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COMPLEX SYMMETRIC TRIANGULAR OPERATORS

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Abstract. In this paper we explore complex symmetric operators with eigenvalues. We develop new techniques to give a geometric description of certain complex symmetric triangular operators. This extends a recent result of L. Balayan and S. Garcia concerning finite-dimensional complex symmetric operators. On the other hand, using Apostol's triangular representation for Hilbert space operators, we give a description of the internal structure of complex symmetric operators.

1. Introduction

Throughout this paper, we let $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of complex numbers, the set of integers and the set of positive integers respectively. $\mathscr{H}$ will always denote a complex separable Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$. We let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$.

DEFINITION 1.1. A map C on $\mathscr{H}$ is called an *antiunitary operator* if C is conjugate-linear, invertible and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathscr{H}$; if, in addition, $C^{-1} = C$, then C is called a *conjugation* on $\mathscr{H}$.

DEFINITION 1.2. An operator $T \in \mathscr{B}(\mathscr{H})$ is called a *complex symmetric operator* (CSO, for short) if there exists a conjugation C on $\mathscr{H}$ so that $CTC = T^*$.

Note that $T \in \mathscr{B}(\mathscr{H})$ is complex symmetric if and only if T can be represented as a symmetric matrix relative to some orthonormal basis for $\mathscr{H}$ (see [10, Lem. 1]). CSOs have been studied for many years in the finite-dimensional setting. Garcia and Putinar [10, 11] initiated the general study of complex symmetric operators, which has many motivations in function theory, matrix analysis and other areas. In particular, CSOs are closely related to the study of truncated Toeplitz operators [12, 13], which was initiated in Sarason's seminal paper [23]. Some interesting results concerning complex symmetric operators have been obtained (see [3, 7, 15, 17, 25, 26] for references).

In general, it is difficult to determine whether a given operator is complex symmetric even in finite-dimensional case (see [2, 8, 9, 14]). So people pay more attention

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to special classes of operators such as partial isometries [15, 25], weighted shifts [26] and Foguel operators [6]. In this paper we shall explore complex symmetric operators with eigenvalues.

The first aim of this paper is to give a geometric description of certain complex symmetric triangular operators. This is partially inspired by a recent paper of Balayan and Garcia [2], which provides a geometric characterization for a finite-dimensional operator with distinct eigenvalues to be complex symmetric. The present aim of this paper is to extend the preceding result to infinite-dimensional Hilbert space. To proceed, we first introduce some notation and terminology.

Recall that an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *triangular* if

$$\bigvee_{\lambda\in\mathbb{C},n\geqslant 1} \ker(T-\lambda)^n = \mathscr{H},$$

where $\vee$ denotes closed linear span. We remark that T is triangular if and only if T admits an upper triangular matrix representation

$$T = \begin{bmatrix} \lambda_1 & * & * & \cdots \\ & \lambda_2 & * & \cdots \\ & & \lambda_3 & \cdots \\ & & & \ddots \end{bmatrix}$$

with respect to some orthonormal basis of $\mathscr{H}$, where each omitted entry is zero. The class of triangular operators contain many important operators such as the well-known Cowen-Douglas operators which are closely related to complex geometry [4].

It is obvious that each operator on finite-dimensional Hilbert space is triangular. However, this is not the case in infinite-dimensional space since there exists T acting on infinite-dimensional Hilbert space with $\sigma_p(T) = \emptyset$. Here and in what follows $\sigma_p(T)$ denotes the point spectrum of T. For example, the forward unilateral shift has no any eigenvalue and hence is not triangular. However triangular operators are universal in the sense of approximation; more precisely, given $T \in \mathscr{B}(\mathscr{H})$ and $\varepsilon > 0$, there exist $K \in \mathscr{B}(\mathscr{H})$ with $\|K\| < \varepsilon$ and triangular operators A, B such that $T+K$ is similar to $A \oplus B^*$. The reader is referred to [1] or [21, Thm. 6.1] for more details.

When an operator T and its adjoint are both triangular (in general, with respect to different orthonormal bases), T is called *bitriangular*. This class contains all algebraic operators, diagonal normal operators and block diagonal operators. Obviously, every operator on finite-dimensional Hilbert space is bitriangular. As indicated in [5, 20, 22], the class of bitriangular operators provide the best infinite-dimensional analogues of finite-dimensional operators. There exist triangular operators which are not bitriangular. The adjoint of the forward unilateral shift is such an example. However, each complex symmetric triangular operator must be bitriangular.

LEMMA 1.3. *If $T \in \mathscr{B}(\mathscr{H})$ is complex symmetric and triangular, then T is bitriangular.*

Proof. Since T is complex symmetric, there is a conjugation C on $\mathscr{H}$ such that $T^*C = CT$. Hence $(T^* - \overline{\lambda})^n C = C(T-\lambda)^n$ and $C(\ker(T-\lambda)^n) = \ker(T^* - \overline{\lambda})^n$ for all $\lambda \in \mathbb{C}$ and $n \geqslant 1$.

Note that

$$\bigvee_{\lambda \in \mathbb{C}, n \geqslant 1} \ker(T-\lambda)^n = \mathscr{H}.$$

Since C is a conjugation, it follows that

$$\mathscr{H} = C(\mathscr{H}) = \bigvee_{\lambda \in \mathbb{C}, n \geqslant 1} C(\ker(T-\lambda)^n) = \bigvee_{\lambda \in \mathbb{C}, n \geqslant 1} \ker(T^* - \overline{\lambda})^n.$$

Hence T^* is triangular. $\square$

REMARK 1.4. Let $T \in \mathscr{B}(\mathscr{H})$ be complex symmetric. From the proof of Lemma 1.3, one can see that $\dim\ker(T-\lambda) = \dim\ker(T-\lambda)^*$ for $\lambda \in \mathbb{C}$; in particular, $\lambda \in \sigma_p(T)$ if and only if $\overline{\lambda} \in \sigma_p(T^*)$.

In [5], it is proved that every bitriangular operator is quasisimilar to a direct sum of Jordan blocks and hence quasisimilar to a CSO [10, Ex. 4]. In this paper, we concentrates on those triangular operators $T \in \mathscr{B}(\mathscr{H})$ with distinct eigenvalues $\{\lambda_i : i \geqslant 1\}$ satisfying

$$\bigvee_{i \geqslant 1} \ker(T-\lambda_i) = \mathscr{H} \quad \text{and} \quad \dim\ker(T-\lambda_i) = 1, \quad \forall i \geqslant 1.$$

By the preceding lemma, if T is complex symmetric, then

$$\bigvee_{i \geqslant 1} \ker(T-\lambda_i)^* = \mathscr{H} \quad \text{and} \quad \dim\ker(T-\lambda_i)^* = 1, \quad \forall i \geqslant 1.$$

The main result of this paper is the following result.

