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C–SYMMETRIC OPERATORS AND REFLEXIVITY

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Abstract. We study subspaces of all C-symmetric operators. Description of the preanihilator of all C-symmetric operators is given. It is shown that the subspace of all C-symmetric operators is transitive and 2-hyperreflexive.

1. Introduction and preliminaries

Let $\mathscr{H}$ be a complex separable Hilbert space with an inner product $\langle\cdot,\cdot\rangle$. Let C be an isometric antilinear involution in $\mathscr{H}$. By isometric it is meant that $\langle f,g\rangle = \langle Cg, Cf\rangle$ for all $f,g\in\mathscr{H}$. Since C is an involution, $C^2 = I$. A bounded operator $T\in B(\mathscr{H})$ is called C-*symmetric*, if $CTC = T^*$. This is equivalent to the symmetry of T with respect to the bilinear form $[f,g] = \langle f, Cg\rangle$. Let us denote the set of all C-symmetric operators by $\mathscr{C} = \{T\in B(\mathscr{H}) : CTC = T^*\}$.

C-symmetric operators and the whole set $\mathscr{C}$ was intensively studied in [3]. There were given many examples of C-symmetric operators such as Jordan blocks, truncated Toeplitz operators, Hankel operators ect.. The aim of the paper is to study the space of C-symmetric operators from reflexivity–transitivity point of view, for definitions see bellow. It is shown that the subspace of all C-symmetric operators is transitive and 2-reflexive or even 2-hyperreflexive. It means that the preanihilator of $\mathscr{C}$ does not contain any rank-one operators and rank-two operators are dense in the preanihilator. Moreover, we describe all rank-two operators in this preanihilator.

The set of all trace class operators on $\mathscr{H}$ will be denoted by τc with the norm $\|\cdot\|_1$, (this class of operators is often also denoted by $\mathscr{C}_1$, see [8], or $\mathscr{B}_1$, see [2]). The dual action between τc and $B(\mathscr{H})$ is given by trace, i.e. $\langle A,t\rangle = \mathrm{tr}(At)$ for $A\in B(\mathscr{H})$, $t\in\tau c$. For $k\in\mathbb{N}$, F_k stands for the set of operators on $\mathscr{H}$ of rank at most k. Every rank-one operator may be written as $x\otimes y$, for $x,y\in\mathscr{H}$, and $(x\otimes y)z = \langle z,y\rangle x$ for $z\in\mathscr{H}$. Moreover, $\langle T, x\otimes y\rangle = \mathrm{tr}(T(x\otimes y)) = \langle Tx, y\rangle$ for any $T\in B(\mathscr{H})$.

Recall that *the reflexive closure* of a subspace $\mathscr{S}\subset B(\mathscr{H})$ is given by

$$\mathrm{Ref}\,\mathscr{S} = \{T\in B(\mathscr{H}) : Tx\in[\mathscr{S}x] \text{ for all } x\in\mathscr{H}\},$$

where $[\cdot]$ denotes the norm-closure. A subspace $\mathscr{S}$ is called *reflexive*, if $\mathscr{S} = \mathrm{Ref}\,\mathscr{S}$ and $\mathscr{S}$ is called *transitive*, if $\mathrm{Ref}\,\mathscr{S} = B(\mathscr{H})$. Transitivity means that there are no

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rank-one operators in the preanihilator. Reflexivity means, in contrast, that we have "a lot" of rank-one operators in the preanihilator. Namely, due to [7] we know that when $\mathscr{S}$ is a weak* closed subspace of $B(\mathscr{H})$, then $\mathscr{S}$ is reflexive if and only if $\mathscr{S}_\perp$ is a closed linear span of rank-one operators contained in $\mathscr{S}_\perp$ (i.e., $\mathscr{S}_\perp = [\mathscr{S}_\perp \cap F_1]$). A subspace $\mathscr{S} \subset B(\mathscr{H})$ is called *k-reflexive* if $\mathscr{S}^{(k)} = \{S^{(k)} : S \in \mathscr{S}\}$ is reflexive in $B(\mathscr{H}^{(k)})$, where $S^{(k)} = S \oplus \ldots \oplus S$ and $\mathscr{H}^{(k)} = \mathscr{H} \oplus \ldots \oplus \mathscr{H}$. In [6, Theorem 2.1] it was proved that a weak* closed subspace $\mathscr{S} \subset B(\mathscr{H})$ is k-reflexive if and only if $\mathscr{S}_\perp$ is a closed linear span of rank-k operators contained in $\mathscr{S}_\perp$ (i.e., $\mathscr{S}_\perp = [\mathscr{S}_\perp \cap F_k]$).

