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COMMUTING TRACES ON INVERTIBLE AND SINGULAR OPERATORS

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Abstract. Let $m \geqslant 1$ be a natural number, and let $B(H)$ be the Banach space of all bounded operators from a infinite dimensional separable complex (real) Hilbert space H to itself. We describe traces of m-additive maps $G : B(H)^m \to B(H)$ such that $[G(T,\dots,T),T] = 0$ for all invertible or singular $T \in B(H)$.

Let $\mathbb{K}$ be either the field of the real or complex numbers. Denote by H the infinite dimensional separable Hilbert space over $\mathbb{K}$ and by $\{\phi_1,\phi_2,\dots\}$ a fixed orthonormal system for H, that is, $x = \sum_{i=1}^{\infty} < x,\phi_i > \phi_i$ for each $x \in H$, where $< . >$ denotes the inner product in H. As usual $B(H)$ stands for the Banach space of all bounded operators from H to itself. Observe that the operator $e_{ij}(x) =< x,\phi_i > \phi_j \in B(H)$ for each $i, j \in \mathbb{N}$. The spectrum $\sigma(T)$ of $T \in B(H)$ is defined by

$$\sigma(T) = \{\lambda \in \mathbb{K} \mid \lambda I - T \quad \text{is not invertible}\}.$$

The resolvent set $\nu(T)$ is defined by $\nu(T) = \mathbb{K} \setminus \sigma(T)$. It is well known that the spectrum $\sigma(T)$ of T is a compact set in $\mathbb{K}$ bounded by $||T||$. In particular, the resolvent $\nu(T)$ is an unbounded open set that contains $\{\varepsilon \in \mathbb{K} \mid \varepsilon > ||T||\}$.

Now, let $m \geqslant 1$ be a natural number. In the following discussion, we fix an m-additive map $G : B(H)^m \to B(H)$. This means that G is additive in each component, that is,

$$G(T_1,\dots,T_i + S_i,\dots,T_m) = G(T_1,\dots,T_i,\dots,T_m) + G(T_1,\dots,S_i,\dots,T_m)$$

for all $T_i, S_i \in B(H)$, and $i \in \{1,\dots,m\}$. The map $F : B(H)^m \to B(H)$ defined by $F(T) = G(T,\dots,T)$ is known as the *trace* of G. We call F *commuting* if for each $T \in B(H)$ the equality $G(T,\dots,T)T = TG(T,\dots,T)$ holds. Using the commutator form we can rewrite the latter as $[G(T,\dots,T),T] = G(T,\dots,T)T - TG(T,\dots,T) = 0$.

In [1] the author describes all commuting traces of an m-additive map $G : B(H)^m \to B(H)$ such that $[G(x,\dots,x),x] = 0$ for all invertible or singular $x \in B(H)$ in the finite dimensional setting and $m \geqslant 2$. The test case for $m = 1$ has been covered in the author's paper [2]. Recently, Liu [4] characterized centralizing maps on invertible (singular) matrices over division rings. Precisely, Liu proved that if $f : M_n(\mathbb{D}) \to M_n(\mathbb{D})$

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is an additive map satisfying $f(x)x - xf(x) \in Z$ for all invertible $x \in M_n(\mathbb{D})$, where $M_n(\mathbb{D})$ denotes the ring of all $n \times n$ matrices over a division ring D and Z is the center of $M_n(\mathbb{D})$, then there exist $\lambda \in Z$ and an additive map $\mu : M_n(\mathbb{D}) \to Z$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in M_n(\mathbb{D})$ except when $D \cong \mathbb{Z}_2$, the Galois field of two elements. A map $f : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ is called centralizing on a subset $S \subset M_n(\mathbb{D})$ if $f(x)x - xf(x) \in Z$ for all $x \in S$. Centralizing additive maps on the set of singular matrices is also obtained in [4, Theorem 1.2].

The purpose of this article is to characterize commuting traces of an m-additive map $G : B(H)^m \to B(H)$ such that $[G(x,\ldots,x),x] = 0$ for all invertible or singular $x \in B(H)$, when H is an infinite dimensional separable Hilbert space and $m \geqslant 1$.

In this work we may assume that the m-additive map $G : B(H)^m \to B(H)$ is symmetric. For instance, consider $G^{'}(T_1,\ldots,T_m) = \sum_{\sigma \in S_m} G(T_{\sigma(1)},\ldots,T_{\sigma(m)})$. It is clear that $G^{'}$ is symmetric and $G^{'}(T,\ldots,T) = m!G(T,\ldots,T)$ for all $T \in B(H)$. Clearly, we see that for $T \in B(H)$, $[G^{'}(T,\ldots,T),T] = 0$ if and only if $[G(T,\ldots,T),T] = 0$. Also, we have that G is m-linear over $\mathbb{Q}$. This fact will be largely used in this paper.

