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MORE ON THE MINIMUM SKEW–RANK OF GRAPHS

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Abstract. The minimum (maximum) skew-rank of a simple graph G over real field is the smallest (largest) possible rank among all skew-symmetric matrices over real field whose ij-th entry is nonzero whenever v_iv_j is an edge in G and is zero otherwise. In this paper we obtain more results about the minimum skew-rank of graphs. Further we get a lower (upper) bound for minimum (maximum) skew-rank of unicyclic graph of order n with girth k, and characterize unicyclic graphs attaining the extremal values. Moreover, we characterize the unicyclic graphs with skew-rank 4 or 6, respectively. Finally we consider the non-singularity of skew-symmetric matrices described by unicyclic graphs.

1. Introduction

An $n \times n$ matrix A is symmetric (resp. skew-symmetric) if $A^T = A$ $(resp.\ A^T = -A)$. The minimum (symmetric) rank problem is to determine the minimum possible rank of all real symmetric matrices that realize a graph G [14]. This problem has been modified to consider all fields [6, 7, 8, 9, 14, 17], and to consider graphs with loops and multiple edges [20]. The problem has also been altered to consider positive definite matrices, Hermitian matrices, Hermitian positive semidefinite matrices and other non-symmetric matrices that realize a graph G [14, 19]. For other developments in this direction, one may refer to [2, 3, 4, 5, 10, 18].

The minimum skew rank problem, to calculate the minimum rank of skew-symmetric matrices which realize a graph, arose after extensive study of the minimum (symmetric) rank problem. This problem attracts much attention recently [11, 12, 13, 19, 21]. In this paper we focus on the problem of determining the minimum rank of real skew-symmetric matrices described by a unicyclic graph over real field $\mathbf{R}$.

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. An *oriented graph* G^σ is a graph with an orientation, which assigns to each edge of G a direction so that G^σ becomes a directed graph. A *weighted oriented graph* G_w^σ is a pair (G^σ, w) where G^σ is an oriented graph with arc set $E(G^\sigma)$ and w is a weight function from the arc set $E(G^\sigma)$ to the set of positive real numbers. The

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skew-adjacency matrix of the weighted oriented graph G_w^σ of order n is the real matrix $S(G_w^\sigma) = (w_{ij})_{n\times n}$ such that

$$w_{ij} = \begin{cases} w(v_iv_j), & \text{if there is an arc from } v_i \text{ to } v_j; \\ -w(v_iv_j), & \text{if there is an arc from } v_j \text{ to } v_i; \\ 0, & \text{otherwise.} \end{cases}$$

The rank of $S(G_w^\sigma)$ is called the *skew-rank* of G_w^σ, denoted by $sr(G_w^\sigma)$.

For an $n\times n$ real skew-symmetric matrix $A = (a_{ij})$, there exists a graph corresponding to A, denoted by $\mathscr{G}(A)$, with vertex set $\{v_1, v_2, \cdots, v_n\}$, edge set $\{v_iv_j : a_{ij} \neq 0, 1 \leqslant i < j \leqslant n\}$. In fact there exits a bijection between the set of real skew-symmetric matrices and the set of weighted oriented graphs. The set of skew-symmetric matrices over real field $\mathbf{R}$ described by G is

$$\mathscr{S}^-(G) = \{A \in \mathbf{R}^{n\times n} : A^T = -A, \mathscr{G}(A) = G\}.$$

It should be mentioned that when calculating the minimum (symmetric) rank, a matrix can have zero or nonzero diagonal entries; the diagonal is unconstrained. In the skew-symmetric case, for $A \in \mathscr{S}^-(G)$ each diagonal entry $a_{ii} = -a_{ii}$, and thus each diagonal entry must be zero. The minimum skew-rank of a graph G over $\mathbf{R}$ is defined to be

$$mr^-(G) = \min\{rank(A) : A \in \mathscr{S}^-(G)\},$$

and the maximum skew nullity of G over real field $\mathbf{R}$ is defined to be

$$M^-(G) = \max\{null(A) : A \in \mathscr{S}^-(G)\},$$

where $null(A)$ is the nullity of A. Obviously, $mr^-(G) + M^-(G) = n$. The maximum skew-rank of a graph is

$$MR^-(G) = \max\{rank(A) : A \in \mathscr{S}^-(G)\}.$$

A unicyclic graph is a connected graph with equal vertex number and edge number. For a vertex $v \in V(G)$, $G - v$ denotes the graph obtained from G by deleting vertex v and all edges incident with v. A vertex of a graph G is called *pendant* if it is only adjacent to one vertex, and is called *quasi-pendant* if it is adjacent to a pendant vertex. A set M of edges in G is a *matching* if every vertex of G is incident with at most one edge in M. It is *perfect matching* if every vertex of G is incident with exactly one edge in M. We denote by $\beta(G)$ the *matching number* of G (i.e. the number of edges of a maximum matching in G). For a graph G on at least two vertices, a vertex $v \in V(G)$ is called *mismatched* in G if there exists a maximum matching M of G in which no edge is incident with v; otherwise, v is called *matched* in G.

