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SPECTRALLY TWO–UNIFORM FRAMES FOR ERASURES

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Abstract. We continue to work on the problem of characterizing erasure-optimal frames when spectral radius is used as a measurement of the error operator. Spectrally optimal (N,n)-frames for one erasures are the ones that the minimal spectral error n/N can be achieved. This class of frames was completely characterized in [28] in terms of the connectivity property and the redundancy distributions of the involved frames. We show that the best spectral error for the two erasures is always greater than or equal to $\frac{n}{N}+(\frac{Nn-n^2}{N^2(N-1)})^{1/2}$. We characterize all the frames such that the above lower bound can be achieved. Different characterizations are also obtained for the case that when $N=n+1$ or $n+2$. We show that in these special cases, spectrally 2-erasure optimal frames are related to the n-independence property of frames.

1. Introduction

In recent years frame theory has become an active research area because of the redundancy features of frames which are desirable in many applications. For instance, this feature of frames is particularly useful in applications of recovering signals with erasure corrupted data. In a signal transmission process, a signal is encoded with frame coefficients of the signal vector, and then decoded with a dual frame. During the transmission process of the encoded data, some coefficients may get erased. In this case, a full recovering (or even a good approximation) of the original signal is almost impossible if the encoding frame is a basis (i.e., linearly independent frame for the signal space). However, if (carefully chosen) redundant frames are used for encoding and decoding, then we can reduce the approximation errors dramatically and in many cases a perfect reconstruction is possible.

When dealing with signal reconstruction from erasure-corrupted frame coefficients, one of the mostly studied approach is to find optimal frames (c.f. [2, 3, 4, 5, 8, 10, 18, 19, 20]) that minimize the maximal erasure errors occured at all the possible locations. The other method is to find optimal dual frames for a fixed frame that has been selected for encoding (c.f. [22, 23, 24, 25, 26, 27, 28]). Most of the research related to the first approach is based on finding optimal Parseval (or tight) frames. It is known that uniform Parseval frames and equiangular frames are one-erasure and two-erasure optimal, respectively, among all the Parseval frames ([19]). Since these are the frames with nice

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geometric structures, they may not be suitable in some particular applications. This leads to the investigation of optimal dual frames for decoding when a frame is given for encoding ([25] etc.) In all these studies, operator (matrix) norm is commonly used as the measurement of the error operators. However, some other measurements maybe more suitable or accurate when different mechanisms are used in the reconstruction (or approximation) of the signal. For example, if we are allowed to do iterations in the approximation process, spectral radius of the error operator provides more accurate error bound estimate ([28]). Spectral radius measurement was suggested first by Holmes and Paulsen ([19]) where optimal Parseval frames were investigated. The spectral radius measurement is the same as the norm measurement if we only use the standard dual frame. However, the outcomes are quite different if we consider using alternate duals (a key point of using redundant frames instead of using Riesz bases). Here we are interested in the following two natural questions:

(i) Given a frame, what can we say about the spectrally optimal dual frames (see definitions in section 2) of the given frame? What is the maximal spectral radius of all the error operators for a spectrally optimal dual pair?

(ii) For a given pair (N,n) of positive integers with $n<N$, characterize the k-erasure spectrally optimal frame of length N for $\mathbb{C}^n$.

In [28] we completely answered these two questions for the one-erasure case. The one-erasure spectrally optimal frames are precisely those frame that admit a dual frame with constant spectral radius for all the (rank-one) error operators. They are completely characterized in terms of the redundance distribution of the frame. With these characterization, one-erasure spectrally optimal frames can be easily constructed. When a frame is not one-erasure spectrally optimal, the minimum of the maximal spectral radius of all the error operators for all possible duals is the largest value of the redundance distribution set. The problems get more complicated and subtle when we deal with higher erasures due to the complexity of spectral radius information for high rank matrices. In this article we are able to characterize a special class of 2-erasure spectral optimal frames (namely, *spectrally 2-uniform frames*). As a consequence we obtain that all such frames must be linearly connected. We also obtain some alternate necessary and/or sufficient conditions in terms of n-independent property of the frame when N is relatively small. Spectrally 2-uniform frames may not exist for some choices of N. So characterizing 2-erasure spectrally optimal frames still remains to be a challenging problem.

2. Preliminaries and the main results

Let H be an n-dimensional (real or complex) Hilbert space. A finite sequence $F=\{f_i\}_{i=1}^N$ in H is called a *frame* for H if there are two constants $0<A\leqslant B$ such that

$$A\|f\|^2 \leqslant \sum_{i=1}^{N} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2$$

holds for every $f\in H$. When $A=B$, F is called a *tight frame*, and if $A=B=1$, it is called *Parseval frame*. A frame F consisting of equal norm vectors is called *uniform* *frame* and if additionally this norm is one, it is called a *unit norm frame*. If every set of k vector in a frame F is linearly independent then we call F as k*-independent*. The linear map $\Theta_F : H \to \mathbb{C}^N$ defined by

$$\Theta_F(f) = \sum_{i=1}^{N} \langle f, f_i \rangle e_i \quad \text{for all f } \in H,$$

is the analysis operator, where $\{e_i\}_{i=1}^N$ is the standard orthonormal basis for $\mathbb{C}^N$ (or $\mathbb{R}^n$).

The frame operator S is defined by

$$Sf = \Theta_F^* \Theta_F f = \sum_i \langle f, f_i \rangle f_i$$

which is a positive invertible operator on H and leads to the reconstruction formula:

$$f = \sum_{i=1}^{N} \langle f, S^{-1} f_i \rangle f_i = \sum_{i=1}^{N} \langle f, S^{-1/2} f_i \rangle S^{-1/2} f_i, \quad \text{for all } f \in H.$$

In this case the frame $\{S^{-1} f_i\}_{i=1}^N$ is called the *canonical or standard dual frame* of F. In addition to the canonical dual frames, when $N > n$ there exist infinitely many frames $G = \{g_i\}_{i=1}^N$ that also give us a reconstruction formula

$$f = \sum_{i=1}^{N} \langle f, g_i \rangle f_i = \sum_{i=1}^{N} \langle f, f_i \rangle g_i \quad (\text{for all } f \in H), \quad \text{i.e.}\, \Theta_G^* \Theta_F = I.$$

Here the frame $G = \{g_i\}_{i=1}^N$ is called an *alternate dual frame* or *dual frame* for F. It is known that $G = \{g_i\}_{i=1}^N$ is a dual frame for F if and only if there exists a sequence $U = \{u_i\}_{i=1}^N$ such that $\sum_{i=1}^N \langle f, f_i \rangle u_i = 0$ for all $f \in H$ (i.e. $\Theta_U^* \Theta_F = 0$) and $\{g_i\}_{i=1}^N = \{S^{-1} f_i + u_i\}_{i=1}^N$. Such a sequence U is called *orthogonal or strongly disjoint* with F (c.f. [15, 16]).

