}_{od}\cdot\mathscr{F} : \left[L^2_+(\mathbb{R})\right]^2 \to \left[L^2(\mathbb{R}_+)\right]^2$$

*where $\widehat{\Phi}_{od}$ has the matricial representation*

$$\widehat{\Phi}_{od}(\xi) = \begin{bmatrix} \zeta^{\frac{1}{2}+\varepsilon}(\xi)e^{-ia\xi} & 0 \\ \zeta^{\frac{1}{2}-\frac{1}{2}m+\varepsilon}(\xi) & -\zeta^{\frac{1}{2}-m+\varepsilon}(\xi)e^{ia\xi} \end{bmatrix}. \tag{3.11}$$

For the even case, we have similar results to those presented in Theorems 3.3 and 3.4. Thus, we present them omitting the proofs.

THEOREM 3.5. *Let $m$ be an even number. The finite interval convolution type operator defined by*

$$W_{t^{m-1},\Sigma} = r_\Sigma\mathscr{F}^{-1}t^{m-1}\cdot\mathscr{F} : \widetilde{\mathscr{H}}^{-\frac{1}{2}+\varepsilon}(\Sigma) \to \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\Sigma)$$

*is algebraically equivalent after extension to the Wiener-Hopf operator*

$$W_{\Phi_{ev},\mathbb{R}_+} : \widetilde{\mathscr{H}}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+)$$

$$W_{\Phi_{ev},\mathbb{R}_+} = r_+\mathscr{F}^{-1}\Phi_{ev}\cdot\mathscr{F}, \tag{3.12}$$

*where* $\Phi_{ev}$ *is the Fourier symbol*

$$\Phi_{ev}(\xi) = \begin{bmatrix} e^{-ia\xi} & 0 \\ t^{m-1}(\xi) & -e^{ia\xi} \end{bmatrix}.$$

THEOREM 3.6. *Let* $m$ *be an even number. The Wiener-Hopf operator* $\mathrm{W}_{\Phi_{ev},\mathbb{R}_+}$ *in (3.12) is equivalent to the Wiener-Hopf operator*

$$\widetilde{W}_{\widetilde{\Phi}_{ev},\mathbb{R}_+} = r_+\mathscr{F}^{-1}\widetilde{\Phi}_{ev}\cdot\mathscr{F} : \left[L^2_+(\mathbb{R})\right]^2 \to \left[L^2(\mathbb{R}_+)\right]^2$$

*where* $\widehat{\Phi}_{ev}$ *has the matricial representation*

$$\widetilde{\Phi}_{ev}(\xi) = \begin{bmatrix} \zeta^{-\frac{1}{2}+\varepsilon}(\xi)e^{-ia\xi} & 0 \\ \zeta^{-\frac{1}{2}m+\varepsilon}(\xi) & -\zeta^{\frac{1}{2}-m+\varepsilon}(\xi)e^{ia\xi} \end{bmatrix}.$$

## 4. Fredholm analysis

The objective is now to study and characterize the Fredholm property improving the results presented in [13] by introducing the Fredholm index of the finite interval convolution type operators $\mathrm{W}_{t^m,\Sigma}$ and $\mathrm{W}_{t^{m-1},\Sigma}$, respectively, for general $\varepsilon$. In view of this, we will use the operators introduced in the last section together with different factorization procedures. We start by recalling the definition of Fredholm operator.

DEFINITION 4.1. Let $X$, $Y$ be two Banach spaces and $A : X \to Y$ a bounded linear operator with closed image. The operator $A$ is called a Fredholm operator if

$$n(A) := \dim Ker A < \infty,$$

and

$$d(A) : \dim Y/Im A < \infty.$$

If $A$ is a Fredholm operator, then the Fredholm index of $A$ is the integer defined by $Ind A = n(A) - d(A)$.

THEOREM 4.1. *Let* $\widehat{\Phi}_{od}$ *be defined by (3.11). The above defined operator* $\widehat{\mathrm{W}}_{\widehat{\Phi}_{od},\mathbb{R}_+}$ *admits the factorization*

$$\widehat{\mathrm{W}}_{\widehat{\Phi}_{od},\mathbb{R}_+} = \widehat{\mathrm{W}}_{\widehat{\Phi}_-,\mathbb{R}_+}\mathscr{W}_{\Upsilon_{od},\mathbb{R}_+}\widehat{\mathrm{W}}_{\widehat{\Phi}_+,\mathbb{R}_+} \tag{4.1}$$

*where* $\widehat{\mathrm{W}}_{\widehat{\Phi}_-,\mathbb{R}_+}$ *and* $\widehat{\mathrm{W}}_{\widehat{\Phi}_+,\mathbb{R}_+}$ *are invertible operators with the Fourier symbols*

$$\widehat{\Phi}_-(\xi) = \begin{bmatrix} 1 & e^{-ia\xi}\tau^-(\xi) \\ 0 & 1 \end{bmatrix}, \qquad \widehat{\Phi}_+(\xi) = \begin{bmatrix} 0 & 1 \\ -1 & e^{ia\xi}\tau^+(\xi) \end{bmatrix},$$