THEOREM 1.5. *Let $T \in \mathscr{B}(\mathscr{H})$. Suppose that $\{\lambda_i : 1 \leqslant i < \infty\}$ are distinct eigenvalues of T, u_i is a normalized eigenvector of T corresponding to λ_i and v_i is a normalized eigenvector of T^* corresponding to $\overline{\lambda_i}$ for $i \geqslant 1$. If*

$$\dim\ker(T-\lambda_i) = 1 = \dim\ker(T-\lambda_i)^*, \quad \forall i \geqslant 1,$$

and $\vee\{u_i : i \geqslant 1\} = \mathscr{H} = \vee\{v_i : i \geqslant 1\}$, then the following are equivalent:

(i) T is complex symmetric;

(ii) there exist unimodular constants $\{\alpha_i : i \geqslant 1\}$ such that

$$\alpha_i \langle u_i, u_j \rangle = \alpha_j \langle v_j, v_i \rangle, \quad \forall i, j \geqslant 1;$$

(iii) the condition

$$\langle u_{i_1},u_{i_2}\rangle\langle u_{i_2},u_{i_3}\rangle\cdots\langle u_{i_{n-1}},u_{i_n}\rangle\langle u_{i_n},u_{i_1}\rangle$$
$$=\langle v_{i_2},v_{i_1}\rangle\langle v_{i_3},v_{i_2}\rangle\cdots\langle v_{i_n},v_{i_{n-1}}\rangle\langle v_{i_1},v_{i_n}\rangle$$

holds for any $n\in\mathbb{N}$ *and any* n*-tuple* $(i_1,i_2,\cdots,i_n)$ *in* $\mathbb{N}$.

As an application of Theorem 1.5, we obtain the following result.

THEOREM 1.6. *Let* $T\in\mathscr{B}(\mathscr{H})$. *Suppose that* $\{\lambda_i : 1\leqslant i<\infty\}$ *are distinct eigenvalues of* T, u_i *is a normalized eigenvector of* T *corresponding to* λ_i *and* v_i *is a normalized eigenvector of* T^* *corresponding to* $\overline{\lambda_i}$ *for* $i\geqslant 1$. *If*

$$\dim\ker(T-\lambda_i)=1=\dim\ker(T-\lambda_i)^*,\quad \forall i\geqslant 1,$$

$\vee\{u_i : i\geqslant 1\}=\mathscr{H}=\vee\{v_i : i\geqslant 1\}$ *and* $\langle u_i,u_j\rangle\neq 0$ *for all* $i,j\geqslant 1$, *then* T *is complex symmetric if and only if the condition*

$$\langle u_i,u_j\rangle\langle u_j,u_k\rangle\langle u_k,u_i\rangle=\overline{\langle v_i,v_j\rangle\langle v_j,v_k\rangle\langle v_k,v_i\rangle} \tag{1.1}$$

holds for any triad (i,j,k) *with* $i\leqslant j\leqslant k$.

The other aim of this paper is to give Apostol's triangular representation for CSOs. First let us give a brief introduction to Apostol's triangular representation for Hilbert space operators.

An operator $A\in\mathscr{B}(\mathscr{H})$ is called a *semi-Fredholm* operator, if ran A is closed and either nul A or nul A^* is finite, where nul $A:=\dim\ker A$ and nul $A^*:=\dim\ker A^*$; in this case, ind $A:=$ nul $A-$ nul A^* is called the *index* of A. In particular, if $-\infty<$ ind $A<\infty$, then A is called a *Fredholm operator*. The *Wolf spectrum* $\sigma_{lre}(A)$ is defined as

$$\sigma_{lre}(A):=\{\lambda\in\mathbb{C} : A-\lambda \text{ is not semi-Fredholm}\}.$$

The set $\rho_{s-F}(A):=\mathbb{C}\setminus\sigma_{lre}(A)$ is called the *semi-Fredholm domain* of A. For $\lambda\in\rho_{s-F}(A)$, the *minimal index* of $A-\lambda$ is defined by

$$\min\cdot\text{ind }(A-\lambda)=\min\{\text{nul }(A-\lambda),\text{nul }(A-\lambda)^*\}.$$

The function $\lambda\mapsto\min\cdot\text{ind }(A-\lambda)$ is constant on every component of $\rho_{s-F}(A)$ except for an at most denumerable subset $\rho^s_{s-F}(A)$ without limits in $\rho_{s-F}(A)$. Each $\lambda\in\rho^s_{s-F}(A)$ is called a *singular point* of the semi-Fredholm domain of A, and the set $\rho^r_{s-F}(A)=\rho_{s-F}(A)\setminus\rho^s_{s-F}(A)$ is the set of *regular points*. The reader is referred to [21, Chap. 1] for more details.

Given $T\in\mathscr{B}(\mathscr{H})$, let

$$\mathscr{H}_r(T)=\bigvee\{\ker(\lambda-T) : \lambda\in\rho^r_{s-F}(T)\},$$

$$\mathscr{H}_l(T)=\bigvee\{\ker(\lambda-T)^* : \lambda\in\rho^r_{s-F}(T)\}$$

and $\mathscr{H}_0(T)$ be the orthogonal complement of $\mathscr{H}_r(T)+\mathscr{H}_l(T)$. Denote the compressions of T to $\mathscr{H}_r(T),\mathscr{H}_l(T)$ and $\mathscr{H}_0(T)$ by T_r,T_l and T_0, respectively. As seen in [21, Thm. 3.38], $\mathscr{H}_r(T)$ is orthogonal to $\mathscr{H}_l(T)$. Noting that $\mathscr{H}_r(T)$ is hyperinvariant for T and $\mathscr{H}_l(T)$ is hyperinvariant for T^*, T can be written as

$$T=\begin{bmatrix} T_r & E & G \\ 0 & T_0 & F \\ 0 & 0 & T_l \end{bmatrix} \begin{matrix} \mathscr{H}_r(T) \\ \mathscr{H}_0(T), \\ \mathscr{H}_l(T) \end{matrix} \tag{1.2}$$

where T_r and T_l^* are triangular operators with perfect spectrum. The upper triangular operator matrix (1.2) is called Apostol's triangular representation for T, and its basic properties are established in [21, Thm. 3.38]. This triangular representation, which describes the internal structure of a general operator, is a very useful tool in operator theory.

In general, for given $T\in\mathscr{B}(\mathscr{H})$, T_r,T_0 and T_l are independent, and each of them can be absent. For example, if S is the classical forward unilateral shift on l^2 and $T=(S+2)^*\oplus(S-2)$, then (i) S_r,S_0 are absent, $S_l=S$, and (ii) T_0 is absent, $T_r=(S+2)^*$ and $T_l=(S-2)$.

Employing the idea of Apostol's triangular representation, we shall describe the upper triangular matrix representation for general CSOs (see Theorem 3.3). As we shall see later, if T is complex symmetric and admits the representation (1.2), then T_r is unitarily equivalent to a transpose of T_l (see Definition 3.1); in particular, $\sigma(T_r)=\sigma(T_l)$.

