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C*–ENVELOPES OF JORDAN OPERATOR SYSTEMS

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Abstract. We determine the boundary representations and the C*-envelope of operator systems of the form Span $\{1,T,T^*\}$, where T is a Jordan operator.

Introduction

If $\mathscr{F}$ is a function system on a compact Hausdorff space Ω – that is, a vector space of continuous functions $f:\Omega\to\mathbb{C}$ that contains the constants, separates the points of Ω, and contains $\overline{f}$ for every $f\in\mathscr{F}$ – then the Šilov boundary of $\mathscr{F}$ is the smallest compact subset $\partial_{\mathrm{S}}\mathscr{F}$ of Ω for which

$$\max_{\omega\in\Omega}|f(\omega)| = \max_{\zeta\in\partial_{\mathrm{S}}\mathscr{F}}|f(\zeta)|,$$

for every $f\in\mathscr{F}$. Put differently, $\partial_{\mathrm{S}}\mathscr{F}$ is the smallest compact subset Z of Ω for which the map $f\mapsto f_{|Z}$ is a linear isometry $\mathscr{F}\to C(Z)$, where $C(Z)$ is the unital abelian C*-algebra of continuous complex-valued functions on Z. Inspired by this fact, W. Arveson [2, 3] significantly advanced the study of Hilbert space operators by introducing the concept of Šilov boundary (and Choquet boundary, as well) to certain vector spaces of bounded linear operators called *operator systems*, which are linear submanifolds of $\mathscr{B}(\mathscr{H})$ that contain the identity operator and are closed under the adjoint map $T\mapsto T^*$. Here, $\mathscr{H}$ is a complex Hilbert space and $\mathscr{B}(\mathscr{H})$ is the C*-algebra of bounded linear operators acting on $\mathscr{H}$.

For a given function system $\mathscr{F}$, the algebra $C(\partial_{\mathrm{S}}\mathscr{F})$ is an enveloping C*-algebra for $\mathscr{F}$, and the operator-theoretic analogue of this notion is called the *C^*-envelope*. If $\mathscr{S}$ is an operator system, then a pair $(\mathscr{A},\iota)$ consisting of a unital C*-algebra $\mathscr{A}$ and a unital completely isometric linear map $\iota:\mathscr{S}\to\mathscr{A}$ such that $\mathscr{A}=\mathrm{C}^*(\iota(\mathscr{S}))$ (meaning that the operator system $\iota(\mathscr{S})$ generates the C*-algebra $\mathscr{A}$) is a C*-envelope for $\mathscr{S}$ if for every unital completely isometric linear map $\psi:\mathscr{S}\to\mathscr{B}(\mathscr{H}_\psi)$ there is a surjective C*-algebra homomorphism $\pi:\mathrm{C}^*(\psi(\mathscr{S}))\to\mathscr{A}$ such that $\pi\circ\psi=\iota$. Because any two C*-envelopes of $\mathscr{S}$ are isomorphic, we use the notation $(\mathrm{C}^*_{\mathrm{e}}(\mathscr{S}),\iota_{\mathrm{e}})$ to denote the C*-envelope of $\mathscr{S}$. The basic facts about C*-envelopes are treated Paulsen's monograph [15].

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Recall that a point $\zeta_0 \in \Omega$ is in the Choquet boundary $\partial_C \mathscr{F}$ of a function system $\mathscr{F}$ if the only Radon probability measure μ on the Borel sets of Ω in which $\int_\Omega f\,d\mu = f(\zeta_0)$ for every $f \in \mathscr{F}$ is the point-mass measure $\mu = \delta_{\zeta_0}$. Because the Choquet boundary is topologically dense in the Šilov boundary, one can determine the enveloping C^*-algebra of $\mathscr{F}$ by computing – if possible – the Choquet boundary of $\mathscr{F}$. The operator theoretic analogue is somewhat more delicate, and is reviewed below.

For an operator system $\mathscr{S}$, every unital representation $\rho : C^*(\mathscr{S}) \to \mathscr{B}(\mathscr{H}_\rho)$ of the C^*-algebra induces a unital completely positive (ucp) linear map $\varphi : \mathscr{S} \to \mathscr{B}(\mathscr{H}_\rho)$ by restricting the domain of ρ to $\mathscr{S}$ – that is, $\varphi = \rho_{|\mathscr{S}}$. Hence, ρ is just one of potentially many ucp extensions of the ucp map $\varphi : \mathscr{S} \to \mathscr{B}(\mathscr{H}_\rho)$ to $C^*(\mathscr{S})$. In particular, a unital representation $\rho : C^*(\mathscr{S}) \to \mathscr{B}(\mathscr{H}_\rho)$ is a *boundary representation for* $\mathscr{S}$ if

1. ρ is irreducible and
2. $\rho_{|\mathscr{S}}$ has a unique ucp extension to $C^*(\mathscr{S})$ (namely, ρ itself).

Recent results of Arveson [5] and Davidson and Kennedy [8] show that every operator system $\mathscr{S}$ admits sufficiently many boundary representations in the sense that if $p \in \mathbb{N}$ and $X = [X_{ij}]_{i,j}$ is any $p \times p$ matrix with entries $X_{ij} \in \mathscr{S}$, then

$$\|X\| = \sup\left\{ \|[\rho(X_{ij})]_{i,j}\| \,:\, \rho \text{ is a boundary representation for } \mathscr{S} \right\}.$$

The ideal $\mathfrak{S}_{\mathscr{S}}$ of $C^*(\mathscr{S})$ consisting of all $a \in C^*(\mathscr{S})$ for which $\rho(a) = 0$ for every boundary representation ρ of $\mathscr{S}$ is called the *Šilov ideal for* $\mathscr{S}$. The Šilov ideal $\mathfrak{S}_{\mathscr{S}}$ is the largest ideal $\mathscr{K}$ of $C^*(\mathscr{S})$ for which the canonical quotient map $q_{\mathscr{K}} : C^*(\mathscr{S}) \to C^*(\mathscr{S})/\mathscr{K}$ is completely isometric on $\mathscr{S}$. The quotient C^*-algebra $C^*(\mathscr{S})/\mathfrak{S}_{\mathscr{S}}$ together with the unital completely isometric embedding of $\mathscr{S}$ into $C^*_e(\mathscr{S})$ induced by the quotient homomorphism is a C^*-envelope for $\mathscr{S}$, which is to say that $C^*_e(\mathscr{S}) = C^*(\mathscr{S})/\mathfrak{S}_{\mathscr{S}}$ and $\iota_e = q_{\mathfrak{S}_{\mathscr{S}}|\mathscr{S}}$. If $\mathscr{S}$ has a trivial Šilov ideal, then necessarily $C^*_e(\mathscr{S}) = C^*(\mathscr{S})$ and, therefore, the operator system $\mathscr{S}$ is said to be *reduced* [6].