*which admit bounded analytic extensions in* $\Im m\,\xi < 0$, $\Im m\,\xi > 0$, *respectively, and with*

$$\tau^-(\xi) = \frac{1-S(\xi)}{2} + \frac{1+S(\xi)}{2}e^{im\pi}$$

*and*

$$\tau^{+}(\xi)=\frac{1-S(\xi)}{2}+\frac{1+S(\xi)}{2}e^{-im\pi}$$

*where* $S:\mathbb{C}\to\mathbb{C}$ *is the normalized sine-integral function defined by*

$$S(\xi)=\frac{2}{\pi}\int_{0}^{\xi}\frac{\sin x}{x}dx.$$

*The Fourier symbol* $\Upsilon_{od}$ *belongs to* $PC^{2\times 2}(\dot{\mathbb{R}})$, *the space of two by two matrix functions with piecewise continuous entries on* $\dot{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$, *and is given by*

$$\Upsilon_{od}(\xi)=\zeta^{\frac{1}{2}+\varepsilon}(\xi)\times \begin{bmatrix} \left[1-\tau^{-}(\xi)\zeta^{-\frac{1}{2}m}(\xi)\right]\tau^{+}(\xi)+\tau^{-}(\xi)\zeta^{-m}(\xi) & e^{-ia\xi}\left[\tau^{-}(\xi)\zeta^{-\frac{1}{2}m}(\xi)-1\right] \\ e^{ia\xi}\left[\zeta^{-\frac{1}{2}m}(\xi)\tau^{+}(\xi)-\zeta^{-m}(\xi)\right] & -\zeta^{-\frac{1}{2}m}(\xi) \end{bmatrix}. \quad (4.2)$$

*Proof.* The factorization can be directly obtained by computing the identity (4.1) where we agree that $\lim_{\xi\to-\infty}\zeta^{\sigma}(\xi)=1$, and $\lim_{\xi\to+\infty}\zeta^{\sigma}(\xi)=e^{i2\pi\sigma}$ for $\sigma\in\mathbb{R}$. We also have in such a case that

$$\lim_{\xi\to\pm\infty}\zeta^{\frac{1}{2}+\varepsilon}(\xi)\left[\tau^{-}(\xi)\zeta^{-\frac{1}{2}m}(\xi)-1\right]=0 \quad (4.3)$$

and

$$\lim_{\xi\to\pm\infty}\zeta^{\frac{1}{2}+\varepsilon}(\xi)\left[\zeta^{-\frac{1}{2}m}(\xi)\tau^{+}(\xi)-\zeta^{-m}(\xi)\right]=0. \quad (4.4)$$

□

In order to continue we need to introduce some auxiliary results necessary to study the Fredholm property of the operators associated with the problems of wave diffraction.

For $\Upsilon\in PC^{n\times n}(\dot{\mathbb{R}})$, let us consider the function $\overline{\Upsilon}:\dot{\mathbb{R}}\times[0,1]\to\mathbb{C}^{n\times n}$ defined by

$$\overline{\Upsilon}(\xi,\mu):=(1-\mu)\Upsilon(\xi-0)+\mu\Upsilon(\xi+0),\quad (\xi,\mu)\in\dot{\mathbb{R}}\times[0,1]$$

where $\Upsilon(\infty-0):=\Upsilon(+\infty)$ and $\Upsilon(\infty+0):=\Upsilon(-\infty)$. Note that for each $\xi\in\dot{\mathbb{R}}$, the set $\{\det\overline{\Upsilon}(\xi,\mu):\mu\in[0,1]\}$ is the line segment joining $\det\Upsilon(\xi-0)$ to $\det\Upsilon(\xi+0)$.

THEOREM 4.2. [2, Theorem 5.9] *For* $\Upsilon\in PC^{n\times n}(\dot{\mathbb{R}})$, *it follows that*

$$\det\overline{\Phi}(\xi,\mu)\neq 0 \quad \textit{for all} \quad (\xi,\mu)\in\dot{\mathbb{R}}\times[0,1]$$

*if and only if the operator*

$$\mathscr{W}_{\Upsilon,\mathbb{R}_{+}}=r_{+}\mathscr{F}^{-1}\Upsilon\cdot\mathscr{F}$$

*is a Fredholm operator. Additionally, in case of Fredholm property, the Fredholm index of $\mathscr{W}_{\Upsilon,\mathbb{R}_+}$ is given by*

$$Ind\,\mathscr{W}_{\Upsilon,\mathbb{R}_+} = -wind(\det\overline{\Upsilon})$$

*where wind denotes the winding number.*

We are now in position to improve the result presented in [13] for the Fredholm study to our operator $\mathrm{W}_{t^m,\Sigma}$ in (3.2) with the corresponding Fredholm index and consequently to our initial problem in the case where $m$ is an odd number.