The rest of this paper is organized as follows. In Section 2, we shall give the proofs of Theorems 1.5 and 1.6. The proof of Theorem 3.3 shall be provided in Section 3. Also we shall give several concrete examples.

2. Proofs of Theorems 1.5 and 1.6

We first give several auxiliary results.

LEMMA 2.1. *Let $T\in\mathscr{B}(\mathscr{H})$. Assume that $\lambda_1,\lambda_2\in\mathbb{C}$ with $\lambda_1\neq\lambda_2$ and $u\in\ker(T-\lambda_1), v\in\ker(T-\lambda_2)^*$. Then $\langle u,v\rangle=0$.*

Proof. Compute to see

$$\lambda_1\langle u,v\rangle=\langle Tu,v\rangle=\langle u,T^*v\rangle=\lambda_2\langle u,v\rangle.$$

Since $\lambda_1\neq\lambda_2$, it follows that $\langle u,v\rangle=0$. $\square$

THEOREM 2.2. *Let $T\in\mathscr{B}(\mathscr{H})$. Suppose that $\{\lambda_i : i\geqslant 1\}$ are distinct eigenvalues of T and $u_i\in\ker(T-\lambda_i)$ is a unit vector for $i\geqslant 1$. If $\vee\{u_i : i\geqslant 1\}=\mathscr{H}$, then T is complex symmetric if and only if there exist unit vectors $\{v_i : i\geqslant 1\}$ with $v_i\in\ker(T-\lambda_i)^*$ for $i\geqslant 1$ such that $\vee\{v_i : i\geqslant 1\}=\mathscr{H}$ and $\langle u_i,u_j\rangle=\langle v_j,v_i\rangle$ for any $i,j\geqslant 1$.*

Proof. "$\Longrightarrow$". Assume that C is a conjugation on $\mathscr{H}$ satisfying $CTC = T^*$. For each $i \geqslant 1$, set $v_i = Cu_i$. Note that

$$T^* v_i = T^* C u_i = CT u_i = \overline{\lambda_i} C u_i = \overline{\lambda_i} v_i.$$

It follows that each v_i is a normalized eigenvector of T^* corresponding to $\overline{\lambda_i}$. Moreover, we have

$$\vee\{v_i : i \geqslant 1\} = \vee\{Cu_i : i \geqslant 1\} = C(\vee\{u_i : i \geqslant 1\}) = C(\mathscr{H}) = \mathscr{H}.$$

For $i, j \geqslant 1$, since C is a conjugation, it follows that

$$\langle v_j, v_i \rangle = \langle Cu_j, Cu_i \rangle = \langle u_i, u_j \rangle.$$

This proves the necessity.

"$\Longleftarrow$". Assume that v_i is a normalized eigenvector of T^* corresponding to $\overline{\lambda_i}$ for $i \geqslant 1$, $\vee\{v_i : i \geqslant 1\} = \mathscr{H}$ and

$$\langle u_i, u_j \rangle = \langle v_j, v_i \rangle, \quad \forall i, j \geqslant 1.$$

We shall construct a conjugation C on $\mathscr{H}$ such that $CTC = T^*$.

Denote by $\mathscr{H}_0$ the set of all finite linear combinations of u_i's, and by $\mathscr{H}_1$ the set of all finite linear combinations of v_i's. By the hypothesis, $\mathscr{H}_i$ is a dense linear manifold of $\mathscr{H}$, $i = 1, 2$.

For each $x \in \mathscr{H}_0$ with $x = \sum_{i=1}^n \alpha_i u_i$, define $Cx = \sum_{i=1}^n \overline{\alpha_i} v_i$. If $y \in \mathscr{H}_0$ and $y = \sum_{j=1}^n \beta_j u_j$, one can check that

$$\begin{aligned}
\langle Cx, Cy \rangle &= \Big\langle \sum_{i=1}^n \overline{\alpha_i} v_i, \sum_{j=1}^n \overline{\beta_j} v_j \Big\rangle \\
&= \sum_{i,j=1}^n \overline{\alpha_i} \beta_j \langle v_i, v_j \rangle \\
&= \sum_{i,j=1}^n \overline{\alpha_i} \beta_j \langle u_j, u_i \rangle \\
&= \Big\langle \sum_{j=1}^n \beta_j u_j, \sum_{i=1}^n \alpha_i u_i \Big\rangle \\
&= \langle y, x \rangle.
\end{aligned}$$

It follows that the map $C : \mathscr{H}_0 \to \mathscr{H}_1$ is conjugate-linear, isometric and hence well defined. Moreover, C admits a continuous extension to $\mathscr{H}$, denoted by C again. It is obvious that C is surjective and hence invertible. In particular, we have

$$\langle Cx, Cy \rangle = \langle y, x \rangle, \quad \forall x, y \in \mathscr{H}. \tag{2.1}$$

We claim that C is a conjugation. Now it suffices to prove that C is involutive, that is, $C^2 = I$. Since $\vee\{u_i : i \geqslant 1\} = \mathscr{H}$, we need only check that $C^2 u_i = u_i$ for each $i \geqslant 1$.

Now fix an $i \geqslant 1$. Since $\vee\{v_j : j \geqslant 1\} = \mathscr{H}$, it follows that $\dim\{v_j : j \neq i\}^\perp \leqslant 1$. By Lemma 1.3, $\langle u_i, v_j\rangle = 0$ for all $j \neq i$, we deduce that $\{v_j : j \neq i\}^\perp = \vee\{u_i\}$. On the other hand, since $\langle v_i, u_j\rangle = 0$ for all $j \neq i$, in view of (2.1), we obtain

$$\langle Cv_i, v_j\rangle = \langle Cv_i, Cu_j\rangle = \langle u_j, v_i\rangle = 0, \quad \forall j \neq i.$$

Hence $Cv_i \in \vee\{u_i\}$, that is, $Cv_i = \alpha u_i$ for some unimodular constant α. So

$$\langle u_i, v_i\rangle = \langle Cv_i, Cu_i\rangle = \langle \alpha u_i, v_i\rangle.$$

Noting that $\vee\{v_j : j \geqslant 1\} = \mathscr{H}$ and $\langle u_i, v_j\rangle = 0$ for all $j \neq i$, it follows that $\langle u_i, v_i\rangle \neq 0$. Hence we have $\alpha = 1$ and $C^2 u_i = Cv_i = u_i$. Thus we have proved that C is a conjugation.