The present paper is organized as follows. In Section 2 we further study the skew-rank of graphs and give several formula for calculating the skew-rank of graphs. In Section 3, we consider the minimum skew-rank of unicyclic graphs. Firstly get a lower bound for minimum skew-rank of unicyclic graphs of order n with fixed girth and characterize unicyclic graphs attaining the minimum value. Then we characterize the unicyclic graphs with skew-rank 4 or 6, respectively. In Section 4, we consider the non-singularity of the skew-symmetric matrices described by unicyclic graphs.

2. Preliminaries

Let G^{σ} be an oriented unicyclic graph of order n with skew-adjacency matrix $S(G^{\sigma}) = (s_{ij})_{n\times n}$. Let $C_k^{\sigma} = u_1u_2\cdots u_ku_{k+1}(=u_1)$ be the unique oriented cycle of G^{σ}. The *sign* of the cycle C_k^{σ} is defined as $sgn(C_k^{\sigma}) = \prod_{i=1}^{k} s_{u_iu_{i+1}}$. The graph G^{σ} with an even oriented cycle C_k^{σ} is called *evenly oriented* (*oddly oriented*) if $sgn(C_k^{\sigma})$ is positive (negative). An oriented graph H^{σ} is called an *elementary oriented graph* if H^{σ} is K_2^{σ} or an oriented cycle with even length.

The weight of a weighted elementary oriented graph H^{σ} is defined as the square of the weight of the unique arc if H^{σ} is K_2^{σ}; or the product of all weights of those arcs if H^{σ} is an even cycle. An oriented graph H^{σ} is called a *linear oriented graph* if each component of H^{σ} is an elementary oriented graph. The weight of a linear oriented graph H^{σ}, denoted by $w(H^{\sigma})$, is the product of all weights of those elementary oriented graphs contained in it.

LEMMA 2.1. [16] *Let G_w^{σ} be a weighted oriented graph of order n with skew adjacency matrix $S(G_w^{\sigma})$ and its characteristic polynomial*

$$\phi(G_w^{\sigma},\lambda) = \sum_{i=0}^{n}(-1)^i a_i\lambda^{n-i} = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + (-1)^{n-1}a_{n-1}\lambda + (-1)^n a_n.$$

Then

$$a_i = \sum_{H^{\sigma}}(-1)^{c^+}2^c w(H^{\sigma}) \qquad \text{if } i \text{ is even,}$$

where the summation is over all linear oriented subgraphs H^{σ} of G_w^{σ} having i vertices, and c^+, c are respectively the numbers of evenly oriented even cycles and even cycles contained in H^{σ}. In particular, $a_i = 0$ if i is odd.

The IMA-ISU research group [19] obtained the following result by means of the pfaffian of a matrix. Here we present an alternative and concise proof.

LEMMA 2.2. [19] *Let T be a tree with matching number $\beta(T)$. Then*

$$mr^-(T) = MR^-(T) = 2\beta(T).$$

Proof. It is suffices to verify that $sr(T_w^{\sigma}) = 2\beta(T)$ for any weighted oriented tree T_w^{σ}. It is natural that any elementary oriented subgraph in T_w^{σ} is K_2^{σ}. If $i > \beta(T)$, there exists no elementary oriented subgraph with $2i$ vertices and $a_{2i} = 0$. Therefore we suppose $0 \leqslant i \leqslant \beta(T)$. From Lemma 2.1, we have $a_{2i} = \sum_H \prod_{e\in H}(w(e))^2$. So $a_{2\beta(T)}$ is the last nonzero coefficient of $\phi(G_w^{\sigma},\lambda)$, which yields the result. □

LEMMA 2.3. *Let G be a graph containing a pendant vertex v with the unique neighbor u. Then $mr^-(G) = mr^-(G-u-v)+2$, $MR^-(G) = MR^-(G-u-v)+2$.*

