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HIGHER RANK NUMERICAL HULLS OF MATRICES AND MATRIX POLYNOMIALS

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Dedicated to our teacher:
Professor Abbas Salemi

(*Communicated by C.-K. Li*)

Abstract. In this paper, some properties of the higher rank numerical hulls, as a generalization of higher rank numerical ranges and polynomial numerical hulls, of matrices are investigated. In particular, the higher rank numerical hulls of Pauli matrices are characterized. Moreover, the notion of higher rank numerical hulls of matrix polynomials is introduced, and some algebraic properties of this notion are investigated. The higher rank numerical hulls of the basic A-factor block circulant matrix, which is the block companion matrix of the matrix polynomial $Q(\lambda) = \lambda^s I_n - A$, are also studied.

1. Introduction and preliminaries

Let $M_{n\times m}$ be the vector space of all $n\times m$ complex matrices. For the case $n = m$, $M_{n\times n}$ is denoted by M_n, namely, the algebra of all $n\times n$ complex matrices. Throughout the paper, k,m and n are considered as positive integers, and $k \leqslant n$. Moreover, I_k denotes the $k\times k$ identity matrix, and $\mathscr{I}_{n,k}$ is the set of all $n\times k$ isometry matrices, i.e., $\mathscr{I}_{n,k} = \{X \in M_{n\times k} : X^*X = I_k\}$. Motivated by the study of convergence of iterative methods in solving linear systems, e.g., see [23], researchers studied the *polynomial numerical hull of order m* of a matrix $A \in M_n$, which is defined and denoted by

$$V^m(A) = \{\lambda \in \mathbb{C} : \ |p(\lambda)| \leqslant \|p(A)\| \text{ for all } p \in \mathbb{P}_m\},$$

where $\mathbb{P}_m$ is the set of all scalar polynomials of degree m or less and $\|.\|$ is the spectral matrix norm (i.e., the matrix norm subordinate to the Euclidean vector norm). This is a set designed to give more information than the spectrum alone can provide about the behavior of the matrix A under the action of polynomials and other functions. For more information see [7], [8], [12], [13] and [24].

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In the context of quantum information theory, if the quantum states are represented as matrices in M_n, then a quantum channel is a trace preserving completely positive map $L : M_n \longrightarrow M_n$ with the operator sum representation:

$$L(A) = \sum_{j=1}^{r} E_j^* A E_j,$$

where $E_1,\ldots,E_r \in M_n$ satisfy $\sum_{j=1}^{r} E_j E_j^* = I_n$. The matrices $E_1,\ldots,E_r$ are known as the error operators of the quantum channel L. Let V be a k-dimensional subspace of $\mathbb{C}^n$, and P be the orthogonal projection of $\mathbb{C}^n$ onto V. Then, the k-dimensional subspace V is a quantum error correction code for the channel L if and only if there are scalars $\gamma_{ij} \in \mathbb{C}$ with $i,j \in \{1,\ldots,r\}$ such that $PE_i^* E_j P = \gamma_{ij} P$; for more information, see [16], [17] and [18]. In this connection, the *rank-k numerical range* of $A \in M_n$ is defined and denoted by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : X^* A X = \lambda I_k \text{ for some } X \in \mathscr{I}_{n,k}\}.$$

The sets $\Lambda_k(A)$, where $k \in \{1,\ldots,n\}$, are generally called *higher rank numerical ranges of* A; see [5], [6], [19], [20] and [21] for more information.

Recently, the notion of *rank-k numerical hull of order m* of a matrix $A \in M_n$, as a generalization of $V^m(A)$ and $\Lambda_k(A)$, is introduced by A. Salemi in [26] and is denoted by:

$$\mathscr{X}_k^m(A) = \{\lambda \in \mathbb{C} : (\lambda,\lambda^2,\ldots,\lambda^m) \in conv(\Lambda_k(A,A^2,\ldots,A^m))\},$$

where $conv(S)$ denotes the *convex hull* of $S \subseteq \mathbb{C}$, and

$$\Lambda_k(A_1,A_2,\ldots,A_m) = \{(\lambda_1,\ldots,\lambda_m) \in \mathbb{C}^m : \exists X \in \mathscr{I}_{n,k} \ s.t.\ X^* A_j X = \lambda_j I_k,\ j = 1,\ldots,m\}$$

is the *joint rank-k numerical range* of $(A_1,A_2,\ldots,A_m) \in \underbrace{M_n \times \cdots \times M_n}_{m-times}$. The joint rank-1 numerical range of $(A_1,A_2,\ldots,A_m)$ is the *joint numerical range*; namely,

$$\Lambda_1(A_1,A_2,\ldots,A_m) = W(A_1,A_2,\ldots,A_m) = \{(x^*A_1x, x^*A_2x,\ldots,x^*A_mx) : x \in \mathbb{C}^n,\ x^*x = 1\}.$$

The sets $\mathscr{X}_k^m(A)$, where $k \in \{1,2,\ldots,n\}$ and $m \in \mathbb{N}$, are generally called *higher rank numerical hulls* of A. For the case $k = m = 1$, $\mathscr{X}_k^m(A)$ reduces to the classical numerical range of A; namely,

$$\mathscr{X}_1^1(A) = V^1(A) = \Lambda_1(A) = W(A) := \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\},$$

which is useful in studying and understanding of matrices and operators, and has many applications in numerical analysis, differential equations, systems theory, etc; e.g., see [9, 14, 15] and references cited there. The *rank-k spectrum* of a matrix $A \in M_n$, as a generalization of the spectrum of A, is defined and denoted, see [26], by $\sigma_k(A) = \{\lambda \in \mathbb{C} : dim(ker(\lambda I_n - A)) \geqslant k\}$.

Next, we list some properties of the higher rank numerical hulls and rank-k spectrum of matrices which will be useful in our discussion. One may see [26] for more details.