Just as one is interested in determining the Choquet and Šilov boundaries of a function system, there has been considerable effort to determine the boundary representations and C^*-envelopes of various operator systems, and some very beautiful results have been obtained in this direction – for example, [4, 7, 12, 13, 14]. Nevertheless, there remains a need for tractable interesting examples, as the issue of determining the boundary representations and C^*-envelope of a given operator system is generally quite difficult. To this end, the operator systems we study are among the most basic of operator systems: namely, complex vector spaces of the form

$$\mathrm{OS}\,(T) = \mathrm{Span}\,\{1, T, T^*\},$$

for some $T \in \mathscr{B}(\mathscr{H})$. Such operator systems can be viewed as operator-theoretic versions of the function system $\mathscr{F} = \mathrm{Span}\{1, z, \overline{z}\}$ in the unital C^*-algebra $C(\Omega)$, for $\Omega \subset \mathbb{C}$. In this case, there are classical function-theoretic results for normal (and subnormal) operators, and we recently considered the case of operator systems generated by an irreducible periodic weighted unilateral shift [1]. The purpose of the present paper is to determine the boundary representations and the C*-envelope of operator systems of the form $\mathrm{OS}(J)$ generated by a Jordan operator or matrix J.

A completely positive linear bijection $\varphi : \mathscr{S} \to \mathscr{T}$ of operator systems is a *complete order isomorphism* if φ^{-1} is completely positive. We will use the following well-known elementary lemma often and without mention: if $\varphi : \mathscr{S} \to \mathscr{T}$ is a completely contractive bijective linear map of operator systems such that φ^{-1} is completely contractive, then φ is a complete isometry.

The dimension of $\mathrm{OS}(T)$ is 1, 2, or 3, depending on the choice of T. The cases of dimensions 1 and 2 are easily determined: up to complete order isomorphism there is exactly one operator system of dimension 1 (namely, $\mathbb{C}$) and exactly one operator system of dimension 2 (namely, $\mathbb{C}\oplus\mathbb{C}$; see Proposition 2.4). However, the situation is very different for dimension 3, and in this case we are far from classifying such operator systems up to complete order isomorphism – even for operator systems acting on finite-dimensional Hilbert spaces.

The main results on Jordan operator systems are contained in Section 2 and make use of the matricial range of an operator. Therefore, we begin with a preliminary section on the Choquet boundary and the matricial range, and we point out the role of the matricial range in determining boundary representations.

1. Choquet boundary, matricial ranges, and direct sums of matrix algebras

The set $\partial_{\mathrm{C}}\mathrm{OS}(T)$ of all boundary representations for $\mathrm{OS}(T)$ is called the *Choquet boundary* for $\mathrm{OS}(T)$. The C*-envelope $\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}(T))$ of $\mathrm{OS}(T)$, defined in the introduction as a quotient algebra, arises in a different guise, which is useful for applications:

THEOREM 1.1. (Arveson [5]) *If $T \in \mathscr{B}(\mathscr{H})$, then*

$$\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}(T)) = \left(\prod_{\rho\in\partial_{\mathrm{C}}\mathrm{OS}(T)} \rho\right)(\mathrm{C}^*(\mathrm{OS}(T))).$$

In particular, the map $\prod_{\rho\in\partial_{\mathrm{C}}\mathrm{OS}(T)}\rho$ is completely isometric on $\mathrm{OS}(T)$.

It is worth noting that although $\mathrm{OS}(T)$ is finite-dimensional and $\mathrm{C}^*(T)$ is separable, it is possible for $\mathrm{OS}(T)$ to admit uncountably many non-equivalent boundary representations. An example of this is given by the unilateral shift; it is well-known that its boundary representations are given by the point evaluations in $C(\mathbb{T})$ (this was already known to Arveson; details can be found in [1, Corollary 3.6]).

DEFINITION 1.2. The *matricial range* of $T \in \mathscr{B}(\mathscr{H})$ is the set

$$\mathbb{W}(T) = \bigcup_{n\in\mathbb{N}} \mathbb{W}_n(T),$$

where

$$\mathbb{W}_n(T) = \{\varphi(T) : \ \varphi : \mathrm{OS}(T) \to M_n(\mathbb{C}) \ \text{ is ucp}\}$$

The matricial range is a generalisation of the classical numerical range in operator theory. Recall that the convex set

$$W_{\mathrm{s}}(T) = \{\langle T\xi,\xi\rangle : \xi \in \mathscr{H},\ \|\xi\| = 1\}$$

is called the *numerical range* of T. We denote the closure of $W_{\mathrm{s}}(T)$ by $W(T)$ and again refer to $W(T)$ as the "numerical range" of T. It is well-known that $W(T)$ is the convex compact set

$$W(T) = \{\varphi(T) : \varphi \text{ is a state on } \mathrm{OS}(T)\};$$

hence, $\mathbb{W}_1(T) = W(T)$. Note that $W(T) = W_{\mathrm{s}}(T)$ if $\mathscr{H}$ has finite dimension.

We denote by $\bigoplus_j M_{k_j}(\mathbb{C})$ the ℓ^∞-sum. The compressions π_ℓ of $\bigoplus_j M_{k_j}(\mathbb{C})$ to the ℓ^{th} direct summand are irreducible representations of $\bigoplus_j M_{k_j}(\mathbb{C})$, and if the sum is finite then these are all the irreducible representations (up to unitary equivalence).

DEFINITION 1.3. Let $m \in \mathbb{N} \cup \{\infty\}$. The family $\{T_j : j \in \{1,\dots,m\}\}$ of operators $T_j \in M_{k_j}(\mathbb{C})$ is an *irreducible family* if $\mathrm{C}^*(\bigoplus_j T_j)$ contains $\left(\bigoplus_{j=1}^m M_{k_j}(\mathbb{C})\right) \oplus 0$ for all $m \in \mathbb{N}$.

The notation $\bigoplus_{j\neq k} T_j$ will be taken to mean the operator in $\bigoplus_j M_{k_j}(\mathbb{C})$ such that the entry corresponding to k in the direct sum is equal to zero.

THEOREM 1.4. *Let $T = \bigoplus_j T_j \in \bigoplus_j M_{k_j}(\mathbb{C})$, with $T_j \in M_{k_j}(\mathbb{C})$ such that $\{T_j\}_j$ is an irreducible family. Let π_ℓ be the irreducible representation given by compression to the ℓ^{th} block. Then the following statements are equivalent:*

1. *π_ℓ is a boundary representation for $\mathrm{OS}(T)$;*
2. *$T_\ell \notin \mathbb{W}_{k_\ell}(\bigoplus_{j\neq\ell} T_j)$.*

Proof. For notational simplicity we will take $\ell = 1$; this does not affect generality as we can achieve permutation of blocks by unitary conjugation, which is a complete isometry.

Assume first that $T_1 \in \mathbb{W}_{k_1}(\bigoplus_{j>1} T_j)$. We will show that this implies that π_1 is not boundary. By assumption there exists a ucp map $\varphi : \mathrm{OS}\left(\bigoplus_{j>1} T_j\right) \to M_{k_1}(\mathbb{C})$ that maps $\bigoplus_{j>1} T_j \mapsto T_1$. We can use this φ to construct a ucp inverse to the restriction of the canonical compression $\rho : \bigoplus_j X_j \mapsto \bigoplus_{j>1} X_j$ to $\mathrm{OS}(T)$. Namely, the map $\rho' : X \mapsto \varphi(X) \oplus X$ is ucp and $\rho\rho'(X) = X$, $\rho'\rho(Y) = Y$, for $X \in \mathrm{OS}\left(\bigoplus_{j>1} T_j\right)$, $Y \in \mathrm{OS}(T)$. So ρ is a complete isometry on $\mathrm{OS}(T)$. We can see ρ as the quotient map induced by the ideal $\mathscr{K}_1 = M_{k_1}(\mathbb{C}) \oplus 0$. So we have proven that $\mathscr{K}_1$ is a boundary ideal for $\mathrm{OS}(T)$. As such, it is contained in the Šilov ideal and thus in the kernel of any boundary representation; in particular $\pi_j(Z \oplus 0) = 0$ for all j such that $\pi_j \in \partial_{\mathrm{C}}\mathrm{OS}(T)$ and all $Z \in M_{k_1}(\mathbb{C})$. As $\pi_1(Z \oplus 0) = Z$ for all such Z, we conclude that π_1 is not a boundary representation. In other words, if π_1 is boundary then $T_1 \notin \mathbb{W}_{k_1}(\bigoplus_{j>1} T_j)$.

