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NONCOHERENCE OF THE MULTIPLIER ALGEBRA OF THE DRURY–ARVESON SPACE H^2_n FOR $n \geqslant 3$

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Abstract. Let H^2_n denote the Drury-Arveson Hilbert space on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, and let $\mathscr{M}(H^2_n)$ be its multiplier algebra. We show that for $n \geqslant 3$, the ring $\mathscr{M}(H^2_n)$ is not coherent.

1. Introduction

The aim of this article is to investigate a certain algebraic property of rings, called *coherence*, which is a generalization of the property of being Noetherian, for a particular algebra of holomorphic functions in the unit ball in $\mathbb{C}^n$.

DEFINITION 1.1. (Coherent ring) Let R be a unital commutative ring, and for an $n \in \mathbb{N} := \{1,2,3,\cdots\}$, let $R^n = R \times \cdots \times R$ (n times). If $\mathbf{f} \in R^n$, say $\mathbf{f} = (f_1,\cdots,f_n)$, then a *relation* $\mathbf{g}$ on $\mathbf{f}$, written $\mathbf{g} \in \mathbf{f}^\perp$, is an n-tuple $\mathbf{g} = (g_1,\cdots,g_n) \in R^n$ such that $g_1 f_1 + \cdots + g_n f_n = 0$. The ring R is said to be *coherent* if for each $n \in \mathbb{N}$ and each $\mathbf{f} \in R^n$, the R-module $\mathbf{f}^\perp$ is finitely generated.

A property which is equivalent to coherence is that the intersection of any two finitely generated ideals in R is finitely generated, and the annihilator of any element is finitely generated [4]. We refer the reader to the monograph [7] for the relevance of the property of coherence in homological algebra. All Noetherian rings are coherent, but not all coherent rings are Noetherian. For example, the polynomial ring $\mathbb{C}[x_1,x_2,x_3,\cdots]$ is not Noetherian (because the sequence of ideals $\langle x_1\rangle \subset \langle x_1,x_2\rangle \subset \langle x_1,x_2,x_3\rangle \subset \cdots$ is ascending and not stationary), but $\mathbb{C}[x_1,x_2,x_3,\cdots]$ is coherent [7, Corollary 2.3.4].

For algebras of holomorphic functions in the unit disk

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

in $\mathbb{C}$, it is known that the Hardy algebra $H^\infty(\mathbb{D})$, consisting of all bounded and holomorphic functions on $\mathbb{D}$ with pointwise operations, is coherent, while the disk algebra $A(\mathbb{D})$ (of all functions in $H^\infty(\mathbb{D})$ that admit a continuous extension to the closure of $\mathbb{D}$

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in $\mathbb{C}$) is not coherent [8]. For $n \geqslant 3$, Amar [1] showed that the Hardy algebra $\mathrm{H}^\infty(\mathbb{B}_n)$, consisting of all bounded and holomorphic functions in the unit ball

$$\mathbb{B}_n := \{\mathbf{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\},$$

is not coherent. Related results about some other subalgebras of holomorphic functions in the ball and the polydisk were also obtained in [1]. Whether or not the Hardy algebra $\mathrm{H}^\infty(\mathbb{D}^2)$ (of the bidisk $\mathbb{D}^2$) and $\mathrm{H}^\infty(\mathbb{B}_2)$ are coherent does not seem to be known.

The aim of this article is to prove the noncoherence of the multiplier algebra of the Drury-Arveson space in $\mathbb{C}^n$ with $n \geqslant 3$, and our main result is the following.

THEOREM 1.2. *For $n \geqslant 3$, $\mathscr{M}(\mathrm{H}^2_n)$ is not coherent.*

We give the pertinent definitions and notation below.

A multivariable analogue of the classical Hardy space on $\mathbb{D}$ in $\mathbb{C}$ is the Drury-Arveson space H^2_n on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ [2], [5]. The space H^2_n is a Hilbert function space that has a natural n-tuple of operators acting on it, giving it the structure of a Hilbert module, and has been the object of intensive study in the last decade or so owing to its relation to multivariable operator theory (for example the von Neumann inequality for commuting row contractions [5]) and multivariable function theory (for instance Nevanlinna-Pick interpolation [3]).