PROPOSITION 1.1. *Let $A \in M_n$. Then the following assertions are true:*

(i) $\sigma_k(A) \subseteq \sigma_{k-1}(A) \subseteq \cdots \subseteq \sigma_1(A) = \sigma(A)$;

(ii) $\sigma_k(A) \subseteq \mathscr{X}_k^m(A) \subseteq \mathscr{X}_{k-1}^m(A) \subseteq \cdots \subseteq \mathscr{X}_1^m(A) = V^m(A) \subseteq V^{m-1}(A) \subseteq \cdots \subseteq V^1(A) = W(A)$;

(iii) $\sigma_k(A) \subseteq \mathscr{X}_k^m(A) \subseteq \mathscr{X}_k^{m-1}(A) \subseteq \cdots \subseteq \mathscr{X}_k^1(A) = \Lambda_k(A) \subseteq \Lambda_{k-1}(A) \subseteq \cdots \subseteq \Lambda_1(A) = W(A)$;

(iv) $\mathscr{X}_k^m(\alpha A + \beta I_n) = \alpha \mathscr{X}_k^m(A) + \beta$; *where* $\alpha, \beta \in \mathbb{C}$;

(v) *If A is Hermitian and $m \geqslant 2$, then* $\mathscr{X}_k^m(A) = \sigma_k(A)$;

(vi) *If A is unitary, then* $\mathscr{X}_k^m(A) \cap \sigma(A) = \sigma_k(A)$;

(vii) *If $m,k \geqslant 2$, $n = 2k$, and A is a unitary matrix with distinct eigenvalues, then* $\mathscr{X}_k^m(A) = \emptyset$.

At the end of this section, we give some information about matrix polynomials. Notice that matrix polynomials arise in many applications and their spectral analysis is very important when studying linear systems of ordinary differential equations with constant coefficients [11]. Suppose that

$$Q(\lambda) = A_s\lambda^s + \cdots + A_1\lambda + A_0 \tag{1}$$

is a matrix polynomial, where $A_i \in M_n$ $(i = 0, 1, \ldots, s)$, $A_s \neq 0$ and λ is a complex variable. The numbers s and n are referred to as the *degree* and the *order* of $Q(\lambda)$, respectively. The matrix polynomial $Q(\lambda)$, as in (1), is called *selfadjoint* if all coefficients A_i are Hermitian. It is called a *monic matrix polynomial* if $A_s = I_n$. A scalar $\lambda_0 \in \mathbb{C}$ is an *eigenvalue* of $Q(\lambda)$ if the system $Q(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $Q(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $Q(\lambda)$ is said to be the *spectrum* of $Q(\lambda)$; namely, $\sigma[Q(\lambda)] = \{\mu \in \mathbb{C} : \det(Q(\mu)) = 0\}$. The *(classical) numerical range* of $Q(\lambda)$, as in (1), is defined as:

$$W[Q(\lambda)] := \{\mu \in \mathbb{C} : x^*Q(\mu)x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\},$$

which is closed and contains $\sigma[Q(\lambda)]$; see [22] for more information. The numerical range of matrix polynomials plays an important role in the study of overdamped vibration systems with finite number of degrees of freedom, and it is also related to the stability theory; e.g., see [11] and [22]. One generalization of the classical numerical range of $Q(\lambda)$, as in (1), is the *polynomial numerical hull of order m*, which is defined and denoted, see [1] and [27], by

$$V^m[Q(\lambda)] = \{\mu \in \mathbb{C} : |p(0)| \leqslant \|p(Q(\mu))\| \text{ for all } p \in \mathbb{P}_m\}.$$

Recently, the notion of *rank-k numerical range* of $Q(\lambda)$, as another generalization of the classical numerical range of $Q(\lambda)$ was introduced by Aretki and Maroulas [2] as

$$\Lambda_k[Q(\lambda)] = \{\mu \in \mathbb{C} : X^*Q(\mu)X = 0I_k \text{ for some } X \in \mathscr{I}_{n,k}\}.$$

It is known that $V^1[Q(\lambda)] = W[Q(\lambda)] = \Lambda_1[Q(\lambda)]$. Also, for the case $Q(\lambda) = \lambda I_n - A$, where $A \in M_n$, we have $V^m[Q(\lambda)] = V^m(A)$ and $\Lambda_k[Q(\lambda)] = \Lambda_k(A)$. So, $V^m[Q(\lambda)]$ and $\Lambda_k[Q(\lambda)]$ can be considered as generalizations of $V^m(A)$ and $\Lambda_k(A)$, respectively. In this paper, we are going to study some algebraic and geometrical properties of the higher rank numerical hulls of matrices, and we are also going to generalize this notion for matrix polynomials. For this mind, in Section 2, we present some algebraic and geometrical properties of the higher rank numerical hulls of matrices. In Section 3, we characterize the higher rank numerical hulls of Pauli matrices. In Section 4, we introduce and study the notion of higher rank numerical hulls of matrix polynomials. The higher rank numerical hulls of the basic A-factor block circulant matrix, denoted by π_A, which is the block companion matrix of the matrix polynomial $Q(\lambda) = \lambda^s I_n - A$, are also studied.

2. Some properties of higher rank numerical hulls of matrices

At first, we state a result about joint higher rank numerical ranges of matrices which will be useful in our discussion.

THEOREM 2.1. *Let $A_1, A_2, \ldots, A_m \in M_n$. Then the following assertions are true:*

(i) $\Lambda_k(A_1, A_2, \ldots, A_m) \subseteq \Lambda_k(A_1 \oplus B_1, A_2 \oplus B_2, \ldots, A_m \oplus B_m)$, *where $B_1, B_2, \ldots, B_m \in M_{n'}$;*

(ii) $\Lambda_k(A_1, A_2, \ldots, A_m) \subseteq \bigcap_{X \in \mathscr{I}_{n,n-k+1}} W(X^*A_1X, \ldots, X^*A_mX)$, *and for the case $k = 1$, the equality holds.*

Proof. Let $(\lambda_1, \ldots, \lambda_m) \in \Lambda_k(A_1, A_2, \ldots, A_m)$. So, there exists a $X \in \mathscr{I}_{n,k}$ such that $X^*A_iX = \lambda_i I_k$ for $i = 1, 2, \ldots, m$. By setting $Y := \begin{pmatrix} X \\ 0 \end{pmatrix} \in M_{(n+n') \times k}$, we have $Y \in \mathscr{I}_{n+n',k}$ and $Y^*(A_i \oplus B_i)Y = X^*A_iX = \lambda_i I_k$, for $i = 1, 2, \ldots, m$. Hence, $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \Lambda_k(A_1 \oplus B_1, A_2 \oplus B_2, \ldots, A_m \oplus B_m)$ and so, the result in (i) holds.