Conversely, if π_1 is not a boundary representation, then Theorem 1.1 implies that the map induced by $T \mapsto \bigoplus_{j>1} T_j$ is completely isometric on $\mathrm{OS}(T)$. Indeed, given a boundary representation $\pi : \bigoplus_j M_{k_j}(\mathbb{C}) \to \mathscr{B}(\mathscr{H})$, $\pi(I_{k_1} \oplus 0)$ is necessarily $I_{\mathscr{H}}$ or 0. If it is the former, then $\dim \mathscr{H} = k_1$ and π would be unitarily equivalent with π_1, a contradiction. So $\pi(Z \oplus 0) = 0$ for any $Z \in M_{k_1}(\mathbb{C})$ and any boundary representation π. Then Theorem 1.1 justifies the assertion at the beginning of the paragraph.

In conclusion, there exists a ucp inverse ψ that maps $\bigoplus_{j>1} T_j \mapsto T$. Combining this with the compression to the first coordinate we get a ucp map with

$$\bigoplus_{j>1} T_j \mapsto T \mapsto T_1,$$

and so $T_1 \in \mathbb{W}_{k_1}(\bigoplus_{j>1} T_j)$. We have thus shown that if $T_1 \notin \mathbb{W}_{k_\ell}(\bigoplus_{j>1} T_j)$, then π_1 is a boundary representation. $\square$

It is important to notice that Theorem 1.4 does not deal with all possible boundary representations in the case of infinite sums (see Proposition 2.2).

EXAMPLE 1.5. For each $\lambda \in \mathbb{C}$, let $T_\lambda \in M_3(\mathbb{C})$ be given by

$$T_\lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Then

$$C_e^*(\mathrm{OS}(T_\lambda)) = \left\{ \begin{matrix} M_2(\mathbb{C}) & \text{if } |\lambda| \leqslant 1/2 \\ M_2(\mathbb{C}) \oplus \mathbb{C} & \text{if } |\lambda| > 1/2 \end{matrix} \right\}.$$

Proof. The C^*-algebra generated by $\mathrm{OS}(T_\lambda)$ is $M_2(\mathbb{C}) \oplus \mathbb{C}$. We have only two (classes of) irreducible representations, i.e. π_1 is compression to the upper-left 2×2 block, and π_2 is compression to the $(3,3)$-entry.

One can tell right away that π_1 is a boundary representation, because the range of π_2 is one-dimensional and thus has no room to fit the 3-dimensional operator system in. But we can also deduce the same from Theorem 1.4. Because $\mathbb{W}_2(\lambda) = \{\lambda I_2\}$, we see that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin \mathbb{W}_2(\lambda),$$

and so by Theorem 1.4 π_1 is a boundary representation. As

$$\mathbb{W}_1\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = B_{1/2}(0),$$

the irreducible representation π_2 will be a boundary representation precisely when $\lambda \notin B_{1/2}(0)$, i.e. when $|\lambda| > 1/2$. $\square$

To conclude this section we show below that the numerical range $W(T)$ and spectrum $\sigma(T)$ of T capture information about the one-dimensional boundary representations of $\mathrm{OS}(T)$. We already found in [1] that convexity plays a crucial role in understanding boundary representations. Here is more evidence of this relation:

PROPOSITION 1.6. *Let $T \in \mathscr{B}(\mathscr{H})$, $\lambda \in \mathbb{C}$.*

1. *If $\lambda = \rho(T)$ for some boundary representation ρ for* $\mathrm{OS}(T)$, *then $\lambda \in \sigma(T) \cap \partial W(T)$, and λ is an extreme point of $W(T)$.*

2. *Assume that $\lambda \in \sigma(T) \cap \partial W(T)$. If λ is an extreme point of $W(T)$ and if T is hyponormal, then $\lambda = \rho(T)$ for some boundary representation ρ for* $\mathrm{OS}(T)$.

Proof. To prove (1), note first that we have $\rho(T - \lambda I) = 0$. As ρ is multiplicative, this shows that $\lambda \in \sigma(T)$. Also, since $\rho(T)$ is scalar, we deduce that ρ is a state on $\mathrm{OS}(T)$, and thus $\lambda \in W(T)$. After we prove that λ is an extreme point of $W(T)$, we will know that $\lambda \in \partial W(T)$.

Let $\varphi = \rho|_{\mathrm{OS}(T)}$. Suppose that $\lambda_1, \lambda_2 \in W(T)$ and that $\lambda = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2$. As every state on $\mathrm{OS}(T)$ extends to a state on $\mathrm{C}^*(\mathrm{OS}(T))$ (by the Hahn–Banach Theorem, coupled with the fact that a linear functional is positive if and only if it is unital and contractive), there are states φ_1 and φ_2 on $\mathrm{C}^*(\mathrm{OS}(T))$ such that $\lambda_j = \varphi_j(T)$, $j = 1, 2$. Thus, the state $\psi = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$ is an extension of φ to $\mathrm{C}^*(\mathrm{OS}(T))$. Because ρ is a boundary representation for $\mathrm{OS}(T)$, $\psi = \rho$. That is, $\rho = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$. But since ρ is a pure state (because it is multiplicative), we deduce that $\varphi_1 = \varphi_2 = \rho$; hence, $\lambda_1 = \lambda_2 = \lambda$, which implies that λ is an extreme point of $W(T)$.

For the proof of (2), the hypothesis $\lambda \in \sigma(T) \cap \partial W(T)$ implies that there is a homomorphism $\rho : \mathrm{C}^*(\mathrm{OS}(T)) \to \mathbb{C}$ (that is, a 1-dimensional representation ρ) such that $\lambda = \rho(T)$ [2, Theorem 3.1.2]. Assume that λ is an extreme point of $W(T)$ and that T is hyponormal. As mentioned above, the hypothesis $\lambda \in \sigma(T) \cap \partial W(T)$ implies that there is a homomorphism $\rho : \mathrm{C}^*(\mathrm{OS}(T)) \to \mathbb{C}$ such that $\lambda = \rho(T)$. Let $\varphi = \rho|_{\mathrm{OS}(T)}$ and define

$$C_\varphi = \{\varphi : \varphi \text{ is a state on } \mathrm{C}^*(\mathrm{OS}(T)) \text{ such that } \varphi|_{\mathrm{OS}(T)} = \rho|_{\mathrm{OS}(T)}\}.$$