DEFINITION 1.3. (The Drury-Arveson space H^2_n) The *Drury-Arveson space* H^2_n is a reproducing kernel Hilbert space of holomorphic functions on $\mathbb{B}_n$ with the kernel

$$K(\mathbf{z}, \mathbf{w}) = \frac{1}{1 - \langle \mathbf{z}, \mathbf{w} \rangle}, \quad \mathbf{z}, \mathbf{w} \in \mathbb{B}_n.$$

We will use the standard multi-index notation: For $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n_+$, where $\mathbb{Z}_+ := \{0, 1, 2, 3, \cdots\}$,

$$\boldsymbol{\alpha}! := \alpha_1!\, \alpha_2! \cdots \alpha_n!, \quad |\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n, \quad \boldsymbol{\zeta}^{\boldsymbol{\alpha}} := \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}.$$

DEFINITION 1.4. (The multiplier algebra $\mathscr{M}(\mathrm{H}^2_n)$) A holomorphic function f on $\mathbb{B}_n$ is called a *multiplier* for H^2_n if $f \cdot \mathrm{H}^2_n \subset \mathrm{H}^2_n$.

$\mathscr{M}(\mathrm{H}^2_n)$ is the ring of all multipliers on H^2_n with pointwise operations.

If f is a multiplier, then the multiplication operator $M_f : \mathrm{H}^2_n \to \mathrm{H}^2_n$ corresponding to f defined by

$$M_f(g) := fg, \quad g \in \mathrm{H}^2_n,$$

is necessarily bounded on H^2_n [2], and the multiplier norm of f in $\mathscr{M}(\mathrm{H}^2_n)$ is defined to be the operator norm of M_f. Then $\mathscr{M}(\mathrm{H}^2_n)$ is a strict sub-algebra of $\mathrm{H}^\infty(\mathbb{B}_n)$ if $n \geqslant 2$ [2]. If $n = 1$, then $\mathrm{H}^2_n = \mathrm{H}^2_1$ is the usual Hardy space of the disk, and $\mathscr{M}(\mathrm{H}^2_n) = \mathrm{H}^\infty(\mathbb{D})$, the Hardy algebra on the disk $\mathbb{D}$.

The proof of our main result, Theorem 1.2, is an adaption to the case of $\mathscr{M}(\mathrm{H}^2_n)$ of the proof given in Amar [1] for showing the noncoherence of $\mathrm{H}^\infty(\mathbb{D}^n)$, $n \geqslant 3$.

2. Preliminaries

The following result is shown along the same lines as the calculation done in [6, Lemma 2.3], where it was shown that

$$\frac{z_2}{1-sz_1} \in \mathscr{M}(\mathrm{H}^2_n)$$

for all real $s \in (0,1)$ and $n \geqslant 2$.

LEMMA 2.1. *Let* $\alpha \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ *and* $n \geqslant 2$. *Then the function* $G_\alpha : \mathbb{B}_n \to \mathbb{C}$, *given by*

$$G_\alpha(\mathbf{z}) = \frac{z_2}{(1-\alpha z_1^2)^{1/4}}, \quad \mathbf{z} = (z_1, \cdots, z_n) \in \mathbb{B}_n,$$

belongs to $\mathscr{M}(H^2_n)$.

Before proving this result, we need some preliminaries from [6, Section 2], reproduced here for the convenience of the reader, as they will play an essential role in the justification of Lemma 2.1. Let

$$\mathscr{B} := \{(0, \beta_2, \cdots, \beta_n) : \beta_2, \cdots, \beta_n \in \mathbb{Z}_+\} \subset \mathbb{Z}^n_+.$$

We will denote as before, the components of $\mathbf{z}$ by $z_1, \cdots, z_n$. For each $\boldsymbol{\beta} \in \mathscr{B}$, define the closed linear subspace

$$H_{\boldsymbol{\beta}} = \overline{\operatorname{span}\{z_1^k \mathbf{z}^{\boldsymbol{\beta}} : k \geqslant 0\}}$$

of H^2_n. Then we have the orthogonal decomposition

$$\mathrm{H}^2_n = \bigoplus_{\boldsymbol{\beta} \in \mathscr{B}} H_{\boldsymbol{\beta}}.$$

For each $\boldsymbol{\beta} \in \mathscr{B}$, we have an orthonormal basis $\{e_{k,\boldsymbol{\beta}} : k \geqslant 0\}$ for $H_{\boldsymbol{\beta}}$, where

$$e_{k,\boldsymbol{\beta}}(\mathbf{z}) = \sqrt{\frac{(k+|\boldsymbol{\beta}|)!}{k!\,\boldsymbol{\beta}!}}\, z_1^k \mathbf{z}^{\boldsymbol{\beta}}. \tag{2.1}$$

Then $H_{\mathbf{0}} = \mathrm{H}^2_1$, the Hardy space of the unit disk $\mathbb{D}$. For the proof of Lemma 2.1, we need to identify each $H_{\boldsymbol{\beta}}$, $\boldsymbol{\beta} \neq \mathbf{0}$, as a weighted Bergman space on the unit disk.