THEOREM 4.3. *Let $m$ be an odd number. The finite interval convolution type operator $\mathrm{W}_{t^m,\Sigma}$ in (3.2) is a Fredholm operator with zero Fredholm index if and only if*

$$\varepsilon \neq q+\frac{m}{2} \quad for \quad q\in\mathbb{Z}. \tag{4.5}$$

*Proof.* First of all, we notice that from Theorems 3.3–4.1 we have that the operator $\mathrm{W}_{t^m,\Sigma}$ is algebraically equivalent after extension to the operator

$$\mathscr{W}_{\Upsilon_{od},\mathbb{R}_+} = r_+\mathscr{F}^{-1}\Upsilon_{od}\cdot\mathscr{F} : \left[L^2_+(\mathbb{R})\right]^2 \to \left[L^2(\mathbb{R}_+)\right]^2$$

where $\Upsilon_{od}$ is given by (4.2). Therefore, to prove that $\mathrm{W}_{t^m,\Sigma}$ is a Fredholm operator we are going to prove that its algebraically equivalent after extension operator $\mathscr{W}_{\Upsilon_{od},\mathbb{R}_+}$ is a Fredholm operator. For more details see [1, 26].

Letting $\overline{\Upsilon}_{od}(\xi,\mu) = (1-\mu)\Upsilon_{od}(\xi-0)+\mu\Upsilon_{od}(\xi+0)$, and $\Upsilon_{od}(\infty\pm 0) := \Upsilon_{od}(\mp\infty)$, by Theorem 4.2, if $\det\overline{\Upsilon}_{od}(\xi,\mu)\neq 0$ for $(\xi,\mu)\in\dot{\mathbb{R}}\times[0,1]$, then the operator $\mathscr{W}_{\Upsilon_{od},\mathbb{R}_+}$ is Fredholm.

From Theorem 4.1 we already know that the Fourier symbol $\Upsilon_{od}$ can be written as

$$\Upsilon_{od}(\xi) = \widehat{\Phi}_-^{-1}(\xi)\widehat{\Phi}_{od}(\xi)\widehat{\Phi}_+^{-1}(\xi).$$

For any $\xi\in\mathbb{R}$ we have

$$\det\Upsilon_{od}(\xi\pm 0) = \det\widehat{\Phi}_{od}(\xi)$$

because $\widehat{\Phi}_{od}(\xi)$ has no discontinuities on the real line, $\det\widehat{\Phi}_\pm^{-1}$ also have no discontinuities on the real line and, moreover $\det\widehat{\Phi}_\pm^{-1}\equiv 1$. Therefore,

$$\det\overline{\Upsilon}_{od}(\xi,\mu) = \det\left[(1-\mu)\widehat{\Phi}_{od}(\xi)+\mu\widehat{\Phi}_{od}(\xi)\right] = \det\widehat{\Phi}_{od}(\xi) = -\zeta^{1-m+2\varepsilon}(\xi)\neq 0,$$

in the case of $\xi\in\mathbb{R}$.

For $\xi=\infty$, we have

$$\det\overline{\Upsilon}_{od}(\infty,\mu) = \det\left[(1-\mu)\Upsilon_{od}(+\infty)+\mu\Upsilon_{od}(-\infty)\right].$$

Appealing to the limits (4.3) and (4.4), we obtain

$$\Upsilon_{od}(+\infty)=\begin{bmatrix} e^{i\pi(1-m+2\varepsilon)} & 0\\ 0 & -e^{i\pi(1-m+2\varepsilon)}\end{bmatrix},\qquad \Upsilon_{od}(-\infty)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{4.6}$$

Thus

$$\begin{aligned}\det\overline{\Upsilon}_{od}(\infty,\mu)&=\det\begin{bmatrix}(1-\mu)e^{i\pi(1-m+2\varepsilon)}+\mu & 0\\ 0 & -(1-\mu)e^{i\pi(1-m+2\varepsilon)}-\mu\end{bmatrix}\\ &=-\left[(1-\mu)e^{i\pi(1-m+2\varepsilon)}+\mu\right]^2.\end{aligned}$$

As a consequence, $\mathscr{W}_{\Upsilon_{od},\mathbb{R}_+}$ is a Fredholm operator if and only if

$$(1-\mu)e^{i\pi(1-m+2\varepsilon)}+\mu\neq 0,\qquad \mu\in[0,1]. \tag{4.7}$$

Since the set

$$\mathscr{S}=\left\{(1-\mu)e^{i\pi(1-m+2\varepsilon)}+\mu:\ \mu\in[0,1]\right\}$$

defines the line segment joining 1 to $e^{i\pi(1-m+2\varepsilon)}$, to obtain the inequality (4.7), we need that $e^{i\pi(1-m+2\varepsilon)}\notin\mathbb{R}_-$. Thus $\pi(1-m+2\varepsilon)\neq\pi+2\pi q$, $q\in\mathbb{Z}$, i.e., $\varepsilon\neq q+\frac{m}{2}$, $q\in\mathbb{Z}$.