To prove the result in (ii), let $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \Lambda_k(A_1, A_2, \ldots, A_m)$ be given. So, there exists a $Y \in \mathscr{I}_{n,k}$ such that $Y^*A_jY = \alpha_j I_k$ for $j = 1, \ldots, m$. Now, let $X \in \mathscr{I}_{n,n-k+1}$ be given. The dimensions of the column spaces of Y and X are k and $n-k+1$, respectively. Hence, there exists a unit vector w in the intersection of the column spaces of Y and X. Let $Y = [y_1, y_2, \ldots, y_k]$. Since $X^*X = I_{n-k+1}$, there exists a unit vector $z \in \mathbb{C}^n$ such that $Xz = w$. If $w = \beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_k y_k$, where $\beta_i \in \mathbb{C}$ and $\sum_{i=1}^k |\beta_i|^2 = 1$, then for every $j = 1, \ldots, m$, we have $z^*X^*A_jXz = w^*A_jw = |\beta_1|^2 y_1^* A_j y_1 + \cdots + |\beta_k|^2 y_k^* A_j y_k = \alpha_j$. So, $(\alpha_1, \ldots, \alpha_m) \in W(X^*A_1X, X^*A_2X, \ldots, X^*A_mX)$, and hence, the proof of $\subseteq$ is complete. Since $W(U^*A_1U, \ldots, U^*A_mU) = W(A_1, \ldots, A_m)$ for any unitary matrix $U \in M_n$, the second assertion is also true. $\square$

The following example shows that the set equality in Theorem 2.1(ii) does not hold in general.

EXAMPLE 2.2. Consider the following matrices in M_4:

$$A_1 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},\ \text{and } A_3 = \begin{pmatrix} 0 & \imath I_2 \\ -\imath I_2 & 0 \end{pmatrix},$$

where $\imath = \sqrt{-1}$. By [19, Example 2.6], we have $\Lambda_2(A_1,A_2,A_3) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2+x_2^2+x_3^2 = 1\}$, which is not convex. Also $W(X^*A_1X, X^*A_2X, X^*A_3X)$ is convex for every $X \in \mathscr{I}_{4,3}$. So, $\bigcap\limits_{X\in\mathscr{I}_{4,3}} W(X^*A_1X, X^*A_2X, X^*A_3X)$ is convex, and can not be equal to $\Lambda_2(A_1,A_2,A_3)$.

Now, in the following proposition, we are going to state some basic properties of the higher rank numerical hulls of matrices.

PROPOSITION 2.3. *Let $A \in M_n$. Then the following assertions are true:*

(i) $\mathscr{X}_k^m(A)$ *is a compact set;*

(ii) $\mathscr{X}_k^m(A^*) = \overline{\mathscr{X}_k^m(A)}$. *Consequently, if A is Hermitian, then $\mathscr{X}_k^m(A) \subseteq \mathbb{R}$;*

(iii) $\mathscr{X}_k^m(U^*AU) = \mathscr{X}_k^m(A)$, *where $U \in M_n$ is unitary;*

(iv) $\mathscr{X}_k^m(A) \cup \mathscr{X}_k^m(B) \subseteq \mathscr{X}_k^m(A\oplus B)$, *where $B \in M_{n'}$ and $k \leqslant min\{n,n'\}$.*

Proof. It is clear that $\Lambda_k(A,A^2,\ldots,A^m)$ is compact, and hence, $conv(\Lambda_k(A,A^2,\ldots,A^m))$ is also compact by [4, Lemma 2.7]. So, $\mathscr{X}_k^m(A)$ is a closed subset of the compact set $W(A)$ and hence the result in (i) holds.

We have that $\lambda \in \mathscr{X}_k^m(A^*)$ if and only if there exist $l \in \mathbb{N}$ and positive real numbers t_i $(i=1,\ldots,l)$ with $\sum_{i=1}^l t_i = 1$, and $X_i \in \mathscr{I}_{n,k}$ such that $\lambda^j I_k = \sum_{i=1}^l t_i X_i^*(A^*)^j X_i = (\sum_{i=1}^l t_i X_i^* A^j X_i)^*$, for $j = 1,\ldots,m$. This is equivalent to $\bar{\lambda}^j I_k = \sum_{i=1}^l t_i X_i^* A^j X_i$, for $j = 1,\ldots,m$; or equivalently, $\bar{\lambda} \in \mathscr{X}_k^m(A)$. So the result in (ii) also holds.

The result in (iii) is derived easily from this fact that $\Lambda_k(U^*A_1U,\ldots,U^*A_mU) = \Lambda_k(A_1,\ldots,A_m)$, for every $A_1,\ldots,A_m \in M_n$, and for any unitary matrix $U \in \mathbb{M}_n$.

For (iv), let $\lambda \in \mathscr{X}_k^m(A) \cup \mathscr{X}_k^m(B)$. Without loss of generality, we assume that $\lambda \in \mathscr{X}_k^m(A)$. Then $(\lambda,\lambda^2,\ldots,\lambda^m) \in conv(\Lambda_k(A,A^2,\ldots,A^m)) \subseteq conv(\Lambda_k(A\oplus B, A^2\oplus B^2,\ldots,A^m\oplus B^m))$, in which the inclusion follows from Theorem 2.1(i). This means that $\lambda \in \mathscr{X}_k^m(A\oplus B)$. $\square$

Notice that $\mathscr{X}_k^m(\cdot)$ can be an empty or a nonempty set in $\mathbb{C}$. For example, let $A = diag(1,-1)$. Then by Proposition 1.1(v), $\mathscr{X}_2^2(A) = \sigma_2(A) = \emptyset$; while $\mathscr{X}_2^2(A\oplus A) = \sigma_2(A\oplus A) = \{-1,1\} \neq \emptyset$. Moreover, by [20, Theorem 3] and Proposition 1.1(iii), we have the following result.

PROPOSITION 2.4. *If $k \geqslant n/3+1$, then there exists $A \in M_n$ such that $\mathscr{X}_k^m(A) = \emptyset$.*

In the following theorem, we study the higher rank numerical hulls of nilpotent matrices.