For each $i \geqslant 1$, compute to see that

$$CTu_i = C(\lambda_i u_i) = \overline{\lambda_i} Cu_i = \overline{\lambda_i} v_i = T^* v_i = T^* Cu_i,$$

which implies that $CT = T^*C$. Hence T is complex symmetric. $\square$

PROPOSITION 2.3. *Let $\{u_i, v_i : i \geqslant 1\}$ be unit vectors in $\mathscr{H}$. Then there exist unimodular constants $\{\alpha_i : i \geqslant 1\}$ such that $\alpha_i \langle u_i, u_j\rangle = \alpha_j \langle v_j, v_i\rangle$ for all $i, j \geqslant 1$ if and only if the condition*

$$\begin{aligned}&\langle u_{i_1}, u_{i_2}\rangle\langle u_{i_2}, u_{i_3}\rangle \cdots \langle u_{i_{n-1}}, u_{i_n}\rangle\langle u_{i_n}, u_{i_1}\rangle\\ &= \langle v_{i_2}, v_{i_1}\rangle\langle v_{i_3}, v_{i_2}\rangle \cdots \langle v_{i_n}, v_{i_{n-1}}\rangle\langle v_{i_1}, v_{i_n}\rangle\end{aligned}$$

holds for any $n \in \mathbb{N}$ and any n-tuple $(i_1, i_2, \cdots, i_n)$ of positive integers.

Proof. The necessity is obvious. We need only prove the sufficiency.

"$\Longleftarrow$". For $i, j \in \mathbb{N}$, we define $i \sim j$ if there exist $i_1, i_2, \cdots, i_n \in \mathbb{N}$ such that

$$\langle u_i, u_{i_1}\rangle\langle u_{i_1}, u_{i_2}\rangle \cdots \langle u_{i_{n-1}}, u_{i_n}\rangle\langle u_{i_n}, u_j\rangle \neq 0.$$

One can verify that $\sim$ is an equivalence relation on $\mathbb{N}$. Denote $\mathbb{N}/\sim = \{\Lambda_m : m \in \Gamma\}$. Thus $\Lambda_{m_1} \cap \Lambda_{m_2} = \emptyset$ for all $m_1, m_2 \in \Gamma$ with $m_1 \neq m_2$.

By the hypothesis, we have

$$|\langle u_i, u_j\rangle| = |\langle v_i, v_j\rangle|, \quad \forall i, j \geqslant 1. \tag{2.2}$$

For convenience, we denote

$$\Upsilon(i_1, i_2, \cdots, i_k) = \frac{\langle u_{i_1}, u_{i_2}\rangle\langle u_{i_2}, u_{i_3}\rangle \cdots \langle u_{i_{k-1}}, u_{i_k}\rangle}{\langle v_{i_2}, v_{i_1}\rangle\langle v_{i_3}, v_{i_2}\rangle \cdots \langle v_{i_k}, v_{i_{k-1}}\rangle}$$

for $k \geqslant 2$ and k-tuple $(i_1, i_2, \cdots, i_k)$ in $\mathbb{N}$. By (2.2), if $\langle u_{i_l}, u_{i_{l+1}}\rangle \neq 0$ for all $1 \leqslant l \leqslant k-1$, then $|\Upsilon(i_1, i_2, \cdots, i_k)| = 1 = \Upsilon(i_1, i_2, \cdots, i_k, i_1)$.

Let $m \in \Gamma$ be fixed. Arbitrarily choose an $l_m \in \Lambda_m$ and set $\alpha_{l_m} = 1$. For each $j \in \Lambda_m$, by the hypothesis, there exist $i_1, i_2, \cdots, i_n \in \Lambda_m$ such that

$$\langle u_{l_m}, u_{i_1} \rangle \langle u_{i_1}, u_{i_2} \rangle \cdots \langle u_{i_{n-1}}, u_{i_n} \rangle \langle u_{i_n}, u_j \rangle \neq 0.$$

By the preceding discussion, $\Upsilon(l_m, i_1, i_2, \cdots, i_n, j) \in \mathbb{C}$ with modulus 1. Set

$$\alpha_j = \Upsilon(l_m, i_1, i_2, \cdots, i_n, j).$$

We need to prove the definition of α_j is unique. Assume that there also exist $j_1, j_2, \cdots, j_p \in \Lambda_m$ such that

$$\langle u_{l_m}, u_{j_1} \rangle \langle u_{j_1}, u_{j_2} \rangle \cdots \langle u_{j_{p-1}}, u_{j_p} \rangle \langle u_{j_p}, u_j \rangle \neq 0.$$

Then we have to check that

$$\Upsilon(l_m, i_1, i_2, \cdots, i_n, j) = \Upsilon(l_m, j_1, j_2, \cdots, j_p, j).$$

By the hypothesis, we have

$$\begin{aligned} &\Upsilon(l_m, i_1, i_2, \cdots, i_n, j)\Upsilon(j, j_p, j_{p-1}, \cdots, j_1, l_m) \\ &= \Upsilon(l_m, i_1, i_2, \cdots, i_n, j, j_p, j_{p-1}, \cdots, j_1, l_m) = 1, \end{aligned}$$

and hence

$$\begin{aligned} \Upsilon(l_m, i_1, i_2, \cdots, i_n, j) &= \overline{\Upsilon(j, j_p, j_{p-1}, \cdots, j_1, l_m)} \\ &= \Upsilon(l_m, j_1, j_2, \cdots, j_p, j). \end{aligned}$$

This shows that α_j is well defined. Now we have defined α_i for all $i \in \mathbb{N}$. Also we note that $|\alpha_i| = 1$ for all $i \in \mathbb{N}$.

Arbitrarily choose $i, j \in \mathbb{N}$. It remains to prove that $\alpha_i \langle u_i, u_j \rangle = \alpha_j \langle v_j, v_i \rangle$. In view of (2.2), we may directly assume that $\langle u_i, u_j \rangle \neq 0$. Thus $i \sim j$. We further assume that $i, j \in \Lambda_m$, $i \neq j$ and

$$\alpha_i = \Upsilon(l_m, i_1, i_2, \cdots, i_n, i),$$

where

$$\langle u_{l_m}, u_{i_1} \rangle \langle u_{i_1}, u_{i_2} \rangle \cdots \langle u_{i_{n-1}}, u_{i_n} \rangle \langle u_{i_n}, u_i \rangle \neq 0.$$

Then

$$\langle u_{l_m}, u_{i_1} \rangle \langle u_{i_1}, u_{i_2} \rangle \cdots \langle u_{i_{n-1}}, u_{i_n} \rangle \langle u_{i_n}, u_i \rangle \langle u_i, u_j \rangle \neq 0$$

and

$$\alpha_j = \Upsilon(l_m, i_1, i_2, \cdots, i_n, i, j).$$

A direct calculation shows that

$$\begin{aligned} \alpha_i \langle u_i, u_j \rangle &= \Upsilon(l_m, i_1, i_2, \cdots, i_n, i) \langle u_i, u_j \rangle \\ &= \Upsilon(l_m, i_1, i_2, \cdots, i_n, i) \frac{\langle u_i, u_j \rangle}{\langle v_j, v_i \rangle} \langle v_j, v_i \rangle \\ &= \Upsilon(l_m, i_1, i_2, \cdots, i_n, i, j) \langle v_j, v_i \rangle \\ &= \alpha_j \langle v_j, v_i \rangle. \end{aligned}$$

This completes the proof. □

Now we are ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. The equivalence "(ii)$\Longleftrightarrow$(iii)" follows from Proposition 2.3.