In this paper we always assume that a frame F consists of only nonzero vectors and $N > n$. When k-erasures occur during the data transmission, we define the error operator E_Λ by

$$E_\Lambda f = \Theta_G^* D \Theta_F f = \sum_{i \in \Lambda} \langle f, f_i \rangle g_i,$$

where Λ is the set of indices corresponding to the erased coefficients, D is an $N \times N$ diagonal matrix with $d_{ii} = 1$ for $i \in \Lambda$ and zero otherwise. Using the received data we get an estimated reconstruction $\tilde{f} = \sum_{i \notin \Lambda} \langle f, f_i \rangle g_i = f - E_\Lambda f$. Using spectral radius as a measurement, the maximum error when k-erasures occur is defined by

$$r_{F,G}^{(k)} = \max\{r(E_\Lambda) : |\Lambda| = k\}$$

and minimal maximal error is defined by

$$r_F^{(k)} = \min\{r_{F,G}^{(k)} : G \text{ is a dual frame of } F\},$$

where $|\Lambda|$ denotes the cardinality of Λ and $r(E_\Lambda)$ is the spectral radius of E_Λ. A dual frame G of F is called 1-*erasure spectrally optimal* if $r_{F,G}^{(1)} = r_F^{(1)}$. We say that G is k-*erasure spectrally optimal* if it is $(k-1)$-erasure spectrally optimal and $r_{F,G}^{(k)} = r_F^{(k)}$.

It is true that $r_{F,G}^{(1)} \geqslant \frac{n}{N}$ for any dual frame pair (F,G), and consequently we have $r_F^{(1)} \geqslant \frac{n}{N}$. This lower bound can be always achieved if F is a uniform-length Parseval frame. All the frames F with $r_F^{(1)} = \frac{n}{N}$ (we say that such a frame is *spectrally one-uniform*) were characterized in [28] in terms of the redundancy distribution of F.

We say that two vectors f_i and f_j in a sequence F of vectors are *linearly F-connected (or simply, connected)* if there exist vectors $\{f_{k_1},\dots,f_{k_\ell}\}$ from F such that $\{f_j, f_{k_1},\dots,f_{k_\ell}\}$ are linearly independent and $f_i = cf_j + \sum_{m=1}^{\ell} c_m f_{k_m}$ with c, c_m all nonzero.

DEFINITION 2.1. Let $F = \{f_i\}_{i=1}^N$ be a sequence of nonzero vectors in H. We say that F

(i) is *linearly connected* if every two vectors in F are F-connected.

(ii) has the *intersection dependent property* if $H_\Lambda \cap H_{\Lambda^c} \neq \{0\}$ holds for every proper subset Λ of $\{1,\dots,N\}$, where H_Λ is the subspace spanned by $\{f_i : i \in \Lambda\}$.

(ii) is *k-independent* if every k vectors in F are linearly independent.

In [28], we proved that (i) and (ii) are equivalent. This led to the following:

PROPOSITION 2.1. *Let $F = \{f_i\}_{i=1}^N$ be a frame for H. Then there exists a (unique up to permutations) partition $\{\Lambda_j\}_{j=1}^J$ of $\{1,2,\dots,N\}$ such that each $\{f_i\}_{i\in\Lambda_j}$ is linearly connected, and H is the direct sum of the subspaces $H_j = span\{f_i : i \in \Lambda_j\}$.*

Let H_j, Λ_j be as in the above proposition. Then the *redundancy distribution* of F is defined to be $\left\{\frac{\dim H_j}{|\Lambda_j|}\right\}_{1\leqslant j\leqslant J}$. We say that F has the *uniform redundancy distribution* if $\frac{\dim H_j}{|\Lambda_j|}$ is a constant for all j.

It was proved in [28] that a frame F is spectrally one-uniform if and only if it has the uniform redundancy distribution, which is also equivalent to the condition that there exists a dual G such that $\langle f_i, g_i\rangle = n/N$ for all $i = 1,\dots,N$. Moreover, for any general frame F we have $r_F^{(1)} = \max\{\alpha_j\}_{1\leqslant j\leqslant J}$, where $\{\alpha_j\}_{1\leqslant j\leqslant J}$ is the redundancy distribution of F. For a spectrally one-uniform frame F we will show that $r_F^{(2)} \geqslant \frac{n}{N} + \sqrt{\frac{Nn-n^2}{N^2(N-1)}}$. Here we are interested in characterizing those frames such that this lower (best) error bound can be achieved. So we propose the following definition:

DEFINITION 2.2. Let F be an (N,n) frame. We say that F is *spectrally two-uniform* if there exists a dual frame G of F such that $r_{F,G}^{(1)} = n/N$ and $r_{F,G}^{(2)} = \frac{n}{N} + \sqrt{\frac{Nn-n^2}{N^2(N-1)}}$.

The following tells us that spectrally 2-uniform frames are ones that there exists a dual frame with the property that all the 2×2 error operators have the same spectral radius. So this also justifies the use of the terminology of 2-uniformity.

THEOREM 2.2. Let $F=\{f_i\}_{i=1}^N$ be a spectrally one-uniform frame for an n-dimensional Hilbert space H. Then the following are equivalent:

(i) F is spectrally 2-uniform;

(ii) There exists a one-erasure spectrally optimal dual G of F such that $\langle g_j,f_i\rangle\langle g_i,f_j\rangle$ is constant for all $i\neq j$;

(iii) There exists a one-erasure spectrally optimal dual G of F such that $r(E_\Lambda)=\frac{n}{N}+\sqrt{\frac{Nn-n^2}{N^2(N-1)}}$ for all subset Λ of $\{1,\dots,N\}$ with $|\Lambda|=2$.

As an application, we get the following necessary condition for spectrally 2-uniform frames. The condition is not sufficient in general (see Example 3.1).

COROLLARY 2.3. If a frame F is spectrally 2-uniform, then it is linearly connected.

More can be said in the case that when N is relatively small (comparing to the dimension n). For example, the following theorem gives the characterization of spectrally two-uniform frames when $N=n+1$ and a stronger necessary condition on such frames for $N=n+2$.

THEOREM 2.4. Let $F=\{f_i\}_{i=1}^N$ be a spectrally one-uniform frame for an n-dimensional H.

(i) If $N=n+1$, then F is a spectrally two-uniform frame if and only if F is n-independent.

(ii) If $N=n+2$ and F is spectrally two-uniform, then F is n-independent.