Now we recall the definition of stronger property than reflexivity. Suppose that $\mathscr{S} \subseteq B(\mathscr{H})$ is a subspace. By $d(A,\mathscr{S})$ we denote the standard distance from an operator A to the subspace $\mathscr{S}$, i.e., $d(A,\mathscr{S}) = \inf\{\|A - T\| : T \in \mathscr{S}\}$. In [1] Arveson defines an algebra $\mathscr{W}$ as *hyperreflexive* if there is a constant κ such that

$$d(A,\mathscr{W}) \leqslant \kappa \sup\{\|P^\perp AP\| : P \in \operatorname{Lat}\mathscr{W}\} \text{ for all } A \in B(\mathscr{H}).$$

As it was shown in [6] the supremum on the right hand side of the inequality above is equal to $\sup\{|\langle A, g\otimes h\rangle| : g\otimes h \in \mathscr{W}_\perp, \|g\otimes h\|_1 \leqslant 1\}$. It is known that when $\mathscr{S}$ is weak* closed, then $d(A,\mathscr{S}) = \sup\{|tr(Af)| : f \in \mathscr{S}_\perp, \|f\|_1 \leqslant 1\}$. Now we can generalize the definition of hyperreflexivity for k-hyperreflexivity not only for algebras but also for subspaces, see [4],[5]. For an operator $A \in B(\mathscr{H})$ and $k \in \mathbb{N}$ we consider the following quantity

$$\alpha_k(A,\mathscr{S}) = \sup\{|\langle A,t\rangle| : t \in \mathscr{S}_\perp \cap F_k, \|t\|_1 \leqslant 1\},$$

where $\langle A,t\rangle = \operatorname{tr}(At)$. Recall that $d(A,\mathscr{S}) \geqslant \alpha_k(A,\mathscr{S})$ for every $A \in B(\mathscr{H})$. The subspace $\mathscr{S}$ is called k*-hyperreflexive* if there is a constant κ such that

$$d(A,\mathscr{S}) \leqslant \kappa\,\alpha_k(A,\mathscr{S}), \quad A \in B(\mathscr{H}). \tag{1}$$

It was noted in [4] that property of k-hyperreflexivity is stronger than k–reflexivity.

For more properties of C-symmetric operators we refer the reader to [3]. Recall only that the set of all C-symmetric operators $\mathscr{C} = \{T \in B(\mathscr{H}) : CTC = T^*\} \subset B(\mathscr{H})$ is a subspace, which is closed in norm, weak and strong operator topology. In the same manner it can be proved that $\mathscr{C}$ is also weak* closed.

2. Transitivity

Let start with the following:

THEOREM 2.1. *Let $\mathscr{H}$ be a complex separable Hilbert space with an antilinear involution C. Let $\mathscr{C}$ be the set of C-symmetric operators. The subspace $\mathscr{C}$ is transitive.*

Proof. Let $\{e_n\}$ be an orthonormal basis of $\mathscr{H}$ such that $Ce_n = e_n$ (see [3, Lemma 1]). Let us consider a rank-one operator $x\otimes y \in \mathscr{C}_\perp$. By [3, Lemma 2] the operator $u\otimes Cu \in \mathscr{C}$ for all $u \in \mathscr{H}$. Hence $e_i\otimes e_i \in \mathscr{C}$, $i \in \mathbb{N}$. Thus

$$0 = \langle e_i\otimes e_i, x\otimes y\rangle = \langle (e_i\otimes e_i)x, y\rangle = \langle x, e_i\rangle\langle e_i, y\rangle.$$

Hence $x \perp e_i$ or $y \perp e_i$ for all $i \in \mathbb{N}$. Let $k \in \mathbb{N}$ be the smallest number such that $\langle x, e_k \rangle \neq 0$ and $l \in \mathbb{N}$ be the smallest number such that $\langle y, e_l \rangle \neq 0$. Clearly $k \neq l$ and $\langle x, e_l \rangle = 0$, $\langle y, e_k \rangle = 0$.

