Operators
and
Matrices
Volume 9, Number 3 (2015), 571–596 doi:10.7153/oam-09-35

FUNCTIONAL DECOMPOSITION THEOREMS FOR C^*–MATRIX OPERATOR SPACES

TITARII WOOTIJIRATTIKAL, SING-CHEONG ONG AND JITTI RAKBUD

(*Communicated by D. Hadwin*)

Abstract. Let S be a nonempty set; and let $\mathscr{A}$ be a fixed C^*-algebra with state space $s(\mathscr{A})$ equipped with the relative weak* topology inherited from the dual space $\mathscr{A}^\#$ of $\mathscr{A}$. Let $\mathscr{X}$ be the space of all functions $\mathbf{x}: S \to \mathscr{A}$ such that $\varphi \circ (\mathbf{x}^*\mathbf{x}) \in \ell^1(S)$ for all $\varphi \in s(\mathscr{A})$, and the map $\varphi \to \varphi \circ (\mathbf{x}^*\mathbf{x})$ is weak* to norm continuous from $s(\mathscr{A})$ to $\ell^1(S)$. Elementary methods are used to show that the space $\mathscr{M}$ of $\mathscr{A}$-valued functions on $S \times S$ that define bounded operators on $\mathscr{X}$ contains a closed subspace $\mathscr{K}$ such that the annihilator $\mathscr{K}^\perp$ is an ℓ^1 direct summand of the dual space $\mathscr{M}^\#$ of $\mathscr{M}$; i.e., $\mathscr{M}$ contains an M-ideal. When $\mathscr{A}$ is specialized to the complex field, this is a classical theorem of Dixmier. An analogue of the trace formula $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ for a trace class operator A and a bounded operator B on a Hilbert space is proved.

1. Introduction

As defined in [1], a closed subspace J of a Banach space X is called an M-ideal if the annihilator $J^\perp$ of J is an ℓ^1 direct summand in the dual space $X^\#$ of X. That is each bounded linear functional f on X has a unique ℓ^1 decomposition $f = g + h$, where $g = f\big|_J$, $h\big|_J \equiv 0$, and $\|f\| = \|g\| + \|h\|$. Dixmier [2] proved that the compact operators form an M-ideal in the algebra of bounded operators on a Hilbert space. In [4] it is proved that same is true for operators on the sequence spaces ℓ^p, $1 < p < \infty$, and c_0. Many more examples have been constructed over the years. Most are related to operators. Smith and Ward [7] proved that each M-ideal in a C^*-algebra is in fact an ideal, and an M-ideal in a Banach algebra must be a subalgebra. Much of the recent work on M-ideals can be found in [3]. With a fixed C^*-algebra $\mathscr{A}$, we will use elementary methods to construct a Banach algebra of $\mathscr{A}$-matrix operators on a certain $\mathscr{A}$-valued function space that contains an M-ideal. The Banach algebra constructed is not a C^*-algebra.

Mathematics subject classification (2010): Primary 46A20; Secondary 47L50, 47L10, 47L75.
Keywords and phrases: C^*-matrix operator, Banach algebra, M-ideal.
[1] This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.
[2] Work done during a sabbatical leave from Central Michigan University, visiting Silpakorn University and Ubon Ratchathani University. I thank J. Rakbud and T. Wootijirattikal and their department for their warm hospitality during my visit.

Paper OaM-09-35

For a fix a nonempty set S, denote by $\mathscr{F}(S)$, or simply $\mathscr{F}$ if no ambiguity, the family of all finite subsets of S directed by set inclusion. For a function $\mathbf{x}$ from S to a Banach space X, the sum $\sum_{s\in S}\mathbf{x}(s)$ is said to *converge to* $x \in X$ ([5, p. 25]) if the net of finite partial sums, $\left\{\sum_{s\in F}\mathbf{x}(s)\right\}_{F\in\mathscr{F}(S)}$, converges to x. When this is the case we write $\sum_{s\in S}\mathbf{x}(s) = x$. That is,

$$x = \sum_{s\in S}\mathbf{x}(s) \quad \text{iff} \quad \lim_{F\in\mathscr{F}(S)}\left\|x - \sum_{s\in F}\mathbf{x}(s)\right\| = 0.$$

All the classical sequence spaces ℓ^p have their generalized versions $\ell^p(S)$ of spaces of real- or complex-valued functions defined on S.

Fix a C^*-algebra $\mathscr{A}$ with identity 1 and state space $s(\mathscr{A})$ (consisting of all *states* on $\mathscr{A}$, that is all positive linear functionals φ with $\|\varphi\| = \varphi(1) = 1$ [5, p. 257]). With the relative weak* topology it inherits from the dual space $\mathscr{A}^\#$ of $\mathscr{A}$, $s(\mathscr{A})$ is a compact Hausdorff space [5, p. 257]. Let $\mathscr{X}$ be the Banach space $\ell^2_{*u}(S,\mathscr{A})$ of $\mathscr{A}$-valued functions $\mathbf{x} : S \to \mathscr{A}$ such that the map $\varphi \mapsto \varphi\circ(\mathbf{x}^*\mathbf{x})$ is weak* to norm continuous from $s(\mathscr{A})$ to $\ell^1(S)$ [10]. A function $A : S\times S \to \mathscr{A}$ is said to *define an operator on* $\mathscr{X}$ if for each $\mathbf{x} \in \mathscr{X}$,

$$(A\mathbf{x})(s) := \sum_{t\in S}A(s,t)\mathbf{x}(t) \text{ converges in } \mathscr{A}, \text{ for each } s\in S; \quad \text{and} \quad A\mathbf{x}\in\mathscr{X}.$$

An $\mathscr{A}$-valued function A on $S\times S$ that defines an operator on $\mathscr{X}$ is called an $\mathscr{A}$-*matrix operator*. Each $\mathscr{A}$-matrix operator is automatically bounded, and the space $\mathscr{M} := \mathscr{M}(\mathscr{X})$ of all $\mathscr{A}$- matrix operators is a Banach algebra [10, Theorem 3.4]. We will show that $\mathscr{M}$ contains an M-ideal. (There are Banach spaces of $\mathscr{A}$-valued functions constructed from operators which contain M-ideals [8, 9]. But elements in those spaces are not operators and there are no apparent way of defining product of the elements.)

2. Notation and preliminaries

With a fixed nonempty set S, for each $p\in[1,\infty)$, denote by $\ell^p(S) := \ell^p(S,\mathbb{C})$ the space of complex-valued functions on S that are p-th power absolutely summable over S. The norm on $\ell^p(S)$ is given by,

$$\|x\|_p = \left[\sum_{s\in S}|x(s)|^p\right]^{1/p} \qquad x\in\ell^p(S).$$

The proofs for the classical ℓ^p spaces can be easily adapted to show that each $\ell^p(S)$ is a Banach space with this norm.