THEOREM 2.5. *Let $A \in M_n$ be a nilpotent matrix and s be the geometric multiplicity of its zero eigenvalue. Let t be the smallest positive integer number such that $A^t = 0$. If $m \geqslant t$, then $\mathscr{X}_k^m(A) \subseteq \{0\}$, and for the case $k \leqslant s$, the equality holds.*

Proof. It is clear that:

$$\begin{aligned}\mathscr{X}_k^m(A) &= \{\lambda \in \mathbb{C} : (\lambda, \lambda^2, \ldots, \lambda^m) \in conv(\Lambda_k(A, A^2, \ldots, A^{t-1}, \underbrace{0, \ldots, 0}_{(m-t+1)-\text{ times}}))\} \\ &= \{\lambda \in \mathbb{C} : (\lambda, \lambda^2, \ldots, \lambda^m) \in conv(\Lambda_k(A, A^2, \ldots, A^{t-1})) \times \underbrace{\{0\} \times \cdots \times \{0\}}_{(m-t+1)-\text{ times}}\}.\end{aligned}$$

Now, if $\mathscr{X}_k^m(A) \neq \emptyset$ and $\lambda \in \mathscr{X}_k^m(A)$, then by this fact that $t \leqslant m$, $\lambda^t = 0$ and hence $\lambda = 0$. So $\mathscr{X}_k^m(A) \subseteq \{0\}$. If $k \leqslant s$, then by Proposition 1.1((ii) or (iii)), $\{0\} = \sigma_k(A) \subseteq \mathscr{X}_k^m(A) \subseteq \{0\}$. Hence, $\mathscr{X}_k^m(A) = \{0\}$. $\square$

COROLLARY 2.6. *Let $A \in M_n$ be a nilpotent matrix and t be the smallest positive integer number such that $A^t = 0$. If $m \geqslant t$, then $V^m(A) = \{0\}$.*

The following example shows that the result in Theorem 2.5 for the case $t > m$ does not hold.

EXAMPLE 2.7. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. One can easily see that $A^2 = 0$ and

$$\mathscr{X}_1^1(A) = W(A) = \{z \in \mathbb{C} : |z| \leqslant 1/2\} \nsubseteq \{0\}.$$

At the end of this section, we are going to study the higher rank numerical hulls of tensor products of matrices. For this mind, we need the following lemma. Before, we recall that for the compact set $S \subseteq \mathbb{C}$, the *polynomially convex hull of degree m of S* (e.g., see [10]), is the following set:

$$pconv_m(S) = \{\lambda \in \mathbb{C} : |p(\lambda)| \leqslant \max_{z \in S} |p(z)| \text{ for all } p \in \mathbb{P}_m\}.$$

LEMMA 2.8. *Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_l \in M_n$, where $A_i \in M_{n_i}$ $(n_1 + n_2 + \cdots + n_l = n)$. If all matrices A_i are normal such that $\sigma(A_i) = \sigma_k(A_i)$, then*

$$V^m(A) = pconv_m(\bigcup_{i=1}^{l} \mathscr{X}_k^m(A_i)).$$

Proof. Since A_i's are normal for $i = 1, 2, \ldots, l$, in view of Proposition 2.3(iii), we assume, without loss of generality, that they are diagonal. Note that the property $\sigma(A_i) = \sigma_k(A_i)$ implies that $\mathscr{X}_k^m(A_i) \neq \emptyset$ for $i = 1, 2, \ldots, l$. Now, by setting $S = \bigcup_{i=1}^{l} \mathscr{X}_k^m(A_i)$ and using [10, Theorem 1(v)], it is enough to show that $\|p(A)\| = max_{z \in S} |p(z)|$ for all $p \in \mathbb{P}_m$. Let $p \in \mathbb{P}_m$ be given. It is known that $\|p(A)\| =$

$max_{1\leqslant i\leqslant l}\|p(A_i)\|$. Since, $\mathscr{X}_k^m(A_i)\subseteq V^m(A_i)$ for all i, $S\subseteq\bigcup_{i=1}^{l}V^m(A_i)$, and hence, $max_{1\leqslant i\leqslant l}\|p(A_i)\|\geqslant max_{z\in S}|p(z)|$. So $\|p(A)\|\geqslant max_{z\in S}|p(z)|$.

Conversely, we assume that $\|p(A)\|=|p(\alpha)|$, where $\alpha\in\sigma(A_j)$ for some $1\leqslant j\leqslant l$. Since $\sigma(A_j)=\sigma_k(A_j)$, Proposition 1.1 (ii) implies that $\alpha\in\mathscr{X}_k^m(A_j)\subseteq S$, and hence $\|p(A)\|=|p(\alpha)|\leqslant max_{z\in S}|p(z)|$. So the proof is complete. $\square$

THEOREM 2.9. *Let $A\in M_{n_1}$, $B\in M_{n_2}$, and $1\leqslant k_1\leqslant n_1$ and $1\leqslant k_2\leqslant n_2$ be two positive integers. Then the following assertions are true:*

(i) $\mathscr{X}_{k_1}^m(A)\mathscr{X}_{k_2}^m(B)\subseteq\mathscr{X}_{k_1k_2}^m(A\otimes B)$;

(ii) *If A and B are normal matrices, and $\sigma_{k_1}(A)=\sigma(A)$ and $\sigma_{k_2}(B)=\sigma(B)$, then*

$$V^m(A\otimes B)=pconv_m(\mathscr{X}_{k_1}^m(A)\mathscr{X}_{k_2}^m(B));$$

(iii) *If A and B are Hermitian matrices and $m\geqslant 2$, then*

$$\mathscr{X}_{k_1k_2}^m(A\otimes B)=\sigma_{k_1k_2}(A\otimes B)\supseteq\sigma_{k_1}(A)\sigma_{k_2}(B).$$

Proof. Let $\lambda\in\mathscr{X}_{k_1}^m(A)$ and $\mu\in\mathscr{X}_{k_2}^m(B)$ be given. Then there exists nonnegative real numbers $t_1,\ldots,t_p$ and $s_1,\ldots,s_q$ with $\sum_{i=1}^{p}t_i=1$ and $\sum_{i=1}^{q}s_i=1$, and $X_1,\ldots,X_p\in\mathscr{I}_{n_1,k_1}$, and $Y_1,\ldots,Y_q\in\mathscr{I}_{n_2,k_2}$ such that for $i=1,2,\ldots,m$, $\lambda^iI_{k_1}=\sum_{j=1}^{p}t_j(X_j^*A^iX_j)$ and $\mu^iI_{k_2}=\sum_{j=1}^{q}s_j(Y_j^*B^iY_j)$. Now, for $i=1,2,\ldots,m$, we have:

$$\begin{aligned}(\lambda\mu)^iI_{k_1k_2}&=(\lambda^iI_{k_1})\otimes(\mu^iI_{k_2})\\&=\sum_{j=1}^{p}\sum_{l=1}^{q}t_js_l(X_j\otimes Y_l)^*(A\otimes B)^i(X_j\otimes Y_l).\end{aligned}$$

Since $\sum_{j=1}^{p}\sum_{l=1}^{q}t_js_l=1$ and $X_j\otimes Y_l\in\mathscr{I}_{n_1n_2,k_1k_2}$,

$$(\lambda\mu,(\lambda\mu)^2,\ldots,(\lambda\mu)^m)\in conv(\Lambda_{k_1k_2}((A\otimes B),(A\otimes B)^2,\ldots,(A\otimes B)^m)),$$

and hence $\lambda\mu\in\mathscr{X}_{k_1k_2}^m(A\otimes B)$. So, the result in (i) holds.