Therefore, from the operator identities provided by both the above mentioned algebraic and topological equivalence relations, given in Theorems 3.3, 3.4 and 4.1, we conclude that $\widetilde{\mathscr{C}}_{\Phi_{\widetilde{\mathscr{C}}},\mathbb{R}_+}$ and $C_{\Phi_C,\Sigma}$ are Fredholm operators if and only if condition (4.5) holds, and that the corresponding defect spaces of these operators have the same dimensions, [1, 26]. From this, and since by [1, Theorem 3] Fredholm operators in Banach spaces are equivalent after extension if and only if their corresponding defect spaces have equal dimensions, we even arrive at the conclusion that $\widetilde{\mathscr{C}}_{\Phi_{\widetilde{\mathscr{C}}},\mathbb{R}_+}$ and $C_{\Phi_C,\Sigma}$ are not only algebraically equivalent after extension but also topologically equivalent after extension.

Finally, jointing the last conclusion with Theorem 4.2, we obtain the following formula for the Fredholm index of $W_{t^m,\Sigma}$,

$$\begin{aligned} ind\, W_{t^m,\Sigma}&=ind\,\widetilde{W}_{\Upsilon_{od},\mathbb{R}_+}\\ &=-wind\left(\det\overline{\Upsilon}_{od}(\xi,\mu)\right)\\ &=-\frac{1}{2\pi}\left(\left[\arg\det\overline{\Upsilon}_{od}(\xi,\mu)\right]_{\mathbb{R}}+\left[\arg\det\overline{\Upsilon}_{od}(\infty,\mu)\right]_{[0,1]}\right)\\ &=-\frac{1}{2\pi}\left(\left[\arg\det\Upsilon_{od}(\xi)\right]_{\mathbb{R}}+\left[\arg\det\overline{\Upsilon}_{od}(\infty,\mu)\right]_{[0,1]}\right)\end{aligned}$$

where $[f(\xi)]_{\mathbb{R}}$ denotes the increment of $f(\xi)$ when $\xi$ varies through $\mathbb{R}$ from $-\infty$ to $+\infty$ and $[f(\infty,\mu)]_{[0,1]}$ is the increment of $f(\infty,\mu)$ when $\mu$ varies through $\mathbb{R}$ from 0 to

1 . Directly, we obtain

$$\begin{aligned}[\arg\det\Upsilon_{od}(\xi)]_{\mathbb{R}} &= \arg\det\Upsilon_{od}(+\infty)-\arg\det\Upsilon_{od}(-\infty)\\ &= \arg e^{i\pi+i2\pi(1-m+2\varepsilon)}-\arg e^{i\pi}\\ &= \pi+2\pi(1-m+2\varepsilon)-\pi\\ &= 2\pi(1-m+2\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\left[\arg\det\overline{\Upsilon}_{od}(\infty,\mu)\right]_{[0,1]} &= \arg e^{i\pi}-\arg e^{i\pi+i2\pi(1-m+2\varepsilon)}\\ &= \pi-\pi-2\pi(1-m+2\varepsilon)\\ &= -2\pi(1-m+2\varepsilon).\end{aligned}$$

So, we have $ind\, W_{t^m,\Sigma}=0$. □

For the even case we have a corresponding result.

THEOREM 4.4. *Let m be an even number. The finite interval convolution type operator* $\mathrm{W}_{t^{m-1},\Sigma}$ *in (3.8) is a Fredholm operator with zero Fredholm index if and only if*

$$\varepsilon\neq q+\frac{1+m}{2}\quad for\quad q\in\mathbb{Z}.$$

*Proof.* Proceeding similarly to the above result, the operator

$$\mathrm{W}_{t^{m-1},\Sigma}=r_\Sigma\mathscr{F}^{-1}t^{m-1}\cdot\mathscr{F}:\widetilde{\mathscr{H}}^{-\frac{1}{2}+\varepsilon}(\Sigma)\to\mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\Sigma)$$

is algebraically equivalent after extension to the operator

$$\mathscr{W}_{\Upsilon_{ev},\mathbb{R}_+}=r_+\mathscr{F}^{-1}\Upsilon_{ev}\cdot\mathscr{F}:\left[L^2_+(\mathbb{R})\right]^2\to\left[L^2(\mathbb{R}_+)\right]^2$$

where $\Upsilon_{ev}$ is given by

$$\Upsilon_{ev}(\xi)=\zeta^{\varepsilon}(\xi)\begin{bmatrix}\Upsilon_{ev,11}(\xi) & \Upsilon_{ev,12}(\xi)\\ \Upsilon_{ev,21}(\xi) & \Upsilon_{ev,22}(\xi)\end{bmatrix}\tag{4.8}$$