In order to prove our main results, we need a simple reduction based on the equivalence of frames. Recall that two frames $F=\{f_i\}_{i=1}^N$ and $F'=\{f_i'\}_{i=1}^N$ for H are called *similar* if there exists an invertible operator T such that $Tf_i=f_i'$ for all i.

DEFINITION 2.3. Two frames $F=\{f_i\}_{i=1}^N$ and $F'=\{f_i'\}_{i=1}^N$ are called *equivalent* if one of them can be obtained from the other through either the similarity and/or permutation (i.e., there exists a permutation $\sigma:\{1,\dots,N\}\to\{1,\dots,N\}$ such that $f_i'=f_{\sigma(i)}$).

PROPOSITION 2.5. Let F be a frame and k be a positive integer. Then $r_F^{(k)}$ is equivalent invariant.

Proof. The permutation clearly preserves $r_F^{(k)}$. Let $F=\{f_i\}_{i=1}^N$ and $F'=\{f_i'\}_{i=1}^N$ be similar frames for H and $G=\{g_i\}_{i=1}^N$ be a dual frame of F. Let T be the invertible operator such that $f_i'=Tf_i$. Let $G'=\{g_i'\}_{i=1}^N$ with $g_i'=(T^{-1})^*g_i$. Then G' is a dual frame of F'. Clearly, $\langle f_i',g_j'\rangle=\langle f_i,g_j\rangle$ for all $i.j$. Thus we get $r_F^{(k)}=r_{F'}^{(k)}$. $\square$

3. Proofs of the main results

Let F be a spectrally one-uniform frame and G be a dual frame of F such that $\langle g_i, f_i\rangle = n/N$ for all i. Consider the error operator E_Λ for $\Lambda = \{i, j\}$. Then the spectral radius of error operator E_Λ is

$$r(E_\Lambda) = r(\Theta_G^* D_\Lambda \Theta_F) = r(\Theta_G^* D_\Lambda^* D_\Lambda \Theta_F) = r(D_\Lambda \Theta_F \Theta_G^* D_\Lambda^*).$$

where Θ_F and Θ_G are analysis operators of F and G, respectively, and D_Λ is an N by N diagonal matrix with $d_{ii} = 1$ for $i \in \Lambda$ and zero otherwis. Note that

$$D_\Lambda \Theta_F \Theta_G^* D_\Lambda^* = \begin{pmatrix} \langle g_i, f_i\rangle & \langle g_j, f_i\rangle \\ \langle g_i, f_j\rangle & \langle g_j, f_j\rangle \end{pmatrix} = \begin{pmatrix} \frac{n}{N} & \langle g_j, f_i\rangle \\ \langle g_i, f_j\rangle & \frac{n}{N} \end{pmatrix},$$

for $i \neq j$ and $i, j \in \{1, \dots, n\}$. For the spectral radius of E_Λ, we consider the characteristic polynomial

$$\left(\frac{n}{N} - \lambda\right)^2 - \langle g_j, f_i\rangle\langle g_i, f_j\rangle.$$

So the eigenvalues are given by

$$\lambda = \frac{n}{N} \pm \sqrt{\langle g_j, f_i\rangle\langle g_i, f_j\rangle}. \tag{3.1}$$

LEMMA 3.1. Assume that F is one-uniform frame. Then:

(i) $r_F^{(2)} \geqslant \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$.

(ii) $r_F^{(2)} = \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$ if and only if there exists a one-erasure spectrally optimal dual G of F such that $\langle g_j, f_i\rangle\langle g_i, f_j\rangle = c$ for all $i \neq j$ where c is a constant.

Proof. (i) Let G be a one-erasure spectrally optimal dual G of F. Then we have $\langle g_i, f_i\rangle = \frac{n}{N}$ for all i since F is spectrally one-uniform. Note that since $\Theta_F\Theta_G^* = \Theta_F\Theta_G^*\Theta_F\Theta_G^* = (\Theta_F\Theta_G^*)^2$ by $\Theta_G^*\Theta_F = I$, we have

$$n = \sum_{i=1}^{N} \langle g_i, f_i\rangle = tr(\Theta_F\Theta_G^*) = tr\big((\Theta_F\Theta_G^*)^2\big) = \sum_{i,j=1}^{N} \langle g_i, f_j\rangle\langle g_j, f_i\rangle. \tag{3.2}$$

Note also that since

$$\sum_{i,j=1}^{N} \langle g_i, f_j\rangle\langle g_j, f_i\rangle = \sum_{i\neq j} \langle g_i, f_j\rangle\langle g_j, f_i\rangle + \sum_{i=1}^{N} |\langle g_i, f_i\rangle|^2 = \sum_{i\neq j} \langle g_i, f_j\rangle\langle g_j, f_i\rangle + \frac{n^2}{N},$$

by (3.2), we have

$$\sum_{i\neq j} \langle g_i, f_j\rangle\langle g_j, f_i\rangle = n - \frac{n^2}{N} = \frac{nN - n^2}{N}. \tag{3.3}$$

Now set $c_{ij} = \langle g_i, f_j\rangle\langle g_j, f_i\rangle$ and $c = \frac{nN-n^2}{N^2(N-1)}$. Then, by (3.3), $\sum_{i\neq j} Re(c_{ij}) = \frac{nN-n^2}{N}$. Let $\alpha = \{(i,j) : Re(c_{ij}) \in \mathbb{R}^+\}$. Then $\sum_{(i,j)\in\alpha} Re(c_{ij}) \geqslant \frac{nN-n^2}{N}$, which implies that there exist $(i_0, j_0) \in \alpha$ such that $Re(c_{i_0j_0}) \geqslant c > 0$. Write $c_{i_0j_0} = |c_{i_0j_0}|e^{i\theta} = |c_{i_0j_0}|\cos\theta + i|c_{i_0j_0}|\sin\theta$. Then, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ because $Re(c_{i_0j_0}) = |c_{i_0j_0}|\cos\theta \in \mathbb{R}^+$. Thus,

$$\begin{aligned}\max\left\{\left|\frac{n}{N} + (c_{i_0j_0})^{1/2}\right|, \left|\frac{n}{N} - (c_{i_0j_0})^{1/2}\right|\right\} &\geqslant \frac{n}{N} + Re(c_{i_0j_0})^{1/2} = \frac{n}{N} + |c_{i_0j_0}|^{1/2}\cos\frac{\theta}{2} \\ &\geqslant \frac{n}{N} + |c_{i_0j_0}|^{1/2}(\cos\theta)^{1/2} \\ &= \frac{n}{N} + \big(Re(c_{i_0j_0})\big)^{1/2} \geqslant \frac{n}{N} + c^{1/2}.\end{aligned}$$