We start with commuting traces of m-additive maps on the set of invertible operators. First, we need an auxiliary result.

PROPOSITION 1. *Let $m \geqslant 1$ be a natural number. Let $G : B(H)^m \to B(H)$ be a symmetric m-additive map such that*

$$[G(T,\ldots,T),T] = 0 \quad \textit{for all invertible } T \in B(H). \tag{1}$$

Then $G(kI,\ldots,kI) \in Z$ for all $k \in \mathbb{K}$, where $Z = \mathbb{K} \cdot I$.

Proof. First of all observe that the result holds trivially when $k = 0$. Now, fix $k \in \mathbb{K}^*$, and let e_{ij}, $i \neq j$ the operator $e_{ij}(x) =< x, \phi_i > \phi_j \in B(H)$. Let s be the smallest even number greater or equal than m, that is, $s = m$ if m is even, and $s = m+1$ if m is odd. We will show that $[G(kI,\ldots,kI), e_{ij}] = 0$.

For each $a \in \mathbb{K}^*$, let $y_a = akI + u$, where $u = (I + e_{ij})$. Note that $y_a = akI + u$ is invertible if and only if $-ak \in \nu(u) = \mathbb{K} \setminus \{1\}$. Therefore y_a is invertible if $a \neq -\frac{1}{k}$. So, we can find a nonzero rational number b such that y_a is invertible for all $a \in \{\pm b, \pm 2b, \ldots, \pm \frac{s}{2} b\}$ (take b satisfying $|b| > \frac{1}{k}$). It follows from (1) that $0 = [G(u,\ldots,u),u] = [G(kI,\ldots,kI),kI] = [G(y_a,\ldots,y_a),y_a]$, and this last bracket can be written as

$$[G(y_a,\ldots,y_a),u] = 0, \tag{2}$$

since $y_a = akI + u$. For $m = 1$ we conclude from (2) and $[G(u),u] = 0$ that $[G(kI),u] = [G(kI),e_{ij}] = 0$ because $u = (I + e_{ij})$. It remains to prove that $[G(kI,\ldots,kI),e_{ij}] = 0$ for $m \geqslant 2$. Using (2) one more time, we see that $[G(y_a,\ldots,y_a) + G(y_{-a},\ldots,y_{-a}),u] = 0$ for all $a \in \{b, 2b, \ldots, \frac{s}{2} b\}$. Now, since G is symmetric, m-additive, and $y_a = akI + u$ we can obtain for each $a \in \{b, 2b, \ldots, \frac{s}{2} b\}$ that:

$$G(y_a,\ldots,y_a) = \sum_{\zeta=0}^{m} a^{m-\zeta} \binom{m}{\zeta} G(kI,\ldots,kI,\underbrace{u,\ldots,u}_{\zeta}), \tag{3}$$

and

$$G(y_{-a},\ldots,y_{-a}) = \sum_{\zeta=0}^{m}(-1)^{m-\zeta}a^{m-\zeta}\binom{m}{\zeta}G(kI,\ldots,kI,\underbrace{u,\ldots,u}_{\zeta}). \tag{4}$$

By keeping in mind the equations (3), (4), and the relation $[G(u,\ldots,u),u]=0$ we see that $[G(y_a,\ldots,y_a)+G(y_{-a},\ldots,y_{-a}),u]=0$ becomes:

$$\sum_{\zeta=0}^{\frac{s-2}{2}} a^{m-2\zeta}\binom{m}{2\zeta}[G(kI,\ldots,kI,\underbrace{u,\ldots,u}_{2\zeta}),u]=0, \quad \text{when } m \text{ is even,} \tag{5}$$

and

$$\sum_{\zeta=0}^{\frac{s-4}{2}} a^{m-(2\zeta+1)}\binom{m}{2\zeta+1}[G(kI,\ldots,kI,\underbrace{u,\ldots,u}_{2\zeta+1}),u]=0, \quad \text{when } m \text{ is odd.} \tag{6}$$

With (6), the identity $[G(y_a,\ldots,y_a),u]=0$ becomes:

$$\sum_{\zeta=0}^{\frac{s-2}{2}} a^{m-2\zeta}\binom{m}{2\zeta}[G(kI,\ldots,kI,\underbrace{u,\ldots,u}_{2\zeta}),u]=0, \quad \text{when } m \text{ is odd.}$$