with

$$\begin{aligned}\Upsilon_{ev,11}(\xi)&=\left[\zeta^{-\frac{1}{2}}(\xi)-\tau^-(\xi)\zeta^{-\frac{1}{2}m}(\xi)\right]\tau^+(\xi)+\tau^-(\xi)\zeta^{\frac{1}{2}-m}(\xi)\\ \Upsilon_{ev,12}(\xi)&=e^{-ia\xi}\left[\tau^-(\xi)\zeta^{-\frac{1}{2}m}(\xi)-\zeta^{-\frac{1}{2}}(\xi)\right]\\ \Upsilon_{ev,21}(\xi)&=e^{ia\xi}\left[\zeta^{-\frac{1}{2}m}(\xi)\tau^+(\xi)-\zeta^{\frac{1}{2}-m}(\xi)\right]\\ \Upsilon_{ev,22}(\xi)&=-\zeta^{-\frac{1}{2}m}(\xi)\end{aligned}$$

and where

$$\tau^{-}(\xi)=\frac{1-S(\xi)}{2}+\frac{1+S(\xi)}{2}e^{i\pi(-1+m)},\qquad \tau^{+}(\xi)=\frac{1-S(\xi)}{2}+\frac{1+S(\xi)}{2}e^{i\pi(1-m)}.$$

Therefore, the operator $\mathscr{W}_{\Upsilon_{ev},\mathbb{R}_+}$ is Fredholm if and only if $\det\overline{\Upsilon}_{ev}(\xi,\mu)\neq 0$ for $(\xi,\mu)\in \dot{\mathbb{R}}\times[0,1]$, where

$$\overline{\Upsilon}_{ev}(\xi,\mu)=(1-\mu)\Upsilon_{ev}(\xi-0)+\mu\Upsilon_{ev}(\xi+0),\qquad \Upsilon_{ev}(\infty\pm 0):=\Upsilon_{ev}(\mp\infty).$$

For any $\xi\in\mathbb{R}$ we have $\det\overline{\Upsilon}_{ev}(\xi,\mu)=-\zeta^{-m+2\varepsilon}(\xi)\neq 0$, $\xi\in\mathbb{R}$.
For $\xi=\infty$, we have

$$\det\overline{\Upsilon}_{ev}(\infty,\mu)=-\left[(1-\mu)e^{i\pi(-m+2\varepsilon)}+\mu\right]^2.$$

This shows that $\mathrm{W}_{t^{m-1},\Sigma}$ is a Fredholm operator if and only if

$$(1-\mu)e^{i\pi(-m+2\varepsilon)}+\mu\neq 0,\qquad \mu\in[0,1]$$

and therefore the result is concluded from

$$e^{i\pi(-m+2\varepsilon)}\notin\mathbb{R}_-\Leftrightarrow \pi(-m+2\varepsilon)\neq\pi+2\pi q\Leftrightarrow \varepsilon\neq q+\frac{1+m}{2},\qquad q\in\mathbb{Z}.$$

For the Fredholm index of $W_{t^{m-1},\Sigma}$ we have

$$ind\,W_{t^{m-1},\Sigma}=-\frac{1}{2\pi}\left(\left[\arg\det\Upsilon_{ev}(\xi)\right]_{\mathbb{R}}+\left[\arg\det\overline{\Upsilon}_{ev}(\infty,\mu)\right]_{[0,1]}\right).$$

Directly, we obtain

$$\left[\arg\det\Upsilon_{ev}(\xi)\right]_{\mathbb{R}}=2\pi(-m+2\varepsilon)$$

and

$$\left[\arg\det\overline{\Upsilon}_{ev}(\infty,\mu)\right]_{[0,1]}=-2\pi(-m+2\varepsilon).$$

So, we have $ind\,W_{t^{m-1},\Sigma}=0$. □

## 5. Image normalization of Wiener-Hopf operators

We conclude this paper by presenting a technique widely used in the study of the Fredholm property of Wiener-Hopf operators associated with diffraction problems. We refer to the *operator normalization*. This technique relies on the fact that it is possible to make an extension of the image space or a restriction of the domain so that an operator which is not Fredholm, which is equivalent to say that is not normally solvable, [33], passes to enjoy this property. By physical reasons, we will choose to do the normalization without changing simultaneously both space $X_0$ and $Y_0$. We will carry out the image normalization [7], [11], [30], [31]–[33].

We know, by Theorem 4.2, that if the Fourier symbol $\Upsilon \in PC^{n\times n}(\dot{\mathbb{R}})$ fulfils the condition

$$\det \overline{\Upsilon}_{od}(\xi,\mu) = \det[(1-\mu)\Upsilon_{od}(\xi-0)+\mu\Upsilon_{od}(\xi+0)] = 0, \tag{5.1}$$

to some $(\xi,\mu) \in \dot{\mathbb{R}} \times [0,1]$, then $W_{\Upsilon,\mathbb{R}_+} = r_+\mathscr{F}^{-1}\Upsilon\cdot\mathscr{F}$ is not a Fredholm operator.

We also know that if we have the condition (5.1) with $\Upsilon \in PC^{n\times n}(\dot{\mathbb{R}})$, then the operator associated with the diffraction problem is not normally solvable, i.e., $Im W$ is not closed although we have $dim Ker W < \infty$ and $dim Y_0/\overline{Im W} < \infty$ (considering $W : X_0 \to Y_0$), [7, Teorema 5.7].