The inequalities follow from the fact that $\cos\frac{\theta}{2} \geqslant (\cos\theta)^{1/2}$ and $Re(c_{i_0j_0})^{1/2} > 0$ since $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \geqslant \frac{\cos\theta+\cos\theta}{2}$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Therefore,

$$r_F^{(2)} = \min_G\{r_{F,G}^{(2)}\} = \min_G \max_\Lambda\{r(E_\Lambda) : |\Lambda| = 2\} \geqslant \frac{n}{N} + \sqrt{c} = \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}.$$

(ii) Let G be a dual frame of F with $\langle g_i, f_i\rangle = \frac{n}{N}$ for all i. Assume that $c_{ij} = \langle g_i, f_j\rangle\langle g_j, f_i\rangle = c$ for all $i \neq j$, where c is constant. Then by (3.3), we have $c = \frac{nN-n^2}{N^2(N-1)}$. Then, by (3.1),

$$r_{F,G}^{(2)} = \max_\Lambda\{r(E_\Lambda) : |\Lambda| = 2\} = \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}.$$

Thus, since we have $r_F^{(2)} \geqslant \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$ in part (i), we conclude $r_F^{(2)} = \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$.

For the other direction assume that $r_F^{(2)} = \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$. Let G be a dual such that $r_F^{(2)} = r_{F,G}^{(2)}$, and $\langle g_i, f_i\rangle = \frac{n}{N}$ for all i. We claim that for all $i \neq j$, $Re(c_{i,j}) = \frac{nN-n^2}{N^2(N-1)}$ and $Im(c_{i,j}) = 0$. To prove the first claim, we assume that $Re(c_{i,j}) \neq \frac{nN-n^2}{N^2(N-1)}$ for some $i \neq j$. Then there exist (i_0, j_0) such that $Re(c_{i_0j_0}) > \frac{nN-n^2}{N^2(N-1)}$ since $\sum_{i\neq j} Re(c_{ij}) = \frac{nN-n^2}{N}$ by (3.3). Using the same argument as in the proof of part (i), we obtain

$$\max\left\{\left|\frac{n}{N} + (c_{j_0i_0})^{1/2}\right|, \left|\frac{n}{N} - (c_{j_0i_0})^{1/2}\right|\right\} > \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}.$$

Thus, $r_F^{(2)} = r_{F,G}^{(2)} > \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$. This contradicts to the assumption that $r_F^{(2)} = \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$. Therefore, $Re(c_{ij})$ is constant for all $i \neq j$.

To prove the last claim, we assume that there exist (i_0, j_0) such that $Im(c_{i_0,j_0}) \neq 0$. Note that if we write $c_{i_0,j_0} = |c_{i_0,j_0}|e^{i\theta}$, then $\theta \neq 0$ and $\theta \neq \pi$. Then, we have

$$
\begin{aligned}
&\max\left\{\left|\frac{n}{N}+(c_{i_0j_0})^{1/2}\right|,\left|\frac{n}{N}-(c_{i_0j_0})^{1/2}\right|\right\}\\
&=\left(\left(\frac{n}{N}+Re(c_{i_0j_0})^{1/2}\right)^2+\left(Im(c_{i_0j_0})^{1/2}\right)^2\right)^{1/2}\\
&=\left(\left(\frac{n}{N}+|c_{i_0j_0}|^{1/2}\cos\frac{\theta}{2}\right)^2+\left(|c_{i_0j_0}|^{1/2}\sin\frac{\theta}{2}\right)^2\right)^{1/2}\\
&\geqslant\left(\left(\frac{n}{N}+|c_{i_0j_0}|^{1/2}(\cos\theta)^{1/2}\right)^2+\left(|c_{i_0j_0}|^{1/2}\sin\frac{\theta}{2}\right)^2\right)^{1/2}\\
&=\left(\left(\frac{n}{N}+\sqrt{\frac{nN-n^2}{N^2(N-1)}}\right)^2+\left(|c_{i_0j_0}|^{1/2}\sin\frac{\theta}{2}\right)^2\right)^{1/2}\\
&>\frac{n}{N}+\sqrt{\frac{nN-n^2}{N^2(N-1)}}
\end{aligned}
$$

The inequalities, again, follow from the fact that $\cos\frac{\theta}{2} \geqslant (\cos\theta)^{1/2}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $Re(c_{i_0j_0}) = \frac{nN-n^2}{N^2(N-1)}$. Thus, $r_F^{(2)} = r_{F,G}^{(2)} > \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$ which contradicts to the assumption that $r_F^{(2)} = \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$. Therefore, $Im(c_{ij}) = 0$ for all $i \neq j$. Thus, $c_{ij} = \frac{nN-n^2}{N^2(N-1)}$ for all $i \neq j$. $\square$

Proof of Theorem 2.2. Clearly, (iii) implies (ii). The equivalence of (i) and (ii) has been established by Lemma 3.1. For $(ii) \Rightarrow (iii)$, assume that $\langle g_j, f_i\rangle\langle g_i, f_j\rangle = c$ is constant for all $i \neq j$. Then, by the proof of Lemma 3.1 (i), we have that $c = \frac{nN-n^2}{N^2(N-1)}$. So $r(E_\Lambda)$ is the largset module of the solutions of $(\lambda - \frac{n}{N})^2 = \langle g_j, f_i\rangle\langle g_i, f_j\rangle = \frac{nN-n^2}{N^2(N-1)}$. Hence $r(E_\Lambda) = \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}$ for all subsets Λ of $\{1,\dots,N\}$ with $|\Lambda| = 2$. $\square$

Proof of Corollary 2.3. Assume, to the contrary, that F is not linearly connected. Then, by 2.1, there exists a partition $\{I_j\}_{j=1}^J$ $(J > 1)$ of $\{1,2,\dots,N\}$ such that each $\{f_i\}_{i\in\Lambda_j}$ is linearly connected, and H is the direct sum of the subspaces $H_j = span\{f_i : i \in \Lambda_j\}$. Since F is spectrally 2-uniform, from Theorem 2.2 there exists a dual frame $G = \{g_i\}_{i=1}^N$ for F such that $\langle f_i, g_i\rangle = \frac{n}{N}$ for all i, and $\langle f_i, g_j\rangle \cdot \langle f_j, g_i\rangle = c = \frac{nN-n^2}{N^2(N-1)}$.