A C^*-algebra $\mathscr{A}$ with identity 1 and state space $s(\mathscr{A})$ will also be fixed along with the set S. Each $\varphi \in s(\mathscr{A})$ defines a semi-inner product: $\langle a, b\rangle_\varphi = \varphi(b^* a)$, for $a, b \in \mathscr{A}$ [5, p. 256]. The induced semi-norm is $\|a\|_\varphi = \sqrt{\langle a, a\rangle_\varphi}$, for $a \in \mathscr{A}$. Given functions $\mathbf{x}, \mathbf{y} : S \to \mathscr{A}$, the product $\mathbf{xy}$ is defined pointwise: $\mathbf{xy}(s) = \mathbf{x}(s)\mathbf{y}(s)$ for $s \in S$. So is the involution * (the unary adjoint operation on $\mathscr{A}$): $\mathbf{x}^*(s) = (\mathbf{x}(s))^*$ for $s \in S$. For each $G \subseteq S$, $\mathbf{x}_G$ denotes the function $\mathbf{x}_G(s) = \mathbf{x}(s)$ for $s \in G$ and $\mathbf{x}_G(s) = 0$ for $s \in S \setminus G$, i.e., $\mathbf{x}_G = \chi_G \mathbf{x}$, where χ_G is the characteristic function of G.

We summarize results from [10] that will be used here. Let $\mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$ be the set of all functions $\mathbf{x} : S \to \mathscr{A}$ such that $\varphi \circ (\mathbf{x}^*\mathbf{x}) \in \ell^1(S)$ for all $\varphi \in s(\mathscr{A})$, and the map $\varphi \mapsto \varphi \circ (\mathbf{x}^*\mathbf{x})$ from $s(\mathscr{A})$ to $\ell^1(S)$ is weak* to norm continuous. (This is equivalent to uniformity (in $\varphi \in s(\mathscr{A})$) of the convergence of the sum of the functions $\varphi \circ (\mathbf{x}^*\mathbf{x})$; thus the subscript u in the notation.) Then, $\mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$ is a Banach space with the norm

$$\|\mathbf{x}\|^2 := \sup_{\varphi \in s(\mathscr{A})} \left\| \varphi \circ (\mathbf{x}^*\mathbf{x}) \right\|_{\ell^1(S)} = \sup_{\varphi \in s(\mathscr{A})} \left(\sum_{s \in S} \|\mathbf{x}(s)\|^2_\varphi \right).$$

The larger space of all functions $\mathbf{x} : S \to \mathscr{A}$ such that

$$\sqrt{\varphi \circ (\mathbf{x}^*\mathbf{x})} = \|\mathbf{x}(\cdot)\|_\varphi \in \ell^2(S) \qquad \text{for all } \varphi \in s(\mathscr{A})$$

(without continuity), is denoted by $\ell^2_*(S, \mathscr{A})$, which is also a Banach space with the same norm above. It is clear from the definition that $\ell^2_*(S, \mathscr{A}) \supseteq \mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$, and the inclusion is in fact proper. Alternate descriptions of memberships of the spaces $\ell^2_*(S, \mathscr{A})$ and $\mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$ are given below.

THEOREM 1. [10, Propositions 5.1-2] *Let $\mathbf{x} \in \mathscr{A}^S$ (the space of functions from S to $\mathscr{A}$). Then*

(i) $\mathbf{x} \in \ell^2_*(S, \mathscr{A})$ *iff* $\sup_{F \in \mathscr{F}} \left\| \sum_{s \in F} (\mathbf{x}^*\mathbf{x})(s) \right\| < \infty$; *and*

(ii) $\mathbf{x} \in \ell^2_{*u}(S, \mathscr{A}) = \mathscr{X}$ *iff* $\sum_{s \in S} (\mathbf{x}^*\mathbf{x})(s)$ *converges in* $\mathscr{A}$.

The following proposition shows some resemblance of the pairs $(\ell^2_{*u}(S, \mathscr{A}), \ell^2_*(S, \mathscr{A}))$ and (ℓ^1, ℓ^∞), in that each bounded linear functional on $\mathscr{X}$ has a unique Hahn-Banach extension to all of $\ell^2_*(S, \mathscr{A})$.

PROPOSITION 2. *For each $g \in \mathscr{X}^\# = [\ell^2_{*u}(S, \mathscr{A})]^\#$ (the dual space of $\mathscr{X}$), there is a function $\widetilde{g} : S \to \mathscr{A}^\#$ such that*

$$\widehat{g}(\mathbf{x}) = \sum_{s \in S} [\widetilde{g}(s)](\mathbf{x}(s)) \qquad \text{converges for all } \mathbf{x} \in \ell^2_*(S, \mathscr{A}).$$

Furthermore $\widehat{g} \in [\ell^2_*(S,\mathscr{A})]^\#$ *with* $\widehat{g}\Big|_{\mathscr{X}} = g$, *and* $\|\widehat{g}\| = \|g\|$.

Proof. Let $g \in \mathscr{X}^\#$. For each $s \in S$, define $[\widetilde{g}(s)](a) = g(\mathbf{e}_s(a))$, where $(\mathbf{e}_s(a))(s) = a$ and $(\mathbf{e}_s(a))(t) = 0$ for $t \neq s$. Then

$$|[\widetilde{g}(s)](a)| \leqslant \|g\| \, \|\mathbf{e}_s(a)\| = \|g\| \, \|a\|,$$

and hence $\widetilde{g}(s) \in A^\#$.

Since for each $\mathbf{x} \in \mathscr{X}$, we have $\lim_{F\in\mathscr{F}(S)} \|\mathbf{x} - \mathbf{x}_F\| = 0$, thus, by continuity of g on $\mathscr{X}$ and the definition of sums over the set S,

$$\begin{aligned} g(\mathbf{x}) &= \lim_{F\in\mathscr{F}(S)} g(\mathbf{x}_F) = \lim_{F\in\mathscr{F}(S)} g\left(\sum_{s\in F} \mathbf{e}_s(\mathbf{x}(s))\right) = \lim_{F\in\mathscr{F}(S)} \sum_{s\in F} g(\mathbf{e}_s(\mathbf{x}(s))) \\ &= \lim_{F\in\mathscr{F}(S)} \sum_{s\in F} \widetilde{g}(\mathbf{x}(s)) = \sum_{s\in S} \widetilde{g}(\mathbf{x}(s)) = \widehat{g}(\mathbf{x}). \end{aligned}$$

That is the sum that defines $\widehat{g}$ converges for all $\mathbf{x} \in \mathscr{X}$ and $\widehat{g} = g$ on $\mathscr{X}$.