If A and B are normal matrices, then $A\otimes B$ is also normal.

Without loss of generality, by Proposition 2.3 (iii), we assume that $A=diag(\alpha_1,\alpha_2,\ldots,\alpha_{n_1})$. So, $A\otimes B=\alpha_1B\oplus\alpha_2B\oplus\cdots\oplus\alpha_{n_1}B$, and hence by Lemma 2.8, $V^m(A\otimes B)=pconv_m(\bigcup_{j=1}^{n_1}(\alpha_j\mathscr{X}_{k_2}^m(B)))$. In view of Proposition 1.1 (ii), $\alpha_j\in\mathscr{X}_{k_1}^m(A)$ $(j=1,2,\ldots,n_1)$ and hence $pconv_m(\bigcup_{j=1}^{n_1}(\alpha_j\mathscr{X}_{k_2}^m(B)))\subseteq pconv_m(\mathscr{X}_{k_1}^m(A)\mathscr{X}_{k_2}^m(B))$. So, $V^m(A\otimes B)\subseteq pconv_m(\mathscr{X}_{k_1}^m(A)\mathscr{X}_{k_2}^m(B))$. On the other hand, by (i) and Proposition 1.1 (iii), we have $\mathscr{X}_{k_1}^m(A)\mathscr{X}_{k_2}^m(B)\subseteq V^m(A\otimes B)$.

Hence, by [10, Theorem 1 (v)], $pconv_m(\mathscr{X}_{k_1}^m(A)\mathscr{X}_{k_2}^m(B))\subseteq V^m(A\otimes B)$. So the result in (ii) also holds.

The set equality in (iii) follows from Proposition 1.1 (v), and also $\supseteq$ is clear. So, the proof is complete. $\square$

By setting $k_1=k_2=1$ in Theorem 2.9, we have the following result.

COROLLARY 2.10. (See also [1, Theorem 3.5]) *Let $A \in M_{n_1}$ and $B \in M_{n_2}$. Then*

(i) $pconv_m(V^m(A)V^m(B)) \subseteq V^m(A \otimes B)$;

(ii) *If A and B are normal matrices, then* $V^m(A \otimes B) = pconv_m(V^m(A)V^m(B))$.

The following example shows that there are matrices in Theorem 2.9(i) such that the set equality holds.

EXAMPLE 2.11. Let $A = I_2$ and $B = [-1] \oplus I_3$. By setting $k_1 = 2$, $k_2 = 3$ and $m = 2$ in Theorem 2.9(i), and using Proposition 1.1 (v), we have $\mathscr{X}_2^2(A) = \sigma_2(A) = \{1\}$, $\mathscr{X}_3^2(B) = \sigma_3(B) = \{1\}$, $\mathscr{X}_{k_1k_2}^m(A \otimes B) = \mathscr{X}_6^2(I_2 \oplus B) = \sigma_6(B) = \{1\}$.

So, $\mathscr{X}_{k_1}^m(A)\mathscr{X}_{k_2}^m(B) = \mathscr{X}_{k_1k_2}^m(A \otimes B)$.

3. Higher rank numerical hulls of Pauli matrices

Four extremely useful matrices in the study of quantum computation and quantum information are known as Pauli matrices, represented as follows:

$$\sigma_0 := I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \text{ and } \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\imath = \sqrt{-1}$. These four matrices form an orthogonal basis for the algebra of 2×2 complex matrices with the Hilbert-Schmidt inner product $\langle A, B \rangle = trace(B^*A)$. Let N be a positive integer and $n = 2^N$. The Pauli group $\mathscr{P}_N$ is defined to consist of all N-fold tensor product of Pauli matrices with multiplicative factors ± 1 and $\pm \imath$, as follows

$$\mathscr{P}_N = \{\alpha\ (\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_N}) : \ i_1, i_2, \ldots, i_N \in \{0,1,2,3\}, \alpha \in \{\pm 1, \pm \imath\}\}.$$

By a Pauli matrix $P \in M_n$ we mean an element of the Pauli group $\mathscr{P}_N$. For more information, see [16] and [25].

In the following theorem, we characterize the higher rank numerical hulls of Pauli matrices.

THEOREM 3.1. *Let $n = 2^N$ and $P = \alpha\ (\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_N}) \in M_n$, where $\alpha \in \{\pm 1, \pm \imath\}$ and $i_1, i_2, \ldots, i_N \in \{0,1,2,3\}$, be a Pauli matrix. Then*

$$\mathscr{X}_k^m(P) = \begin{cases} \{-\alpha, \alpha\} & \text{if } P \neq \alpha I_n,\ k \leqslant n/2 \text{ and } m > 1, \\ [-\alpha, \alpha] & \text{if } P \neq \alpha I_n,\ k \leqslant n/2 \text{ and } m = 1, \\ \emptyset & \text{if } P \neq \alpha I_n,\ k > n/2, \\ \{\alpha\} & \text{if } P = \alpha I_n, \end{cases}$$

where $[-\alpha, \alpha] = \{\alpha x : -1 \leqslant x \leqslant 1\}$.

Proof. We know that both σ_1 and σ_2 are unitarily similar to σ_3. Thus, if $i_j = 0$ for all j, then $P = \alpha I_n$; otherwise, P is unitarily similar to $\alpha(I_{n/2} \oplus -I_{n/2})$ if $i_j \neq 0$ for some j. Now, Proposition 1.1 (v) and [6, Theorem 2.4] yield that

$$\mathscr{X}_k^m(I_{n/2} \oplus -I_{n/2}) = \begin{cases} \sigma_k(I_{n/2} \oplus -I_{n/2}) = \{-1,1\} & \text{if } k \leqslant n/2 \text{ and } m > 1, \\ \Lambda_k(I_{n/2} \oplus -I_{n/2}) = [-1,1] & \text{if } k \leqslant n/2 \text{ and } m = 1, \\ \emptyset & \text{if } k > n/2. \end{cases}$$

Hence, by Propositions 1.1 (iv) and 2.3 (iii), the result holds. $\square$

By setting $k = 1$ or $m = 1$ in Theorem 3.1, we have the following result.