Let dA be the area measure on $\mathbb{D}$ with the normalization $A(\mathbb{D}) = 1$. For each integer $m \geqslant 0$, let

$$\mathrm{B}^{(m)} := \mathrm{L}^2_a\Big(\mathbb{D}, (1-|\zeta|^2)^m dA(\zeta)\Big),$$

the usual weighted Bergman space of weight m. Then

$$\{e_k^{(m)} : k \in \mathbb{Z}_+\}$$

is the standard orthonormal basis for $\mathrm{B}^{(m)}$, where

$$e_k^{(m)}(\zeta) = \sqrt{\frac{(k+m+1)!}{k!m!}}\zeta^k. \tag{2.2}$$

For each $\boldsymbol{\beta} \in \mathscr{B} \setminus \{\mathbf{0}\}$, define the unitary operator $W_{\boldsymbol{\beta}} : H_{\boldsymbol{\beta}} \to \mathrm{B}^{(|\boldsymbol{\beta}|-1)}$ by

$$W_{\boldsymbol{\beta}} e_{k,\boldsymbol{\beta}} = e_k^{(|\boldsymbol{\beta}|-1)}, \quad k \in \mathbb{Z}_+. \tag{2.3}$$

It follows from (2.1) and (2.2) that the weighted shift $M_{z_1}|H_{\boldsymbol{\beta}}$ is unitarily equivalent to M_ζ on $\mathrm{B}^{(|\boldsymbol{\beta}|-1)}$. Thus if $\boldsymbol{\beta} \in \mathscr{B} \setminus \{\mathbf{0}\}$, then

$$W_{\boldsymbol{\beta}} M_{z_1} h_{\boldsymbol{\beta}} = M_\zeta W_{\boldsymbol{\beta}} h_{\boldsymbol{\beta}} \quad \text{for all } h \in H_{\boldsymbol{\beta}}.$$

Note that $M_{z_1}|H_{\mathbf{0}}$ is the unilateral shift.

We will also need the following fact.

LEMMA 2.2. *$|1-\zeta^2|^{-1/2}dA(\zeta)$ is a Carleson measure for the Hardy space H_1^2 of the unit disk $\mathbb{D}$.*

Proof. For $z = e^{i\varphi}$, where $\varphi \in (-\pi, \pi]$, let

$$S_\theta(z) := \{re^{it} : 1-\theta \leqslant r < 1,\ |t-\varphi| \leqslant \theta\}.$$

Then we have

$$\begin{aligned}
\iint_{S_\theta(z)} |1-\zeta^2|^{-1/2} dA(\zeta) &= \int_{\varphi-\theta}^{\varphi+\theta} \int_{1-\theta}^{1} \frac{1}{|1-(re^{it})^2|^{1/2}} r dr dt \\
&= \int_{\varphi-\theta}^{\varphi+\theta} \int_{1-\theta}^{1} \frac{1}{\sqrt[4]{1-2r^2\cos(2t)+r^4}} r dr dt \\
&\leqslant \int_{\varphi-\theta}^{\varphi+\theta} \int_{1-\theta}^{1} \frac{1}{\sqrt[4]{1-2r^2+r^4}} r dr dt \\
&= \int_{\varphi-\theta}^{\varphi+\theta} \int_{1-\theta}^{1} \frac{1}{\sqrt{1-r^2}} r dr dt \\
&= \int_{\varphi-\theta}^{\varphi+\theta} \int_{0}^{1-(1-\theta)^2} \frac{1}{2\sqrt{u}} du dt \ (\text{with } u = 1-r^2) \\
&= \int_{\varphi-\theta}^{\varphi+\theta} \sqrt{u}\Big|_0^{1-(1-\theta)^2} dt \\
&= \int_{\varphi-\theta}^{\varphi+\theta} \sqrt{1-(1-\theta)^2} dt \\
&\leqslant \int_{\varphi-\theta}^{\varphi+\theta} 1 dt = 2\theta.
\end{aligned}$$

This completes the proof. $\square$

We are now ready to prove Lemma 2.1.