The set C_φ is evidently convex and weak*-compact. Thus, to show that C_φ consists of a single point it is sufficient to show that the only extreme point of C_φ is ρ itself. To this end, select an extreme point φ of C_φ. Because $\varphi(T) = \lambda$ is an extreme point of $W(T)$, φ is an extremal state on $\mathrm{C}^*(\mathrm{OS}(T))$; hence, via the GNS decomposition, there are a Hilbert space $\mathscr{H}_\pi$, an irreducible representation $\pi : \mathrm{C}^*(\mathrm{OS}(T)) \to \mathscr{B}(\mathscr{H}_\pi)$, and a unit vector $\xi \in \mathscr{H}_\pi$ such that $\varphi(A) = \langle \pi(A)\xi, \xi \rangle$ for every $A \in \mathrm{C}^*(\mathrm{OS}(T))$. In particular, $\lambda = \langle \pi(T)\xi, \xi \rangle$. Now since the numerical range of $\pi(T)$ is a subset of the numerical range of T, λ is an extreme point of $W(\pi(T))$. Moreover, as T is hyponormal, we have that $[\pi(T)^*, \pi(T)] = \pi([T^*, T])$ is positive and so $W(\pi(T))$ coincides with the convex hull of the spectrum of $\pi(T)$. The equation $\lambda = \langle \pi(T)\xi, \xi \rangle$ and the fact that $\lambda \in \sigma(\pi(T)) \cap \partial W(\pi(T))$ imply that $\pi(T)\xi = \lambda\xi$ and $\pi(T)^*\xi = \overline{\lambda}\xi$ [11, Satz2]. Thus, φ is a homomorphism and agrees with ρ on the generating set $\mathrm{OS}(T)$; hence, $\varphi = \rho$. □

It is interesting to contrast (1) of Proposition 1.6 with Theorem 3.1.2 of [2], which states that if $\lambda \in \sigma(T) \cap \partial W(T)$, then $\lambda = \rho(T)$ for some boundary representation ρ for the nonselfadjoint operator algebra $\mathscr{P}_T \subset \mathscr{B}(\mathscr{H})$ given by the norm closure of all operators of the form $p(T)$, for polynomials $p \in \mathbb{C}[t]$. In this latter assertion, there is no requirement that λ be an extreme point of $W(T)$, and this is one way in which we see that the operator spaces $\mathscr{P}_T$ and $\mathrm{OS}(T)$ differ fundamentally.

2. Jordan operator systems

We consider Jordan operators for several reasons: they are irreducible as operators in their own matrix algebras; they are expressed in terms of fairly simple matrices; they allow us to determine with certain ease when a family is irreducible, and we have information available about their matricial ranges.

DEFINITION 2.1. An operator J on an n-dimensional Hilbert space $\mathscr{H}$ is a *basic Jordan block* if there is an orthonormal basis of $\mathscr{H}$ for which J has matrix representation

$$J = J_n(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{bmatrix}$$

for some $\lambda \in \mathbb{C}$.

Note that a basic Jordan block $J = J_n(\lambda) \in \mathscr{B}(\mathscr{H})$ is an irreducible operator. Thus, $\mathrm{C}^*(\mathrm{OS}(J)) = \mathscr{B}(\mathscr{H})$, which is a simple C^*-algebra. Hence, the Šilov boundary ideal $\mathfrak{S}_{\mathrm{OS}(J)}$ for $\mathrm{OS}(J)$ is trivial, which implies that $\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}(J)) = \mathrm{C}^*(\mathrm{OS}(J)) = \mathscr{B}(\mathscr{H})$. That is, in Arveson's terminology [6], $\mathrm{OS}(J)$ is a reduced operator system.

If, on the other hand, we consider the unilateral shift S on $\ell^2(\mathbb{N})$, we have that $\mathrm{OS}(S)$ is not reduced – since $\mathrm{C}^*_{\mathrm{e}}(S) = C(\mathbb{T})$, which cannot contain compact operators.

Now what is the situation if we form direct sums of basic Jordan blocks of various sizes, but with a fixed eigenvalue λ? Does the C^*-envelope behave like the case of the finite or the infinite-dimensional shift? It turns out that there is a strong dichotomy, depending on how the sizes of the blocks behave.

PROPOSITION 2.2. *If* $J = \bigoplus_{k=1}^{\infty} J_{m_k}(\lambda) \in \mathscr{B}\left(\ell^2(\mathbb{N})\right)$ *and* $m = \sup\{m_k : k \in \mathbb{N}\}$, *then*

$$\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}(J)) = \begin{cases} C(\mathbb{T}), & \text{if } m = \infty; \\ M_m(\mathbb{C}), & \text{if } m < \infty. \end{cases}$$

Proof. We will assume, without loss of generality, that $\lambda = 0$, because J and $J - \lambda I$ generate the same operator system.

We consider first the case $m=\infty$. Recall the following positivity conditions (see, for example, [9, Proposition 5.4]):

$$\alpha 1_k+\beta J_k(0)+\overline{\beta}J_k(0)^* \geqslant 0 \iff \alpha \geqslant 0 \text{ and } |\beta| \leqslant \frac{\alpha}{2\cos\left(\frac{\pi}{k+1}\right)} \tag{2.1}$$

and

$$\alpha 1+\beta S+\overline{\beta}S^* \geqslant 0 \text{ if and only if } \alpha \geqslant 0 \text{ and } |\beta| \leqslant \frac{\alpha}{2}, \tag{2.2}$$

where S is the unilateral shift operator on $\ell^2(\mathbb{N})$. In considering

$$\ell^2(\mathbb{N})=\bigoplus_{k\in\mathbb{N}}\ell^2(\{1,\dots,m_k\}),$$

let $P_k\in\mathscr{B}\left(\ell^2(\mathbb{N})\right)$ be the projection onto the k-th direct summand $\ell^2(\{1,\dots,m_k\})$ and define $\psi:\mathrm{OS}\,(S)\to\mathrm{OS}\,(J)$ by

$$\psi(X) = \bigoplus_{k=1}^{\infty}P_kXP_k\,,\ X\in\mathrm{OS}\,(S)\,.$$

The map ψ is clearly ucp, and it sends S to J.

Now define $\varphi:\mathrm{OS}\,(J)\to\mathrm{OS}\,(S)$ by

$$\varphi(\alpha 1+\beta J+\gamma J^*) = \alpha 1S+\beta S+\gamma S^*.$$

As a linear map, we see that $\varphi^{-1}=\psi$. Hence, we need only show that φ is completely positive. We clearly have

$$\alpha 1+\beta J+\overline{\beta}J^* \geqslant 0 \iff \alpha 1_{m_k}+\beta J_{m_k}(0)+\overline{\beta}J_{m_k}(0)^* \geqslant 0,\ \forall k\in\mathbb{N}.$$

This assertion above means, by (2.1), that

$$\alpha 1+\beta J+\overline{\beta}J^* \geqslant 0 \iff \alpha\geqslant 0 \text{ and } |\beta|\leqslant \frac{\alpha}{2\cos\left(\frac{\pi}{m_k+1}\right)},\forall k\in\mathbb{N}.$$

But $m=\infty$ implies that m_k is arbitrarily large for some suitably chosen k, and so

$$\alpha 1+\beta J+\overline{\beta}J^* \geqslant 0 \text{ if and only if } \alpha\geqslant 0 \text{ and } |\beta|\leqslant\frac{\alpha}{2}.$$

Therefore, by (2.2),

$$\alpha 1+\beta J+\overline{\beta}J^* \geqslant 0 \text{ if and only if } \alpha 1+\beta S+\overline{\beta}S^* \geqslant 0.$$

Thus, the map $\varphi:\mathrm{OS}\,(J)\to\mathrm{OS}\,(S)$ is a unital, positive map. Because $\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}\,(S))=C(\mathbb{T})$, we may view, without loss of generality, $\mathrm{OS}\,(S)$ as an operator subsystem of $C(\mathbb{T})$. In this regard, then, the positive linear map φ maps $\mathrm{OS}\,(T)$ into an abelian C^*-algebra, and thus φ is automatically completely positive [15, Theorem 3.9]. Hence, $\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}\,(T))\simeq\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}\,(S))=C(\mathbb{T})$.