Proof. We shall verify that $sr(G_w^\sigma) = sr(G_w^\sigma - u - v) + 2$ for any weighted oriented graph G_w^σ. Assume that $V(G_w^\sigma) = \{v_1, v_2, \cdots, v_n\}$ with $v_1 = v$, $v_2 = u$. Then the skew-adjacency matrix of G_w^σ can be expressed as

$$S(G_w^\sigma) = \begin{pmatrix} 0 & w_{12} & 0 & \cdots & 0 \\ w_{21} & 0 & w_{23} & \cdots & w_{2n} \\ 0 & w_{32} & 0 & \cdots & w_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & w_{n2} & w_{n3} & \cdots & 0 \end{pmatrix},$$

where the first two rows and columns are labeled by v_1, v_2. Therefore it follows that

$$\begin{aligned} sr(G_w^\sigma) &= sr\begin{pmatrix} 0 & w_{12} & 0 & \cdots & 0 \\ w_{21} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & w_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & w_{n3} & \cdots & 0 \end{pmatrix} \\ &= sr\begin{pmatrix} 0 & w_{12} \\ w_{21} & 0 \end{pmatrix} + sr\begin{pmatrix} 0 & \cdots & w_{3n} \\ \vdots & \ddots & \vdots \\ w_{n3} & \cdots & 0 \end{pmatrix} \\ &= sr\begin{pmatrix} 0 & w_{12} \\ w_{21} & 0 \end{pmatrix} + sr(G_w^\sigma - \{v_1, v_2\}) \\ &= 2 + sr(G_w^\sigma - u - v). \end{aligned}$$

We complete the proof. □

Let u, v be two pendant vertices of a weighted graph G_w. u, v are called pendant twins if they have the same neighbor in G.

LEMMA 2.4. *Let u, v be pendant twins of a graph G. Then $mr^-(G) = mr^-(G - u) = mr^-(G - v)$.*

Proof. It is sufficient to verify that $sr(G_w^\sigma) = sr(G_w^\sigma - u) = sr(G_w^\sigma - v)$. Let u_0 be the unique neighbor of u, v. Then the skew-adjacency matrix of G_w^σ can be expressed as

$$sr(G_w^\sigma) = \left(\begin{array}{cc|c|c} 0 & 0 & s_1 & \multirow{2}{*}{$\mathbf{0}$} \\ 0 & 0 & s_2 & \\ \hline -s_1 & -s_2 & 0 & \alpha \\ \hline \multicolumn{2}{c|}{\mathbf{0}^t} & -\alpha^t & B \end{array}\right),$$

where B is the adjacency matrix of $G_w^\sigma - u - v - u_0$ and the first three rows and columns are labeled by u, v and u_0. So we have

$$sr(G_w^\sigma) = r\left(\begin{array}{cc|c|c} 0 & 0 & s_1 & \multirow{2}{*}{0} \\ 0 & 0 & 0 & \\ \hline -s_1 & 0 & 0 & \alpha \\ \hline \multicolumn{2}{c|}{0^t} & -\alpha^t & B \end{array}\right)$$

$$= r\left(\begin{array}{c|c|c} 0 & s_1 & 0 \\ \hline -s_1 & 0 & \alpha \\ \hline 0^t & -\alpha^t & B \end{array}\right)$$

$$= sr(G_w^\sigma - v).$$

Similarly, we have $sr(G_w^\sigma) = sr(G_w^\sigma - u)$. $\square$

For convenience, we call the transformation in Lemma 2.3 the δ-*transformation*.

LEMMA 2.5. [19] *Let C_n be a cycle of order n. Then*

$$mr^-(C_n) = \begin{cases} n-2, & n \text{ is even,} \\ n-1, & n \text{ is odd.} \end{cases}$$

LEMMA 2.6. [19] *Let H be an induced subgraph of G. Then $mr^-(H) \leqslant mr^-(G)$.*

Let G_1 be a graph containing a vertex u and G_2 be a graph of order n disjoint from G_1. For $1 \leqslant k \leqslant n$, a k–joining graph of G_1 and G_2 with respect to u, denoted by $G_1(u) \odot^k G_2$, is obtained from $G_1 \cup G_2$ by joining u and certain k vertices of G_2 with edges.

LEMMA 2.7. *Let T be a tree with $u \in V(T)$ and G be a graph different from T. Let $T(u) \odot^k G$ be the k–joining graph of T and G with respect to u. Then the following statements hold:*

(1) If u is matched in T, then

$$mr^-(T(u) \odot^k G) = mr^-(G) + mr^-(T). \tag{$*$}$$

(2) If u is mismatched in T, then

$$mr^-(T(u) \odot^k G) = mr^-(T-u) + mr^-(G+u),$$

where $G+u$ is the subgraph of $T(u) \odot^k G$ induced by the vertices of G and u.

