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MAPS PRESERVING THE FIXED POINTS OF SUM OF OPERATORS

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Abstract. Let $\mathscr{B}(\mathscr{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathscr{X}$ with $\dim \mathscr{X} \geqslant 2$. In this paper, we characterize the maps on $\mathscr{B}(\mathscr{X})$ which preserve the fixed points of sum of operators. Moreover, if $\mathscr{X}$ is a finite dimensional Banach space, we also characterize the maps on $\mathscr{B}(\mathscr{X})$ which preserve the dimension of fixed points of sum of operators.

1. Introduction

The study of maps on operator algebras preserving certain properties is a topic which attracts much attention of many authors (see [1]–[18] and the references cited there.) Some of these problems are concerned with preserving a certain property of sum or products of operators (see [1]–[11] and [16]–[18]).

Let $\mathscr{B}(\mathscr{X})$ denote the algebra of all bounded linear operators on a complex Banach space $\mathscr{X}$. Recall that $x \in \mathscr{X}$ is a fixed point of an operator $A \in \mathscr{B}(\mathscr{X})$, whenever we have $Ax = x$. For $A \in \mathscr{B}(\mathscr{X})$, denote by $\mathrm{Lat}A$ and $F(A)$ the lattice of A, that is, the set of all invariant subspaces of A and the set of all fixed points of A, respectively. Authors in [3] characterize the maps $\phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ which satisfy one of the following preserving properties: $\mathrm{Lat}(A+B) = \mathrm{Lat}(\phi(A)+\phi(B))$, or $\mathrm{Lat}(AB) = \mathrm{Lat}(\phi(A)\phi(B))$, or $\mathrm{Lat}(AB+BA) = \mathrm{Lat}(\phi(A)\phi(B)+\phi(B)\phi(A))$, or $\mathrm{Lat}(ABA) = \mathrm{Lat}(\phi(A)\phi(B)\phi(A))$, or $\mathrm{Lat}(AB-BA) = \mathrm{Lat}(\phi(A)\phi(B)-\phi(B)\phi(A))$.

Since $F(A) \in \mathrm{Lat}A$, one can replace the lattice preserving property by the fixed points preserving property. Moreover, one can consider the maps preserving the dimension of fixed points which is a very weaker condition. For usual product, in [16], we characterized the surjective maps $\phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ satisfying $\dim F(AB) = \dim F(\phi(A)\phi(B))$, where $\dim F(T)$ denotes the dimension of $F(T)$.

In this paper, we characterize the maps $\phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ and $\phi : \mathscr{M}_n \to \mathscr{M}_n$ satisfying $F(A+B) = F(\phi(A)+\phi(B))$ and $\dim F(A+B) = \dim F(\phi(A)+\phi(B))$, respectively.

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2. Maps preserving the dimension of fixed points of sum of operators

Recall that two operators A and B are adjacent if $A-B$ is of rank one.

LEMMA 2.1. *Let n be an integer number such that $n \geqslant 2$. Suppose that $\phi : \mathscr{M}_n \longrightarrow \mathscr{M}_n$ is a map which satisfies*

$$\dim F(A+B) = \dim F(\phi(A)+\phi(B)) \quad (A,B \in \mathscr{M}_n).$$

Then the following statements are hold:

(i) *ϕ is injective.*

(ii) *If ϕ is surjective, then ϕ preserves adjacency in both directions.*

Proof. (i) From $n = \dim F(A+I-A) = \dim F(\phi(A)+\phi(I-A))$ we obtain

$$I = \phi(A)+\phi(I-A) \tag{2.1}$$

for every $A \in \mathscr{M}_n$. Let $\phi(A_1) = \phi(A_2)$. By (2.1) we have

$$\begin{aligned} n &= \dim F(\phi(A_1)+I-\phi(A_2)) \\ &= \dim F(\phi(A_1)+\phi(I-A_2)) \\ &= \dim F(A_1+I-A_2), \end{aligned}$$

which implies that $A_1+I-A_2 = I$ and so $A_1 = A_2$.