Proof of Lemma 2.1. It is enough to consider the case when $\alpha = 1$. Let $h_{\boldsymbol\beta} \in H_{\boldsymbol\beta}$, where $\boldsymbol\beta = (0, \beta_2, \cdots, \beta_n)$. Then

$$h_{\boldsymbol\beta}(\mathbf{z}) = \sum_{k=0}^{\infty} c_k z_1^k \mathbf{z}^{\boldsymbol\beta}.$$

First we assume that $\boldsymbol\beta \neq \mathbf{0}$. By (2.3),

$$(W_{\boldsymbol\beta} h_{\boldsymbol\beta})(\zeta) = \sqrt{\frac{\boldsymbol\beta!}{(|\boldsymbol\beta|-1)!}} \sum_{k=0}^{\infty} c_k \zeta^k, \quad \zeta \in \mathbb{D}.$$

Then $W_{\boldsymbol\beta} h_{\boldsymbol\beta} \in \mathrm{B}^{(|\boldsymbol\beta|-1)}$. Denote $\mathbf{e}_2 = (0, 1, \cdots, 0)$. Since $z_2 \mathbf{z}^{\boldsymbol\beta} = \mathbf{z}^{\boldsymbol\beta + \mathbf{e}_2}$, we have

$$(W_{\boldsymbol\beta+\mathbf{e}_2} z_2 h_{\boldsymbol\beta})(\zeta) = \sqrt{\frac{(\boldsymbol\beta+\mathbf{e}_2)!}{|\boldsymbol\beta|!}} \sum_{k=0}^{\infty} c_k \zeta^k, \quad \zeta \in \mathbb{D},$$

and $W_{\boldsymbol\beta+\mathbf{e}_2} z_2 h_{\boldsymbol\beta} \in \mathrm{B}^{|\boldsymbol\beta|}$. Now suppose that

$$h_{\boldsymbol\beta}(\mathbf{z}) = (1 - z_1^2)^{-1/4} f_{\boldsymbol\beta}(\mathbf{z}),$$

where

$$f_{\boldsymbol\beta}(\mathbf{z}) = \sum_{k=0}^{\infty} a_k z_1^k \mathbf{z}^{\boldsymbol\beta}.$$

For $\zeta \in \mathbb{D}$, we have $|1 - \zeta^2| \geqslant 1 - |\zeta|^2$, and so

$$|1 - \zeta^2|^{1/2} \geqslant (1 - |\zeta|^2)^{1/2} \geqslant 1 - |\zeta|^2. \tag{2.4}$$

We have

$$\begin{aligned}
\|z_2(1 - z_1^2)^{-1/4} f_{\boldsymbol\beta}\|_{\mathrm{H}_n^2}^2 &= \|z_2 h_{\boldsymbol\beta}\|_{\mathrm{H}_n^2}^2 = \|W_{\boldsymbol\beta+\mathbf{e}_2} \zeta_2 h_{\boldsymbol\beta}\|_{\mathrm{B}^{(|\boldsymbol\beta|)}}^2 \\
&= \frac{(\boldsymbol\beta+\mathbf{e}_2)!}{|\boldsymbol\beta|!} \int_{\mathbb{D}} \Big| \sum_{k=0}^{\infty} c_k \zeta^k \Big|^2 (1 - |\zeta|^2)^{|\boldsymbol\beta|} dA(\zeta) \\
&= \frac{(\boldsymbol\beta+\mathbf{e}_2)!}{|\boldsymbol\beta|!} \int_{\mathbb{D}} \Big| \frac{1}{(1-\zeta^2)^{1/4}} \sum_{k=0}^{\infty} a_k \zeta^k \Big|^2 (1 - |\zeta|^2)^{|\boldsymbol\beta|} dA(\zeta) \\
&= \frac{(\boldsymbol\beta+\mathbf{e}_2)!}{|\boldsymbol\beta|!} \int_{\mathbb{D}} \Big| \sum_{k=0}^{\infty} a_k \zeta^k \Big|^2 \frac{(1 - |\zeta|^2)^{|\boldsymbol\beta|}}{|1-\zeta^2|^{1/2}} dA(\zeta) \\
&\leqslant \frac{(\boldsymbol\beta+\mathbf{e}_2)!}{|\boldsymbol\beta|!} \int_{\mathbb{D}} \Big| \sum_{k=0}^{\infty} a_k \zeta^k \Big|^2 (1 - |\zeta|^2)^{|\boldsymbol\beta|-1} dA(\zeta) \text{ (using (2.4))} \\
&= \frac{\beta_2+1}{|\boldsymbol\beta|} \frac{\boldsymbol\beta!}{(|\boldsymbol\beta|-1)!} \int_{\mathbb{D}} \Big| \sum_{k=0}^{\infty} a_k \zeta^k \Big|^2 (1 - |\zeta|^2)^{|\boldsymbol\beta|-1} dA(\zeta) \\
&= \frac{\beta_2+1}{|\boldsymbol\beta|} \|W_{\boldsymbol\beta} f_{\boldsymbol\beta}\|_{\mathrm{B}^{(|\boldsymbol\beta|-1)}}^2 = \frac{\beta_2+1}{|\boldsymbol\beta|} \|f_{\boldsymbol\beta}\|_{\mathrm{H}_n^2}^2 \leqslant 2\|f_{\boldsymbol\beta}\|_{\mathrm{H}_n^2}^2.
\end{aligned}$$