Proof. (1). We shall prove the results by applying induction to the matching number $\beta(T)$. If $\beta(T)=1$. Then T is star and u is the center of T. Assume that v is a pendant vertex in T. By Lemmas 2.2 and 2.3, we have

$$\begin{aligned} mr^-(T(u)\odot^k G) &= mr^-(T(u)\odot^k G - v - u) + 2 \\ &= mr^-(G) + 2 \\ &= mr^-(G) + mr^-(T). \end{aligned}$$

If $\beta(T)\geqslant 2$. Assume that the assertion is true when $\beta(T)\leqslant t$. Now we consider the case $\beta(T)=t+1$. Since $\beta(T)\geqslant 2$, T contains a pendant vertex v and its neighbor w such that v,w are both different to u. It is evident that w is matched in T. Let T_1 be a new tree by deleting v and w. Hence $\beta(T_1)=\beta(T)$, or $\beta(T)-1$ since v is a pendant vertex. If $\beta(T_1)=\beta(T)$, then there exists a maximum matching M of T that does not cover w, which contradicts to the fact that w is matched in T. So $\beta(T_1)=\beta(T)-1=t$. Therefore by Lemmas 2.3 and 2.2, it follows that

$$\begin{aligned} mr^-(T(u)\odot^k G) &= mr^-\big(T(u)\odot^k G - v - w\big) + 2 \\ &= mr^-(T_1(u)\odot^k G) + 2 \\ &= mr^-(T_1) + mr^-(G) + 2 \quad \text{by induction} \\ &= mr^-(T - v - w) + mr^-(G) + 2 \\ &= mr^-(T) + mr^-(G). \end{aligned}$$

(2). Let $\{u_1,u_2,\cdots,u_m\}$ be the neighborhood of u in T. $T_1,T_2,\cdots,T_m$ are the components of $T-u$ that contain the vertices $u_1,u_2,\cdots,u_m$, respectively. Therefore each vertex u_i is matched in T_i. Then

$$\begin{aligned} T(u)\odot^k G &= T_1(u_1)\odot^1 \big((T(u)\odot^k G) - T_1\big) \\ &= T_1(u_1)\odot^1 \big[T_2(u_2)\odot^1 \big((T(u)\odot^k G) - \cup_{i=1}^{2} T_i\big)\big] \\ &= \cdots \\ &= T_1(u_1)\odot^1 \big[T_2(u_2)\odot^1 \cdots \odot^1 [T_m(u_m)\odot^1 \big((T(u)\odot^k G) - \cup_{i=1}^{m} T_i\big)]\big] \\ &= T_1(u_1)\odot^1 \big[T_2(u_2)\odot^1 \cdots \odot^1 [T_m(u_m)\odot^1 (G+u)]\big]. \end{aligned}$$

Applying formula $(*)$ repeatedly, we have

$$\begin{aligned} mr^-(T(u)\odot^k G) &= mr^-\Big(T_1(u_1)\odot^1 \big[T_2(u_2)\odot^1 \cdots \odot^1 [T_m(u_m)\odot^1 (G+u)]\big]\Big) \\ &= mr^-(T_1) + mr^-\Big(T_2(u_2)\odot^1 \cdots \odot^1 [T_m(u_m)\odot^1 (G+u)]\Big) \\ &= \cdots \\ &= \sum_{i=1}^{m-1} mr^-(T_i) + mr^-(T_m(u_m)\odot^1 (G+u)) \\ &= \sum_{i=1}^{m} mr^-(T_i) + mr^-(G+u) \\ &= mr^-(T-u) + mr^-(G+u). \end{aligned}$$

This implies the result. $\square$

Let G be a unicyclic graph and C_k be the unique cycle of G. Let G' be the graph obtained from G by deleting the two neighbors of v on C_k and let $G\{v\}$ be the component of G' containing v. Then $G\{v\}$ is a tree rooted at v and an induced subgraph of G.

By Lemma 2.7, we have

COROLLARY 2.8. *Let G be a unicyclic graph and C_k be the unique cycle in G. For each vertex $v \in V(C_k)$, let $G\{v\}$ be the tree rooted at v and containing v. Then the following statements hold:*

(1) If there exists a vertex $v \in V(C_k)$ which is matched in $G\{v\}$, then

$$mr^-(G) = mr^-(G\{v\}) + mr^-(G - G\{v\}).$$

(2) If there exists a vertex $v \in V(C_k)$ which is mismatched in $G\{v\}$, then

$$mr^-(G) = mr^-(C_k) + mr^-(G - C_k).$$

3. Small minimum skew-rank of unicyclic graphs

In this section, we investigate the lower bound for minimum skew-rank of unicyclic graphs and characterize the unicyclic graphs with minimum skew-rank 4 or 6, respectively.