(ii) Let A and B be two matrices such that $A-B$ is of rank one. Preserving property of ϕ together with (2.1) implies that

$$\begin{aligned} n-1 = \dim\ker(A-B) &= \dim F(A-B+I) \\ &= \dim F(\phi(A)+\phi(I-B)) \\ &= \dim F(\phi(A)+I-\phi(B)) \\ &= \dim\ker(\phi(A)-\phi(B)) \end{aligned}$$

which implies that $\phi(A)-\phi(B)$ is of rank one and so ϕ preserves adjacency. Since ϕ^{-1} has the preserving property of ϕ, we can conclude that ϕ preserves adjacency in both directions. □

THEOREM 2.2. *Let n be an integer number such that $n \geqslant 2$. Suppose that $\phi : \mathscr{M}_n \longrightarrow \mathscr{M}_n$ is a surjective map which satisfies*

$$\dim F(A+B) = \dim F(\phi(A)+\phi(B)) \quad (A,B \in \mathscr{M}_n).$$

Then there exists a matrix $R \in \mathscr{M}_n$ and invertible matrices $U,S \in \mathscr{M}_n$ such that $\phi(A) = US^{-1}A_\sigma S+R$ or $\phi(A) = US^{-1}A_\sigma^t S+R$, for every $A \in M_n$, where σ is an automorphism of $\mathbb{C}$ and $A_\sigma = [\sigma(a_{ij})]$ for $A = [a_{ij}]$.

Proof. By Lemma 2.1, ϕ is injective and so bijective and preserves adjacency in both directions. By fundamental theorem of geometry of matrices [8], the forms of bijective adjacency preserving map $\phi : \mathscr{M}_n \longrightarrow \mathscr{M}_n$ is $\phi(A) = TA_\sigma S + R$ or $\phi(A) = TA^t_\sigma S + R$, where R is a matrix, T, S are invertible matrices, σ is an automorphism of the underlying field and $A_\sigma = [\sigma(a_{ij})]$ for $A = [a_{ij}]$.

Let the first case occurs. From (2.1) we have $\phi(0) + \phi(I) = I$ which implies that $R + TI_\sigma S + R = I$. Since $I_\sigma = I$, we obtain $TS = I - 2R$. Setting $U = I - 2R$, we obtain $T = US^{-1}$. Therefore, $\phi(A) = US^{-1}A_\sigma S + R$. Since

$$\dim F(2R)) = \dim F(2\phi(0)) = \dim F(0) = 0,$$

$F(2R) = \ker(I - 2R) = 0$ and so $I - 2R$ is invertible. In a similar way, we can obtain the second case. □

3. Maps preserving the fixed points of sum of operators

We recall some notations. $\mathscr{X}^*$ denotes the dual space of $\mathscr{X}$. For every nonzero $x \in \mathscr{X}$ and $f \in \mathscr{X}^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $\mathscr{X}$ defined by $(x \otimes f)y = f(y)x$ for every $y \in \mathscr{X}$. Note that every rank one operator in $\mathscr{B}(\mathscr{X})$ can be written in this way. The rank one operator $x \otimes f$ is idempotent if and only if $f(x) = 1$. We denote by $\mathscr{F}_1(\mathscr{X})$ and $\mathscr{P}_1(\mathscr{X})$ the set of all rank one operators and the set of all rank one idempotent operators on $\mathscr{X}$, respectively.

Let $x \otimes f$ be a rank one operator. It is easy to check that $x \otimes f$ is an idempotent if and only if $F(x \otimes f) = \langle x \rangle$ (the linear subspace spanned by x). If $x \otimes f$ isn't idempotent, then $F(x \otimes f) = \{0\}$.

Let $x, y \in \mathscr{X}$. We denote by $\mathrm{Gcv}\{x, y\} = \{\lambda x + (1-\lambda)y : \lambda \in \mathbb{C}\}$ the generalized convex combination of x and y.

In order to prove the main results of this section, first we prove some auxiliary lemmas. In the following lemmas assume that $\dim \mathscr{X} \geqslant 3$.

LEMMA 3.1. [5] *Let $A, B \in \mathscr{B}(\mathscr{X})$ be non-scalar operators. Suppose that for every such $x \in \mathscr{X}$ that x and Ax are linearly independent or that $x = Ax$, $Bx \in$* $\mathrm{Gcv}\{x, Ax\}$. *Then $B = \lambda I + (1-\lambda)A$ for some $\lambda \in \mathbb{C} \setminus \{1\}$.*

LEMMA 3.2. *Let $A, B \in \mathscr{B}(\mathscr{X})$ be non-scalar operators. If $F(A+P) = F(B+P)$, for every $P \in \mathscr{P}_1(\mathscr{X})$, $B = \lambda I + (1-\lambda)A$, for some $\lambda \in \mathbb{C} \setminus \{1\}$.*

Proof. By Lemma 3.1, it is enough to consider the following two cases.

Case 1. Let x and Ax be linear independent. So there exists a linear functional f such that $f(x) = 1$ and $f(Ax) = 0$. Setting $P = (x - Ax) \otimes f$ yields that $(A+P)x = x$ which implies that $(B+P)x = x$ and so $Ax = Bx$. Therefore, $Bx \in \mathrm{Gcv}\{x, Ax\}$.