Suppose that $\widehat{g}(\mathbf{x})$ does not converge for some $\mathbf{x} \in \ell^2_*(S,\mathscr{A})$. Then, by the Cauchy criterion, there is an $\varepsilon > 0$ such that

$$\forall\ F \in \mathscr{F}(S),\ \exists\, G \in \mathscr{F}(S \setminus F) \text{ such that } \left|\sum_{s\in G} [\widetilde{g}(s)](\mathbf{x}(s))\right| \geqslant \varepsilon.$$

Thus, inductively, there is a pairwise disjoint sequence $\{G_1, G_2, \ldots\}$ in $\mathscr{F}(S)$ such that

$$\left|\sum_{s\in G_k} [\widetilde{g}(s)](\mathbf{x}(s))\right| \geqslant \varepsilon \qquad \text{for each } k \in \mathbb{N}.$$

Let α_k be the sum in the last expression without absolute value, and $\beta_k = k^{-1}\operatorname{sgn}(\alpha_k)$ (where $\operatorname{sgn}(\zeta) = \overline{\zeta}/|\zeta|$ for $\zeta \in \mathbb{C} \setminus \{0\}$, and $\operatorname{sgn}(0) = 0$). Define $\mathbf{y} : S \to \mathscr{A}$ by

$$\mathbf{y}(s) = \begin{cases} \beta_k \mathbf{x}(s) & \text{if } s \in G_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{if } s \in S \setminus \left[\bigcup_{k=1}^{\infty} G_k\right]. \end{cases}$$

We show that $\mathbf{y} \in \mathscr{X}$. Note that, by Theorem 1 (i), we have

$$M := \sup_{F\in\mathscr{F}} \left\|\sum_{s\in F} (\mathbf{x}^*\mathbf{x})(s)\right\| < \infty.$$

Let $\eta > 0$. From the convergence of

$$\sum_{k=1}^{\infty} |\beta_k|^2 M = \sum_{k=1}^{\infty} \frac{1}{k^2} M < \infty,$$

there is a k_0 such that $\sum_{k=k_0}^{\infty} |\beta_k|^2 M < \eta$. Let

$$F_0 = \bigcup_{k=1}^{k_0} G_k, \quad \text{and} \quad F \in \mathscr{F}(S \setminus F_0).$$

Now the finiteness of F implies the existence of a $\kappa \in \mathbb{N}$ such that

$$F \cap \left(\bigcup_{k=k_0}^{\infty} G_k \right) \subseteq \bigcup_{k=k_0}^{\kappa} G_k.$$

Then we have, from the positivity of $(\mathbf{y}^*\mathbf{y})(s)$ for each $s \in S$,

$$\begin{aligned}\left\| \sum_{s\in F} (\mathbf{y}^*\mathbf{y})(s) \right\| &\leqslant \left\| \sum_{k=k_0}^{\kappa} \sum_{s\in G_k} (\mathbf{y}^*\mathbf{y})(s) \right\| \leqslant \sum_{k=k_0}^{\kappa} \left\| \sum_{s\in G_k} |\beta_k|^2 (\mathbf{x}^*\mathbf{x})(s) \right\| \\ &= \sum_{k=k_0}^{\kappa} |\beta_k|^2 \left\| \sum_{s\in G_k} (\mathbf{x}^*\mathbf{x})(s) \right\| \leqslant \sum_{k=k_0}^{\infty} |\beta_k|^2 M < \eta.\end{aligned}$$

Since $\eta > 0$ is arbitrary, this shows that $\left\{ \sum_{s\in G} (\mathbf{y}^*\mathbf{y})(s) \right\}_{G\in\mathscr{F}(S)}$ is a Cauchy net in $\mathscr{A}$ and hence converges. Thus $\mathbf{y} \in \mathscr{X}$ by Theorem 1 (ii), and hence

$$g(\mathbf{y}) = \widehat{g}(\mathbf{y}) = \sum_{s\in S} \widetilde{g}(\mathbf{y}(s)).$$

In particular, finite partial sums of $g(\mathbf{y})$ are bounded [10]. On the other hand, we also have

$$\lim_{k\to\infty} g(\mathbf{y}_{G_1\cup G_2\cup\ldots\cup G_k}) = \lim_{k\to\infty} \sum_{j=1}^{k} g(\mathbf{y}_{G_j}) = \lim_{k\to\infty} \sum_{j=1}^{k} \beta_j \sum_{s\in G_j} g(\mathbf{x}(s)) \geqslant \lim_{k\to\infty} \sum_{j=1}^{k} \frac{\varepsilon}{j} = \infty.$$

This is a contradiction, and it shows that the sum that defines $\widehat{g}$ converges for every $\mathbf{x} \in \ell^2_*(S,\mathscr{A})$.

The boundedness of $\widehat{g}$ follows from a uniform boundedness argument. Define, for each $F \in \mathscr{F}$,

$$\widehat{g}_F(\mathbf{x}) = \sum_{s\in F} [\widetilde{g}(s)](\mathbf{x}(s)) \quad \text{for all } \mathbf{x} \in \ell^2_*(S,\mathscr{A}).$$

Let $\mathbf{x} \in \ell^2_*(S,\mathscr{A})$ and $F \in \mathscr{F}$. Since $\mathbf{x}_F \in \mathscr{X}$, $g(\mathbf{x}_F) = \widehat{g}_F(\mathbf{x})$, and hence

$$|\widehat{g}_F(\mathbf{x})| = |g(\mathbf{x}_F)| \leqslant \|g\| \, \|\mathbf{x}_F\| \leqslant \|g\| \, \|\mathbf{x}\|.$$

That is

$$\widehat{g}_F \in [\ell^2_*(S,\mathscr{A})]^{\#} \quad \text{and} \quad \|\widehat{g}_F\|_{[\ell^2_*(S,\mathscr{A})]^{\#}} \leqslant \|g\|_{[\mathscr{X}]^{\#}} \quad \text{for all } F \in \mathscr{F}.$$