COROLLARY 3.2. *Let $n = 2^N$ and $P = \alpha\ (\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_N}) \in M_n$, where $\alpha \in \{\pm 1, \pm \imath\}$ and $i_1, i_2, \ldots, i_N \in \{0,1,2,3\}$, be a Pauli matrix. Then*

$$V^m(P) = \begin{cases} \{-\alpha, \alpha\} & \text{if } P \neq \alpha I_n \text{ and } m > 1, \\ [-\alpha, \alpha] & \text{if } P \neq \alpha I_n \text{ and } m = 1, \\ \{\alpha\} & \text{if } P = \alpha I_n, \end{cases}$$

and

$$\Lambda_k(P) = \begin{cases} [-\alpha, \alpha] & \text{if } P \neq \alpha I_n \text{ and } k \leqslant n/2, \\ \emptyset & \text{if } P \neq \alpha I_n \text{ and } k > n/2, \\ \{\alpha\} & \text{if } P = \alpha I_n, \end{cases}$$

where $[-\alpha, \alpha] = \{\alpha x : -1 \leqslant x \leqslant 1\}$.

4. Higher rank numerical hulls of matrix polynomials

In this section, we consider a matrix polynomial $Q(\lambda) = A_s\lambda^s + \cdots + A_1\lambda + A_0$ as in (1), and at first, we introduce the notions of higher rank numerical hulls and rank-k spectrum of $Q(\lambda)$.

DEFINITION 4.1. Let $Q(\lambda)$ be a matrix polynomial as in (1). The *rank-k numerical hull of order m of $Q(\lambda)$* is defined and denoted by

$$\mathscr{X}_k^m[Q(\lambda)] = \{\mu \in \mathbb{C} : 0 \in \mathscr{X}_k^m(Q(\mu))\}.$$

Also, the *rank-k spectrum of $Q(\lambda)$* is defined and denoted by

$$\sigma_k[Q(\lambda)] = \{\mu \in \mathbb{C} : 0 \in \sigma_k(Q(\mu))\}.$$

The sets $\mathscr{X}_k^m[Q(\lambda)]$, where $k \in \{1,2,\ldots,n\}$ and $m \in \mathbb{N}$, are called generally *higher rank numerical hulls* of $Q(\lambda)$.

REMARK 4.2. Let $Q(\lambda) = \lambda I - A$, where $A \in M_n$. By Definition 4.1 and Proposition 1.1 (iv), it is clear that $\mathscr{X}_k^m[Q(\lambda)] = \mathscr{X}_k^m(A)$. Also, $\sigma_k[Q(\lambda)] = \sigma_k(A)$. Thus, the notions of rank-k numerical hull and rank-k spectrum of matrix polynomials are generalizations of the rank-k numerical hull and rank-k spectrum of matrices, respectively.

In the next theorem, we establish some basic properties of the higher rank numerical hulls of matrix polynomials.

THEOREM 4.3. *Let $Q(\lambda)$ be a matrix polynomial as in (1). Then the following assertions are true:*

(i) $\sigma_k[Q(\lambda)] \subseteq \mathscr{X}_k^m[Q(\lambda)] \subseteq \mathscr{X}_{k-1}^m[Q(\lambda)] \subseteq \cdots \subseteq \mathscr{X}_1^m[Q(\lambda)] = V^m[Q(\lambda)] \subseteq V^{m-1}[Q(\lambda)] \subseteq \cdots \subseteq V^1[Q(\lambda)] = W[Q(\lambda)]$;

(ii) $\sigma_k[Q(\lambda)] \subseteq \mathscr{X}_k^m[Q(\lambda)] \subseteq \mathscr{X}_k^{m-1}[Q(\lambda)] \subseteq \cdots \subseteq \mathscr{X}_k^1[Q(\lambda)] = \Lambda_k[Q(\lambda)] \subseteq \Lambda_{k-1}[Q(\lambda)] \subseteq \cdots \subseteq \Lambda_1[Q(\lambda)] = W[Q(\lambda)]$;

(iii) $\mathscr{X}_k^m[Q(\lambda+\alpha)] = \mathscr{X}_k^m[Q(\lambda)] - \alpha$, *where* $\alpha \in \mathbb{C}$;

(iv) $\mathscr{X}_k^m[\alpha Q(\lambda)] = \mathscr{X}_k^m[Q(\lambda)]$, *where* $\alpha \in \mathbb{C} \setminus \{0\}$;

(v) $\mathscr{X}_k^m[U^*Q(\lambda)U] = \mathscr{X}_k^m[Q(\lambda)]$, *where* $U \in M_n$ *is unitary;*

(vi) $\mathscr{X}_k^m[Q(\lambda)] = \mathscr{X}_k^m[(Q(\lambda))^*]$, *where* $(Q(\lambda))^* = A_s^*\lambda^s + \cdots + A_1^*\lambda + A_0^*$;

(vii) *If* $R(\lambda) = \lambda^s Q(\lambda^{-1}) := A_0\lambda^s + A_1\lambda^{s-1} + \cdots + A_{s-1}\lambda + A_s$, *then*

$$\mathscr{X}_k^m[R(\lambda)] \setminus \{0\} = \{\mu^{-1} : \mu \in \mathscr{X}_k^m[Q(\lambda)], \mu \neq 0\};$$

$(viii)$ *If all the powers of λ in $Q(\lambda)$ are even (or all of them are odd), then $\mathscr{X}_k^m[Q(\lambda)]$ is symmetric with respect to the origin.*

Proof. The results in parts (i), (ii), (iii), (iv) and (vii) follows from Definition 4.1 and Proposition 1.1 $((ii),(iii),(iv))$. By Definition 4.1 and Proposition 2.3 $((ii)$ and $(iii))$, the results in (v) and (vi) can be easily verified. For investigating $(viii)$, assume that all the powers of λ in $Q(\lambda)$ are even. Thus, $\mu \in \mathscr{X}_k^m[Q(\lambda)]$ if and only if $0 \in \mathscr{X}_k^m(Q(\mu)) = \mathscr{X}_k^m(Q(-\mu))$; or equivalently, $-\mu \in \mathscr{X}_k^m[Q(\lambda)]$. Another case in $(viii)$ follows from this fact that $Q(-\mu) = -Q(\mu)$ and using the same manner in the proof of the first case. So, the proof is complete. $\square$

It is known, by Proposition 1.1 (ii), that the higher rank numerical hulls of matrices are bounded sets. In the next example, we show that this result is not necessarily true for matrix polynomials.