Case 2. Let $x = Ax$. There exists a vector $z \in \mathscr{X}$ such that x and z are linear independent and so there exists a linear functional f such that $f(x) = 0$ and $f(z) = 1$. Setting $P = z \otimes f$ yields that $(A+P)x = x$ which implies that $(B+P)x = x$ and so $Ax = Bx$. Therefore, $Bx \in \mathrm{Gcv}\{x, Ax\}$. □

LEMMA 3.3. *Let $A, B \in \mathscr{B}(\mathscr{X})$. If $F(A+R) = F(B+R)$, for every $R \in \mathscr{F}_1(\mathscr{X})$, then $A = B$.*

Proof. Let $x \in \mathscr{X}$. If x and Ax is linear independent, then there exists a linear functional f such that $f(x) = f(Ax) = 1$, because $\dim \mathscr{X} \geqslant 3$. Setting $R = (x - Ax) \otimes f$ yields that $x \in F(A+R)$ which implies that $x \in F(B+R)$ and so $Ax = Bx$.

Let $Ax = ax$ for a nonzero complex number a. There exists a linear functional f such that $f(x) = 1$. Setting $R = (1-a)x \otimes f$ yields that $x \in F(A+R)$ which implies that $x \in F(B+R)$ and so $Ax = Bx$. The proof is complete. $\square$

LEMMA 3.4. *Let $\phi : \mathscr{B}(\mathscr{X}) \longrightarrow \mathscr{B}(\mathscr{X})$ be a map which satisfies*

$$F(A+B) = F(\phi(A) + \phi(B)) \quad (A, B \in \mathscr{B}(\mathscr{X})).$$

Then the following statements are hold:

(i) ϕ is injective.

(ii) $\phi(P) = UP + R$ for every rank one idempotent P, where $U = I - 2\phi(0)$ and $R = \phi(0)$.

Proof. (i) From

$$\mathscr{X} = F(A + I - A) = F(\phi(A) + \phi(I - A)),$$

we obtain

$$I = \phi(A) + \phi(I - A), \tag{3.1}$$

for every $A \in \mathscr{B}(\mathscr{X})$. Let $\phi(A_1) = \phi(A_2)$. By (3.1) we have

$$\mathscr{X} = F(\phi(A_1) + I - \phi(A_2)) = F(\phi(A_1) + \phi(I - A_2)) = F(A_1 + I - A_2),$$

which implies that $A_1 + I - A_2 = I$ and so $A_1 = A_2$.

(ii) For every nonzero $x \in \mathscr{X}$ and nonzero $f \in \mathscr{X}^*$, we have

$$\ker(f) = F(x \otimes f + I) = F(\phi(x \otimes f) + \phi(I))$$

which implies that

$$\phi(x \otimes f) + \phi(I) = I \tag{3.2}$$

or there exists a vector $y \in \mathscr{X}$ and a linear functional $g \in \mathscr{X}^*$ such that

$$\phi(x \otimes f) + \phi(I) = y \otimes g + I. \tag{3.3}$$

By (3.1) we have

$$\phi(I) + \phi(0) = I \tag{3.4}$$

and so if (3.2) holds, then $\phi(x \otimes f) = \phi(0)$. This is a contradiction, because ϕ is injective. Thus (3.3) holds and so $\ker(f) = F(y \otimes g + I) = \ker(g)$ which implies that

f and g are linear dependent. Without loss of generality, we can assume that $f = g$ and hence

$$\phi(x \otimes f) + \phi(I) = y \otimes f + I. \tag{3.5}$$

Let $f(x) = 1$. By (3.4) and (3.5) we have

$$\begin{aligned} \langle x \rangle = F(x \otimes f) &= F(\phi(x \otimes f) + \phi(0)) \\ &= F(y \otimes f + I - \phi(I) + \phi(0)) \\ &= F(y \otimes f + 2\phi(0)), \end{aligned}$$

which implies that $(y \otimes f + 2\phi(0))x = x$ and so $y = (I - 2\phi(0))x$. This together with (3.4) and (3.5) yields that $\phi(x \otimes f) = (I - 2\phi(0))x \otimes f + \phi(0)$ which completes the proof. □

THEOREM 3.5. *Let $\mathscr{X}$ be a complex Banach space with* $\dim \mathscr{X} \geqslant 2$. *Suppose that $\phi : \mathscr{B}(\mathscr{X}) \longrightarrow \mathscr{B}(\mathscr{X})$ is a surjective map which satisfies*

$$F(A+B) = F(\phi(A) + \phi(B)) \quad (A, B \in \mathscr{B}(\mathscr{X})).$$

Then $\phi(A) = UA + R$ for every $A \in \mathscr{B}(\mathscr{X})$, where $U = I - 2\phi(0)$ and $R = \phi(0)$.