Thus, for each $\mathbf{x} \in \ell^2_*(S,\mathscr{A})$, we have, by definitions of $\widehat{g}(\mathbf{x})$, and the sum over an arbitrary set,

$$\begin{aligned} |\widehat{g}(\mathbf{x})| &= \lim_{F\in\mathscr{F}} \left| \sum_{s\in F} [\widetilde{g}(s)](\mathbf{x}(s)) \right| = \lim_{F\in\mathscr{F}} |\widehat{g}_F(\mathbf{x})| \\ &\leqslant \limsup_{F\in\mathscr{F}} \|\widehat{g}_F\|_{[\ell^2_*(S,\mathscr{A})]^{\#}} \|\mathbf{x}\| \leqslant \|g\|_{[\mathscr{X}]^{\#}} \|\mathbf{x}\|, \end{aligned}$$

and hence

$$\widehat{g} \in [\ell^2_*(S,\mathscr{A})]^{\#}, \quad \text{and} \quad \|\widehat{g}\|_{[\ell^2_*(S,\mathscr{A})]^{\#}} \leqslant \|g\|_{[\mathscr{X}]^{\#}}.$$

Since $\widehat{g} = g$ on $\mathscr{X}$,

$$\|g\|_{[\mathscr{X}]^{\#}} \leqslant \|\widehat{g}\|_{[\ell^2_*(S,\mathscr{A})]^{\#}}.$$

Therefore equality holds. □

An adaptation of the proof gives the following corollary, which will be used in the proof of Proposition 13.

COROLLARY 3. *Let $h : S \to \mathscr{A}^{\#}$ be such that*

$$f(\mathbf{x}) = \sum_{t\in S} [h(t)](\mathbf{x}^*(t)) \ \textit{converges for all} \ \mathbf{x} \in \mathscr{X}.$$

Then

$$\widehat{f}(\mathbf{y}) = \sum_{t\in S} [h(t)](\mathbf{y}^*(t)) \ \textit{converges for all} \ \mathbf{y} \in \ell^2_*(S,\mathscr{A}),$$

and $\widehat{f}$ is a continuous conjugate linear functional on $\ell^2_*(S,\mathscr{A})$ satisfying

$$\left\|\widehat{f}\right\|_{[\ell^2_*(S,\mathscr{A})]^{\#}} = \|f\|_{[\ell^2_{*u}(S,\mathscr{A})]^{\#}}.$$

Proof. Define $\bar{f}$ by

$$\bar{f}(\mathbf{x}) = \overline{f(\mathbf{x})} = \sum_{s\in S} \overline{(h(s))(\mathbf{x}^*(s))} \quad \text{for all } \mathbf{x} \in \mathscr{X}.$$

Then it is clear that $\bar{f}$ is a linear functional on $\mathscr{X}$. A routine uniform boundedness argument, as in the preceding proof, shows that $\bar{f}$ is a bounded linear functional on $\mathscr{X}$. Clearly h^* given by $h^*(s) = [h(s)]^*$ (where, for each $\psi \in \mathscr{A}^{\#}$, ψ^* is defined by

$\psi^*(a) = \overline{\psi(a^*)}$ for $a \in \mathscr{A}$ [5]) is the representing function (from S to $\mathscr{A}^\#$) of $\bar{f}$ in Proposition 2; and hence

$$\widehat{(\bar{f})}(\mathbf{y}) = \sum_{s\in S}[h^*(s)](\mathbf{y}(s)) \quad \text{converges for all } \mathbf{y} \in \ell^2_*(S,\mathscr{A}).$$

The norm equality follows directly also from the proposition and the fact that $\|f\| = \left\|\bar{f}\right\|$. □

3. $\mathscr{A}$-duality between $\ell^2_*(S,\mathscr{A})$ and $\mathscr{X} = \ell^2_{*u}(S,\mathscr{A})$

The following are analogues of the well known fact that a complex-valued function x on S belongs to $\ell^2(S)$ iff $\sum_{s\in S} x(s)y(s)$ converges for all $y \in \ell^2(S)$.

THEOREM 4. [10, Theorem 5.3] *Let* $\mathbf{a} \in \mathscr{A}^S$. *Then*

(i) $\sum_{s\in S} \mathbf{a}(s)\mathbf{x}(s)$ *converges in* $\mathscr{A}$ $\forall\, \mathbf{x} \in \ell^2_*(S,\mathscr{A})$ *iff* $\mathbf{a}^* \in \ell^2_{*u}(S,\mathscr{A}) = \mathscr{X}$; *and*

(ii) $\sum_{s\in S} \mathbf{a}(s)\mathbf{x}(s)$ *converges in* $\mathscr{A}$ $\forall\, \mathbf{x} \in \mathscr{X} = \ell^2_{*u}(S,\mathscr{A})$ *iff* $\mathbf{a}^* \in \ell^2_*(S,\mathscr{A})$.

Uniform boundedness arguments can be used to show that in each case, if converges, the sum defines a bounded linear operator $T_{\mathbf{a}}$ from the respective space to $\mathscr{A}$, and the operator norm is $\|\mathbf{a}^*\|$. So there is an "$\mathscr{A}$-duality" between the spaces $\ell^2_*(S,\mathscr{A})$ and $\mathscr{X}$. We will further explore this phenomenon. An immediate consequence of this result is that the following definition is meaningful.

DEFINITION 5. For $(\mathbf{x},\mathbf{y}) \in [\ell^2_*(S,\mathscr{A}) \times \mathscr{X}] \cup [\mathscr{X} \times \ell^2_*(S,\mathscr{A})]$, define

$$\langle\!\langle \mathbf{x},\, \mathbf{y} \rangle\!\rangle = \sum_{s\in S} \mathbf{y}^*(s)\mathbf{x}(s).$$

In particular $\langle\!\langle \cdot,\, \cdot \rangle\!\rangle$ is an $\mathscr{A}$-valued inner product on $\mathscr{X} = \ell^2_{*u}(S,\mathscr{A})$. We will see in Lemma 16 that $\mathscr{X}$ with $\langle\!\langle \cdot,\, \cdot \rangle\!\rangle$ is, in fact, a Hilbert C^*-module over $\mathscr{A}$ [6, p. 4].

The *state norm* on $\mathscr{A}$ is defined by

$$\|a\|_\sigma = \sup_{\varphi\in s(\mathscr{A})} |\varphi(a)| \qquad \text{for all } a \in \mathscr{A}.$$

It is well-known ([5, p. 263]) that the state norm is equivalent to the C^*-norm on $\mathscr{A}$:

$$\|a\|_\sigma \leqslant \|a\| \leqslant 2\,\|a\|_\sigma \qquad \text{for all } a \in \mathscr{A}.$$

The following is another duality analogue. (It is routine to verify that this is exactly the well known fact, when $\mathscr{A}$ is $\mathbb{C}$.)