Suppose now that $m < \infty$; that is, $m = \max\{m_k : k \in \mathbb{N}\}$. We may assume without loss of generality that $m = m_1$. Consider the quotient map

$$q : \alpha 1 + \beta J + \gamma J^* \mapsto \alpha 1_{m_1} + \beta J_{m_1}(0) + \gamma J_{m_1}(0)^* \subset M_{m_1}(\mathbb{C}).$$

If P_k is, as above, the compression onto the k^{th} block of size m_k, then $P_k J_{m_1}(0) P_k = J_{m_k}(0)$ (this is where we use $m_k \leqslant m_1$). Now define

$$\psi : \mathrm{OS}\,(J_{m_1}(0)) \to \mathscr{B}(\ell^2(\mathbb{N}))$$

by

$$\psi(X) = \bigoplus_{k=1}^{\infty} P_k X P_k.$$

The map ψ is clearly ucp and, moreover, $\psi \circ q(J) = J$. This is to say that the map $q|_{\mathrm{OS}(J)}$ is ucp and has an ucp inverse; therefore, q is completely isometric. Thus

$$\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}\,(J)) = \mathrm{C}^*(q(\mathrm{OS}\,(J))) = \mathrm{C}^*(J_{m_1}(0)) = M_{m_1}(\mathbb{C}),$$

which completes the argument. $\square$

REMARK 2.3. Proposition 2.2 shows that Theorem 1.4 does not characterise all boundary representations. Indeed, the fact that the C^*-envelope in the unbounded dimension case is $C(\mathbb{T})$ shows that there are boundary representations not coming from the π_k.

We will also need the following basic result, which is very well-known to the specialists; we have not been able, however, to find a reference in the literature. A different proof than the one we provide below can be obtained by means of Proposition 2.6.

PROPOSITION 2.4. *If $T = T^* \in \mathscr{B}(\mathscr{H})$ and $T \notin \mathbb{C}I$, then $\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}\,(T)) = \mathbb{C} \oplus \mathbb{C}$.*

Proof. Since T is selfadjoint, all its irreducible representations are one-dimensional. Proposition 1.6 ensures that the only boundary representations for $\mathrm{OS}\,(T)$ are the two that send T to each of the extreme points of its spectrum. By Theorem 1.1, we conclude that $\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}\,(T)) = \mathbb{C}^2$. $\square$

DEFINITION 2.5. An operator $J \in \mathscr{B}(\mathscr{H})$ is a *Jordan operator* if $J = \bigoplus_j J_{n_j}(\lambda_j)$ for some finite or infinite sequence of basic Jordan operators $J_{n_j}(\lambda_j)$.

In the definition of Jordan operator above, we do not require the n_j nor the λ_j to be distinct. But we do not allow a repetition of the same pair n_j, λ_j: if we are considering a direct sum of d copies of a basic Jordan block $J_n(\lambda)$, then we denote this by $J_n(\lambda) \otimes 1_d$.

Although every operator T on a finite-dimensional Hilbert space is similar to a Jordan operator J, the C^*-envelopes of $\mathrm{OS}(T)$ and $\mathrm{OS}(J)$ may be be quite different. For example, the idempotent $E = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$ acting on $\mathbb{C}^2$, is similar to the orthogonal projection $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, but if $x \neq 0$ then $E^*E \neq EE^*$ implies that $C^*(E) = M_2(\mathbb{C})$, which is simple; thus,

$$C_e^*(\mathrm{OS}(E)) = M_2(\mathbb{C}) \neq \mathbb{C} \oplus \mathbb{C} = C_e^*(\mathrm{OS}(P)).$$

There are a number of subtleties in attempting to determine the C^*-envelope of a Jordan operator J on a finite-dimensional Hilbert space $\mathscr{H}$ in terms of the sizes of the basic Jordan blocks that combine to form J and the geometry of the spectrum of J. We have seen this already in Example 1.5; and Proposition 2.6 and Remark 2.7 below are further illustrations.

It is clear that when T is normal, $C_e^*(T)$ is abelian (being a quotient of the C^*-algebra generated by T). The C^*-envelope can also be abelian for non-normal operators: we have already mentioned that $C_e^*(S) = C(\mathbb{T})$ for the unilateral shift. For finite-dimensional Jordan operators with real eigenvalues, we can characterise precisely when their C^*-envelopes are abelian:

PROPOSITION 2.6. *Assume that* $J = \bigoplus_{k=1}^{n} \left(J_{m_k}(\lambda_k) \otimes 1_{d_k}\right)$ *with each pair* (m_k, λ_k) *unique. If each* $\lambda_k \in \mathbb{R}$*, then the following statements are equivalent:*

1. $C_e^*(\mathrm{OS}(J))$ *is abelian;*

2. $m_1 = \cdots = m_k = 1$.

Proof. If $m_1 = \cdots = m_k = 1$ then J is diagonal. It is then clear that $C^*(\mathrm{OS}(J))$ is abelian and so is any quotient of it; thus, $C_e^*(\mathrm{OS}(J))$ is abelian (we can reach the same conclusion by appealing to Proposition 2.4, since in this case $J = J^*$).

Conversely, assume that $C_e^*(\mathrm{OS}(J))$ is abelian. If $m_j > 1$ for some j, then the numerical range of J contains non-real numbers (this is for instance a consequence of the fact that the numerical range of the 2×2 shift is a disk in the complex plane). The numerical range is preserved under complete isometries, so J would have the same numerical range in its C^*-envelope. But this means that at least one block with $m_j > 1$ would have to correspond to a boundary representation, and this would make $C_e^*(\mathrm{OS}(J))$ non-abelian. This contradiction shows that $m_j = 1$ for all j. $\square$

REMARK 2.7. At first sight the condition of having real eigenvalues in Proposition 2.6 could be seen as a limitation of the technique employed in the proof. This is not the case, however: consider

$$J = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{bmatrix},$$

where $\omega = (-1-i\sqrt{3})/2$. As usual, put $\mathrm{OS}(J) = \mathrm{Span}\{1, J, J^*\}$. It is easy to see that $\mathrm{C}^*(\mathrm{OS}(J)) = \mathbb{C}^3 \oplus M_2(\mathbb{C})$, and so we have four Jordan blocks and four (classes of) irreducible representations, the fourth of which we denote by $\pi_4 : \mathrm{C}^*(\mathrm{OS}(J)) \to M_2(\mathbb{C})$.

Let

$$U = \begin{bmatrix} 1 & 1 & 1 \\ \omega^2 & \omega & 1 \\ \omega & \omega^2 & 1 \end{bmatrix}, \qquad V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Define $\psi : \mathbb{C}^3 \to M_2(\mathbb{C})$ be given by

$$\psi(\alpha, \beta, \gamma) = VU^* \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix} UV^*.$$

This map ψ is linear and ucp by construction, and $\psi(1, \omega, \omega^2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. By Theorem 1.4, π_4 is not a boundary representation.