The normalization problem for a bounded linear operator $W : X_0 \to Y_0$ defined between Banach spaces (and not normally solvable) consists in finding a pair of Banach spaces $X_1$ and $Y_1$ such that

(1) the inclusion $X_0 \cap X_1 \subset X_1$ is dense,

(2) $W$ maps $X_0 \cap X_1$ into $Y_1$

(3) the restriction of $W$ to $X_0 \cap X_1$ admits a continuous extension

$$\overset{\approx}{W} = Ext\, W_{|X_0\cap X_1} : X_1 \longrightarrow Y_1$$

which is normally solvable.

Then we say that the pair $(X_1, Y_1)$ solves the normalization problem for $W$ with $\overset{\approx}{W}$ a Fredholm operator.

Let us consider

$$X_1 = X_0 \quad \text{e} \quad Y_1 \subset Y_0$$

and define the restriction operator

$$\overset{<}{W} = Res\, W : X_0 \longrightarrow Y_1.$$

In order to simplify the notation, for $s \in \mathbb{R}^n$, we consider

$$\mathscr{H}^s(\mathbb{R}_+) = \mathscr{H}^{s_1}(\mathbb{R}_+) \times \ldots \times \mathscr{H}^{s_n}(\mathbb{R}_+)$$

and

$$\widetilde{\mathscr{H}^s}(\mathbb{R}_+) = \widetilde{\mathscr{H}^{s_1}}(\mathbb{R}_+) \times \ldots \times \widetilde{\mathscr{H}^{s_n}}(\mathbb{R}_+).$$

Let us consider the Wiener-Hopf operator $W_{\Phi,\mathbb{R}_+}$ defined by

$$W_{\Phi,\mathbb{R}_+} = r_+\mathscr{F}^{-1}\Phi\cdot\mathscr{F} : \widetilde{\mathscr{H}^r}(\mathbb{R}_+) \to \mathscr{H}^s(\mathbb{R}_+) \tag{5.2}$$

with $r,s \in \mathbb{R}^n$ and admitting the lifted Fourier symbol $\widetilde{\Phi} \in \mathscr{G}\mathscr{C}^{\nu}(\dot{\mathbb{R}})^{n\times n}$ with $\nu \in ]0,1[$ (space of Hölder continuous matricial functions with order $n$, invertible and with exponent $\nu$) and $\det\widetilde{\Phi}(\xi) \neq 0$, $\xi \in \dot{\mathbb{R}}$.

We know that the jump at infinity matrix is defined by $\widetilde{\Phi}^{-1}(+\infty)\widetilde{\Phi}(-\infty)$ and, considering their eigenvalues denoted by $\lambda_1, \lambda_2, \ldots, \lambda_c$ with their multiplicities $l_1, l_2, \ldots, l_c$, respectively, $c \leqslant n$ and $\sum_{j=1}^{c} l_j = n$, we can write it in the normal Jordan form

$$\widetilde{\Phi}^{-1}(+\infty)\widetilde{\Phi}(-\infty) = T^{-1}JT, \tag{5.3}$$

where $T \in \mathscr{G}\mathbb{C}^{n\times n}$ and $J$ is a matrix defined by

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_c \end{bmatrix},$$

with each block $J_j$, $j = 1, \ldots, c$, have order equal to the multiplicity $l_j$ of the correspondent eigenvalues $\lambda_j$, is defined by

$$J_j = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_j \end{bmatrix}.$$

Considering now the following notation for the diagonal elements of $J$,

$$\begin{aligned} diag\, J &= diag(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \\ &= diag(e^{2\pi i\tilde{\omega}_1}, \ldots, e^{2\pi i\tilde{\omega}_n}), \end{aligned}$$

with $\tilde{\omega}_j = \tilde{\sigma}_j + \tilde{\tau}_j i$ such that $\tilde{\sigma}_j \in \left[-\frac{1}{2}, \frac{1}{2}\right[$, $j = 1, \ldots, n$, we can rewrite the following result of [31].

THEOREM 5.1. *The Wiener-Hopf operator defined in (5.2) by*

$$W_{\Phi,\mathbb{R}_+} = r_+\mathscr{F}^{-1}\Phi\cdot\mathscr{F} : \widetilde{\mathscr{H}^r}(\mathbb{R}_+) \to \mathscr{H}^s(\mathbb{R}_+)$$

*where $r,s \in \mathbb{R}^n$ with lifted Fourier symbol $\widetilde{\Phi} \in \mathscr{G}\mathscr{C}^{\nu}(\dot{\mathbb{R}})^{n\times n}$ with $\nu \in ]0,1[$, $\det\widetilde{\Phi}(\xi) \neq 0$, $\xi \in \dot{\mathbb{R}}$ and with the jump at infinity matrix with the normal Jordan form defined as (5.3) is normally solvable if and only if*

$$\tilde{\sigma}_j \neq -\frac{1}{2}, \qquad j = 1, \ldots, n. \tag{5.4}$$

We now reordering the elements of the matrix $J$ by a permutation of the columns of $T$ such that the $m$ eigenvalues that violate the condition (5.4), i.e., the eigenvalues such that

$$\tilde{\sigma}_j = -\frac{1}{2}, \qquad j = 1, \dots, n' \tag{5.5}$$

with $1 \leqslant c \leqslant n$, will be positioned at the entries $[\cdot]_{jj}$, $j = 1, \dots, c$ of $J$.