3.1. Lower bound for minimum skew-rank of unicyclic graphs

Let $H(n,k)$ be a unicyclic graph obtained from C_k by attaching $n-k$ pendant edges to some vertex on C_k. Let U^* be a unicyclic graph obtained from a cycle C_k and a star S_{n-k} by inserting an edge between a vertex on C_k and the center of S_{n-k}.

THEOREM 3.1. *Let G be a unicyclic graph of order n with girth k $(n \geqslant k+1)$. Then*

$$mr^-(G) \geqslant \begin{cases} k, & k \text{ is even,} \\ k+1, & k \text{ is odd.} \end{cases}$$

The equality holds if and only if the following statements hold:

(1) If there exists a vertex $v \in V(C_k)$ which is matched in $G\{v\}$, then $G\{v\}$ is a star, and $\beta(G - G\{v\}) = \begin{cases} \frac{k-2}{2}, & k \text{ is even,} \\ \frac{k-1}{2}, & k \text{ is odd.} \end{cases}$

(2) If there exists a vertex $v \in V(C_k)$ which is mismatched in $G\{v\}$, then $G \cong U^*$.

Proof. Since G must contain $H(k+1,k)$ as an induced subgraph, $mr^-(H(k+1,k)) \leqslant mr^-(G)$ from Lemma 2.6. According to the definition of $H(k+1,k)$, there exists exactly one vertex with degree more than 2, saying u. Let w be a pendant vertex adjacent to u in $H(k+1,k)$. By Lemma 2.3, we have

$$\begin{aligned} mr^-(H(k+1,k)) &= mr^-(H(k+1,k)-u-w)+2 \\ &= mr^-(P_{k-1})+2 \\ &= \begin{cases} k, & k \text{ is even,} \\ k+1, & k \text{ is odd.} \end{cases} \quad \text{by Lemma 2.2} \end{aligned}$$

Therefore the result follows.

For the equality case, we first consider the necessity.

(1). Assume that there exists a vertex $v \in V(C_k)$ which is matched in $G\{v\}$. Note that $G\{v\}$ and $G-G\{v\}$ are two trees. If k is even, by Lemma 2.2 and Corollary 2.8 we have

$$\begin{aligned} k = mr^-(G) &= mr^-(G\{v\}) + mr^-(G-G\{v\}) \\ &= 2\beta(G\{v\}) + 2\beta(G-G\{v\}). \end{aligned}$$

Since $\beta(G\{v\}) \geqslant 1$, $\beta(G-G\{v\}) \geqslant \frac{k-2}{2}$, so $\beta(G\{v\}) = 1$ and $\beta(G-G\{v\}) = \frac{k-2}{2}$, which implies $G\{v\}$ is a star.

Similarly the result holds for the case when k is odd.

(2). Suppose that there exists a vertex $v \in V(C_k)$ which is mismatched in $G\{v\}$. By Corollary 2.8, we have

$$mr^-(G) = mr^-(C_k) + 2\beta(G-C_k).$$

In view of Lemma 2.5, together with the assumption, we have $\beta(G-C_k) = 1$ which implies $G \cong U^*$.

The sufficiency of the equality case is easy to verify. □

By Theorem 3.1, we have

COROLLARY 3.2. *Let G be a unicyclic graph of order n with pendant vertices. Then $mr^-(G) \geqslant 4$.*

3.2. Unicyclic graphs with minimum skew-rank 4

As is well known, the rank of a real skew-symmetric matrix is even. So $mr^-(G)$ is even for any oriented graph. It is observed in [19] that $mr^-(G) = 0$ if and only if G is an empty graph, and $mr^-(G) = 2$ if and only if G is a complete multipartite graph. The authors [19] posed an open question (Question 5.2) to characterize the graphs G such that $mr^-(G) = 4$ over infinite field.

Let $U_1^{r,s}$ $(r,s \geqslant 0,\ r+s = n-3)$, $U_2^{p,q}$ $(p,q \geqslant 0,\ p+q = n-4)$, U_3^{n-4}, U_4^{n-5} be four graphs as depicted in Fig. 3.1.

As a consequence of Theorem 3.1 and Lemma 2.5, we can characterize the unicyclic graphs G with $mr^-(G) = 4$ over real field.