The other three one-dimensional irreducible representations have to be boundary as the quotient needs to have dimension at least 3; so the C^*-envelope of $\mathrm{OS}(J)$ is $\mathbb{C}^3$.

The next proposition plays a key role in the proof of Theorem 2.9.

PROPOSITION 2.8. *Let* $J = \bigoplus_{j=1}^n J_{m_j}(\lambda_j)$, *with* $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ *all distinct. Fix* $k \in \{1, \ldots, n\}$. *Assume that* $\lambda_{s_1}, \ldots, \lambda_{s_r}$ *are the extreme points of* $\mathrm{Conv}\{\lambda_1, \ldots, \lambda_n\}$, *and that* $\min\{m_{s_1}, \ldots, m_{s_r}\} \geqslant m_k$. *If* λ_k *is not an extreme point of* $\mathrm{Conv}\{\lambda_1, \ldots, \lambda_n\}$, *then* π_k *is not a boundary representation of* $\mathrm{OS}(J)$.

Proof. Let $P_t : \mathbb{C}^{m_{s_t}} \to \mathbb{C}^{m_k}$, $t = 1, \ldots, r$ be the operators defined on the canonical basis by

$$P_t e_j = \begin{cases} e_j & \text{if } j \leqslant m_k \\ 0 & \text{otherwise} \end{cases}$$

Straightforward computations show that

$$P_t J_{m_{s_t}}(\lambda) P_t^* = J_{m_k}(\lambda), \quad t = 1, \ldots, r$$

for any number λ (this is where one uses the hypothesis that $m_{s_t} \geqslant m_k$). By hypothesis, we can find convex coefficients $a_t \geqslant 0$ with $\lambda_k = \sum_{t=1}^r a_t \lambda_{s_t}$, $\sum_{t=1}^r a_t = 1$.

Now we define $\psi : M_{m_{s_1}}(\mathbb{C}) \oplus \cdots \oplus M_{m_{s_r}}(\mathbb{C}) \longrightarrow M_{m_k}(\mathbb{C})$ by

$$\psi\left(\bigoplus_{t=1}^{r} A_{s_t}\right) = \sum_{t=1}^{r} a_t\, P_t A_{s_t} P_t^*.$$

It is clear that ψ is ucp, since $P_{k,t}P_{k,t}^* = 1_{m_k}$ and ψ is made up of conjugations, convex combinations, and direct sums.

As

$$\sum_{t=1}^{r} a_t\, P_t J_{m_{s_t}}(\lambda_{s_t}) P_t^* = \sum_{t=1}^{r} a_t\, J_{m_k}(\lambda_{s_t}) = J_{m_k}\Big(\sum_{t=1}^{r} a_t\, \lambda_{s_t}\Big) = J_{m_k}(\lambda_k),$$

we have $\psi\left(\bigoplus_{t=1}^{r} J_{m_{s_t}}(\lambda_{s_t})\right) = J_{m_k}(\lambda_k)$. Thus

$$J_{m_k}(\lambda_k) \in \mathbb{W}_{m_k}\left(\bigoplus_{t=1}^{r} J_{m_{s_t}}(\lambda_{s_t})\right).$$

By Theorem 1.4, π_k is not a boundary representation. $\square$

We are now in position to determine the boundary representations of the operator system generated by a finite-dimensional Jordan operator with real eigenvalues.

THEOREM 2.9. *Assume that $\lambda_1 > \cdots > \lambda_n$ in $\mathbb{R}$, and that $J = \bigoplus_{j=1}^{n} J_{m_j}(\lambda_j)$.*

1. *If $k \in \{2,\ldots,n-1\}$, then the following statements are equivalent:*
   (a) *π_k is a boundary representation of* $\mathrm{OS}(J)$*;*
   (b) *At least one of the following inequalities holds:*
      i. $\max\{m_1,\ldots,m_{k-1}\} < m_k$;
      ii. $\max\{m_{k+1},\ldots,m_n\} < m_k$.
2. *If $k \in \{1,n\}$ and π_k is a boundary representation of* $\mathrm{OS}(J)$*, then one of the following assertions holds:*
   (a) $m_k > 1$;
   (b) *$m_k = 1$ and $\lambda_k \notin W(J_j(\lambda_j))$, for all $j \in \{1,\ldots,n\} \setminus \{k\}$.*

   Conversely, if condition (2b) holds, then π_k is a boundary representation.

Proof. (1a) $\Longrightarrow$ (1b) If (1b) fails, then we are in the conditions of Proposition 2.8 and so π_k is not a boundary representation.

(1b) $\Longrightarrow$ (1a) Theorem 1 in [10] implies that

$$\lambda_j + i\cos\frac{\pi}{m_j+1} \in \mathbb{W}_1(J_{m_j}(\lambda_j)) \subset B_{\cos\frac{\pi}{m_j+1}}(\lambda_j).$$

Indeed, Haagerup and de la Harpe prove that $\cos\pi/(n+1) \in \mathbb{W}_1(J_n(0))$; as $iJ_n(0)$ is unitarily equivalent to $J_n(0)$, one can construct a state φ with $\varphi(J_n(0)) = i\cos\pi/(n+1)$, and it follows that $\varphi(J_n(\lambda)) = \lambda + i\cos\pi/(n+1)$.

Note that $\lambda_k + i\cos\frac{\pi}{m_k+1} \notin \operatorname{Conv}\bigcup_{j\neq k} B_{\cos\frac{\pi}{m_j+1}}(\lambda_j)$. Indeed, by hypothesis all points in the convex combination will have imaginary part less than $\max\{\cos\frac{\pi}{m_j+1} : j \neq k\} < \cos\frac{\pi}{m_k+1}$.

This implies that $J_{m_k}(\lambda_k) \notin \mathbb{W}_{m_k}(\bigoplus_{j\neq k} J_{m_j}(\lambda_j))$ (otherwise, evaluating on states would contradict the previous paragraph). By Theorem 1.4, π_k is a boundary representation.

(2) We will assume that π_1 is a boundary representation (the argument for π_n is entirely similar). If $m_1 = 1$, then by Theorem 1.4 we have

$$\lambda_1 \notin W(\bigoplus_{j\neq 1} J_{m_j}(\lambda_j)) = \operatorname{Conv}\bigcup_{j\neq 1} W(J_{m_j}(\lambda_j)).$$

In particular, λ_1 fails to be in each of the individual numerical ranges.

(2b) $\implies$ π_1 is a boundary representation. If $m_k = 1$ and $\lambda_k \notin W(J_{m_j}(\lambda_j))$, for all $j \in \{1,\dots,n\}\setminus\{k\}$, then $\lambda_k \notin \operatorname{Conv}\bigcup_{j\neq k} W(J_{m_j}(\lambda_j))$; indeed, since λ_k is at the extreme of the list $\lambda_1,\dots,\lambda_n$, if λ_k were in the convex hull then there would exist a fixed $j \neq k$ with $\lambda_1 \in W(\lambda_{m_j}(\lambda_j))$ (these are all discs with centre on the real line); as it is not, it fails to be in the convex hull. Then Theorem 1.4 guarantees that π_k is a boundary representation. $\square$

REMARK 2.10. The ideas in Theorem 2.9 can certainly be applied to cases where the λ_j are allowed to be complex. But the possibilities seem much harder to consider – as the example in Remark 2.7 already illustrates. Note also that for complex eigenvalues one has no control over the positions of the balls considered in the proof of (1b) $\implies$ (1a) in Theorem 2.9.