PROPOSITION 6. *For each* $\mathbf{x} \in \ell^2_*(S, \mathscr{A})$, *we have*

$$\|\mathbf{x}\| = \sup\left\{\|\langle\!\langle \mathbf{x},\, \mathbf{y}\rangle\!\rangle\|_\sigma : \mathbf{y} \in \mathscr{X}, \|\mathbf{y}\| \leqslant 1\right\}.$$

Proof. For each $F \in \mathscr{F}$, since $\mathbf{x}_F \in \mathscr{X}$, we have

$$\|\langle\!\langle \mathbf{x},\, \mathbf{x}_F\rangle\!\rangle\|_\sigma = \sup_{\varphi \in s(\mathscr{A})} \varphi\left(\sum_{s \in F} \mathbf{x}^*(s)\mathbf{x}(s)\right) = \sup_{\varphi \in s(\mathscr{A})} \sum_{s \in F} \|\mathbf{x}\|_\varphi^2 = \|\mathbf{x}_F\|^2,$$

and hence

$$\|\mathbf{x}\| = \sup_{F \in \mathscr{F}} \|\mathbf{x}_F\| \leqslant \sup\left\{\|\langle\!\langle \mathbf{x},\, \mathbf{y}\rangle\!\rangle\|_\sigma : \mathbf{y} \in \mathscr{X}, \|\mathbf{y}\| \leqslant 1\right\}.$$

But for each $\mathbf{y} \in \mathscr{X}$, we have

$$\begin{aligned}
\|\langle\!\langle \mathbf{x},\, \mathbf{y}\rangle\!\rangle\|_\sigma &= \sup_{\varphi \in s(\mathscr{A})} \left|\varphi\left(\sum_{s \in S} \mathbf{y}^*(s)\mathbf{x}(s)\right)\right| \leqslant \sup_{\varphi \in s(\mathscr{A})} \sum_{s \in S} \left|\langle \mathbf{x}(s),\, \mathbf{y}(s)\rangle_\varphi\right| \\
&\leqslant \sup_{\varphi \in s(\mathscr{A})} \sum_{s \in s} \|\mathbf{x}(s)\|_\varphi \|\mathbf{y}(s)\|_\varphi \leqslant \sup_{\varphi \in s(\mathscr{A})} \left[\sum_{s \in S} \|\mathbf{x}(s)\|_\varphi^2\right]^{1/2} \left[\sum_{s \in S} \|\mathbf{y}(s)\|_\varphi^2\right]^{1/2} \\
&\leqslant \|\mathbf{x}\|\,\|\mathbf{y}\|. \qquad (1)
\end{aligned}$$

This implies that

$$\|\mathbf{x}\| \geqslant \sup\left\{\|\langle\!\langle \mathbf{x},\, \mathbf{y}\rangle\!\rangle\|_\sigma : \mathbf{y} \in \mathscr{X}, \|\mathbf{y}\| \leqslant 1\right\}.$$

This together with the opposite inequality above, we have the equality. $\square$

Since $\mathscr{X} \subseteq \ell^2_*(S, \mathscr{A})$, Proposition 6 holds in particular for $\mathbf{x} \in \mathscr{X}$. As an immediate consequence we also have the following.

COROLLARY 7. *The map* $(\mathbf{x}, \mathbf{y}) \mapsto \langle\!\langle \mathbf{x},\, \mathbf{y}\rangle\!\rangle$ *is continuous from* $\ell^2_*(S, \mathscr{A}) \times \mathscr{X}$ *to* $\mathscr{A}$.

Proof. For $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \ell^2_*(S, \mathscr{A}) \times \mathscr{X}$, we have from inequality (1) above,

$$\begin{aligned}
\left\|\langle\!\langle \mathbf{x},\, \mathbf{y}\rangle\!\rangle - \langle \mathbf{x}',\, \mathbf{y}'\rangle\right\| &\leqslant \left\|\langle\!\langle \mathbf{x},\, \mathbf{y}\rangle\!\rangle - \langle \mathbf{x}',\, \mathbf{y}\rangle\right\| + \left\|\langle \mathbf{x}',\, \mathbf{y}\rangle - \langle \mathbf{x}',\, \mathbf{y}'\rangle\right\| \\
&= \left\|\langle \mathbf{x} - \mathbf{x}',\, \mathbf{y}\rangle\right\| + \left\|\langle \mathbf{x}',\, \mathbf{y} - \mathbf{y}'\rangle\right\| \\
&\leqslant 2(\left\|\langle \mathbf{x} - \mathbf{x}',\, \mathbf{y}\rangle\right\|_\sigma + \left\|\langle \mathbf{x}',\, \mathbf{y} - \mathbf{y}'\rangle\right\|_\sigma) \\
&\leqslant 2(\left\|\mathbf{x} - \mathbf{x}'\right\| \|\mathbf{y}\| + \left\|\mathbf{x}'\right\| \left\|\mathbf{y} - \mathbf{y}'\right\|). \quad \square
\end{aligned}$$

A function $A : S \times S \to \mathscr{A}$ is said to define an operator on $\mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$, if for each $\mathbf{x} \in \mathscr{X}$ and each $s \in S$, the sum

$$(A\mathbf{x})(s) := \sum_{t \in S} A(s,t)\mathbf{x}(t) \qquad (2)$$

converges in $\mathscr{A}$ and the function $A\mathbf{x}$, as defined in equation (2), is also in $\mathscr{X}$. Such a function A will be called an $\mathscr{A}$*-matrix operator* on $\mathscr{X}$. It follows from the uniform boundedness principle that such an operator is automatically bounded. Denote by $\mathscr{M}$ the space of all $\mathscr{A}$-matrix operators on $\mathscr{X}$. Then $\mathscr{M}$ is a Banach algebra of bounded operators on $\mathscr{X}$ [10]. The following is an analogue of the adjoint of a bounded operator.