So we have shown that for $\boldsymbol{\beta} \neq \mathbf{0}$, the norm of the restriction of the operator of multiplication by $z_2(1-z_1^2)^{-1/4}$ to $H_{\boldsymbol{\beta}}$ does not exceed $\sqrt{2}$.

Next we consider the case when $\boldsymbol{\beta} = \mathbf{0}$. We know that $H_{\mathbf{0}} = \mathrm{H}_1^2$, the Hardy space on $\mathbb{D}$. Let $h \in H_{\mathbf{0}}$. Then

$$h(\mathbf{z}) = \sum_{k=0}^{\infty} c_k z_1^k.$$

we have

$$(W_{\mathbf{e}_2} z_2 h)(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k, \quad \zeta \in \mathbb{D}$$

and $W_{\mathbf{e}_2} z_2 h$ belongs to the Bergman space $\mathrm{B}^{(0)}$. Now suppose

$$h(\mathbf{z}) = (1-z_1^2)^{-1/4} f(\mathbf{z}),$$

for some

$$f(\mathbf{z}) = \sum_{k=0}^{\infty} a_k z_1^k.$$

Then

$$\begin{aligned}
\|z_2(1-z_1^2)^{-1/4} f\|_{\mathrm{H}_n^2}^2 &= \|W_{\mathbf{e}_2} z_2 h\|_{\mathrm{B}^{(0)}}^2 \\
&= \int_{\mathbb{D}} \left| \sum_{k=0}^{\infty} c_k \zeta^k \right|^2 dA(\zeta) \\
&= \int_{\mathbb{D}} \left| \frac{1}{(1-\zeta^2)^{1/4}} \sum_{k=0}^{\infty} a_k \zeta^k \right|^2 dA(\zeta) \\
&= \int_{\mathbb{D}} \left| \sum_{k=0}^{\infty} a_k \zeta^k \right|^2 |1-\zeta^2|^{-1/2} dA(\zeta) \\
&\leqslant C \|f\|_{\mathrm{H}_1^2}^2,
\end{aligned}$$

where the last inequality follows from the fact that $|1-\zeta^2|^{-1/2} dA(\zeta)$ is a Carleson measure for H_1^2 (Lemma 2.2 above). So we have shown that the norm of the restriction of the operator of multiplication by $z_2(1-z_1^2)^{-1/4}$ to $H_{\mathbf{0}}$ does not exceed $\sqrt{C}$.

If $\boldsymbol{\beta} \neq \boldsymbol{\beta}'$, $f_{\boldsymbol{\beta}} \in H_{\boldsymbol{\beta}}$, and $f_{\boldsymbol{\beta}'} \in H_{\boldsymbol{\beta}'}$, then

$$\frac{z_2}{(1-z_1^2)^{1/4}} f_{\boldsymbol{\beta}} \perp \frac{z_2}{(1-z_1^2)^{1/4}} f_{\boldsymbol{\beta}'}.$$

Thus it follows from the two paragraphs above that the multiplication operator M_{G_α} corresponding to

$$G_\alpha = \frac{z_2}{(1-z_1^2)^{1/4}}$$

is a continuous linear map on H_n^2, that is, $G_\alpha \in \mathscr{M}(\mathrm{H}_n^2)$. This completes the proof. $\square$

3. Noncoherence of $\mathscr{M}(\mathbf{H}_n^2)$

Proof of Theorem 1.2. We will prove the claim by contradiction. Suppose that $\mathscr{M}(\mathrm{H}_n^2)$ is a coherent ring. Let $\mathbf{f}=(f_1,f_2)\in(\mathscr{M}(\mathrm{H}_n^2))^2$, where $f_1:=z_1$ and $f_2:=z_2$. As $\mathscr{M}(\mathrm{H}_n^2)$ is coherent, $\mathbf{f}^\perp$ will be finitely generated, say by $\mathbf{h}_1,\cdots,\mathbf{h}_k$ in $(\mathscr{M}(\mathrm{H}_n^2))^2$. For $\alpha\in\mathbb{T}$, define $\mathbf{g}_\alpha=(g_{1,\alpha},g_{2,\alpha})$ by

$$g_{1,\alpha}(\mathbf{z}) := \frac{z_2}{(1-\alpha z_3^2)^{1/4}},$$
$$g_{2,\alpha}(\mathbf{z}) := \frac{-z_1}{(1-\alpha z_3^2)^{1/4}},$$

for $\mathbf{z}=(z_1,\cdots,z_n)\in\mathbb{B}_n$. Note that by Lemma 2.1, we know that $\mathbf{g}_\alpha$ is in $(\mathscr{M}(\mathrm{H}_n^2))^2$ for each $\alpha\in\mathbb{T}$.