In the concrete case where eigenvalues are real and the blocks corresponding to the extreme eigenvalues are the largest, we can calculate the C*-envelope very explicitly. We first prove a very well-known lemma.

LEMMA 2.11. *For any* $m \in \mathbb{N}$, $m \geqslant 2$, $\mathbb{W}_1(J_m(0))$ *is the ball of radius* $\cos\frac{\pi}{m+1}$ *centered at* 0.

Proof. We know by [10] that $\cos\frac{\pi}{m+1} \in \mathbb{W}_1(J_m(0))$. Select $\theta \in [0,2\pi)$ and let $W = \sum_{k=1}^n e^{-i(k-1)\theta} E_{kk}$ so that $W^*J_m(0)W = e^{i\theta}J_m(0)$. Thus, if the state φ satisfies $\varphi(J_m(0)) = \cos\frac{\pi}{m+1}$, then the state $\varphi_\theta = \varphi(W^*\cdot W)$ satisfies $\varphi_\theta(J_m(0)) = e^{i\theta}J_m(0)$. Now consider the state $\psi(A) = A_{11}$. Then $\psi(J_m(0)) = 0$. For any $r \in [0,1]$, the state $(1-r)\psi + r\varphi_\theta$ sends $J_m(0)$ to $r\cos\frac{\pi}{m+1}e^{i\theta}$. $\square$

COROLLARY 2.12. *If* $J = \bigoplus_{k=1}^{n} (J_{m_k}(\lambda_k) \otimes 1_{d_k})$, *where* $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ *are real, if* $\min\{m_1, m_n\} = 1$, *and if* $\max\{m_2, \ldots, m_{n-1}\} \leqslant \max\{m_1, m_n\}$, *then*

$$\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}(J)) = \begin{cases} \mathbb{C} \oplus \mathbb{C} & \text{if } m_1 = m_n = 1 \\ M_{m_n}(\mathbb{C}) & \text{if } m_1 = 1,\ m_n \geqslant 2,\ |\lambda_1 - \lambda_n| \leqslant \cos\frac{\pi}{(m_n+1)} \\ \mathbb{C} \oplus M_{m_n}(\mathbb{C}) & \text{if } m_1 = 1,\ m_n \geqslant 2,\ |\lambda_1 - \lambda_n| > \cos\frac{\pi}{(m_n+1)} \\ M_{m_1}(\mathbb{C}) & \text{if } m_1 \geqslant 2,\ m_n = 1,\ |\lambda_1 - \lambda_n| \leqslant \cos\frac{\pi}{(m_1+1)} \\ M_{m_1}(\mathbb{C}) \oplus \mathbb{C} & \text{if } m_1 \geqslant 2,\ m_n = 1,\ |\lambda_1 - \lambda_n| > \cos\frac{\pi}{(m_1+1)} \end{cases}$$

Proof. It is easy to see that

$$\mathrm{C}^*(\mathrm{OS}(J)) = \bigoplus_{k=1}^{n} \left(M_{m_k}(\mathbb{C}) \otimes 1_{d_k}\right)$$

It is also clear that $\mathrm{OS}(J)$ is completely order isomorphic to the operator system generated by

$$J' = \bigoplus_{k=1}^{n} J_{m_k}(\lambda_k), \tag{2.3}$$

and so we can eliminate the multiplicities from our computations. Thus, without loss of generality, we assume that J is of the form (2.3), and $\mathrm{C}^*(\mathrm{OS}(J)) = \bigoplus_{k=1}^{n} M_{m_k}(\mathbb{C})$; this, because the condition on the eigenvalues guarantees that the family is irreducible.

By Theorem 2.9, the only possible boundary representations are π_1 and π_n.

Case 1: $m_1 = m_n = 1$. We are in the situation of Proposition 2.4, so $\mathrm{C}^*_{\mathrm{e}}(\mathrm{OS}(J)) = \mathbb{C} \oplus \mathbb{C}$ (i.e. both π_1 and π_n are boundary representations).

Case 2: $m_1 = 1$, $m_n \geqslant 2$, $|\lambda_1 - \lambda_n| \leqslant \cos\pi/(m_n+1)$. Note that by Lemma 2.11 this last condition is the same as $\lambda_1 \in W(J_{m_n}(\lambda_n))$. Then Theorem 2.9 implies that π_1 is not boundary.

Case 3: $m_1 = 1$, $m_n \geqslant 2$, $|\lambda_1 - \lambda_n| > \cos\pi/(m_n+1)$. So $\lambda_1 \notin W(J_{m_n}(\lambda_n))$. The condition $m_1 = 1$ implies that $m_j = 1$ for all $j \neq n$, and in particular $W(J_{m_j}(\lambda_j)) = \{\lambda_j\}$ for all $j \neq n$. We are in a situation similar to Case 2, but in this case $\lambda_1 \notin W(\bigoplus_{j\neq 1} J_{m_j}(\lambda_j))$. So Theorem 2.9 implies that π_1 is boundary.

Cases 4 and 5: $m_1 \geqslant 2$, $m_n = 1$. We did not use that $\lambda_1 > \lambda_n$ in Cases 2 and 3 (only that it was at the extreme of the list), so the same proofs apply with the roles of 1 and n reversed. □

In trying to classify the irreducible representations of a singly generated operator system of the form $\mathrm{OS}(T)$ with $T = \bigoplus_j T_j$ for an irreducible family, recall that Theorem 1.4 gives us a characterisation of the boundary representations, namely π_ℓ is a boundary representation if and only $T_\ell \notin \mathbb{W}_{m_\ell}(\bigoplus_{j\neq\ell} T_j)$. So in principle one could go testing this condition starting with T_1, then T_2, etc., and determining which blocks do not correspond to boundary representations. After "erasing" those blocks we end up with a reduced operator system. But how can we be sure that if we perform this procedure in any order we will obtain the same result? After all, one could imagine that $T_1 \in \mathbb{W}(\bigoplus_{j\geqslant 2} T_j)$ in a way that the ucp map that realises T_1 depends essentially on T_2; and that $T_2 \in \mathbb{W}(\bigoplus_{j\neq 2} T_j)$ in a way that the ucp map that realises T_2 depends essentially on T_1. Is there a contradiction? We show below that no contradiction arises.