From

$$f_1 = \sum_{i\in I_1}\langle f_1, g_i\rangle f_i + \sum_{i\notin I_1}\langle f_1, g_i\rangle f_i,$$

and $H = \sum_{j=1}^{J} \oplus H_j$, we get that $f_1 = \sum_{i \in I_1} \langle f_1, g_i \rangle f_i$. Thus we obtain that

$$\frac{n}{N} = \sum_{i \in I_1} \langle f_1, g_i \rangle \langle f_i, g_1 \rangle = \frac{n^2}{N^2} + c(N_1 - 1),$$

where N_1 is the cardinality of I_1. This implies that $c = \frac{nN-n^2}{N^2(N_1-1)} > \frac{nN-n^2}{N^2(N-1)}$ since $N_1 < N$. Therefore F must be linearly conneced. $\square$

Proof of Theorem 2.4. (i) First assume that F is an n-independent frame and we will show that F is a spectrally two-uniform frame. By Proposition 2.5, we can assume that $F = \{f_1, \dots, f_n, f_{n+1}\} = \{e_1, \dots, e_n, \sum_{i=1}^{n} a_i e_i\}$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis for H and $a_i \neq 0$ for $i = 1, \dots, n$. Because F is spectrally one-uniform frame, it has a dual frame G such that $\langle g_i, f_i \rangle = \frac{n}{n+1}$ for $i = 1, \dots, n+1$. Then, by the reconstruction formula for frames, we have

$$e_j = \sum_{i=1}^{n+1} \langle e_j, f_i \rangle g_i = \langle e_j, e_j \rangle g_j + \langle e_j, a_j e_j \rangle g_{n+1} = g_j + a_j g_{n+1}, \qquad \text{for } j = 1, \dots, n. \tag{3.4}$$

Then we get

$$\langle e_j, e_j \rangle = \langle g_j + a_j g_{n+1}, e_j \rangle = \langle g_j, e_j \rangle + a_j \langle g_{n+1}, e_j \rangle = 1 \qquad \text{for } j = 1, \dots, n, \tag{3.5}$$
$$\langle e_j, e_i \rangle = \langle g_j + a_j g_{n+1}, e_i \rangle = \langle g_j, e_i \rangle + a_j \langle g_{n+1}, e_i \rangle = 0 \qquad \text{for } i \neq j. \tag{3.6}$$

Therefore, for $k \neq \ell$ $(k, \ell = 1, \dots, n)$,

$$\begin{aligned}
\langle g_\ell, f_k \rangle \langle g_k, f_\ell \rangle &= \langle g_\ell, e_k \rangle \langle g_k, e_\ell \rangle \\
&= -a_\ell \langle g_{n+1}, e_k \rangle \cdot -a_k \langle g_{n+1}, e_\ell \rangle \qquad \text{by (3.6)} \\
&= a_\ell \frac{1 - \langle g_k, e_k \rangle}{a_k} \cdot a_k \frac{1 - \langle g_\ell, e_\ell \rangle}{a_\ell} \qquad \text{by (3.5)} \\
&= \left(1 - \frac{n}{n+1}\right)\left(1 - \frac{n}{n+1}\right) = \frac{1}{(n+1)^2}.
\end{aligned} \tag{3.7}$$

Furthermore, for $k = 1, \dots, n$, we have

$$\begin{aligned}
\langle g_{n+1}, f_k \rangle \langle g_k, f_{n+1} \rangle &= \langle g_{n+1}, e_k \rangle \langle g_k, \sum_{i=1}^{n} a_i e_i \rangle \\
&= \langle g_{n+1}, e_k \rangle \left(\sum_{i=1}^{n} a_i \langle g_k, e_i \rangle \right) \\
&= \frac{1 - \langle g_k, e_k \rangle}{a_k} \left(\sum_{i \neq k} a_i \langle g_k, e_i \rangle + a_k \langle g_k, e_k \rangle \right) \qquad \text{by (3.5)} \\
&= \frac{1 - \langle g_k, e_k \rangle}{a_k} \left(\sum_{i \neq k} -a_k a_i \langle g_{n+1}, e_i \rangle + a_k (1 - a_k \langle g_{n+1}, e_k \rangle) \right) \\
&\qquad \text{by (3.5) and (3.6)}
\end{aligned}$$

$$= \frac{1-\langle g_k, e_k\rangle}{a_k} a_k \left(1-\sum_{i=1}^{n} a_i \langle g_{n+1}, e_i\rangle\right)$$
$$\frac{1}{a_k(n+1)} a_k \left(1-\frac{n}{n+1}\right) = \frac{1}{(n+1)^2}. \tag{3.8}$$

The last equality follows from the fact that $\sum_{i=1}^{n} a_i \langle g_{n+1}, e_i\rangle = \langle g_{n+1}, f_{n+1}\rangle = n/(n+1)$. Thus, $\langle g_i, f_j\rangle\langle g_j, f_i\rangle = \frac{1}{(n+1)^2} = \frac{nN-n^2}{N^2(N-1)}$ for all $i \neq j$ by (3.7) and (3.8). Hence, by Theorem 2.2, we conclude that F is spectrally two-uniform.

For the other direction of the proof, assume that F is not n-independent. Again, by Proposition 2.5, we can assume that $F = \{f_1, \ldots, f_n, f_{n+1}\} = \{e_1, \ldots, e_n, \sum_{i=1}^{s} a_i e_i\}$ for some s with the property $s < n$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis for H and $a_i \neq 0$ for $i = 1, \ldots, s$. Let G be a dual frame of F such that $\langle g_i, f_i\rangle = \frac{n}{n+1}$. Then again by the reconstruction formula for frames, since $s < n$, we have

$$e_j = \sum_{i=1}^{n+1} \langle e_j, f_i\rangle g_i = \langle e_j, e_j\rangle g_j = g_j \qquad \text{for } j = s+1, \ldots, n.$$

So, $\langle g_j, e_i\rangle = \langle e_j, e_i\rangle = 0$ for $j = s+1, \ldots, n$, $i \neq j$. This implies that $\langle g_n, e_j\rangle = 0$ for $j \neq n$. Hence, since $s < n$

$$\langle g_n, f_{n+1}\rangle = \sum_{i=1}^{s} a_i \langle g_n, e_i\rangle = 0.$$

Thus, $\langle g_{n+1}, f_n\rangle\langle g_n, f_{n+1}\rangle = 0$. Therefore, by Theorem 2.2, F is not spectrally two-uniform frame.