The rest of the proof is the same, mutatis mutandis, as the proof given in [1, Section 1, pages 69-71]. We repeat it here making sure that the implicit but straightforward changes needed in that proof to adapt it to our different situation, are made explicit here for the convenience of the reader.

We have

$$f_1g_{\alpha,1}+f_2g_{\alpha,2}=z_1\cdot\frac{z_2}{(1-\alpha z_3^2)^{1/4}}+z_2\cdot\frac{-z_1}{(1-\alpha z_3^2)^{1/4}}=0,$$

and so $\mathbf{g}_\alpha=(g_{1,\alpha},g_{2,\alpha})\in\mathbf{f}^\perp$. Thus there exist $\gamma_{\alpha,i}\in\mathscr{M}(\mathrm{H}_n^2)$ such that

$$\mathbf{g}_\alpha=\sum_{i=1}^k\gamma_{\alpha,i}\mathbf{h}_i. \tag{3.1}$$

If $\mathbf{h}_i=:(r_i,s_i)\in(\mathscr{M}(\mathrm{H}_n^2))^2$, then we have

$$z_1r_i+z_2s_i=0.$$

So if $z_2=0$, then $z_1r_i=0$. Thus $r_i=0$ on

$$\{\mathbf{z}=(z_1,\cdots,z_n)\in\mathbb{B}_n : z_2=0\}.$$

Hence there exist t_i, holomorphic in $\mathbb{B}_n$ such that

$$r_i(\mathbf{z})=z_2t_i(\mathbf{z}),\quad i=1,\cdots,k,\quad \mathbf{z}\in\mathbb{B}_n.$$

So it now follows from (3.1) that

$$\frac{z_2}{(1-\alpha z_3^2)^{1/4}}=\sum_{i=1}^k\gamma_{\alpha,i}(\mathbf{z})z_2t_i(\mathbf{z}),$$

that is,

$$\varepsilon_\alpha(\mathbf{z}):=\frac{1}{(1-\alpha z_3^2)^{1/4}}=\sum_{i=1}^k\gamma_{\alpha,i}(\mathbf{z})t_i(\mathbf{z}),\quad \alpha\in\mathbb{T},\ \mathbf{z}\in\mathbb{B}_n.$$

Let $\alpha_1,\cdots,\alpha_k,\alpha_*$ be $k+1$ distinct points in $\mathbb{T}$. The equation

$$\varepsilon_\alpha(\mathbf{z})=\sum_{i=1}^{k}\gamma_{\alpha,i}(\mathbf{z})t_i(\mathbf{z}) \tag{3.2}$$

for these $k+1$ choices of α can be rewritten in matricial form as follows:

$$\begin{bmatrix}\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,k} & \varepsilon_{\alpha_1}\\ \vdots & & \vdots & \vdots\\ \gamma_{\alpha_k,1} & \cdots & \gamma_{\alpha_k,k} & \varepsilon_{\alpha_k}\\ \gamma_{\alpha_*,1} & \cdots & \gamma_{\alpha_*,k} & \varepsilon_{\alpha_*}\end{bmatrix}\underbrace{\begin{bmatrix}t_1\\ \vdots\\ t_k\\ -1\end{bmatrix}}_{\neq 0}=0. \tag{3.3}$$

Since (3.3) is solvable, we must have

$$\det\begin{bmatrix}\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,k} & \varepsilon_{\alpha_1}\\ \vdots & & \vdots & \vdots\\ \gamma_{\alpha_k,1} & \cdots & \gamma_{\alpha_k,k} & \varepsilon_{\alpha_k}\\ \gamma_{\alpha_*,1} & \cdots & \gamma_{\alpha_*,k} & \varepsilon_{\alpha_*}\end{bmatrix}=0.$$