"(i)$\Longrightarrow$(ii)". Assume that C is a conjugation on $\mathscr{H}$ such that $CTC = T^*$. It follows that $C(T-\lambda_i)C = T^* - \overline{\lambda_i}$ for $i \geqslant 1$. Since $u_i \in \ker(T-\lambda_i)$ and $\|u_i\| = 1$, we have $Cu_i \in \ker(T-\lambda_i)^*$ and $\|Cu_i\| = 1$. For each $i \geqslant 1$, noting that nul $(T-\lambda_i)^* = 1$, $v_i \in \ker(T-\lambda_i)^*$ and $\|v_i\| = 1$, we deduce that $Cu_i = \alpha_i v_i$ for some unimodular constant α_i. Since C is a conjugation, it is easy to see that

$$\langle u_i, u_j\rangle = \langle Cu_j, Cu_i\rangle = \langle \alpha_j v_j, \alpha_i v_i\rangle = \overline{\alpha_i}\alpha_j\langle v_j, v_i\rangle.$$

This proves "(i)$\Longrightarrow$(ii)".

"(ii)$\Longrightarrow$(i)". For each $i \geqslant 1$, set $w_i = \alpha_i v_i$. By the hypothesis, we have

$$\langle w_j, w_i\rangle = \langle \alpha_j v_j, \alpha_i v_i\rangle = \overline{\alpha_i}\alpha_j\langle v_j, v_i\rangle = \langle u_i, u_j\rangle.$$

By the proof for the sufficiency of Theorem 2.2, one can see that T is complex symmetric. □

EXAMPLE 2.4. Choose a bounded sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers. For each $n \geqslant 1$, let $T_n \in \mathscr{B}(\mathbb{C}^2)$ be the operator induced by the following matrix

$$T_n = \begin{bmatrix} \frac{1}{n} & a_n \\ 0 & \frac{1}{n} + \frac{1}{2^n} \end{bmatrix}$$

with respect to the canonical orthonormal basis for $\mathbb{C}^2$. Set $T = \oplus_{n=1}^{\infty} T_n$. Then one can check that $\{\frac{1}{n} : n \geqslant 1\} \cup \{\frac{1}{n} + \frac{1}{2^n} : n \geqslant 1\}$ are distinct eigenvalues of T. Denote by $\lambda_1, \lambda_2, \lambda_3, \cdots$ these eigenvalues. It is easy to see that nul $(T-\lambda_i) = 1 =$ nul $(T-\lambda_i)^*$ for all $i \geqslant 1$ and $\vee_{i\geqslant 1}\ker(T-\lambda_i) = \vee_{i\geqslant 1}\ker(T-\lambda_i)^*$ equals the underlying space of T.

By [16, Cor. 1], each T_i is complex symmetric and hence T is also complex symmetric. For each $i \geqslant 1$, if $u_i \in \ker(T-\lambda_i), v_i \in \ker(T-\lambda_i)^*$ and $\|u_i\| = 1 = \|v_i\|$, then, by Theorem 1.5, the condition

$$\begin{aligned} &\langle u_{i_1}, u_{i_2}\rangle\langle u_{i_2}, u_{i_3}\rangle \cdots \langle u_{i_{n-1}}, u_{i_n}\rangle\langle u_{i_n}, u_{i_1}\rangle \\ &= \langle v_{i_2}, v_{i_1}\rangle\langle v_{i_3}, v_{i_2}\rangle \cdots \langle v_{i_n}, v_{i_{n-1}}\rangle\langle v_{i_1}, v_{i_n}\rangle. \end{aligned}$$

holds for any $n \geqslant 2$ and any n-tuple $(i_1, i_2, \cdots, i_n)$.

Now we can give the proof of Theorem 1.6.

Proof of Theorem 1.6. By Theorem 1.5, we need only prove the sufficiency. For convenience, we denote

$$\Upsilon(i_1, i_2, \cdots, i_n) = \frac{\langle u_{i_1}, u_{i_2}\rangle\langle u_{i_2}, u_{i_3}\rangle \cdots \langle u_{i_{n-1}}, u_{i_n}\rangle}{\langle v_{i_2}, v_{i_1}\rangle\langle v_{i_3}, v_{i_2}\rangle \cdots \langle v_{i_n, v_{i_{n-1}}}\rangle}$$

for $n \geqslant 2$ and n-tuple $(i_1, i_2, \cdots, i_n)$. Using Theorem 1.5 again, we need only prove that $\Upsilon(i_1, i_2, \cdots, i_n, i_1) = 1$ for any $n \geqslant 1$ and any n-tuple $(i_1, i_2, \cdots, i_n)$ in $\mathbb{N}$.

We shall proceed by induction. By (1.1), for i, j with $i \leqslant j$, we have

$$\langle u_i, u_i \rangle = \langle v_i, v_i \rangle$$

and

$$\begin{aligned}\langle u_i, u_j \rangle \langle u_j, u_i \rangle &= \langle u_i, u_i \rangle \langle u_i, u_j \rangle \langle u_j, u_i \rangle \\ &= \langle v_i, v_i \rangle \langle v_j, v_i \rangle \langle v_i, v_j \rangle = \langle v_j, v_i \rangle \langle v_i, v_j \rangle.\end{aligned}$$

It follows that $|\langle u_i, u_j \rangle| = |\langle v_i, v_j \rangle|$ for any i, j. Hence

$$\Upsilon(i_1, i_1) = 1, \quad \Upsilon(i_1, i_2, i_1) = \frac{\langle u_{i_1}, u_{i_2} \rangle \langle u_{i_2}, u_{i_1} \rangle}{\langle v_{i_2}, v_{i_1} \rangle \langle v_{i_1}, v_{i_2} \rangle} = 1$$

for any $i_1, i_2 \in \mathbb{N}$.

For $i, j, k \in \mathbb{N}$ with $i \leqslant j \leqslant k$, by (1.1), we have $\Upsilon(i, j, k, i) = 1$. Noting that

$$\begin{aligned}\Upsilon(k, i, j, k) &= \Upsilon(j, k, i, j) = \Upsilon(i, j, k, i) \\ &= \overline{\Upsilon(i, k, j, i)} = \overline{\Upsilon(k, j, i, k)} \\ &= \overline{\Upsilon(j, i, k, j)},\end{aligned}$$

this shows that $\Upsilon(i_1, i_2, i_3, i_1) = 1$ for any triad (i_1, i_2, i_3).