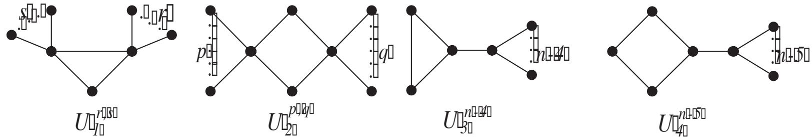

Figure 1: *Four unicyclic graphs* $U_1^{r,s}$, $U_2^{p,q}$, U_3^{n-4}, U_4^{n-5}

COROLLARY 3.3. *Let G be a unicyclic graph of order n with $mr^-(G) = 4$ and C_k be the cycle in G. Then*

(1) If $G = C_k$, then $G = C_5$, or C_6.

(2) If $G \neq C_k$, then the following statements hold:

(a) If there exists a vertex $v \in V(C_k)$ which is matched in $G\{v\}$, then $G \cong U_1^{r,s}$ or $U_2^{p,q}$.

(b) If there exists a vertex $v \in V(C_k)$ which is mismatched in $G\{v\}$, then $G \cong U_3^{n-4}$ or U_4^{n-5}.

3.3. Unicyclic graphs with minimum skew-rank 6

Next we shall characterize all unicyclic graphs with minimum skew-rank 6. From Lemma 2.4, it suffices to characterize the unicyclic graphs among all graphs without pendant twins. For convenience, we give some notations. Let $\mathscr{U}^*$ be a set of unicyclic graphs without pendant twins. Let G' (resp. G'') be the graph obtained from C_8 (resp. C_7) by attaching a pendant edge on a vertex of C_8 (resp. C_7).

THEOREM 3.4. *Let $G \in \mathscr{U}^*$ be a unicyclic graph with girth k and $mr^-(G) = 6$. Then $k \leqslant 8$ and the following statements hold:*

(i). If $k = 8$, then $G \cong C_8$.

(ii). If $k = 7$, then $G \cong C_7$.

(iii). If $k = 6$, then G is one of the graphs G_i's ($i = 1,2,3,4$) (as depicted in Fig.2).

(iv). If $k = 5$, then G is one of the graphs G_i's ($i = 5,6,\cdots,9$) (as depicted in Fig.3).

(v). If $k = 4$, then G is one of the graphs G_i's ($i = 10,11,\cdots,26$) (as depicted in Fig.4).

(vi). If $k = 3$, then G is one of the graphs G_i's ($i = 43,44,\cdots,57$) (as depicted in Fig.6).

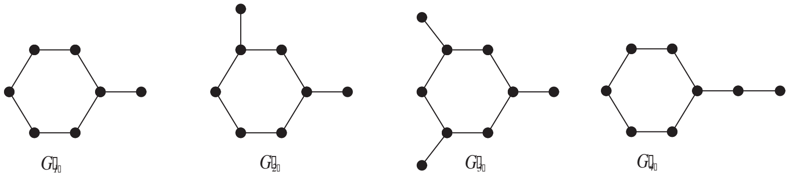

Figure 2: *Four graphs with girth 6 in Theorem 3.4*

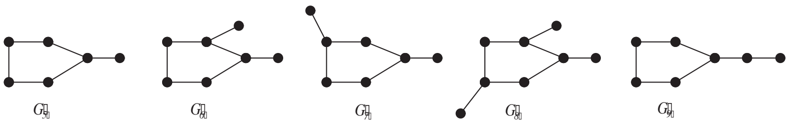

Figure 3: *Five graphs with girth 5 in Theorem 3.4*

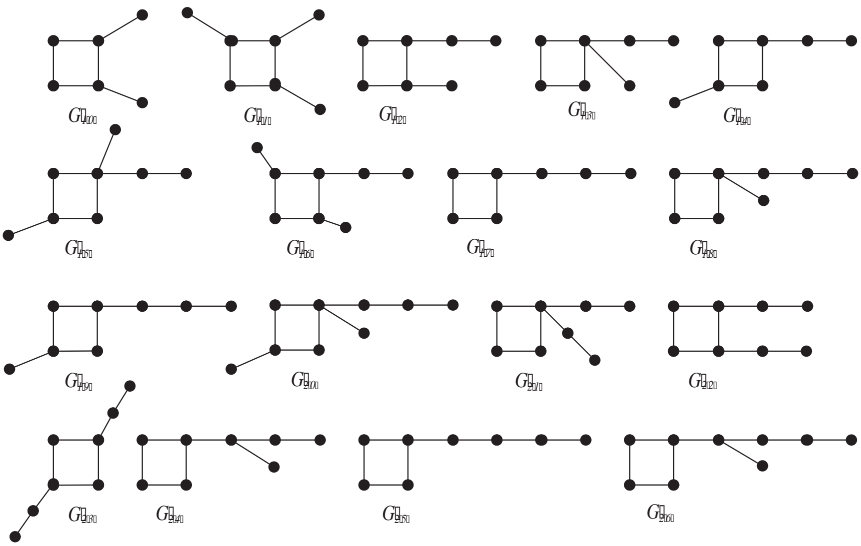

Figure 4: *Seventeen graphs with girth 4 in Theorem 3.4*

Proof. If $k \geqslant 9$, then G must contain P_8 as an induced subgraph. From Lemmas 2.2 and 2.6 we have $mr^-(G) \geqslant 8$ which is a contradiction.