(ii) Suppose that F is not n-independent. Prom Proposition (2.5), we can assume that $F = \{f_1, \ldots, f_n, f_{n+1}, f_{n+2}\} = \{e_1, \ldots, e_n, \sum_{i=1}^{s} a_i e_i, \sum_{i=1}^{n} b_i e_i\}$ for $s < n$ and $a_i \neq 0$ for $i = 1, \ldots, s$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis for H. Since F is spectrally one-uniform frame, then there is a dual frame G of F such that $\langle g_i, f_i\rangle = \frac{n}{n+2}$ for $i = 1, \ldots, n+2$. From the frame reconstruction formula, for $i = s+1, \ldots, n$ we have,

$$e_i = \sum_{j=1}^{n+2} \langle e_i, f_j\rangle g_j = \langle e_i, e_i\rangle g_i + \langle e_i, b_i e_i\rangle g_{n+2} = g_i + b_i g_{n+2}.$$

Moreover, we have

$$\langle e_i, e_i\rangle = \langle g_i + b_i g_{n+2}, e_i\rangle = \langle g_i, e_i\rangle + b_i\langle g_{n+2}, e_i\rangle = 1 \qquad \text{for } i = s+1, \ldots, n. \tag{3.9}$$
$$\langle e_i, e_j\rangle = \langle g_i + b_i g_{n+2}, e_j\rangle = \langle g_i, e_j\rangle + b_i\langle g_{n+2}, e_j\rangle = 0 \qquad \text{for } i = s+1, \ldots, n,\ j \neq i. \tag{3.10}$$

Note here that $b_i \neq 0$ for $i = s+1, \ldots, n$ (Indeed, if $b_i = 0$ then $\langle g_i, e_i\rangle = 1$ which contradicts to the assumption that $\langle g_i, e_i\rangle = \langle g_i, f_i\rangle = \frac{n}{n+2}$ for $i = 1, \ldots, n$). Note also that

$$\langle g_{n+2}, f_{n+2}\rangle = \sum_{i=1}^{n} b_i \langle g_{n+2}, e_i\rangle = \frac{n}{n+2}. \tag{3.11}$$

Thus, by (3.9), we have

$$\langle g_{n+2}, f_{s+1}\rangle = \langle g_{n+2}, e_{s+1}\rangle = \frac{1-n/(n+2)}{b_{s+1}} = \frac{2}{b_{s+1}(n+2)}. \tag{3.12}$$

Since $\sum_{i\neq s+1} b_i\langle g_{n+2}, e_i\rangle = n/(n+2) - b_{s+1}\langle g_{n+2}, e_{s+1}\rangle$ by (3.11), we also have

$$\begin{aligned}
\langle g_{s+1}, f_{n+2},\rangle &= \sum_{i=1}^{n} b_i\langle g_{s+1}, e_i\rangle = \sum_{i\neq s+1} b_i\langle g_{s+1}, e_i\rangle + b_{s+1}\langle g_{s+1}, e_{s+1}\rangle \\
&= \sum_{i\neq s+1} b_i\langle g_{s+1}, e_i\rangle + b_{s+1}\frac{n}{n+2} \\
&= \sum_{i\neq s+1} b_i(-b_{s+1}\langle g_{n+2}, e_i\rangle) + b_{s+1}\frac{n}{n+2} \qquad \text{by (3.10)} \\
&= b_{s+1}\Big(n/(n+2) - \sum_{i\neq s+1} b_i\langle g_{n+2}, e_i\rangle\Big) \\
&= b_{s+1}\Big(n/(n+2) - (n/(n+2) - b_{s+1}\langle g_{n+2}, e_{s+1}\rangle)\Big) \\
&= b_{s+1}\frac{2}{n+2} \qquad \text{by (3.12).}
\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we get

$$\langle g_{n+2}, f_{s+1}\rangle\langle g_{s+1}, f_{n+2}\rangle = \frac{2}{b_{s+1}(n+2)} b_{s+1}\frac{2}{n+2} = \frac{4}{(n+2)^2}.$$

However, if F were a two-uniform frame then we would have $\frac{4}{(n+2)^2} = \frac{n(n+2)-n^2}{(n+2)^2(n+1)}$. But this is impossible since $2n \neq 4n+4$. Hence, F is not a two-uniform. □

From the proof of Theorem 2.4(i), we have the following:

COROLLARY 3.2. If F with $n+1$ vectors is n-independent, then every spectrally one-erasure optimal dual is also spectrally two-erasure optimal.

Finally, we remark that it is well known that there is an upper bound for N in order to have an (N,n)-equiangular uniform length Parseval frame: in the complex case $N \leqslant n^2$ and in the real case $N \leqslant n(n+1)/2$ (c.f. [29]). We conjecture that we may have similar restrictions on N for spectrally two-uniform frames. We provide the following example as a supporting evidence

EXAMPLE 3.1. There is no two-uniform frame with 4 vectors in $\mathbb{R}^2$.

Proof. Let F be a one-uniform frame and by Proposition 2.1 we can assume that $F = \{f_1, f_2, f_3, f_4\} = \{e_1, e_2, a_1e_1 + a_2e_2, b_1e_1 + b_2e_2\}$, where $\{e_1, e_2\}$ is an orthonormal basis for $\mathbb{R}^2$. Note that if one of $a_i = 0$, then F is not two independent. Indeed, if $a_1 = 0$ then $\{f_2, f_3\} = \{e_2, a_2e_2\}$ is not independent. Thus, by Theorem 2.4(ii), F is not two uniform. So, we assume that $a_i, b_i \neq 0$ for $i = 1, 2$. Since F is one-uniform, we let $G = \{g_1, g_2, g_3, g_4\}$ be a dual frame of F such that $\langle g_i, f_i \rangle = \frac{1}{2}$ for $i = 1, 2, 3, 4$. In the frame reconstruction formula letting $f_1 = e_1$ and $f_2 = e_2$, we have

$$e_1 = \sum_{i=1}^{4} \langle e_1, f_i \rangle g_i = g_1 + a_1 g_3 + b_1 g_4 \quad \text{and} \quad e_2 = \sum_{i=1}^{4} \langle e_2, f_i \rangle g_i = g_2 + a_2 g_3 + b_2 g_4.$$

Then we have

$$\langle e_1, e_1 \rangle = \langle g_1 + a_1 g_3 + b_1 g_4, e_1 \rangle = \langle g_1, e_1 \rangle + a_1 \langle g_3, e_1 \rangle + b_1 \langle g_4, e_1 \rangle = 1 \tag{3.14}$$

$$\langle e_2, e_2 \rangle = \langle g_2 + a_2 g_3 + b_2 g_4, e_2 \rangle = \langle g_2, e_2 \rangle + a_2 \langle g_3, e_2 \rangle + b_2 \langle g_4, e_2 \rangle = 1 \tag{3.15}$$

$$\langle e_1, e_2 \rangle = \langle g_1 + a_1 g_3 + b_1 g_4, e_2 \rangle = \langle g_1, e_2 \rangle + a_1 \langle g_3, e_2 \rangle + b_1 \langle g_4, e_2 \rangle = 0 \tag{3.16}$$

$$\langle e_2, e_1 \rangle = \langle g_2 + a_2 g_3 + b_2 g_4, e_1 \rangle = \langle g_2, e_1 \rangle + a_2 \langle g_3, e_1 \rangle + b_2 \langle g_4, e_1 \rangle = 0 \tag{3.17}$$

$$\langle g_3, f_3 \rangle = \langle g_3, a_1 e_1 + a_2 e_2 \rangle = a_1 \langle g_3, e_1 \rangle + a_2 \langle g_3, e_2 \rangle = \frac{1}{2} \tag{3.18}$$

$$\langle g_4, f_4 \rangle = \langle g_4, b_1 e_1 + b_2 e_2 \rangle = b_1 \langle g_4, e_1 \rangle + b_2 \langle g_4, e_2 \rangle = \frac{1}{2}. \tag{3.19}$$