Now suppose we have proved that a positive integer $k \geqslant 3$ exists so that

$$\Upsilon(j_1, j_2, \cdots, j_m, j_1) = 1$$

for any $1 \leqslant m \leqslant k$ and any m-tuple $(j_1, j_2, \cdots, j_m)$. Given a $(k+1)$-tuple $(i_1, i_2, \cdots, i_{k+1})$, by the induction hypothesis, we have

$$\begin{aligned}\Upsilon(i_1, i_2, \cdots, i_{k+1}, i_1) &= \Upsilon(i_1, i_2, \cdots, i_k) \Upsilon(i_k, i_{k+1}, i_1) \\ &= \frac{\Upsilon(i_1, i_2, \cdots, i_k, i_1) \Upsilon(i_k, i_{k+1}, i_1, i_k)}{\Upsilon(i_k, i_1, i_k)} = 1.\end{aligned}$$

This completes the proof. $\square$

3. Apostol's triangular representation for CSOs

In this section, we shall describe Apostol's triangular representation for CSOs. First we make some preparation.

DEFINITION 3.1. Let $T \in \mathscr{B}(\mathscr{H})$. An operator $A \in \mathscr{B}(\mathscr{H})$ is called a *transpose* of T, if $A = CT^*C$ for some conjugation C on $\mathscr{H}$.

Note that if $T \in \mathscr{B}(\mathscr{H})$ is complex symmetric, then $T = CT^*C$ for some conjugation C on $\mathscr{H}$; so T is a transpose of itself. In general, an operator has more than one transpose [24, Ex. 2.2]. However, any two transposes of an operator are unitarily equivalent.

LEMMA 3.2. *Let $T \in \mathscr{B}(\mathscr{H})$ and A be a transpose of T. Then*

(i) $\sigma(A) = \sigma(T)$, $\rho_{s-F}(T) = \rho_{s-F}(A)$, $\mathrm{ind}\,(T-\lambda) = -\mathrm{ind}\,(A-\lambda)$ *and*

$$\min\cdot\mathrm{ind}\,(T-\lambda) = \min\cdot\mathrm{ind}\,(A-\lambda), \quad \forall \lambda \in \rho_{s-F}(T);$$

moreover, $\rho^r_{s-F}(T) = \rho^r_{s-F}(A)$.

(ii) If B is also a transpose of T, then $A \cong B$, where $\cong$ denotes unitary equivalence.

Proof. Since A is a transpose of T, we can choose a conjugations C on $\mathscr{H}$ such that $A = CT^*C$.

(i) For $\lambda \in \mathbb{C}$, we have $A - \lambda = C(T-\lambda)^*C$. Note that C is invertible. Then one can see the desired results from direct verification.

(ii) Since B is a transpose of T, we can choose a conjugation D on $\mathscr{H}$ such that $B = DT^*D$. Set $U = DC$ and $V = CD$. Then $UV = VU = I$. Since D,C are conjugate-linear and isometric, we deduce that $U \in \mathscr{B}(\mathscr{H})$ is unitary and $U^{-1} = V$. So $UA = DT^*C = (DT^*D)(DC) = BU$, that is, $A \cong B$. $\square$

We often write T^t to denote a transpose of T. In general, there is no ambiguity especially when we write $T \cong T^t$.

Given a conjugation C on $\mathscr{H}$, we denote $S_C(\mathscr{H}) = \{X \in \mathscr{B}(\mathscr{H}) : CXC = X^*\}$.

The following theorem is the main result of this section.

THEOREM 3.3. *Let $T \in \mathscr{B}(\mathscr{H})$ and Ω be an open subset of $\rho^r_{s-F}(T)$. Denote $\mathscr{H}_r(\Omega) = \vee\{\ker(\lambda - T) : \lambda \in \Omega\}$ and $\mathscr{H}_l(\Omega) = \vee\{\ker(\lambda - T)^* : \lambda \in \Omega\}$. Let $\mathscr{H}_0(\Omega)$ be the orthogonal complement of $\mathscr{H}_r(\Omega) + \mathscr{H}_l(\Omega)$. Then*

$$\mathscr{H} = \mathscr{H}_r(\Omega) \oplus \mathscr{H}_0(\Omega) \oplus \mathscr{H}_l(\Omega),$$

and with respect to this orthogonal decomposition T can be written as

$$T = \begin{bmatrix} A_r & E & G \\ 0 & A_0 & F \\ 0 & 0 & A_l \end{bmatrix}. \tag{3.1}$$

Furthermore, if T is complex symmetric, then

(i) $A_l \cong (A_r)^t$ *and* $\|E\| = \|F\|$*;*

(ii) both A_0 and the following operator

$$\begin{bmatrix} A_r & G \\ 0 & A_l \end{bmatrix}$$

are complex symmetric;

(iii) there is a conjugation C on $\mathscr{H}_r(\Omega)$, $G_1 \in S_C(\mathscr{H}_r(\Omega))$ and a conjugation D on $\mathscr{H}_0(\Omega)$ such that $A_0 \in S_D(\mathscr{H}_0(\Omega))$ and

$$T \cong \begin{bmatrix} A_r & E & G_1 \\ 0 & A_0 & DE^*C \\ 0 & 0 & CA_r^*C \end{bmatrix} \begin{matrix} \mathscr{H}_r(\Omega) \\ \mathscr{H}_0(\Omega). \\ \mathscr{H}_r(\Omega) \end{matrix}$$

Proof. Note that $\mathscr{H}_r(\Omega) \subset \mathscr{H}_r(T)$, $\mathscr{H}_l(\Omega) \subset \mathscr{H}_l(T)$ and, by [21, Thm. 3.38], $\mathscr{H}_r(T)$ is orthogonal to $\mathscr{H}_l(T)$. So $\mathscr{H} = \mathscr{H}_r(\Omega) \oplus \mathscr{H}_0(\Omega) \oplus \mathscr{H}_l(\Omega)$. It is obvious that $\mathscr{H}_r(\Omega)$ is hyperinvariant for T and $\mathscr{H}_l(\Omega)$ is hyperinvariant for T^*. Then we may assume that T admits the upper triangular matrix (3.1).

Suppose that T is complex symmetric and $C_0TC_0 = T^*$ for some conjugation C_0 on $\mathscr{H}$. For $\lambda \in \Omega$, note that $C_0(T-\lambda)C_0 = (T-\lambda)^*$. It follows that $C_0(\ker(T-\lambda)) \subset \ker(T-\lambda)^*$ and $C_0(\ker(T-\lambda)^*) \subset \ker(T-\lambda)$. Since C_0 is a conjugation, we have $C_0(\ker(T-\lambda)) = \ker(T-\lambda)^*$. It follows immediately that

$$C_0(\mathscr{H}_r(\Omega)) = \mathscr{H}_l(\Omega), \quad C_0(\mathscr{H}_l(\Omega)) = \mathscr{H}_r(\Omega) \quad \text{and} \quad C_0(\mathscr{H}_0(\Omega)) = \mathscr{H}_0(\Omega).$$

Thus C_0 can be written as

$$C_0 = \begin{bmatrix} 0 & 0 & C_2 \\ 0 & D & 0 \\ C_1 & 0 & 0 \end{bmatrix}. \tag{3.2}$$

Since $C_0^{-1} = C_0$, it follows that $D^{-1} = D$ and $C_1^{-1} = C_2$. Thus D is a conjugation on $\mathscr{H}_0(\Omega)$.