PROPOSITION 8. *If $A \in \mathscr{M}$ and $A^{\#} \in \mathscr{A}^{S\times S}$ is defined by $A^{\#}(s,t) = (A(t,s))^{*}$ for all $(s,t) \in S\times S$, then $A^{\#}$ is a bounded linear operator on $\ell^2_*(S,\mathscr{A})$, and $\|A\| = \left\|A^{\#}\right\|$.*

Proof. For each $t \in S$, since $\mathbf{e}_t(1) \in \mathscr{X}$, $A(\mathbf{e}_t(1)) \in \mathscr{X}$. If $\mathbf{z} = A(\mathbf{e}_t(1))$, then $\mathbf{z}$ is the function $\mathbf{z}(s) = A(s,t)$ for $s \in S$. For each $\mathbf{y} \in \ell^2_*(S,\mathscr{A})$, by Theorem 4 (i),

$$\sum_{s\in S} A^{\#}(t,s)\mathbf{y}(s) = \sum_{s\in S}(A(s,t))^{*}\mathbf{y}(s) = \sum_{s\in S}(\mathbf{z}(s))^{*}\mathbf{y}(s) \quad \text{converges in } \mathscr{A}.$$

That is, for each $\mathbf{y} \in \ell^2_*(S,\mathscr{A})$, $A^{\#}\mathbf{y}$ defined by

$$(A^{\#}\mathbf{y})(t) = \sum_{s\in S}(A^{\#}(t,s))\mathbf{y}(s) \quad \text{for all } t \in S$$

is a well-defined function from S to $\mathscr{A}$. Now we show that $A^{\#}\mathbf{y} \in \ell^2_*(S,\mathscr{A})$ for all $\mathbf{y} \in \ell^2_*(S,\mathscr{A})$. Let $\mathbf{x} \in \mathscr{X}$ and $\mathbf{y} \in \ell^2_*(S,\mathscr{A})$. Since $\lim\limits_{F\in\mathscr{F}} \|\mathbf{x}-\mathbf{x}_F\| = 0$, and A is continuous, and $\langle\!\langle\cdot,\,\cdot\rangle\!\rangle$ is continuous in both variables (Corollary 7),

$$\begin{aligned}
\langle\!\langle A\mathbf{x},\,\mathbf{y}\rangle\!\rangle &= \lim_{F\in\mathscr{F}}\langle\!\langle A\mathbf{x}_F,\,\mathbf{y}\rangle\!\rangle = \lim_{F\in\mathscr{F}}\sum_{s\in S}\mathbf{y}^{*}(s)\sum_{t\in F}(A(s,t))\mathbf{x}(t)\\
&= \lim_{F\in\mathscr{F}}\sum_{s\in S}\sum_{t\in F}[(A(s,t))^{*}\mathbf{y}(s)]^{*}\mathbf{x}(t) = \lim_{F\in\mathscr{F}}\sum_{s\in S}\sum_{t\in F}[A^{\#}(t,s)\mathbf{y}(s)]^{*}\mathbf{x}(t)\\
&= \lim_{F\in\mathscr{F}}\sum_{t\in F}\sum_{s\in S}[A^{\#}(t,s)\mathbf{y}(s)]^{*}\mathbf{x}(t) = \lim_{F\in\mathscr{F}}\sum_{t\in F}\left[\sum_{s\in S}(A^{\#}(t,s)\mathbf{y}(s))\right]^{*}\mathbf{x}(t)\\
&= \sum_{t\in S}(A^{\#}\mathbf{y})^{*}(t)\mathbf{x}(t) = \left\langle\!\!\left\langle \mathbf{x},\,A^{\#}\mathbf{y}\right\rangle\!\!\right\rangle \qquad \text{(converges)}
\end{aligned}$$

It follows from Theorem 1 (i) that $A^{\#}\mathbf{y} \in \ell^2_*(S,\mathscr{A})$, and hence $\mathscr{A}^{\#}$ is a bounded $\mathscr{A}$-matrix operator on $\ell^2_*(S,\mathscr{A})$. Furthermore, we also have

$$\begin{aligned}
\|A\| &= \sup_{\|\mathbf{x}\|\leqslant 1}\|A\mathbf{x}\| = \sup_{\|\mathbf{x}\|\leqslant 1}\sup_{\|\mathbf{y}\|\leqslant 1}\|\langle\!\langle A\mathbf{x},\,\mathbf{y}\rangle\!\rangle\|_{\sigma} = \sup_{\|\mathbf{x}\|\leqslant 1}\sup_{\|\mathbf{y}\|\leqslant 1}\left\|\left\langle\!\!\left\langle \mathbf{x},\,A^{\#}\mathbf{y}\right\rangle\!\!\right\rangle\right\|_{\sigma}\\
&= \sup_{\|\mathbf{y}\|\leqslant 1}\sup_{\|\mathbf{x}\|\leqslant 1}\left\|\left\langle\!\!\left\langle \mathbf{x},\,A^{\#}\mathbf{y}\right\rangle\!\!\right\rangle\right\|_{\sigma} = \sup_{\|\mathbf{y}\|\leqslant 1}\left\|A^{\#}\mathbf{y}\right\| = \left\|A^{\#}\right\|. \quad \square
\end{aligned}$$

For each $A \in \mathscr{M}$ and each $G \subseteq S$, denote by $A_{G|}$ the function given by

$$A_{G|}(s,t) = \begin{cases} A(s,t) & \text{if } t \in G, \\ 0 & \text{if } t \notin G. \end{cases}$$

Similarly, $A_{\underline{G}}$ is defined by

$$A_{\underline{G}}(s,t) = \begin{cases} A(s,t) & \text{if } s \in G, \\ 0 & \text{if } s \notin G. \end{cases}$$

We will also use $A_{\underline{G}|}$ to denote $(A_{\underline{G}})_{G|}$; that is $(A_{\underline{G}|})(s,t) = A(s,t)$ if $s,t \in G$ and $(A_{\underline{G}|})(s,t) = 0$ if $(s,t) \in (S \times S) \setminus (G \times G)$.

For each $A \in \mathscr{M}$ and each $s \in S$, denote by $A(s,\cdot)$ the function from S to $\mathscr{A}$ given by $t \mapsto A(s,t)$. The function $A(\cdot,t)$ is similarly defined for each $t \in S$.

LEMMA 9. *Let $A \in \mathscr{M}$, and $G \subseteq H \subseteq S$. Then*

(i) $A_{\underline{G}} \in \mathscr{M}$, $\left\|A_{\underline{G}}\right\| \leqslant \|A_{\underline{H}}\| \leqslant \|A\|$;

(ii) $A_{G|} \in \mathscr{M}$, $\left\|A_{G|}\right\| \leqslant \left\|A_{H|}\right\| \leqslant \|A\|$; *and*

(iii) $A_{\underline{G}|} \in \mathscr{M}$, $\left\|A_{\underline{G}|}\right\| \leqslant \left\|A_{\underline{H}|}\right\| \leqslant \|A\|$.

Proof. (i) For $\mathbf{x} \in \mathscr{X}$, since $(A_{\underline{G}})\mathbf{x} = (A\mathbf{x})_G$, and $\left\|\mathbf{x}_G\right\| \leqslant \|\mathbf{x}_H\| \leqslant \|\mathbf{x}\|$ by the definition of the norm, we have $A_{\underline{G}} \in \mathscr{M}$ with $\left\|A_{\underline{G}}\right\| \leqslant \left\|A_{\underline{H}}\right\| \leqslant \|A\|$.