Therefore, for each $a\in\{b,2b,\ldots,\frac{s}{2}b\}$ we have obtained an equation of the form (5) when m is either even or odd. It means that we got $\frac{s}{2}$ equations in $\frac{s}{2}$ unknowns, namely $\binom{m}{2\zeta}[G(kI,\ldots,kI,u,\ldots,u),u]$, where u appears exactly in 2ζ components of G, and $\zeta\in\{0,1,\ldots,\frac{s-2}{2}\}$. Using matrix notation we can rewrite these systems in the following way:

$$\begin{pmatrix} b^m & b^{m-2} & b^{m-4} & \ldots & b^{m-(s-2)} \\ (2b)^m & (2b)^{m-2} & (2b)^{m-4} & \ldots & (2b)^{m-(s-2)} \\ (3b)^m & (3b)^{m-2} & (3b)^{m-4} & \ldots & (3b)^{m-(s-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\frac{s}{2}b)^m & (\frac{s}{2}b)^{m-2} & (\frac{s}{2}b)^{m-4} & \ldots & (\frac{s}{2}b)^{m-(s-2)} \end{pmatrix} \begin{pmatrix} \binom{m}{0}[G(kI,\ldots,kI),u] \\ \vdots \\ \binom{m}{2\zeta}[G(kI,\ldots,kI,u,\ldots,u),u] \\ \vdots \\ \binom{m}{s-2}[G(kI,u,\ldots,u),u] \end{pmatrix} = 0.$$

Because the determinant of the Vandermonde matrix formed by the coefficients of the system is not zero, we get that $[G(kI,\ldots,kI),u]=0$. As $u=(I+e_{ij})$, we conclude that $[G(kI,\ldots,kI),e_{ij}]=0$, when $i\neq j$. Finally, we see that

$$\begin{aligned}[G(kI,\ldots,kI),e_{ii}] &= [G(kI,\ldots,kI),e_{ij}e_{ji}] = \\ &= [G(kI,\ldots,kI),e_{ij}]e_{ji}+e_{ij}[G(kI,\ldots,kI),e_{ji}]=0.\end{aligned}$$

It means that $G(kI,\ldots,kI)$ commutes with all finite rank operators of the form $e_{ij} = < x,\phi_i > \phi_j$. Therefore, $G(kI,\ldots,kI) \in Z$ for all $k\in\mathbb{K}$. $\square$

THEOREM 2. *Let $m \geqslant 1$ be a natural number. Let $G : B(H)^m \to B(H)$ be an m-additive map such that*

$$[G(T,\dots,T),T] = 0 \quad \textit{for all invertible } T \in B(H). \tag{7}$$

Then, there exist $\mu_0 \in Z$ and maps $\mu_i : B(H) \to Z$, $i \in \{1,\dots,m\}$, such that each μ_i is the trace of an i-additive map and $G(T,\dots,T) = \mu_0 T^m + \mu_1(T)T^{m-1} + \dots + \mu_{m-1}(T)T + \mu_m(T)$ for all $T \in B(H)$, where $Z = \mathbb{K} \cdot I$.

Proof. Without loss of generality, we may assume that G is symmetric. Once again, let s be the smallest even number greater or equal than m, that is, $s = m$ if m is even, and $s = m+1$ if m is odd. Our goal is to show that $[G(T,\dots,T),T] = 0$ for all $T \in B(H)$. Fix $T \in B(H)$. Since $\{\varepsilon \in \mathbb{K} \mid \varepsilon > ||T||\} \subset \nu(T)$, we can find a nonzero number $\lambda \in \mathbb{K}$ such that $y_a = T + a\lambda I$ is invertible for all $a \in \{\pm 1,\dots,\pm \frac{s}{2}\}$. For $m = 1$ we obtain after employing the Proposition 1 in the identity $[G(T+\lambda I),T] = [G(T+\lambda I),T+\lambda I] = 0$ (equation (7)) that $[G(T),T] = 0$. From now on, we may take $m \geqslant 2$. It follows from (7) and $y_a = T + a\lambda I$ that $[G(y_a,\dots,y_a),T] = [G(y_a,\dots,y_a),y_a] = 0$ for all $a \in \{\pm 1,\dots,\pm \frac{s}{2}\}$. Consequently,

$$[G(y_a,\dots,y_a) + G(y_{-a},\dots,y_{-a}),T] = 0 \quad \text{for all} \quad a \in \{1,\dots,\frac{s}{2}\}. \tag{8}$$

Now, since G is symmetric, m-additive and $y_a = T + a\lambda I$, we conclude that

$$G(y_a,\dots,y_a) = \sum_{r=0}^{m} a^r \binom{m}{r} G(\underbrace{\lambda I,\dots,\lambda I}_{r},T,\dots,T), \tag{9}$$

for each $a \in \{\pm 1,\dots,\pm \frac{s}{2}\}$. Thus, if we take into the account that $G(\lambda I,\dots,\lambda I) \in Z$ (Proposition 1) and the equation (9) we can derive from (8) that:

$$[G(T,T),T] = 0 \quad \text{if} \quad m = 2,$$

and

$$\sum_{r=0}^{\frac{s-2}{2}} a^{2r} \binom{m}{2r} [G(\underbrace{\lambda I,\dots,\lambda I}_{2r},T,\dots,T),T] = 0 \quad \text{if} \quad m \geqslant 3.$$