Set $\langle g_4, e_1 \rangle = x$ and $\langle g_4, e_2 \rangle = y$. Then, by (3.19), we have $x = \frac{1}{2b_1} - \frac{b_2}{b_1} y$. So,

$$\langle g_4, e_1 \rangle = \frac{1}{2b_1} - \frac{b_2}{b_1} y \tag{3.20}$$

Moreover, by (3.14) and (3.15), we obtain

$$\langle g_3, e_1 \rangle = \frac{1}{2a_1} - \frac{b_1}{a_1} x = \frac{1}{2a_1} - \frac{b_1}{a_1}\Big(\frac{1}{2b_1} - \frac{b_2}{b_1} y\Big) = \frac{b_2}{a_1} y. \tag{3.21}$$

$$\langle g_3, e_2 \rangle = \frac{1}{2a_2} - \frac{b_2}{a_2} y. \tag{3.22}$$

and by (3.16) and (3.17), we have

$$\langle g_1, e_2 \rangle = -a_1 \langle g_3, e_2 \rangle - b_1 y = -a_1 \Big(\frac{1}{2a_2} - \frac{b_2}{a_2} y\Big) - b_1 y = -\frac{a_1}{2a_2} + y\Big(\frac{a_1 b_2}{a_2} - b_1\Big) \tag{3.23}$$

$$\langle g_2, e_1 \rangle = -b_2 x - a_2 \langle g_3, e_1 \rangle = -b_2 \Big(\frac{1}{2b_1} - \frac{b_2}{b_1} y\Big) - a_2 \frac{b_2}{a_1} y = -\frac{b_2}{2b_1} + y\Big(\frac{b_2^2}{b_1} - \frac{a_2 b_2}{a_1}\Big) \tag{3.24}$$

Then, (3.23) and (3.24) imply that

$$\begin{aligned} \langle g_2, f_1 \rangle \langle g_1, f_2 \rangle =& \langle g_2, e_1 \rangle \langle g_1, e_2 \rangle \\ =& \Bigg(-\frac{b_2}{2b_1} + y\Big(\frac{b_2^2}{b_1} - \frac{a_2 b_2}{a_1}\Big)\Bigg)\Bigg(-\frac{a_1}{2a_2} + y\Big(\frac{a_1 b_2}{a_2} - b_1\Big)\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \frac{-a_1b_2 - 2a_2b_1b_2y + 2a_1b_2^2y}{2a_1b_1}\frac{2a_1b_2y - 2a_2b_1y - a_1}{2a_2} \\
&= \frac{-2a_1^2b_2^2y + 2a_1a_2b_1b_2y + a_1^2b_2 - 4a_1a_2b_1b_2^2y^2 + 4a_2^2b_1^2b_2y^2}{4a_1a_2b_1} \\
&\quad + \frac{2a_1a_2b_1b_2y + 4a_1^2b_2^3y^2 - 4a_1a_2b_1b_2^2y^2 - 2a_1^2b_2^2y}{4a_1a_2b_1} \\
&= \frac{(4a_2^2b_1^2b_2 + 4a_1^2b_2^3 - 8a_1a_2b_1b_2^2)y^2 + (4a_1a_2b_1b_2 - 4a_1^2b_2^2)y + a_1^2b_2}{4a_1a_2b_1}.
\end{aligned} \tag{3.25}
$$

Now we continue computing the rest of the pairs $\langle g_j, f_i\rangle\langle g_i, f_j\rangle$, $i \neq j$ so that we equate all to get a spectrally two-uniform frame F.

By (3.21) and (3.23), we have

$$
\begin{aligned}
\langle g_3, f_1\rangle\langle g_1, f_3\rangle &= \langle g_3, e_1\rangle\langle g_1, a_1e_1 + a_2e_2\rangle \\
&= \langle g_3, e_1\rangle(a_1\langle g_1, e_1\rangle + a_2\langle g_1, e_2\rangle) \\
&= \frac{b_2}{a_1}y\left(\frac{a_1}{2} + a_2\left(-\frac{a_1}{2a_2} + y\left(\frac{a_1b_2}{a_2} - b_1\right)\right)\right) \\
&= \frac{b_2}{a_1}y^2(a_1b_2 - a_2b_1).
\end{aligned} \tag{3.26}
$$

By (3.20) and (3.23), we have

$$
\begin{aligned}
&\langle g_4, f_1\rangle\langle g_1, f_4\rangle \\
&= \langle g_4, e_1\rangle\langle g_1, b_1e_1 + b_2e_2\rangle = \left(\frac{1}{2b_1} - \frac{b_2}{b_1}y\right)(b_1\langle g_1, e_1\rangle + b_2\langle g_1, e_2\rangle) \\
&= \left(\frac{1}{2b_1} - \frac{b_2}{b_1}y\right)\left(\frac{b_1}{2} + b_2\left(-\frac{a_1}{2a_2} + y\left(\frac{a_1b_2}{a_2} - b_1\right)\right)\right) \\
&= \frac{1 - 2b_2y}{2b_1}\frac{a_2b_1 - a_1b_2 + 2a_1b_2^2y - 2a_2b_1b_2y}{2a_2} \\
&= \frac{a_2b_1 - a_1b_2 + 2a_1b_2^2y - 2a_2b_1b_2y - 2a_2b_1b_2y + 2a_1b_2^2y - 4a_1b_2^3y^2 + 4a_2b_1b_2^2y^2}{4a_2b_1} \\
&= \frac{(4a_2b_1b_2^2 - 4a_1b_2^3)y^2 + (4a_1b_2^2 - 4a_2b_1b_2)y + a_2b_1 - a_1b_2}{4a_2b_1}.
\end{aligned} \tag{3.27}
$$