(ii) First note that $(A_{G|})\mathbf{x} = A(\mathbf{x}_G) = (A_{H|})(\mathbf{x}_G)$ for each $\mathbf{x} \in \mathscr{X}$ and $G \subseteq H \subseteq S$. Let $\varepsilon > 0$. There is a unit vector $\mathbf{x} \in \mathscr{X}$ such that $\left\|A_{G|}\right\| - \varepsilon < \left\|(A_{G|})\mathbf{x}\right\|$. Thus

$$\left\|A_{G|}\right\| - \varepsilon < \left\|(A_{G|})\mathbf{x}\right\| = \|A(\mathbf{x}_G)\| = \left\|(A_{H|})(\mathbf{x}_G)\right\| \leqslant \left\|A_{H|}\right\| \|\mathbf{x}_G\| \leqslant \left\|A_{H|}\right\|.$$

(iii) For each $\varepsilon > 0$, there is a unit vector $\mathbf{x} \in \mathscr{X}$ such that the following first inequality holds, and hence the ones that come after it by definitions and routine verifications:

$$\begin{aligned} \left\|A_{\underline{G}|}\right\| - \varepsilon &< \left\|(A_{\underline{G}|})\mathbf{x}\right\| = \left\|(A_{\underline{G}|})(\mathbf{x}_G)\right\| = \left\|[A(\mathbf{x}_G)]_G\right\| \leqslant \left\|[A(\mathbf{x}_G)]_H\right\| \\ &= \left\|(A_{\underline{H}})(\mathbf{x}_G)\right\| = \left\|(A_{\underline{H}})[(\mathbf{x}_G)]_H\right\| = \left\|(A_{\underline{H}|})(\mathbf{x}_G)\right\| \leqslant \left\|A_{\underline{H}|}\right\|. \quad \square \end{aligned}$$

4. The space $\mathscr{K}$

We introduce the subclass $\mathscr{K}$ of the class $\mathscr{M}$ of $\mathscr{A}$-matrix operators and prove some elementary properties of $\mathscr{K}$ in this section. Analogous to the special case $\mathscr{A} = \mathbb{C}$, we define

$$\mathscr{K} := \left\{ A \in \mathscr{M} : \lim_{F \in \mathscr{F}(S)} \left\|A - A_{\underline{F}|}\right\| = 0 \right\}.$$

Notice that this is a coordinate dependent equivalent formulation of the compact operators on a Hilbert space, when the C^*-algebra is taken to be $\mathbb{C}$. Now we establish some of the familiar properties of the compact operators for $\mathscr{K}$ that will be used later.

LEMMA 10.

(i) The subspace $\mathscr{K}$ is (operator norm) closed in $\mathscr{M}$.

(ii) If $A \in \mathscr{M}$ and $t \in S$, then $A_{\{t\}|} \in \mathscr{K}$.

(iii) If $A \in \mathscr{M}$ and $G \in \mathscr{F}$, then $A_{G|} \in \mathscr{K}$.

Proof. (i) Let $\{A_n\}$ be a sequence in $\mathscr{K}$ such that $\|A_n - A\| \to 0$ for some $A \in \mathscr{M}$. Let $\varepsilon > 0$. There is an N such that

$$\|A_n - A\| < \frac{\varepsilon}{3} \qquad \text{for all } n \geqslant N.$$

Since $A_N \in \mathscr{K}$, there is an $F_0 \in \mathscr{F}$ such that

$$\left\|(A_N)_{F\rfloor} - A_N\right\| < \frac{\varepsilon}{3} \qquad \text{for all } F_0 \subseteq F \subseteq \mathscr{F}.$$

Let $F_0 \subseteq F \in \mathscr{F}$. We have

$$\begin{aligned}\left\|A_{F\rfloor} - A\right\| &\leqslant \left\|A_{F\rfloor} - (A_N)_{F\rfloor}\right\| + \left\|(A_N)_{F\rfloor} - A_N\right\| + \|A_N - A\| \\ &< \left\|(A - A_N)_{F\rfloor}\right\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.\end{aligned}$$

Thus $\lim_{F\in\mathscr{F}} \left\|A_{F\rfloor} - A\right\| = 0$, and hence $A \in \mathscr{K}$.

(ii) Since $\mathbf{y} := (A_{\{t\}|})(\mathbf{e}_t(1)) = A(\mathbf{e}_t(1)) \in \mathscr{X}$, we have

$$\lim_{F\in\mathscr{F}} \|\mathbf{y} - \mathbf{y}_F\| = 0.$$

Thus for each $\varepsilon > 0$, there is an $F_\varepsilon \in \mathscr{F}$ such that

$$\|\mathbf{y} - \mathbf{y}_F\| < \varepsilon \qquad \text{for all } F_\varepsilon \subseteq F \in \mathscr{F}.$$

Let $F_\varepsilon \cup \{t\} \subseteq F \in \mathscr{F}$; and let $\mathbf{x} \in \mathscr{X}$. Then

$$\left\|\left[A_{\{t\}|} - \left(A_{\{t\}|}\right)_{F\rfloor}\right]\mathbf{x}\right\| = \|(\mathbf{y} - \mathbf{y}_F)(\mathbf{x}(t))\| \leqslant \|\mathbf{y} - \mathbf{y}_F\|\,\|\mathbf{x}(t)\| < \varepsilon\,\|\mathbf{x}\|,$$

and hence $\left\|A_{\{t\}|} - \left(A_{\{t\}|}\right)_{F\rfloor}\right\| \leqslant \varepsilon$ for all $F_\varepsilon \cup \{t\} \subseteq F \in \mathscr{F}(S)$. Therefore $A_{\{t\}|} \in \mathscr{K}$.