EXAMPLE 4.4. Let $A = diag(1,-1,\imath,0)$. Then, by Proposition 1.1 (ii) and [8, Theorem 2.5], we have

$$\mathscr{X}_1^2(A) = V^2(A) = \sigma(A) \cup \{\imath s : 0 \leqslant s \leqslant 1\}.$$

So, by Theorem 4.3 (vii) and the fact that $0 \notin \mathscr{X}_1^2(I)$, we have

$$\mathscr{X}_1^2[A\lambda - I] = \{\mu^{-1} : \mu \in \mathscr{X}_1^2(A), \mu \neq 0\} = \{-1, 1, -\imath s : s \geqslant 1\},$$

which is an unbounded set in the complex plane.

By [2, Proposition 7] and Theorem 4.3 (ii), we have the following result.

PROPOSITION 4.5. *Let $L(\lambda) = A\lambda + B$ be a selfadjoint linear pencil. If A is a positive semidefinite matrix such that $0 \in \sigma_k(A)$, and B is a positive (or negative) definite matrix, then $\mathscr{X}_k^m[\lambda A + B] = \emptyset$ for any $k = 2, 3, \ldots n$.*

At the end of this section, we study the higher rank numerical hulls of block companion matrix of the monic matrix polynomial $Q(\lambda) = I_n\lambda^s - A$, where $A \in M_n$ and $s \geqslant 2$ (to avoid trivial consideration), which is called the basic A-factor block circulant matrix and denoted by

$$\pi_A = \begin{pmatrix} 0 & I_n & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_n & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I_n & 0 \\ 0 & 0 & 0 & \ldots & 0 & I_n \\ A & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \in M_{ns}. \tag{2}$$

Basic A-factor block circulant matrices have important applications in vibration analysis and differential equations; e.g., see [3] and references therein.

It is known, see [1, Remark 3.2], that

$$\pi_A^{ps+q} = \begin{pmatrix} 0 & I_{s-q} \otimes A^p \\ I_q \otimes A^{p+1} & 0 \end{pmatrix} \in M_{ns},$$

where p and q are two nonnegative integers with $q \leqslant s$, I_0 is a vacuous matrix and $A^0 = I_n$. By Proposition 1.1 (iv) and Definition 4.1, we have $\mathscr{X}_k^m[Q(\lambda)] = \sqrt[s]{\mathscr{X}_k^m(A)}$, where for $T \subseteq \mathbb{C}$, $\sqrt[s]{T} := \{\mu \in \mathbb{C} : \mu^s \in T\}$. In the following theorem, we state the relationship between $\mathscr{X}_k^m[Q(\lambda)]$ and the higher rank numerical hulls of π_A.

THEOREM 4.6. *Let $A \in M_n$ and π_A, as in (2), be the basic A-factor block circulant matrix. Then*

$$\sqrt[s]{\mathscr{X}_k^m(A)} \subseteq \mathscr{X}_k^{ms}(\pi_A).$$

Proof. Let $\mu \in \sqrt[s]{\mathscr{X}_k^m(A)}$ be given. Then $\mu^s \in \mathscr{X}_k^m(A)$, and hence, there are nonnegative real numbers $t_1, t_2, \ldots, t_l$ summing to 1, and isometry matrices $X_1, X_2, \ldots, X_l \in \mathscr{I}_{n,k}$ such that

$$\mu^{sj} I_k = \sum_{i=1}^{l} t_i (X_i^* A^j X_i); \qquad j = 0, 1, \ldots, m. \tag{3}$$

Now we consider the following isometries:

$$Y_i = \frac{1}{\sqrt{\sum_{r=0}^{s-1} |\mu|^{2r}}} \begin{pmatrix} X_i \\ \mu X_i \\ \vdots \\ \mu^{s-1} X_i \end{pmatrix} \in \mathscr{I}_{ns,k}; \qquad i = 1,2,\ldots,l.$$

Let $j \in \mathbb{N}$ with $j \leqslant ms$ be given. Then there are integers $0 \leqslant p \leqslant m-1$ and $0 \leqslant q \leqslant s$ such that $j = ps + q$. So, for any $i = 1,2,\ldots,l$, we have:

$$\begin{aligned}\sum_{r=0}^{s-1} |\mu|^{2r}(Y_i^* \pi_A^j Y_i) &= (\sum_{r=0}^{s-1} |\mu|^{2r})(Y_i^* \pi_A^{ps+q} Y_i) \\ &= (\sum_{r=q}^{s-1} \mu^r \bar{\mu}^{r-q})(X_i^* A^p X_i) + (\sum_{r=0}^{q-1} \mu^r \bar{\mu}^{s-q+r})(X_i^* A^{p+1} X_i),\end{aligned}$$

where for the case $q = 0$ we only consider the left summation and also for the case $q = s$ we only consider the right summation. Therefore, by (3), we have

$$\begin{aligned}(\sum_{r=0}^{s-1} |\mu|^{2r})(\sum_{i=1}^{l} t_i(Y_i^* \pi_A^j Y_i)) &= \sum_{i=1}^{l} t_i[(\sum_{r=0}^{s-1} |\mu|^{2r}) Y_i^* \pi_A^j Y_i] \\ &= \sum_{i=1}^{l} t_i[(\sum_{r=q}^{s-1} \mu^r \bar{\mu}^{r-q})(X_i^* A^p X_i) + (\sum_{r=0}^{q-1} \mu^r \bar{\mu}^{r+s-q})(X_i^* A^{p+1} X_i)] \\ &= (\sum_{r=q}^{s-1} \mu^r \bar{\mu}^{r-q}) \sum_{i=1}^{l} t_i X_i^* A^p X_i + (\sum_{r=0}^{q-1} \mu^r \bar{\mu}^{r+s-q}) \sum_{i=1}^{l} t_i X_i^* A^{p+1} X_i \\ &\overset{(3)}{=} \sum_{r=q}^{s-1} \mu^r \bar{\mu}^{r-q} \mu^{ps} I_k + \sum_{r=0}^{q-1} \mu^r \bar{\mu}^{r+s-q} \mu^{s(p+1)} I_k \\ &= \sum_{r=q}^{s-1} \mu^{r-q} \bar{\mu}^{r-q} \mu^{ps+q} I_k + \sum_{r=0}^{q-1} \mu^{r+s-q} \bar{\mu}^{r+s-q} \mu^{ps+q} I_k \\ &= (\sum_{r=q}^{s-1} |\mu|^{2(r-q)}) \mu^{ps+q} I_k + (\sum_{r=0}^{q-1} |\mu|^{2(r+s-q)}) \mu^{ps+q} I_k \\ &= (\sum_{r=0}^{s-1} |\mu|^{2r}) \mu^j I_k.\end{aligned}$$