As in the proof of the Proposition 1, for each $m \geqslant 3$ we have:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2^2 & 2^4 & \dots & 2^{s-2} \\ 1 & 3^2 & 3^4 & \dots & 3^{s-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (\frac{s}{2})^2 & (\frac{s}{2})^4 & \dots & (\frac{s}{2})^{s-2} \end{pmatrix} \begin{pmatrix} [G(T,\dots,T),T] \\ \binom{m}{2} [G(\lambda I,\lambda I,T,\dots,T),T] \\ \vdots \\ \binom{m}{s-2} [G(\lambda I,\dots,\lambda I,\underbrace{T,\dots,T}_{m-(s-2)}),T] \end{pmatrix} = 0.$$

Therefore, $[G(T,\dots,T),T]=0$ for all $T\in B(H)$. With this in hand, the desired result now follows from [3, Theorem 3.1]. □

Our next goal is to study commuting traces of m-additive maps on the set of singular operators.

THEOREM 3. *Let $m\geqslant 1$ be a natural number. Let $G:B(H)^m\to B(H)$ be a symmetric m-additive map such that*

$$[G(T,\dots,T),T]=0 \quad \textit{for all singular } T\in B(H). \tag{10}$$

Then, there exist $\mu_0\in Z$ and maps $\mu_i:B(H)\to Z$, $i\in\{1,\dots,m\}$, such that each μ_i is the trace of an i-additive map and $G(T,\dots,T)=\mu_0T^m+\mu_1(T)T^{m-1}+\dots+\mu_{m-1}(T)T+\mu_m(T)$ for all $T\in B(H)$, where $Z=\mathbb{K}\cdot I$.

Proof. We shall proceed as we did in the proof of the Theorem 2, that is, we will show that $[G(T,\dots,T),T]=0$ for all $T\in B(H)$. Fix $T\in B(H)$. Let us define the finite rank operator $S\in B(H)$ as the following:

$$S=\sum_{n=1}^{m+2}-\left(\frac{1}{n}\right)< x,\phi_n > T(\phi_n). \tag{11}$$

By construction, we see that $T+jS$ is singular for all $j\in\{1,\dots,m+2\}$, because $(T+jS)(\phi_j)=0$. Thus, $[G(T+jS,\dots,T+jS),T+jS]=0$ (equation 10) for all $j\in\{1,\dots,m+2\}$. Using the symmetricity and the m-additivity of G, we arrive at

$$\begin{gathered}\sum_{h=0}^{m} j^h\binom{m}{h}[G(\underbrace{S,\dots,S}_{h},T,\dots,T),T] \;+\\ \sum_{h=0}^{m} j^{h+1}\binom{m}{h}[G(\underbrace{S,\dots,S}_{h},T,\dots,T),S]=0.\end{gathered} \tag{12}$$

For convenience let us set:

$$\alpha(h)=\binom{m}{h}[G(\underbrace{S,\dots,S}_{h},T,\dots,T),T], \quad \text{where} \quad h\in\{0,\dots,m\},$$

and

$$\gamma(h)=\binom{m}{h}[G(\underbrace{S,\dots,S}_{h},T,\dots,T),S], \quad \text{where} \quad h\in\{0,\dots,m\}.$$

Observe that for each $j\in\{1,\dots,m+2\}$ we have obtained an equation of the form

(12). Thus, using matrix notation we have the following:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2^1 & 2^2 & \dots & 2^{m+1} \\ 1 & 3^1 & 3^2 & \dots & 3^{m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (m+2)^1 & (m+2)^2 & \dots & (m+2)^{m+1} \end{pmatrix} \begin{pmatrix} \alpha(0) \\ \alpha(1)+\gamma(0) \\ \alpha(2)+\gamma(1) \\ \vdots \\ \alpha(m)+\gamma(m-1) \\ \gamma(m) \end{pmatrix} = 0.$$

Therefore, $\alpha(0) = [G(T,\dots,T),T] = 0$, and this is true for all $T \in B(H)$ since T is arbitrary. The result follows now from [3, Theorem 3.1]. □

REFERENCES

[1] W. FRANCA, *Commuting traces of multiadditive maps on invertible and singular matrices*, Linear and Multilinear Algebra, accepted.

[2] W. FRANCA, *Commuting maps on some subsets of matrices that are not closed under addition*, Linear Algebra Appl., **437** (2012), 388–391.

[3] P.-H. LEE, T.-L. WONG, J.-S. LIN, R.-J. WANG, *Commuting trace of multiadditive mappings*, J. Algebra, **193** (1997), 709–723.

[4] C.-K. LIU, *Centralizing maps on invertible or singular matrices over division rings*, Linear Algebra Appl., to appear.

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