PROPOSITION 2.13. *If $T_1,\dots,T_n \in \mathscr{B}(\mathscr{H})$ have the property that $T_1 \in \mathbb{W}(\bigoplus_2^n T_j)$ and $T_2 \in \mathbb{W}(\bigoplus_{j\neq 2} T_j)$, then*

$$\mathrm{OS}\left(\bigoplus_1^n T_j\right) \simeq \mathrm{OS}\left(\bigoplus_2^n T_j\right) \simeq \mathrm{OS}\left(\bigoplus_{j\neq 2} T_j\right).$$

Proof. By hypothesis there exists a ucp map

$$\varphi : \mathrm{OS}\left(\bigoplus_2^n T_j\right) \to \mathrm{OS}\,(T_1)$$

with $\varphi(\bigoplus_2^n T_j) = T_1$. Let

$$P : \mathrm{OS}\left(\bigoplus_1^n T_j\right) \to \mathrm{OS}\left(\bigoplus_2^n T_j\right)$$

be the compression map, i.e. $P(\bigoplus_1^n X_j) = \bigoplus_2^n X_j$, and let

$$Q : \mathrm{OS}\left(\bigoplus_2^n T_j\right) \to \mathrm{OS}\left(\bigoplus_1^n T_j\right)$$

be the map $X \mapsto \varphi(X) \oplus X$. As both P and Q are ucp and they map $\bigoplus_1^n T_j$ to itself, we have that $Q \circ P$ is the identity on $\mathrm{OS}\,(\bigoplus_1^n T_j)$. So both P, Q are completely isometric, and we get the isomorphism $\mathrm{OS}\,(\bigoplus_1^n T_j) \simeq \mathrm{OS}\,(\bigoplus_2^n T_j)$. The other isomorphism is obtained in the same way. $\square$

3. Some Examples

REMARK 3.1. Note that one need not have the isomorphism with $\mathrm{OS}\,(\bigoplus_3^n T_j)$ in Proposition 2.13. For instance, let

$$T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad T_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \qquad T_3 = 1$$

(note that T_1, T_2 are unitary conjugates of each other). Then $\mathrm{OS}\,(T_1 \oplus T_2 \oplus T_3) \simeq \mathrm{OS}\,(T_1) \simeq \mathrm{OS}\,(T_2) \not\simeq \mathrm{OS}\,(T_3)$.

The last isomorphism can occur in adequate examples, as shown below. We will also address the issue that in the conditions of Proposition 2.13, there is no reason to expect that $\mathrm{OS}\,(T_1) \simeq \mathrm{OS}\,(T_2)$. Indeed, let

$$T = 1 \oplus \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then Proposition 2.8 guarantees that the first two blocks are in the matricial range of the last two, so by Theorem 2.9 together with the fact that 0 and 2 are too far away from each other for any of the last two blocks to be in the matricial range of the other,

$$\mathrm{OS}\,(T) \simeq \mathrm{OS}\left(\begin{bmatrix}0&1\\0&0\end{bmatrix} \oplus \begin{bmatrix}2&1\\0&2\end{bmatrix}\right).$$

The first two blocks clearly generate non-isomorphic operator systems, as the first one will have dimension 1, and the second dimension 3.

We show below some examples where one uses the results above to decide whether a given operator system generated by a Jordan operator is reduced.

EXAMPLE 3.2. If

$$J = \begin{bmatrix}0&&&\\&1&1&\\&0&1&\\&&&2\end{bmatrix},$$

then $\mathrm{OS}\,(J)$ is reduced. To verify that $\mathrm{OS}\,(J)$ is indeed reduced, it is enough to look at the combined matricial ranges. Note first that π_2 is certainly a boundary representation, because if it were not Theorem 1.1 would make the C^*-envelope either $\mathbb{C}$ or $\mathbb{C}\oplus\mathbb{C}$, which cannot contain a 3-dimensional subspace (or we can use Theorem 2.9 and notice that $B_{1/2}(1)$ contains neither 0 or 2; or Theorem 1.4 and notice that the numerical range of the direct sum $0\oplus 2$ is the segment $[0,2]$ that contains no ball). We have

$$2 \notin W\left(0 \oplus \begin{bmatrix}1&1\\0&1\end{bmatrix}\right) = \mathrm{Conv}\left\{W(0) \cup W\left(\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)\right\} = \mathrm{Conv}\{0 \cup B_{1/2}(1)\},$$

so π_3 is a boundary representation. And

$$\begin{aligned}0 \notin W\left(\begin{bmatrix}1&1\\0&1\end{bmatrix} \oplus 2\right) &= \mathrm{Conv}\left\{W\left(\begin{bmatrix}1&1\\0&1\end{bmatrix}\right) \cup W(2)\right\}\\ &= \mathrm{Conv}\{B_{1/2}(1), 2\},\end{aligned}$$

so π_1 is a boundary representation.

EXAMPLE 3.3. For the Jordan operator $J = J_1(3) \oplus J_2(2) \oplus J_2(1) \oplus J_1(0)$, the operator system $\mathrm{OS}\,(J)$ is reduced.

Again we look at the numerical ranges. We have

$$W(J_1(3)) = \{3\},\ \ W(J_2(2)) = B_{1/2}(2),$$
$$W(J_2(1)) = B_{1/2}(1),\ \ W(J_1(0)) = \{0\}.$$

It is easy to check that none of the four sets is in the convex hull of the other three. So none of the four components of J is in the matricial range of the other three; by Theorem 1.4 every irreducible representation is boundary, i.e. $\mathrm{OS}\,(J)$ is reduced.

EXAMPLE 3.4. (Compare with Example 3.3) With the Jordan operator $J = J_1(3) \oplus J_2(2) \oplus J_2(1/2) \oplus J_1(0)$, the operator system $\mathrm{OS}(J)$ is not reduced. Indeed, $W(J_2(1/2))$ is the disk of radius $1/2$ centered at $1/2$, and so $0 \in W(J_2(1/2))$. By Theorem 2.9, π_4 is not a boundary representation.

REFERENCES

[1] M. ARGERAMI AND D. FARENICK, *The* C^* *-envelope of an irreducible periodic weighted unilateral shift*, Integral Equations Operator Theory **77** (2013), 199–210.

[2] W. ARVESON, *Subalgebras of* C^**-algebras*, Acta Math. **123** (1969), 141–224.

[3] W. ARVESON, *Subalgebras of* C^**-algebras. II*, Acta Math. **128** (1972), 271–308.

[4] W. ARVESON, *Subalgebras of* C^**-algebras. III. Multivariable operator theory*, Acta Math. **181** (1998), 159–228.

[5] W. ARVESON, *The noncommutative Choquet boundary*, J. Amer. Math. Soc. **21** (2008), 1065–1084.

[6] W. ARVESON, *The noncommutative Choquet boundary III: operator systems in matrix algebras*, Math. Scand. **106** (2010), 196–210.

[7] K. R. DAVIDSON AND E. KATSOULIS, *Primitive limit algebras and* C^**-envelopes*, Adv. Math. **170** (2002), 181–205.

[8] K. R. DAVIDSON AND M. KENNEDY, *The Choquet boundary of an operator system*, preprint, 2013, arXiv:1303.3252.

[9] D. R. FARENICK, *Pure matrix states on operator systems*, Linear Algebra Appl. **393** (2004), 149–173.

[10] U. HAAGERUP AND P. DE LA HARPE, *The numerical radius of a nilpotent operator on a Hilbert space*, Proc. Amer. Math. Soc. **115** (1992), 371–379.

[11] S. HILDEBRANDT, *Über den numerischen Wertebereich eines Operators*, Math. Ann. **163** (1966), 230–247.

[12] E. KATSOULIS AND D. W. KRIBS, *The* C^**-envelope of the tensor algebra of a directed graph*, Integral Equations Operator Theory **56** (2006), 401–414.

[13] S. MCCULLOUGH AND V. PAULSEN, C^**-envelopes and interpolation theory*, Indiana Univ. Math. J. **51** (2002), 479–505.

[14] P. S. MUHLY AND B. SOLEL, *Tensor algebras over* C^**-correspondences: representations, dilations, and* C^**-envelopes*, J. Funct. Anal. **158** (1998), 389–457.

[15] V. PAULSEN, *Completely bounded maps and operator algebras*, Volume 78 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2002.

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