So, $\sum_{i=1}^{l} t_i(Y_i^* \pi_A^j Y_i) = \mu^j I_k$. Therefore, $(\mu, \mu^2, \ldots, \mu^{ms}) \in conv(\Lambda_k(\pi_A, \pi_A^2, \ldots, \pi_A^{ms}))$, and hence $\mu \in \mathscr{X}_k^{ms}(\pi_A)$. So, the proof is complete. $\square$

PROPOSITION 4.7. *Let $A \in M_n$ and π_A, as in (2), be the basic A-factor block circulant matrix. Then*

$$\sqrt[s]{\Lambda_k(A)} \subseteq \mathscr{X}_k^s(\pi_A),$$

and the equality holds if $k = 1$ or A is a scalar matrix.

Proof. By setting $m=1$ in Theorem 4.6, the inclusion $\subseteq$ holds. For the case $k=1$, the equality holds by [1, Theorem 3.9]. If $A=\alpha I$, for some $\alpha\in\mathbb{C}$, and $\mu\in\mathscr{X}_k^s(\pi_A)$, then $(\mu,\mu^2,\dots,\mu^s)\in conv(\Lambda_k(\pi_A,\pi_A^2,\dots,\pi_A^s))$. So, there exist nonnegative real numbers $t_1,\dots,t_l$ with $\sum_{i=1}^l t_i=1$, and $Y_1,\dots,Y_l\in\mathscr{I}_{ns,k}$, where $l\in\mathbb{N}$, such that

$$\mu^s I_k=\sum_{i=1}^l t_i Y_i^*\pi_A^s Y_i.$$

Since $\pi_A^s=\underbrace{A\oplus\cdots\oplus A}_{s-times}=\alpha I_{ns}$, the above relation implies that $\mu^s I_k=\alpha I_k$ and hence, $\mu^s=\alpha$. So, $\mu\in\sqrt[s]{\{\alpha\}}=\sqrt[s]{\Lambda_k(A)}$, and hence the equality also holds. $\square$

PROPOSITION 4.8. *Let $A\in M_n$, $k\leqslant n$ and $m\leqslant s-1$. Then $0\in\mathscr{X}_k^m(\pi_A)$.*

Proof. Since $m\leqslant s-1$, the $(1,1)$-block of π_A^j, where $j=1,2,\dots,m$, is the zero matrix. Hence, by setting

$$X=\begin{pmatrix} I_{n\times k}\\ \hline 0\\ \hline \vdots\\ \hline 0\end{pmatrix}\in M_{ns\times k},$$

where $I_{n\times k}=\begin{pmatrix} I_k\\ 0\end{pmatrix}\in M_{n\times k}$, we have $X\in\mathscr{I}_{ns,k}$ and $X^*\pi_A^j X=0I_k$ for $j=1,2,\dots,m$. So, $\underbrace{(0,0,\dots,0)}_{m-\text{times}}\in\Lambda_k(\pi_A,\pi_A^2,\dots,\pi_A^m)$, and hence, $0\in\mathscr{X}_k^m(\pi_A)$. $\square$

THEOREM 4.9. *Let $A\in M_n$ and π_A, as in (2), be the basic A-factor block circulant matrix. Then*

(i) $\mathscr{X}_k^m(A)\subseteq\mathscr{X}_{k^2}^m(\pi_A^s)$;

(ii) If $m\geqslant 2$ and A is Hermitian, and $(r-1)s<k\leqslant rs$ for some $1\leqslant r\leqslant n$, then

$$\mathscr{X}_k^m(\pi_A^s)=\sigma_r(A).$$

Proof. Since $\pi_A^s=I_s\otimes A$, by Theorem 2.9 (i), $\mathscr{X}_{k^2}^m(\pi_A^s)\supseteq\mathscr{X}_k^m(A)\mathscr{X}_k^m(I)=\mathscr{X}_k^m(A)$, and hence, the result in (i) holds. To prove the result in (ii), by Proposition 1.1 $((i),(v))$, we have

$$\mathscr{X}_k^m(\pi_A^s)=\mathscr{X}_k^m(I_s\otimes A)=\sigma_k(I_s\otimes A)=\sigma_k(\underbrace{A\oplus\cdots\oplus A}_{s-\text{times}})\supseteq\sigma_{rs}(\underbrace{A\oplus\cdots\oplus A}_{s-\text{times}})=\sigma_r(A).$$

For the converse, let $\lambda\in\sigma_k(\underbrace{A\oplus\cdots\oplus A}_{s-\text{times}})$. Then $dim(ker((\lambda I_n-A))\geqslant k/s$. Since $(r-1)<k/s\leqslant r$, $dim(ker(\lambda I_n-A))\geqslant r$, and hence, $\lambda\in\sigma_r(A)$. Therefore, $\mathscr{X}_k^m(\pi_A^s)=\sigma_k(I_s\otimes A)\subseteq\sigma_r(A)$, and so, the result holds. $\square$

At the final proposition, we study the higher rank numerical hulls of unitary basic A-factor block circulant matrices.

PROPOSITION 4.10. *Let $A \in M_n$ be a unitary matrix. Then*

(i) $\mathscr{X}_k^m(\pi_A) \cap \sigma(\pi_A) = \sigma_k(\pi_A) = \sqrt[s]{\sigma_k(A)}$;

(ii) If all eigenvalues of A are distinct, $k,m \geqslant 2$ and $ns = 2k$, then $\mathscr{X}_k^m(\pi_A) = \emptyset$.

Proof. The result in (i) follows from Proposition 1.1 (vi). It is known that $\sigma(\pi_A) = \sigma[Q(\lambda)] = \{\sqrt[s]{\mu} : \mu \in \sigma(A)\}$. Since eigenvalues of A are distinct, the eigenvalues of π_A are also distinct. By [1, Theorem 3.3], π_A is unitary, and hence, Proposition 1.1 (vii) implies $\mathscr{X}_k^m(\pi_A) = \emptyset$. So, the proof is complete. $\square$

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