Consider vector $\alpha e_l + \beta e_k$ for $\alpha, \beta \neq 0$, then, by antilinearity of C, we have $C(\alpha e_l + \beta e_k) = \overline{\alpha} e_l + \overline{\beta} e_k$. Hence $(\alpha e_l + \beta e_k) \otimes (\overline{\alpha} e_l + \overline{\beta} e_k) \in \mathscr{C}$ for any $\alpha, \beta \neq 0$. Thus

$$\begin{aligned} 0 &= \langle (\alpha e_l + \beta e_k) \otimes (\overline{\alpha} e_l + \overline{\beta} e_k), x \otimes y \rangle \\ &= \langle x, \overline{\alpha} e_l + \overline{\beta} e_k \rangle \langle \alpha e_l + \beta e_k, y \rangle = \beta \langle x, e_k \rangle \alpha \langle e_l, y \rangle. \end{aligned}$$

Since $\alpha, \beta \neq 0$ and $\langle x, e_k \rangle \neq 0$, $\langle e_l, y \rangle \neq 0$ we get the contradiction. Hence $x = 0$ or $y = 0$. $\square$

3. Rank-two operators in the preanihilator of $\mathscr{C}$

In the previous section it was shown that there is no rank-one operator in the preanihilator of the space of all C-symmetric operators. In what follows we describe all rank-two operators in this preanihilator. Namely

THEOREM 3.1. *Let $\mathscr{H}$ be a complex separable Hilbert space with an antilinear involution C. Let $\mathscr{C}$ be the set of all C-symmetric operators. Then*

$$F_2 \cap \mathscr{C}_\perp = \{h \otimes g - Cg \otimes Ch : h, g \in \mathscr{H}\}.$$

To proof the theorem above we will need some lemmas for real Hilbert spaces.

LEMMA 3.2. *Let $\mathscr{H}$ be a real Hilbert space and let $h, h', g, g' \in \mathscr{H}$ have norm 1. Assume that*

$$\langle A, h \otimes g - h' \otimes g' \rangle = 0 \quad \textit{for all} \quad A = A^* \in B(\mathscr{H}), \tag{2}$$

then $h \otimes g = h' \otimes g'$ or $h \otimes g = g' \otimes h'$.

As a special case of the previous lemma we will prove the following:

LEMMA 3.3. *Let $\mathscr{H}$ be a real Hilbert space and let $h, g \in \mathscr{H}$. If $\langle A, h \otimes g \rangle = 0$ for all $A = A^* \in B(\mathscr{H})$, then $h \otimes g = 0$.*

Proof. Assume that $g, h \neq 0$. Note that for selfadjoint operator $h \otimes h$ we have

$$0 = \langle h \otimes h, h \otimes g \rangle = \|h\|^2 \langle h, g \rangle.$$

Thus $h \perp g$. Consider a selfadjoint operator $g \otimes h + h \otimes g$ and observe also that

$$0 = \langle g \otimes h + h \otimes g, h \otimes g \rangle = \|h\|^2 \|g\|^2 + \langle h, g \rangle \langle h, g \rangle = \|h\|^2 \|g\|^2.$$

Thus we get the contradiction. $\square$

Proof of Lemma 3.2. Let $H_0 = span\{h,g\}$ and $H_1 = H_0^{\perp}$. Denote $h_1' = P_{H_1}h'$, $g_1' = P_{H_1}g'$. Then $0 = \langle h_1' \otimes g_1' + g_1' \otimes h_1', h \otimes g\rangle$. Since the operator $h_1' \otimes g_1' + g_1' \otimes h_1'$ is selfadjoint, by (2) we have

$$\begin{aligned}0 &= \langle (h_1' \otimes g_1' + g_1' \otimes h_1'), h' \otimes g'\rangle \\ &= \langle h', g_1'\rangle\langle h_1', g'\rangle + \langle h', h_1'\rangle\langle g_1', g'\rangle = \langle h_1', g_1'\rangle^2 + \|h_1'\|^2\|g_1'\|^2.\end{aligned}$$

Hence $h_1' = 0$ or $g_1' = 0$.

Assume that $h_1' = 0$, i.e. $h' \in H_0$, and decompose $g = \beta h + g_0$, where $g_0 \perp h$. Observe that $\langle g_0 \otimes g_0, h \otimes g\rangle = 0$. Since $g_0 \otimes g_0$ is selfadjoint thus by (2)

$$0 = \langle g_0 \otimes g_0, h' \otimes g'\rangle = \langle h', g_0\rangle\langle g_0, g'\rangle$$

and $h' \perp g_0$ or $g' \perp g_0$. If $h' \perp g_0$ and $h' \in H_0$ thus $h' = \alpha h$. Hence for all selfadjoit $A \in B(\mathscr{H})$ we have $\langle Ah, g\rangle = \langle A\alpha h, g'\rangle$. Thus $\langle Ah, g - \alpha g'\rangle = 0$. By Lemma 3.3, $g = \alpha g'$ and we get $h \otimes g = h' \otimes g'$.