Next we shall verify the six statements.

(i) and (ii): If G is a cycle, the results are obvious from Lemma 2.5.

If G is not a cycle, then it must contain G' or G'' as an induced subgraph. Hence $mr^-(G) \geqslant mr^-(G_1) = 8$ and $mr^-(G) \geqslant mr^-(G_2) = 8$ which contradicts the fact that $mr^-(G) = 6$.

(v): It is evident that graphs G_i $(i = 27, 29, \cdots, 42)$ have minimum skew-rank 8 and graphs G_i $(i = 10, \cdots, 26)$ have minimum skew-rank 6. In the following we consider the following five cases. For convenience, denote by $G^* = G - C_4$.

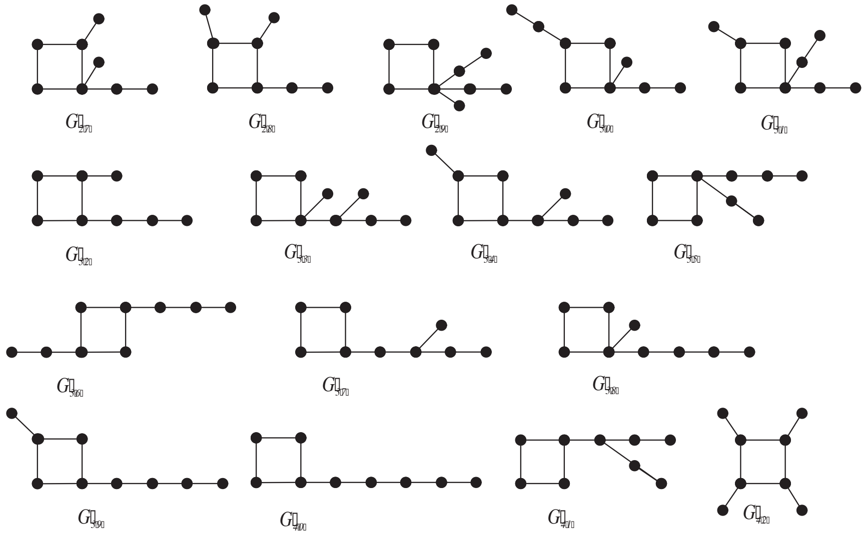

Figure 5: *Sixteen graphs with girth 4 excluded by $mr^-(G) = 3$ in Theorem 3.4*

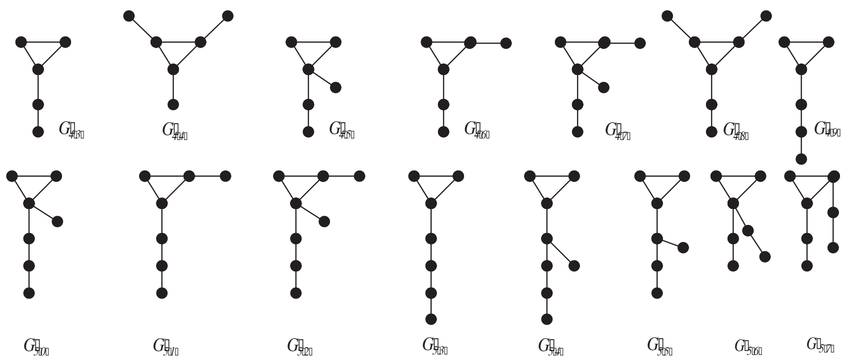

Figure 6: *Fifteen graphs with girth 3 in Theorem 3.4*

Case 1. G^* is a set of isolated vertices.

It is obvious that G is G_{10} or G_{11}.

Case 2. G^* contains P_2, but no P_3, as an induced subgraph.

If $G^* = P_2$, G does not exist.

If G^* is the union of an isolated vertex and P_2, G is one of graphs G_{12}, G_{13} and G_{14}.

If G^* is the union of two isolated vertices and P_2, G is G_{15} or G_{16}.

If G^* is the union of more than two isolated vertices and P_2, G does not exist since it contains G_{27} or G_{28} as an induced subgraph.