With this reordering of the eigenvalues in the diagonal of $J$ we have the following theorem of [31].

THEOREM 5.2. *Let the Wiener-Hopf operator defined in (5.2) by*

$$W_{\Phi,\mathbb{R}_+} = r_+ \mathscr{F}^{-1} \Phi \cdot \mathscr{F} : \widetilde{\mathscr{H}^r}(\mathbb{R}_+) \to \mathscr{H}^s(\mathbb{R}_+)$$

*where $r, s \in \mathbb{R}^n$, with lifted Fourier symbol $\widetilde{\Phi} \in \mathscr{G}\mathscr{C}^{\nu}(\dot{\mathbb{R}})^{n\times n}$ with $\nu \in ]0,1[$, $\det \widetilde{\Phi}(\xi) \neq 0$, $\xi \in \dot{\mathbb{R}}$ and with the jump at infinity matrix with the normal Jordan form checking the condition (5.5).*

*The normalization problem is solvable by the image normalization defined by*

$$Y_1 = r_+ \mathscr{F}^{-1} (\xi - k)^{-s} \cdot \mathscr{F} T l_0 \{ \overset{<}{\mathscr{H}}{}^{-i(\tilde{\tau}_1, \dots, \tilde{\tau}_{n'})}(\mathbb{R}_+) \times \left[ L^2(\mathbb{R}_+) \right]^{n-n'} \},$$

*where $l_0$ denotes the $0$ extension of $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R})$ and*

$$\begin{aligned} \overset{<}{\mathscr{H}}{}^{-i(\tilde{\tau}_1, \dots, \tilde{\tau}_{n'})}(\mathbb{R}_+) &= \overset{<}{\mathscr{H}}{}^{-i\tilde{\tau}_1}(\mathbb{R}_+) \times \dots \times \overset{<}{\mathscr{H}}{}^{-i\tilde{\tau}_{n'}}(\mathbb{R}_+) \\ &= r_+ \Lambda_-^{-i\tilde{\tau}_1} \Lambda_-^{-\frac{1}{2}} \Lambda_+^{\frac{1}{2}} L_+^2(\mathbb{R}) \times \dots \times r_+ \Lambda_-^{-i\tilde{\tau}_{n'}} \Lambda_-^{-\frac{1}{2}} \Lambda_+^{\frac{1}{2}} L_+^2(\mathbb{R}) \end{aligned}$$

*with*

$$\Lambda_\pm^\alpha = \mathscr{F}^{-1} (\xi \pm k)^\alpha \cdot \mathscr{F} : \mathscr{H}^s(\mathbb{R}) \to \mathscr{H}^{s - Re\,\alpha}(\mathbb{R})$$

$\alpha \in \mathbb{C}$, $s \in \mathbb{R}$.

We are able to apply these results to our problem.

THEOREM 5.3. *Let $m$ be an odd number. The Wiener-Hopf operator $W_{\Phi_{od},\mathbb{R}_+}$ defined in (3.10) by*

$$\begin{aligned} &W_{\Phi_{od},\mathbb{R}_+} : \widetilde{\mathscr{H}}{}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathscr{H}}{}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \\ &W_{\Phi_{od},\mathbb{R}_+} = r_+ \mathscr{F}^{-1} \Phi_{od} \cdot \mathscr{F}, \end{aligned}$$

*with Fourier symbol*

$$\Phi_{od}(\xi) = \begin{bmatrix} e^{-ia\xi} & 0 \\ t^m(\xi) & -e^{ia\xi} \end{bmatrix}$$

*is not normally solvable if $\varepsilon = q + \frac{m}{2}$ with $q \in \mathbb{Z}$.*

*Considering $\varepsilon = q + \frac{m}{2}$, the image normalized operator*

$$\overset{<}{W}_{\Phi_{od},\mathbb{R}_+} : \widetilde{\mathscr{H}}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \to Y_1$$

*with*

$$Y_1 = r_+ \mathscr{F}^{-1}(\xi - k)^{\left(-\frac{1}{2}-\varepsilon, -\frac{1}{2}+m-\varepsilon\right)} \cdot \mathscr{F} l_0 \{ \overset{<}{\mathscr{H}}^0(\mathbb{R}_+) \times \overset{<}{\mathscr{H}}^0(\mathbb{R}_+) \},$$

*where*

$$\overset{<}{\mathscr{H}}^0(\mathbb{R}_+) = r_+ \Lambda_-^{-\frac{1}{2}} \Lambda_+^{\frac{1}{2}} L_+^2(\mathbb{R})$$

*solves the normalization problem. Thus, $\overset{<}{W}_{\Phi_{od},\mathbb{R}_+}$ is a normally solvable operator, which leads to obtaining the Fredholm property.*