By (3.22) and (3.24), we have

$$
\begin{aligned}
&\langle g_3, f_2\rangle\langle g_2, f_3\rangle \\
&= \langle g_3, e_2\rangle(a_1\langle g_2, e_1\rangle + a_2\langle g_2, e_2\rangle) \\
&= \left(\frac{1}{2a_2} - \frac{b_2}{a_2}y\right)\left(a_1\left(-\frac{b_2}{2b_1} + y\left(\frac{b_2^2}{b_1} - \frac{a_2b_2}{a_1}\right)\right) + \frac{a_2}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1-2b_2y}{2a_2}\left(-a_2b_2y-a_1b_2\Big(\frac{1}{2b_1}-\frac{b_2}{b_1}y\Big)+\frac{a_2}{2}\right)\\
&=\frac{1-2b_2y}{2a_2}\,\frac{-2a_2b_1b_2y-a_1b_2+2a_1b_2^2y+a_2b_1}{2b_1}\\
&=\frac{-2a_2b_1b_2y-a_1b_2+2a_1b_2^2y+a_2b_1+4a_2b_1b_2^2y^2+2a_1b_2^2y-4a_1b_2^3y^2-2a_2b_1b_2y}{4a_2b_1}\\
&=\frac{(4a_2b_1b_2^2-4a_1b_2^3)y^2+(4a_1b_2^2-4a_2b_1b_2)y+a_2b_1-a_1b_2}{4a_2b_1}. \qquad (3.28)
\end{aligned}
$$

By (3.24), we have

$$
\begin{aligned}
\langle g_4,f_2\rangle\langle g_2,f_4\rangle =&\langle g_4,e_2\rangle(b_1\langle g_2,e_1\rangle+b_2\langle g_2,e_2\rangle)\\
=&y\left(b_1\Big(-\frac{b_2}{2b_1}+y\Big(\frac{b_2^2}{b_1}-\frac{a_2b_2}{a_1}\Big)\Big)+\frac{b_2}{2}\right)\\
=&y\left(-\frac{a_2b_1b_2}{a_1}y-b_1b_2\Big(\frac{1}{2b_1}-\frac{b_2}{b_1}y\Big)+\frac{b_2}{2}\right)\\
=&y\Big(-\frac{a_2b_1b_2}{a_1}y+b_2^2y\Big)\\
=&y^2\frac{b_2}{a_1}(a_1b_2-a_2b_1). \qquad (3.29)
\end{aligned}
$$

Finally, by (3.20), (3.21) and (3.22), we have

$$
\begin{aligned}
&\langle g_4,f_3\rangle\langle g_3,f_4\rangle\\
=&(a_1\langle g_4,e_1\rangle+a_2\langle g_4,e_2\rangle)(b_1\langle g_3,e_1\rangle+b_2\langle g_3,e_2\rangle)\\
=&(a_1x+a_2y)\left(b_1\frac{b_2}{a_1}y+b_2\Big(\frac{1}{2a_2}-\frac{b_2}{a_2}y\Big)\right)\\
=&\Big(\frac{a_1}{2b_1}-\frac{a_1b_2}{b_1}y+a_2y\Big)\frac{a_1b_2-2a_1b_2^2y+2a_2b_1b_2y}{2a_1a_2}\\
=&\frac{a_1+2a_2b_1y-2a_1b_2y}{2b_1}\,\frac{a_1b_2-2a_1b_2^2y+2a_2b_1b_2y}{2a_1a_2}\\
=&\frac{a_1^2b_2-2a_1^2b_2^2y+2a_1a_2b_1b_2y-4a_1a_2b_1b_2^2y^2+2a_1a_2b_1b_2y}{4a_1a_2b_1}\\
&+\frac{4a_2^2b_1^2b_2y^2-2a_1^2b_2^2y+4a_1^2b_2^3y^2-4a_1a_2b_1b_2^2y^2}{4a_1a_2b_1}\\
=&\frac{(4a_2^2b_1^2b_2+4a_1^2b_2^3-8a_1a_2b_1b_2^2)y^2+(4a_1a_2b_1b_2-4a_1^2b_2^2)y+a_1^2b_2}{4a_1a_2b_1}. \qquad (3.30)
\end{aligned}
$$

We observe that equation in (3.25) is equal to equation in (3.30), equation in (3.26) is equal to equation in (3.29) and equation in (3.27) is equal to equation in (3.28). If

(F,G) is two uniform frame pair, then all these six equations has to equal to 1/12 since $\frac{nN-n^2}{N^2(N-1)} = \frac{1}{12}$. Now set equation in (3.26) to 1/12 and solve for y^2, we get

$$y^2 = \frac{a_1}{12b_2(a_1b_2 - a_2b_1)}. \tag{3.31}$$

We note here that $a_1b_2 - a_2b_1 \neq 0$. If it were then F would be not two-independent; thus, by Theorem 2.4, F would be not two-uniform. Substituting y^2 to equation in (3.25) and setting equation in (3.25) to 1/12, we have

$$\frac{\frac{a_1}{12b_2(a_1b_2-a_2b_1)}4b_2(a_1b_2-a_2b_1)^2 + 4a_1b_2(a_2b_1-a_1b_2)y + a_1^2b_2}{4a_1a_2b_1} = \frac{1}{12}.$$

After simplification, we get

$$a_1b_2 - a_2b_1 + 12b_2y(a_2b_1 - a_1b_2) + a_1b_2 = a_2b_1.$$

Solving the equation for y, we get

$$y = \frac{1}{6b_2}. \tag{3.32}$$

By (3.31) and (3.32), we have

$$\frac{a_1}{12b_2(a_1b_2-a_2b_1)} = \frac{1}{36b_2^2}, \quad \text{i.e.,} \quad a_2b_1 = -2a_1b_2.$$

Finally, substituting the values of y^2, y and a_2b_1 into the third equation (3.27), we get

$$\begin{aligned}\langle e_1, g_4\rangle\langle f_4, g_1\rangle &= \frac{y^2 - \frac{a_1}{12b_2(a_1b_2-a_2b_1)}4b_2^2(a_2b_1-a_1b_2) + 4b_2(a_1b_2-a_2b_1)\frac{1}{6b_2} + a_2b_1 - a_1b_2}{4a_2b_1} \\ &= \frac{-\frac{a_1b_2}{3} + \frac{2}{3}3a_1b_2 - 3a_1b_2}{-8a_1b_2} \\ &= \frac{1}{6} \neq \frac{1}{12}.\end{aligned}$$

Hence, we get a contradiction. Therefore, F is not spectrally two uniform. □

CONJECTURES. (i) If there exists a two-uniform frame of length N for an n-dimensional Hilbert space H, then $N \leqslant \frac{n(n+1)}{2}$ if H is a real Hilbert space, and $N \leqslant n^2$ if H is a complex Hilbert space.

(ii) For any N, there exists an n-independent frame F which is spectrally optimal for k-erasures for all k with $1 \leqslant k \leqslant N-n$.

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