Assume now that $g' \perp g_0$ and decompose $g' = \alpha h + g_1$, where $g_1 \perp H_0$. Note that

$$\langle g_1 \otimes g_0 + g_0 \otimes g_1, h \otimes g\rangle = \langle h, g_0\rangle\langle g_1, g\rangle + \langle h, g_1\rangle\langle g_1, g\rangle = 0.$$

Since $g_1 \otimes g_0 + g_0 \otimes g_1$ is selfadjoint thus

$$\begin{aligned}0 &= \langle g_1 \otimes g_0 + g_0 \otimes g_1, h' \otimes g'\rangle \\ &= \langle h', g_0\rangle\langle g_1, g'\rangle + \langle h', g_1\rangle\langle g_0, g'\rangle = \langle h', g_0\rangle\|g_1\|^2.\end{aligned}$$

Hence $h' \perp g_0$ or $g_1 = 0$ thus $h' = \alpha h$ or $g' = \alpha h$. The case $h' = \alpha h$ was considered above. If $g' = \alpha h$, then for selfadjoint A we have $\langle Ah, g\rangle = \langle Ah', \alpha h\rangle = \langle Ah, \alpha h'\rangle$ and as before $g = \alpha h'$ and we get $h \otimes g = g' \otimes h'$.

Since for selfadjoint A we have $\langle Ah', g'\rangle = \langle h', Ag'\rangle = \langle Ag', h'\rangle$ thus (2) is equivalent to

$$\langle A, h \otimes g - g' \otimes h'\rangle = 0 \quad \text{for all} \quad A = A^* \in B(\mathscr{H}).$$

The case $g_1' = 0$ is symmetric. □

Proof of Theorem 3.1. In [3, Lemma 1] it was proved that each $h \in \mathscr{H}$ can be uniquely decomposed to $h = h_R + ih_I$, where $Ch_R = h_R$, $Ch_I = h_I$ and $\|h\|^2 = \|h_R\|^2 + \|h_I\|^2$. In other words, $\mathscr{H} = H_R + iH_I$, where H_R, H_I are real Hilbert spaces.

To show the inclusion "$\supset$" note that for $T \in \mathscr{C}$ we have

$$\begin{aligned}\langle T, h \otimes g - Cg \otimes Ch\rangle &= \langle Th, g\rangle - \langle TCg, Ch\rangle \\ &= \langle Th, g\rangle - \langle C^2h, CTCg\rangle = \langle h, T^*g\rangle - \langle h, CTCg\rangle = 0.\end{aligned}$$

For the converse inclusion "$\subset$" let us take the operator $h \otimes g - h' \otimes g'$ of rank at most 2. Consider the decomposition $h = h_R + ih_I$, $g = g_R + ig_I$, $h' = h_R' + ih_I'$, $g' = g_R' + ig_I'$.

An operator T can be decomposed to $\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ with respect to the decomposition $\mathscr{H} = H_R + iH_I$, where $W\colon H_R \to H_R$, $Z\colon H_I \to H_I$, $X\colon H_I \to H_R$, $Y\colon H_R \to H_I$. It can be easily obtained that T is C-symmetric if and only if $W = W^*$, $Z = Z^*$ and $Y = -X^*$, where the adjoints are taken with respect to the real Hilbert spaces.

If the operator $h \otimes g - h' \otimes g' \in \mathscr{C}_\perp$ then, in particular, $\langle W, h_R \otimes g_R - h'_R \otimes g'_R \rangle = 0$ for all selfadjoint operators W on the real Hilbert space H_R. Thus by Lemma 3.2 we get

$$h_R \otimes g_R = h'_R \otimes g'_R \quad \text{or} \quad h_R \otimes g_R = g'_R \otimes h'_R. \tag{3}$$

Similarly $\langle Z, h_I \otimes g_I - h'_I \otimes g'_I \rangle = 0$ for all selfadjoint operators Z in the real Hilbert space H_I. Thus we get

$$h_I \otimes g_I = h'_I \otimes g'_I \quad \text{or} \quad h_I \otimes g_I = g'_I \otimes h'_I. \tag{4}$$