*Proof.* By Theorems 3.4 and 4.1 we know that $W_{\Phi_{od},\mathbb{R}_+}$ admits the lifted Fourier symbol $\Upsilon_{od} \in PC^{2\times 2}(\dot{\mathbb{R}})$ defined in (4.2). If we consider $\Upsilon_{od}(+\infty)$ and $\Upsilon_{od}(-\infty)$ defined in (4.6), we have the following jump at infinity (in matrix form),

$$\Upsilon_{od}(+\infty)^{-1}\Upsilon_{od}(-\infty) = \begin{bmatrix} e^{i\pi(-1+m-2\varepsilon)} & 0 \\ 0 & e^{i\pi(-1+m-2\varepsilon)} \end{bmatrix}.$$

For $\varepsilon = q + \frac{m}{2}$ with $q \in \mathbb{Z}$ we have

$$\Upsilon_{od}(+\infty)^{-1}\Upsilon_{od}(-\infty) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, we have the eigenvalue $\tilde{\lambda} = -1$ with multiplicity 2 such that $\tilde{\omega} = -\frac{1}{2}$. So, the operator is not normally solvable.

The normalized operator $\overset{<}{W}_{\Phi_{od},\mathbb{R}_+}$ is a direct consequence of Theorem 5.2. □

To the even case we have the following result.

THEOREM 5.4. *Let $m$ be an even number. The Wiener-Hopf operator defined in (3.12) by*

$$W_{\Phi_{ev},\mathbb{R}_+} : \widetilde{\mathscr{H}}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \mathscr{H}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+)$$
$$W_{\Phi_{ev},\mathbb{R}_+} = r_+ \mathscr{F}^{-1} \Phi_{ev} \cdot \mathscr{F},$$

*with Fourier symbol*

$$\Phi_{ev}(\xi) = \begin{bmatrix} e^{-ia\xi} & 0 \\ t^{m-1}(\xi) & -e^{ia\xi} \end{bmatrix}$$

*is not normally solvable if $\varepsilon = q + \frac{1+m}{2}$ with $q \in \mathbb{Z}$.*

*Considering* $\varepsilon = q + \frac{1+m}{2}$, *the image normalized operator*

$$\overset{<}{W}_{\Phi_{ev},\mathbb{R}_+} : \widetilde{\mathscr{H}}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathscr{H}}^{\frac{1}{2}-m+\varepsilon}(\mathbb{R}_+) \to Y_1$$

*with*

$$Y_1 = r_+ \mathscr{F}^{-1}(\xi - k)^{\left(\frac{1}{2}-\varepsilon, -\frac{1}{2}+m-\varepsilon\right)} \cdot \mathscr{F} l_0 \{ \overset{<}{\mathscr{H}}^0(\mathbb{R}_+) \times \overset{<}{\mathscr{H}}^0(\mathbb{R}_+) \},$$

*where*

$$\overset{<}{\mathscr{H}}^0(\mathbb{R}_+) = r_+ \Lambda_-^{-\frac{1}{2}} \Lambda_+^{\frac{1}{2}} L_+^2(\mathbb{R})$$

*solves the normalization problem. Thus,* $\overset{<}{W}_{\Phi_{ev},\mathbb{R}_+}$ *is a normally solvable operator, which leads to obtaining the Fredholm property.*

*Proof.* By the Theorem 3.6 and by the proof of Theorem 4.4 we have that the operator $W_{\Phi_{ev},\mathbb{R}_+}$ admits the lifted Fourier symbol $\Upsilon_{ev} \in PC^{2\times 2}(\dot{\mathbb{R}})$ defined in (4.8). Again, although we have $\Upsilon_{ev} \in PC^{2\times 2}(\dot{\mathbb{R}})$, we can apply Theorem 5.2.

Considering $\Upsilon_{ev}(+\infty)$ and $\Upsilon_{ev}(-\infty)$, we have the following jump at infinity

$$\begin{aligned} \Upsilon_{ev}(+\infty)^{-1}\Upsilon_{ev}(-\infty) &= \begin{bmatrix} e^{i\pi(-m+2\varepsilon)} & 0 \\ 0 & -e^{i\pi(-m+2\varepsilon)} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} e^{i\pi(m-2\varepsilon)} & 0 \\ 0 & e^{i\pi(m-2\varepsilon)} \end{bmatrix}. \end{aligned}$$

For $\varepsilon = q + \frac{1+m}{2}$ with $q \in \mathbb{Z}$, we have

$$\Upsilon_{ev}(+\infty)^{-1}\Upsilon_{ev}(-\infty) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, we have the eigenvalue $\tilde{\lambda} = -1$ with multiplicity 2 such that $\tilde{\omega} = -\frac{1}{2}$. So, the operator is not normally solvable.

The normalized operator $\overset{<}{W}_{\Phi_{ev},\mathbb{R}_+}$ is a direct consequence of Theorem 5.2. □

Finally, we like to remark that the Fredholm index of the normalized operators, in the two last theorems, are both zero. This can be proved by the method presented in [32].

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