Expanding the determinant along the last column gives

$$\underbrace{\det\begin{bmatrix}\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,k}\\ \vdots & & \vdots\\ \gamma_{\alpha_k,1} & \cdots & \gamma_{\alpha_k,k}\end{bmatrix}}_{=:\Delta}\cdot\varepsilon_{\alpha_*}=\sum_{i=1}^{k}\Lambda_{\alpha_*,i}\cdot\varepsilon_{\alpha_i}, \tag{3.4}$$

with $\Lambda_{\alpha_*,i}\in\mathscr{M}(\mathrm{H}^2_n)\subset\mathrm{H}^\infty(\mathbb{B}_n)$ (since the $\gamma_{\alpha_j,i}\in\mathscr{M}(\mathrm{H}^2_n)$). Now we consider the following two possible cases separately:

$\underline{1}^\circ$ For some choice of distinct points $\alpha_1,\cdots,\alpha_k\in\mathbb{T}$, the determinant Δ is not identically 0 on the variety

$$\mathscr{V}:=\Big\{\mathbf{z}=(z_1,\cdots,z_n)\in\mathbb{B}_n : z_k=0 \text{ for all } k\in\{1,\cdots,n\}\setminus\{3\}\Big\}.$$

$\underline{2}^\circ$ For *every* choice of distinct points $\alpha_1,\cdots,\alpha_k\in\mathbb{T}$, $\Delta\equiv 0$ on $\mathscr{V}$.

Let us consider case 1° first. The map $z_3\mapsto\Delta|_{\mathscr{V}}(0,0,z_3,0,\cdots,0):\mathbb{D}\to\mathbb{C}$ is holomorphic and bounded, independent of α_*. As $\Delta|_{\mathscr{V}}$ is not identically zero, there exists a point $\alpha_*\in\mathbb{T}$, which is distinct from $\alpha_1,\cdots,\alpha_k$, such the radial limit of $\Delta|_{\mathscr{V}}(0,0,\cdot,0,\cdots,0)$ is nonzero as $z_3\to\overline{\alpha_*}^{1/2}$. Then z_3^2 approaches $\overline{\alpha_*}$, and we see in (3.4) that the left hand side approaches ∞, while it is not the case that the right hand side approaches ∞ (because the $\Lambda_{\alpha_*,i}$ and the ε_{α_j}, with $\alpha_j\neq\alpha_*$, stay bounded). This contradiction shows that this case can't be possible.

So we now consider case 2°. Set

$$\mathbb{A} := \{(\alpha_1, \cdots, \alpha_k) \in \mathbb{T}^k : \alpha_i \neq \alpha_j \text{ whenever } i \neq j, 1 \leqslant i, j \leqslant k\}.$$

We know that $\Delta = 0$ on $\mathscr{V}$ for every choice of $(\alpha_1, \cdots, \alpha_k) \in \mathbb{A}$. Let ℓ be defined by

$$\ell := \max_{(\alpha_1,\cdots,\alpha_k)\in\mathbb{A}} \max_{\mathbf{z}\in\mathscr{V}} \operatorname{rank} \begin{bmatrix} \gamma_{\alpha_1,1}(\mathbf{z}) & \cdots & \gamma_{\alpha_1,k}(\mathbf{z}) \\ \vdots & & \vdots \\ \gamma_{\alpha_k,1}(\mathbf{z}) & \cdots & \gamma_{\alpha_k,k}(\mathbf{z}) \end{bmatrix}. \tag{3.5}$$

From (3.2), we can deduce that ℓ can't be zero. Indeed, having $\ell = 0$ means that all the $\gamma_{\alpha_j,i} \equiv 0$ on $\mathscr{V}$ and by (3.2), we would have $1/(1-\alpha z^2)^{1/4} = 0$, $z \in \mathbb{D}$, which is clearly impossible. Also $\ell < k$ owing to the fact that $\Delta = 0$ on $\mathscr{V}$. So we know that $1 \leqslant \ell < k$. Now select a particular $(\alpha_1, \cdots, \alpha_k) \in \mathbb{A}$ such that the matrix

$$M := \begin{bmatrix} \gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,k} \\ \vdots & & \vdots \\ \gamma_{\alpha_k,1} & \cdots & \gamma_{\alpha_k,k} \end{bmatrix}$$

has maximal rank ℓ somewhere on the variety $\mathscr{V}$. We can then pick out a square $\ell \times \ell$ submatrix of the matrix M defined above, having the maximal rank ℓ, and after a relabelling (if necessary) of the α_i, we can arrange that the minor

$$\delta := \det \begin{bmatrix} \gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} \\ \vdots & & \vdots \\ \gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell} \end{bmatrix} \not\equiv 0 \text{ on } \mathscr{V}.$$