Proof. If $\dim \mathscr{X} = 2$, from Theorem 2.2 we can conclude that there exists a matrix $R \in \mathscr{M}_2$ and invertible matrices $U, S \in \mathscr{M}_2$ such that $\phi(A) = US^{-1}A_\sigma S + R$ or $\phi(A) = US^{-1}A^t_\sigma S + R$, for every $A \in M_2$, where σ is an automorphism of $\mathbb{C}$. Suppose the first case occurs. This by assumption yields that

$$F(A+B) = F(U(S^{-1}(A_\sigma + B_\sigma)S) + 2R)$$

for every $A, B \in \mathscr{M}_2$. It is easy to see that for arbitrary operators $A, B \in \mathscr{B}(\mathscr{X})$, $F(S^{-1}AS) = S^{-1}(F(A))$ and also $F(A+B) = F(U(A+B) + 2R)$. Therefore, we obtain

$$SF(A+B) = F(A_\sigma + B_\sigma)$$

for every $A, B \in \mathscr{M}_2$. Replacing A and B by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_1 = 0$$

and then

$$A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B_2 = 0$$

yields that $S(x,0) = (x,0)$ and $S(0,y) = (0,y)$, for all $x, y \in \mathbb{C}$ and hence S is the identity operator. Thus we obtain

$$F(A+B) = F(A_\sigma + B_\sigma)$$

for every $A,B\in\mathscr{M}_2$. Let $a\in\mathbb{C}$ be nonzero and set

$$C=\begin{pmatrix}1&0\\a&0\end{pmatrix},D=0.$$

Replacing A and B by C and D yields that

$$\{(x,ax);x\in\mathbb{C}\}=\{(x,\sigma(a)x);x\in\mathbb{C}\}$$

which implies that $\sigma(a)=a$, for every $a\in\mathbb{C}$ and so σ is the identity automorphism. Therefore, $\phi(A)=UA+R$ for every $A\in\mathscr{M}_2$.

The second case can not occur. Otherwise, with a very similar way as above discussion we obtain

$$F(A+B)=F(A^t_\sigma+B^t_\sigma)$$

for every $A,B\in\mathscr{M}_2$. Again replacing A and B by C and D, yields that

$$\{(x,ax);x\in\mathbb{C}\}=\{(x,0);x\in\mathbb{C}\}$$

which implies that $a=0$, a contradiction.

Now let $\dim\mathscr{X}\geqslant 3$. Since $F(2\phi(0))=F(0)=0$, $\ker(I-2\phi(0))=0$ and so $U=I-2\phi(0)$ is injective. Let ψ be a map on $\mathscr{B}(\mathscr{X})$ such that $U\psi=\phi-R$, where $R=\phi(0)$. The injectivity of U yields that ψ is well-defined and that ψ satisfies the preserving property of ϕ and also by Lemma 3.4, $\psi(P)=P$ for every rank one idempotent P. Therefore, without loss of generality, we can assume that $U=I$, $R=0$ and so $\phi(P)=P$ for every rank one idempotent P.

We divide the proof into the following steps.

Step 1. $\phi(aI)=aI$, for every $a\in\mathbb{C}$.

Let $A=aI$, for a nonzero complex number a and set $\phi(A)=B$. Let x be a nonzero arbitrary vector of $\mathscr{X}$. Then there exists a linear functional f such that $f(x)=1$. Setting $P=(x-Bx)\otimes f$ yields that $(B+P)x=x$ which implies that $(A+\phi^{-1}(P))x=x$. So we obtain

$$(A+P)x=x\Rightarrow ax+x-Bx=x\Rightarrow Bx=ax.$$

Step 2. $\phi(A)=A$ for every rank one operator A.

Let $x\in\mathscr{X}$ and $f\in\mathscr{X}^*$ be nonzero. Since $\phi(I)=I$, by (3.5), there exists a $y\in\mathscr{X}$ such that

$$\phi(x\otimes f)=y\otimes f. \tag{3.6}$$

On the other hand, by Lemma 3.2, there exists an $\lambda\in\mathbb{C}\setminus\{1\}$ such that

$$\phi(x\otimes f)=\lambda I+(1-\lambda)x\otimes f, \tag{3.7}$$

because by Step 1, $\phi(x\otimes f)$ is a non-scalar operator. From (3.6) and (3.7) we obtain $\lambda=0$ and $y=x$ and so $\phi(x\otimes f)=x\otimes f$.

Step 3. $\phi(A)=A$ for every operator $A\in\mathscr{B}(\mathscr{X})$.

Assertion follows from Step 2 and Lemma 3.3. □

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