If G^* is two copies of P_2, G is one of G_i $(i = 21, 22, 23)$.

If G^* is the union of some isolated vertices and two P_2's, G does not exist since it contains one of G_i $(i = 27, 28, \cdots, 31)$ as an induced subgraph.

If G^* contains more than two P_2's as its induced subgraph, G does not exist since it must contain one of G_i $(i = 27, 28, 29)$ as an induced subgraph.

Case 3. G^* contains P_3, but no P_4, as an induced subgraph.

If $G^* = P_3$, $G \cong G_{17}$.

If G^* is the union of one isolated vertex and P_3, G is G_{18} or G_{19}.

If G^* is the union of two isolated vertices and P_3, $G \cong G_{20}$.

If G^* is the union of more than two isolated vertices and P_3, G does not exist since it contains G_{31}, G_{32} or G_{33} as an induced subgraph.

If G^* contains the union of P_2 and P_3 as its induced subgraph, G does not exist since it contains G_{31}, G_{34} or G_{35} as an induced subgraph.

Case 4. G^* contains P_4, but no P_5, as an induced subgraph.

In this case $G \cong G_{25}$, G_{26}. The minimum skew-rank of any other graph is more than six since it contains one of G_i $(i = 32, 33, \cdots, 39)$ as an induced subgraph.

Case 5. G^* contains P_5 as an induced subgraph.

In this case G does not exit since it contains one of G_i $(i = 32, 33, \cdots, 41)$ as an induced subgraph.

(iii), (iv) and (vi) can be similarly verified. □

4. Non-singularity of skew-symmetric matrices described by unicyclic graphs

Let $\mathscr{U}_{n,k}$ be the set of unicyclic graphs of order n with girth k. Let $\mathscr{U}_1$ be the set of unicyclic graphs of order n with girth k which can be changed to be an empty graph by finite steps of δ-transformation. Let $\mathscr{U}_2$ be the set of unicyclic graphs of order n with girth k which can be changed to be an cycle C_k or the union of isolated vertices and C_k by finite steps of δ-transformation. Obviously, $\mathscr{U}_{n,k} = \mathscr{U}_1 \cup \mathscr{U}_2$.

In [19], the authors obtained that, for a graph G, $mr^-(G) = n = MR^-(G)$ if and only if G has a unique perfect matching. In this section, we shall consider the case $MR^-(G) = n$.

LEMMA 4.1. [19] *For a graph G, $MR^-(G) = 2\beta(G)$.*

The following result is immediate from Lemma 4.1.

LEMMA 4.2. *Let C_n be a cycle of order n. Then*

$$MR^-(C_n) = \begin{cases} n, & n \text{ is even,} \\ n-1, & n \text{ is odd.} \end{cases}$$

THEOREM 4.3. *Let G be a unicyclic graph of order n with girth k $(k < n)$. Then we have*

(1) If $G \in \mathscr{U}_1$, then $MR^-(G) \leqslant \begin{cases} n, & n \text{ is even,} \\ n-1, & n \text{ is odd.} \end{cases}$

(2) If $G \in \mathscr{U}_2$, *then* $MR^-(G) \leqslant \begin{cases} n-1, & n \text{ is odd and } k \text{ is odd,} \\ n-2, & n \text{ is even and } k \text{ is odd,} \\ n, & n \text{ is even and } k \text{ is even,} \\ n-1, & n \text{ is odd and } k \text{ is even.} \end{cases}$

Proof. If $G \in \mathscr{U}_1$, then by at most $\lfloor \frac{n}{2} \rfloor$ steps of δ-transformation G can be changed to an empty graph. By Lemma 2.3, $MR^-(G) \leqslant 2 \cdot \lfloor \frac{n}{2} \rfloor$.

If $G \in \mathscr{U}_2$, then by at most $\lfloor \frac{n-k}{2} \rfloor$ steps of δ-transformation G can be changed to be the cycle C_k or the union of isolated vertices and C_k. By Lemma 2.3, $MR^-(G) \leqslant 2 \cdot \lfloor \frac{n-k}{2} \rfloor + MR^-(C_k)$. The result holds from Lemma 4.2. □

It is well known that the skew-symmetric matrix must be singular if its order is odd. Therefore the non-singular skew-symmetric matrices must have even order. By Theorem 4.3, we have

THEOREM 4.4. *Let* G *be a unicyclic graph with even order* n. *Then any matrix* $A \in \mathscr{S}^-(G)$ *is nonsingular, i.e.* $MR^-(G) = n$, *if and only if* G *has a perfect matching.*

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