Since $h \otimes g - h' \otimes g' \in \mathscr{C}_\perp$ thus it annihilates all operators with the decomposition $\begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix}$ according to the decomposition $\mathscr{H} = H_R + iH_I$, where $X\colon H_I \to H_R$ is an arbitrary operator. Thus

$$\begin{aligned} 0 &= \langle Xh_I, g_R \rangle - \langle X^* h_R, g_I \rangle - \langle Xh'_I, g'_R \rangle + \langle X^* h'_R, g'_I \rangle \\ &= \langle Xh_I, g_R \rangle - \langle Xg_I, h_R \rangle - \langle Xh'_I, g'_R \rangle + \langle Xg'_I, h'_R \rangle. \end{aligned}$$

Using (3) and (4) we will consider the following cases:

(a) $\quad h'_R = \alpha g_R, \quad g'_R = \frac{1}{\alpha} h_R, \quad h'_I = \beta g_I, \quad g'_I = \frac{1}{\beta} h_I,$

(b) $\quad h'_R = \alpha h_R, \quad g'_R = \frac{1}{\alpha} g_R, \quad h'_I = \beta h_I, \quad g'_I = \frac{1}{\beta} g_I,$

(c) $\quad h'_R = \alpha h_R, \quad g'_R = \frac{1}{\alpha} g_R, \quad h'_I = \beta g_I, \quad g'_I = \frac{1}{\beta} h_I,$

(d) $\quad h'_R = \alpha g_R, \quad g'_R = \frac{1}{\alpha} h_R, \quad h'_I = \beta h_I, \quad g'_I = \frac{1}{\beta} g_I,$

where $\alpha \neq 0$, $\beta \neq 0$.

Let us start with the crucial one (a). For any $X\colon H_I \to H_R$ we have

$$\begin{aligned} 0 &= \langle Xh_I, g_R \rangle - \langle Xg_I, h_R \rangle - \langle X\beta g_I, \tfrac{1}{\alpha} h_R \rangle + \langle X\tfrac{1}{\beta} h_I, \alpha g_R \rangle \\ &= (1 + \tfrac{\alpha}{\beta}) \langle Xh_I, g_R \rangle - (1 + \tfrac{\beta}{\alpha}) \langle Xg_I, h_R \rangle \end{aligned}$$

or equivalently

$$\langle X(\alpha+\beta)h_I, \alpha g_R \rangle = \langle X(\alpha+\beta)g_I, \beta h_R \rangle. \tag{5}$$

If $\beta = -\alpha$, then the equality (5) is fulfilled for any $X \in B(H_I, H_R)$. Thus by (a) we have

$$h \otimes g - h' \otimes g' = (h_R + ih_I) \otimes (g_R + ig_I) - (\alpha g_R - i\alpha g_I) \otimes (\tfrac{1}{\alpha} h_R - i\tfrac{1}{\alpha} h_I)$$

or equivalently

$$h \otimes g - h' \otimes g' = h \otimes g - Cg \otimes Ch. \tag{6}$$

If $\alpha+\beta \neq 0$, then $g_I = h_I$, $g_R = \frac{\beta}{\alpha} h_R$ by (5), since X is an arbitrary operator. Hence, using (a) we get

$$\begin{aligned} h\otimes g - h'\otimes g' &= (h_R + ih_I)\otimes(g_R + ig_I) - (\alpha g_R + i\beta g_I)\otimes(\tfrac{1}{\alpha}h_R + i\tfrac{1}{\beta}h_I) \\ &= (h_R + ih_I)\otimes(\tfrac{\beta}{\alpha}h_R + ih_I) - (\beta h_R + i\beta h_I)\otimes(\tfrac{1}{\alpha}h_R + i\tfrac{1}{\beta}h_I) = 0. \end{aligned}$$

Hence in this case we have inclusion "$\subset$". Considering other cases from (b) to (d) and using similar calculations we obtain either equality (6) or 0 operator. $\square$

Let now consider some examples of C-symmetries given in [3] in the context of Theorem 3.1.