Since $TC_0 = C_0T^*$, a direct matrical calculation shows that

$$A_l = C_1A_r^*C_2, \quad DA_0^* = A_0D, \quad F = DE^*C_2, \quad C_1G = G^*C_2.$$

Thus A_0 is complex symmetric. Since D, C_2 are antiunitary operators, one can see $\|F\| = \|E^*\| = \|E\|$. Arbitrarily choose a conjugation C on $\mathscr{H}_r(\Omega)$ and set $U = C_1C$. Thus $U : \mathscr{H}_r(\Omega) \longrightarrow \mathscr{H}_l(\Omega)$ is unitary and $U^{-1} = CC_1^{-1} = CC_2$. Hence

$$A_l = C_1A_r^*C_2 = (C_1C)(CA_r^*C)(CC_2) = U(CA_r^*C)U^*,$$

that is, $A_l \cong CA_r^*C$.

On the other hand, one can see that the conjugate-linear operator

$$\begin{bmatrix} 0 & C_2 \\ C_1 & 0 \end{bmatrix}$$

is a conjugation on $\mathscr{H}_r(\Omega) \oplus \mathscr{H}_l(\Omega)$ and

$$\begin{aligned} \begin{bmatrix} 0 & C_2 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} A_r & G \\ 0 & A_l \end{bmatrix} &= \begin{bmatrix} 0 & C_2A_l \\ C_1A_r & C_1G \end{bmatrix} = \begin{bmatrix} 0 & A_r^*C_2 \\ A_l^*C_1 & G^*C_2 \end{bmatrix} \\ &= \begin{bmatrix} A_r^* & 0 \\ G^* & A_l^* \end{bmatrix} \begin{bmatrix} 0 & C_2 \\ C_1 & 0 \end{bmatrix}. \end{aligned}$$

That is, the operator

$$\begin{bmatrix} A_r & G \\ 0 & A_l \end{bmatrix}$$

is complex symmetric.

Define

$$V : \mathscr{H}_r(\Omega) \oplus \mathscr{H}_0(\Omega) \oplus \mathscr{H}_r(\Omega) \longrightarrow \mathscr{H},$$
$$(x_1, x_2, x_3) \longmapsto x_1 + x_2 + Ux_3.$$

Then V is unitary. Now compute to see

$$V^*TV = \begin{bmatrix} A_r & E & GU \\ 0 & A_0 & FU \\ 0 & 0 & U^*A_lU \end{bmatrix} = \begin{bmatrix} A_r & E & GU \\ 0 & A_0 & DE^*C \\ 0 & 0 & CA_r^*C \end{bmatrix}.$$

Denote $G_1 = GU$. One can check that

$$CG_1C = CGUC = CGC_1 = CC_2G^* = U^*G^* = G_1^*,$$

which shows that $G_1 \in S_C(\mathscr{H}_r(\Omega))$. □

REMARK 3.4. In Theorem 3.3, if we let $\Omega = \rho_{s-F}^r(T)$, then $A_r = T_r, A_0 = T_0$ and $A_l = T_l$. By Theorem 3.3 (i), T_l is unitarily equivalent to a transpose of T_r; in particular, we have $\|T_r\| = \|T_l\|$ and it follows from Lemma 3.2 that $\sigma(T_r) = \sigma(T_l)$.

Now we give an application of Theorem 3.3.

Let S denote the forward unilateral shift on $\mathscr{H}$ given by $Se_i = e_{i+1}$, $i \geqslant 1$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonromal basis of $\mathscr{H}$. We refer to an operator of the form

$$R_{T,n} = \begin{bmatrix} (S^*)^n & T \\ 0 & S^n \end{bmatrix} \tag{3.3}$$

as a *Foguel operator of order* n, where $T \in \mathscr{B}(\mathscr{H})$ and $n \in \mathbb{N}$.

COROLLARY 3.5. *A Foguel operator $R_{T,n}$ as above is complex symmetric if and only if there is a conjugation C on $\mathscr{H}$ and $T_1 \in S_C(\mathscr{H})$ such that*

$$R_{T,n} \cong \begin{bmatrix} (S^*)^n & T_1 \\ 0 & C(S^*)^nC \end{bmatrix}.$$

Proof. For convenience we denote $A = R_{T,n}$ and write

$$A = \begin{bmatrix} (S^*)^n & T \\ 0 & S^n \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{matrix},$$

where $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$. Since $\sigma_p(S) = \emptyset$, it is easy to see that

$$\rho_{s-F}(A) = \rho_{s-F}^r(A) = \{z \in \mathbb{C} : |z| \neq 1\}$$

and

$$\bigvee_{\lambda\in\mathbb{C},|\lambda|\neq 1} \ker(A-\lambda)=\mathscr{H}_1, \qquad \bigvee_{\lambda\in\mathbb{C},|\lambda|\neq 1} \ker(A-\lambda)^*=\mathscr{H}_2.$$

It follows that $A_r=(S^*)^n$ and $A_l=S^n$. Then the desired result follows readily from Theorem 3.3. □

In the rest, we consider a class of complex symmetric operators constructed in terms of Cowen-Douglas operators.

For $n\in\mathbb{N}$ and a connected open subset Ω of $\mathbb{C}$, let $B_n(\Omega)$ denote the set of operators $T\in\mathscr{B}(\mathscr{H})$ satisfying

(i) $\Omega\subset\sigma(T)$,

(ii) ran $(T-\lambda)=\mathscr{H}$ for $\lambda\in\Omega$,

(iii) $\vee_{\lambda\in\Omega}\ker(T-\lambda)=\mathscr{H}$, and

(iv) nul $(T-\lambda)=n$ for $\lambda\in\Omega$.

Each operator T in $B_n(\Omega)$ is called a *Cowen-Douglas operator* with index n. Note that (iii) can be replaced by the following condition (see [4]).

(iii)' $\vee_{k\geqslant 1}\ker(T-\lambda)^k=\mathscr{H}$ for each $\lambda\in\Omega$.

Let $T\in B_n(\Omega)$ and $\lambda\in\Omega$. Then T can be written as

$$T=\begin{bmatrix}\lambda I_1 & A_1 & * & * & \cdots\\ 0 & \lambda I_2 & A_2 & * & \cdots\\ 0 & 0 & \lambda I_3 & A_3 & \cdots\\ 0 & 0 & 0 & \lambda I_4 & \ddots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\begin{matrix}M_1\\ M_2\\ M_3\\ M_4\\ \vdots\end{matrix},$$

where $M_k=\ker(T-\lambda)^k\ominus\ker(T-\lambda)^{k-1}$ and I_k is the identity operator on M_k for $k\geqslant 1$. Also one can check that $A_k: M_{k+1}\longrightarrow M_k$ is invertible. Moreover, ran $(T-\mu)=\mathscr{H}$ and hence nul $(T-\mu)^*=0$ for all $\mu\in\mathbb{C}$. Thus $\sigma_p(T^*)=\emptyset$. It follows that min$\cdot$ind $(T-\mu)=0$ for $\mu\in\rho_{s-F}(T)$, and hence $\rho^r_{s-F}(T)=\rho_{s-F}(T)\supset\Omega$. Then, by statement (iii), $T=T_r$.