Now from our choice of $(\alpha_1, \cdots, \alpha_k) \in \mathbb{A}$, we keep $\alpha_1, \cdots, \alpha_\ell$ fixed, but start changing $\alpha_{\ell+1}$. Note that it follows from the definition of ℓ that for any choice of $\alpha_{\ell+1}$, which is distinct from $\alpha_1, \cdots, \alpha_\ell$,

$$D_i = \det \begin{bmatrix} \gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} & \gamma_{\alpha_1,i} \\ \vdots & & \vdots & \vdots \\ \gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell} & \gamma_{\alpha_\ell,i} \\ \gamma_{\alpha_{\ell+1},1} & \cdots & \gamma_{\alpha_{\ell+1},\ell} & \gamma_{\alpha_{\ell+1},i} \end{bmatrix} \equiv 0 \text{ on } \mathscr{V} \text{ for all } i \text{ in } \{1, \cdots, k\}.$$

In the rest of the proof, we imagine this $\alpha_{\ell+1}$ to be a variable living in $\mathbb{T} \setminus \{\alpha_1, \cdots, \alpha_\ell\}$. We have

$$\det \begin{bmatrix} \gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} & \varepsilon_{\alpha_1} \\ \vdots & & \vdots & \vdots \\ \gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell} & \varepsilon_{\alpha_\ell} \\ \gamma_{\alpha_{\ell+1},1} & \cdots & \gamma_{\alpha_{\ell+1},\ell} & \varepsilon_{\alpha_{\ell+1}} \end{bmatrix} = \det \left[\begin{array}{ccc|c} \gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} & \\ \vdots & & \vdots & t_1 \begin{bmatrix} \gamma_{\alpha_1,1} \\ \vdots \\ \gamma_{\alpha_\ell,1} \end{bmatrix} + \cdots + t_k \begin{bmatrix} \gamma_{\alpha_1,k} \\ \vdots \\ \gamma_{\alpha_\ell,k} \end{bmatrix} \\ \gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell} & \\ \hline \gamma_{\alpha_{\ell+1},1} & \cdots & \gamma_{\alpha_{\ell+1},\ell} & t_1 \gamma_{\alpha_{\ell+1},1} + \cdots + t_k \gamma_{\alpha_{\ell+1},k} \end{array}\right]$$

$$= \sum_{i=1}^{k} t_i D_i \equiv 0 \text{ on } \mathscr{V}.$$

By expanding the determinant on the left hand side along the last column, we obtain

$$\underbrace{\det \begin{bmatrix} \gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} \\ \vdots & & \vdots \\ \gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell} \end{bmatrix}}_{=\delta} \cdot \varepsilon_{\alpha_{\ell+1}} = \sum_{i=1}^{\ell} \lambda_{\alpha_{\ell+1},i} \cdot \varepsilon_{\alpha_i} \text{ on } \mathscr{V}, \tag{3.6}$$

with $\lambda_{\alpha_{\ell+1},i} \in \mathscr{M}(\mathrm{H}^2_n) \subset \mathrm{H}^\infty(\mathbb{B}_n)$. By the choice of ℓ, we know that it is not the case that $\delta \equiv 0$ on $\mathscr{V}$. Now we repeat the argument in 1° (replacing α_* by $\alpha_{\ell+1}$), as follows to arrive at a contradiction. The map $z_3 \mapsto \delta|_{\mathscr{V}}(0,0,z_3,0,\cdots,0) : \mathbb{D} \to \mathbb{C}$ is holomorphic and bounded, independent of $\alpha_{\ell+1}$. As $\delta|_{\mathscr{V}}$ is not identically zero, there exists a point $\alpha_{\ell+1} \in \mathbb{T}$, which is distinct from $\alpha_1, \cdots, \alpha_\ell$, such the radial limit of $\Delta|_{\mathscr{V}}(0,0,\cdot,0,\cdots,0)$ is nonzero as $z_3 \to \overline{\alpha_{\ell+1}}^{1/2}$. Then z_3^2 approaches $\overline{\alpha_{\ell+1}}$, and we see in (3.6) that the left hand side approaches ∞, while it is not the case that the right hand side approaches ∞ (because the $\lambda_{\alpha_{\ell+1},i}$ and the ε_{α_i}, with $\alpha_i \neq \alpha_{\ell+1}$, stay bounded). This contradiction shows that this case can't be possible.

Consequently, $\mathbf{f}^\perp$ is not finitely generated, and so $\mathscr{M}(\mathrm{H}^2_n)$ is not coherent. $\square$

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