EXAMPLE 3.4. A natural example of a C-symmetry in $l^2(\mathbb{N})$ is given by

$$C(z_0, z_1, z_2, \ldots) = (\bar{z}_0, \bar{z}_1, \bar{z}_2, \ldots).$$

In this case

$$\mathscr{C}_\perp \cap F_2 = \{h\otimes g - \overline{g}\otimes\overline{h} : h, g \in l^2(\mathbb{N})\}.$$

EXAMPLE 3.5. Consider the classical Hardy space H^2 and take a nonconstant inner function u. Denote by $H_u = H^2 \ominus uH^2$. For $f \in H_u$ and $h \in H^2$ the formula

$$Cf = u\overline{zf}$$

defines a C-symmetry on H_u. Then

$$\mathscr{C}_\perp \cap F_2 = \{h\otimes g - u\overline{zg}\otimes u\overline{zh} : h, g \in H_u\}.$$

EXAMPLE 3.6. Let ρ be a bounded, positive continuous weight on the interval $[-1,1]$, symmetric with respect to the midpoint of the interval: $\rho(t) = \rho(-t)$ for $t \in [0,1]$. Then

$$Cf(t) = \overline{f(-t)}$$

defines a C-symmetry on $L^2([-1,1], \rho dt)$. In this case

$$\mathscr{C}_\perp \cap F_2 = \{h(\cdot)\otimes g(\cdot) - \overline{g(-(\cdot))}\otimes\overline{h(-(\cdot))} : h, g \in L^2([-1,1], \rho dt)\}.$$

EXAMPLE 3.7. Consider the isometric antilinear operator

$$C(z_1, z_2) = (\bar{z}_2, \bar{z}_1)$$

on $\mathbb{C}^2$. Then

$$\mathscr{C}_\perp \cap F_2 = \{(h_1,h_2)\otimes(g_1,g_2) - (\overline{g}_2,\overline{g}_1)\otimes(\overline{h}_2,\overline{h}_1) : (h_1,h_2),(g_1,g_2) \in \mathbb{C}^2\}.$$

4. 2-reflexivity and 2-hyperreflexivity

As the straightforward consequence of the previous section we have

THEOREM 4.1. *Let $\mathscr{H}$ be a complex separable Hilbert space with an antilinear involution C. The subspace $\mathscr{C} \subset B(\mathscr{H})$ of all C-symmetric operators is 2-reflexive.*

Proof. If $T \notin \mathscr{C}$, then $\langle T, h\otimes g - Cg \otimes Ch\rangle = \langle h, (T^* - CTC)g\rangle \neq 0$ for some $h, g \in \mathscr{H}$. This means that the rank-two operator $h\otimes g - Cg\otimes Ch$ separates T from $\mathscr{C}$, hence $\mathscr{C}_\perp \cap F_2$ is linearly dense in $\mathscr{C}_\perp$. $\square$

In fact we will prove stronger result for the space of C-symmetric operators than Theorem 4.1.

THEOREM 4.2. *Let $\mathscr{H}$ be a complex separable Hilbert space with an antilinear involution C. The subspace $\mathscr{C}$ of all C-symmetric operators is 2-hyperreflexive with constant 1.*

Proof. Let $A \in B(\mathscr{H})$. Note that by Theorem 3.1 we have

$$\begin{aligned}
\alpha_2(A,\mathscr{C}) &= \sup\{|\mathrm{tr}(A(\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)))| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\}\\
&= \tfrac{1}{2}\sup\{|\langle Ah, g\rangle - \langle ACg, Ch\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\}\\
&= \tfrac{1}{2}\sup\{|\langle h, A^*g\rangle - \langle h, CACg\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\}\\
&= \tfrac{1}{2}\sup\{|\langle h, (A^* - CAC)g\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\}\\
&\geqslant \tfrac{1}{2}\sup\{\||\langle h, (A^* - CAC)g\rangle| : \|h\| \leqslant 1,\ \|g\| \leqslant 1\}\\
&= \tfrac{1}{2}\|A^* - CAC\|.
\end{aligned}$$

Note that

$$C(A + CA^*C)C = CAC + C^2A^*C^2 = CAC + A^*$$

and

$$\begin{aligned}
\langle CACx, y\rangle &= \langle Cy, C^2ACx\rangle = \langle Cy, ACx\rangle\\
&= \langle A^*Cy, Cx\rangle = \langle C^2x, CA^*Cy = \langle x, CA^*Cy\rangle.
\end{aligned}$$

Since $(A + CA^*C)^* = A^* + CAC$, then $A + CA^*C \in \mathscr{C}$, which implies that

$$d(A,\mathscr{C}) \leqslant \|A - \tfrac{1}{2}(A + CA^*C)\| = \tfrac{1}{2}\|A - CA^*C\| \leqslant \alpha_2(A,\mathscr{C}).$$

Hence $\mathscr{C}$ is 2-hyperreflexive with constant 1. $\square$

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