THEOREM 3.6. *Let $A,B\in\mathscr{B}(\mathscr{H})$ be two Cowen-Douglas operators and set $T=A\oplus B^*$. Then T is complex symmetric if and only if $B^*\cong A^t$.*

Proof. "$\Longleftarrow$". If $B^*\cong A^t$, then

$$T=A\oplus B^*\cong A\oplus A^t.$$

By definition, there is a conjugation C on $\mathscr{H}$ such that $A^t=CA^*C$. Set

$$D=\begin{bmatrix}0 & C\\ C & 0\end{bmatrix}.$$

Then D is a conjugation on $\mathscr{H}\oplus\mathscr{H}$ and one can verify that

$$D\begin{bmatrix} A & 0 \\ 0 & CA^*C \end{bmatrix} D = \begin{bmatrix} A^* & 0 \\ 0 & CAC \end{bmatrix}.$$

This proves the sufficiency.

"$\Longrightarrow$". For convenience, we write

$$T = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{matrix},$$

where $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$.

Since T is complex symmetric, there is a conjugation C on $\mathscr{H}\oplus\mathscr{H}$ such that $CTC = T^*$. Then we have $C(T-\lambda) = (T-\lambda)^*C$ for $\lambda\in\mathbb{C}$. It follows that $C(\ker(T-\lambda)) = \ker(T-\lambda)^*$. As a result, we obtain

$$C\left(\bigvee_{\lambda\in\mathbb{C}} \ker(T-\lambda)\right) = \bigvee_{\lambda\in\mathbb{C}} \ker(T-\lambda)^*.$$

Since A is a Cowen-Douglas operator and $\sigma_p(B^*) = \emptyset$, we have

$$\bigvee_{\lambda\in\mathbb{C}} \ker(T-\lambda) = \bigvee_{\lambda\in\mathbb{C}} \ker(A-\lambda) = \mathscr{H}_1.$$

Likewise, one can check that

$$\bigvee_{\lambda\in\mathbb{C}} \ker(T-\lambda)^* = \bigvee_{\lambda\in\mathbb{C}} \ker(B-\overline{\lambda}) = \mathscr{H}_2.$$

So $C(\mathscr{H}_1) = \mathscr{H}_2$, $C(\mathscr{H}_2) = \mathscr{H}_1$ and C can be written as

$$C = \begin{bmatrix} 0 & C_2 \\ C_1 & 0 \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{matrix}.$$

Since $C^{-1} = C$, we have $C_1^{-1} = C_2$. It follows from $CT^* = TC$ that $C_1A^* = B^*C_1$, that is, $C_1A^*C_2 = B^*$.

Choose a conjugation D on $\mathscr{H}$. Compute to see

$$B^* = C_1A^*C_2 = (C_1D)(DA^*D)(DC_2).$$

Noting that $U := DC_2$ is unitary and $U^{-1} = C_1D$, we have $B^* \cong DA^*D$. This completes the proof. □

REMARK 3.7. Let $A,B\in\mathscr{B}(\mathscr{H})$ be Cowen-Douglas operators and set $T = A\oplus B^*$. By the discussion before Theorem 3.6, $A = A_r$ and $B^* = (B^*)_l$. However, it is possible that T_r, T_l are absent. Set $\Omega_1 = \{z\in\mathbb{C} : |z+1| < 1\}$ and $\Omega_2 = \{z\in\mathbb{C} : |z-1| < 1\}$. By [19, Thm. 1.2], we can find Cowen-Douglas operators $A,B\in\mathscr{B}(\mathscr{H})$ satisfying

(i) $A \in B_1(\Omega_1)$, $\sigma(A) = \overline{\Omega_1 \cup \Omega_2}$ and $\sigma_{lre}(A) = \partial\Omega_1 \cup \overline{\Omega_2}$;

(ii) $B \in B_1(\Omega_2)$, $\sigma(B) = \overline{\Omega_1 \cup \Omega_2}$ and $\sigma_{lre}(B) = \partial\Omega_2 \cup \overline{\Omega_1}$.

If $T = A \oplus B^*$, then one can verify that $\sigma(T) = \sigma_{lre}(T) = \overline{\Omega_1 \cup \Omega_2}$. So $\rho_{s-F}(T)$ coincides with the resolvent of T. Hence T_r and T_l are both absent. Thus one can not use Theorem 3.3 to prove Theorem 3.6.

COROLLARY 3.8. *Let $A \in \mathscr{B}(\mathscr{H})$ be a Cowen-Douglas operator and set $T = A \oplus A^*$. Then T is complex symmetric if and only if $A^* \cong A^t$.*

EXAMPLE 3.9. Let $A, B \in \mathscr{B}(\mathscr{H})$ be forward unilateral weighted shifts defined by

$$Ae_i = a_i e_{i+1}, \quad Be_i = b_i e_{i+1}, \quad \forall i \geqslant 1,$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$. Then A^*, B^* are triangular operators. Moreover, we assume that $\inf_i |a_i| > 0$ and $\inf_i |b_i| > 0$. Then one can check that A^*, B^* are Cowen-Douglas operators of index one ([4]).

Set $T = A^* \oplus B$. Then, by Theorem 3.6, T is complex symmetric if and only if $(A^*)^t \cong B$, that is, $A^* \cong B^t$.

Note that A is unitarily equivalent to a forward unilateral weighted shift with positive weights $\{|a_i|\}$ and B is unitarily equivalent to a forward unilateral weighted shift with positive weights $\{|b_i|\}$. We may directly assume that $a_i > 0, b_i > 0$ for all $i \geqslant 1$.

For $x \in \mathscr{H}$ with $x = \sum_{i=1}^{\infty} \alpha_i e_i$, we define $Cx = \sum_{i=1}^{\infty} \overline{\alpha_i} e_i$. One can verify that C is a conjugation on $\mathscr{H}$. Moreover,

$$CBCe_i = CBe_i = C(b_i e_{i+1}) = b_i e_{i+1} = Be_i, \quad \forall i \geqslant 1.$$

So $B = CBC$, that is, $B^* = CB^*C$. This shows that B^* is a transpose of B. Then $A^* \cong B^t$ if and only if $A \cong B$. By [18, Prob. 89], $A \cong B$ if and only if $a_i = b_i$ for all i. Then T is complex symmetric if and only if $a_i = b_i$ for all i.

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