(iii) For each $\mathbf{x} \in \mathscr{X}$, and each $s \in S$,

$$[(A_{G|})\mathbf{x}](s) = \sum_{t\in G} A(s,t)\mathbf{x}(t) = \sum_{t\in G} [A_{\{t\}|}\mathbf{x}](s) = \left[\left(\sum_{t\in G} A_{\{t\}|}\right)\mathbf{x}\right](s),$$

that is $A_{G|} = \sum_{t\in G} A_{\{t\}|}$. Let N be the number of elements in G and $\varepsilon > 0$. By part (ii), for each $t \in G$, there is an $F_t \in \mathscr{F}$ such that

$$\left\| A_{\{t\}|} - \left(A_{\{t\}|}\right)_{\underline{F}} \right\| < \frac{\varepsilon}{N} \qquad \text{for all } F_t \subseteq F \in \mathscr{F}.$$

Let $F_\varepsilon = \left[\bigcup_{t\in G} F_t\right] \cup G$. Then $F_\varepsilon \in \mathscr{F}$, and if $F_\varepsilon \subseteq F \in \mathscr{F}$, we have

$$\left\| A_{G|} - \left(A_{G|}\right)_{\underline{F}|} \right\| = \left\| \sum_{t\in G} \left[A_{\{t\}|} - \left(A_{\{t\}|}\right)_{\underline{F}|} \right] \right\| \leqslant \sum_{t\in G} \left\| A_{\{t\}|} - \left(A_{\{t\}|}\right)_{\underline{F}|} \right\| < \varepsilon.$$

Therefore $A_{G|} \in \mathscr{K}$. $\square$

PROPOSITION 11. *If $\{G_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint sequence in $\mathscr{F}$, $\{A_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathscr{M}$ such that $(A_n)_{G_n|} = A_n$, and $\{\alpha_n\}_{n\in\mathbb{N}}$ is an ℓ^2 sequence, then $A := \sum_{n=1}^{\infty} \alpha_n A_n \in \mathscr{K}$.*

By assumption, each A_n is adjointable, i.e., $A_n^{\#}$ is a matrix operator on $\mathscr{X}$.

Proof. Let $\varepsilon > 0$ and $\sup_{n\in\mathbb{N}} \|A_n\| = \sup_{n\in\mathbb{N}} \left\|A_n^{\#}\right\| < M < \infty$. There is an $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} |\alpha_n|^2 < \left(\frac{\varepsilon}{M}\right)^2$. Let $\mathbf{x} \in \mathscr{X}$ and $m > k \geqslant N$. Let $B = \sum_{n=k}^{m} \alpha_n A_n$.

$$\begin{aligned} \left\| \left[\sum_{n=k}^{m} \alpha_n A_n\right] \mathbf{x} \right\| = \|B\mathbf{x}\| &= \sup_{\substack{\|\mathbf{y}\|\leqslant 1 \\ \mathbf{y}\in\ell_*^2(S,\mathscr{A})}} \|\langle\!\langle B\mathbf{x},\, \mathbf{y}\rangle\!\rangle\|_\sigma = \sup_{\substack{\|\mathbf{y}\|\leqslant 1 \\ \mathbf{y}\in\ell_*^2(S,\mathscr{A})}} \left\| \left\langle\!\!\left\langle \mathbf{x},\, B^{\#}\mathbf{y} \right\rangle\!\!\right\rangle \right\|_\sigma \\ &\leqslant \sup_{\substack{\|\mathbf{y}\|\leqslant 1 \\ \mathbf{y}\in\ell_*^2(S,\mathscr{A})}} \|\mathbf{x}\| \left\|B^{\#}\mathbf{y}\right\|. \end{aligned}$$

For each $n \in \mathbb{N}$ and $\mathbf{y} \in [\ell_*^2(S,\mathscr{A})]_1$, since

$$(A_n^{\#}\mathbf{y})(s) = (A_n^{\#})_{\underline{\{s\}}}\mathbf{y} = ((A_n)_{\{s\}|})^{\#}\mathbf{y} = (0)^{\#}\mathbf{y} = 0 \quad \text{for all } s \in S \setminus G_n,$$

the sequence $\left\{A_n^{\#}\mathbf{y}\right\}_{n\in\mathbb{N}}$ has the pairwise disjoint sequence $\{G_n\}$ as supports, and hence,

$$\begin{aligned} \left\|B^{\#}\mathbf{y}\right\| = \left\| \sum_{n=k}^{m} \bar{\alpha}_n A_n^{\#}\mathbf{y} \right\| &\leqslant \left[\sum_{n=k}^{m} |\alpha_n|^2 \left\|A_n^{\#}\mathbf{y}\right\|^2 \right]^{1/2} \\ &\leqslant \left[\sum_{n=k}^{m} |\alpha_n|^2 \left\|A_n^{\#}\right\|^2 \|\mathbf{y}\|^2 \right]^{1/2} \leqslant \left[\sum_{n=k}^{m} |\alpha_n|^2 \right]^{1/2} M\,\|\mathbf{y}\| < \varepsilon. \end{aligned}$$

From the previous inequality, we have $\|B\mathbf{x}\| \leqslant \|\mathbf{x}\| \varepsilon$. Since $\mathbf{x} \in \mathscr{X}$ is arbitrary, $\|B\| \leqslant \varepsilon$. From arbitrariness of $m > k \geqslant N$, we see that the sequence of partial sums of (the sum that defines) A is a Cauchy sequence and since each partial sum is in $\mathscr{K}$, we have $A \in \mathscr{K}$. $\square$

Note also that if $\|A_n\| \leqslant M$ for all n, then, from the proof we also have the estimate

$$\|A\| \leqslant M \left[\sum_{n=1}^{\infty} |\alpha_n|^2 \right]^{1/2}.$$

5. Extension from $\mathscr{K}$ to $\mathscr{M}$

First we show that each element of $\mathscr{K}^{\#}$ is given by a double sum, and has a unique Hahn-Banach extension to $\mathscr{M}$. (Recall that $\mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$ and $\mathscr{M}$ is the set of all $\mathscr{A}$-matrix operators on $\mathscr{X}$.)

LEMMA 12. *Let $\mathbf{x} \in \ell^2_*(S, \mathscr{A})$ and $s \in S$. Define $B_{s,\mathbf{x}} : S \times S \to \mathscr{A}$ by*

$$B_{s,\mathbf{x}}(u,v) = \begin{cases} (\mathbf{x}(v))^* & \text{if } u = s, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } (u,v) \in S \times S$$

Then

$$B_{s,\mathbf{x}} \in \mathscr{M}, \qquad \text{and} \qquad \left\|B_{s,\mathbf{x}}\right\| \leqslant 2\|\mathbf{x}\|.$$

Proof. For each $\mathbf{y} \in \mathscr{X}$, since

$$[B_{s,\mathbf{x}}\mathbf{y}](u) = \begin{cases} \langle\!\langle \mathbf{y},\, \mathbf{x} \rangle\!\rangle & \text{if } u = s, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } u \in S;$$

from inequalities (1) in the proof of Proposition 6,

$$\left\|B_{s,\mathbf{x}}\mathbf{y}\right\| = \|\langle\!\langle \mathbf{y},\, \mathbf{x} \rangle\!\rangle\| \leqslant 2\,\|\langle\!\langle \mathbf{y},\, \mathbf{x} \rangle\!\rangle\|_{\sigma} \leqslant 2\,\|\mathbf{y}\|\,\|\mathbf{x}\|.$$

Therefore $B_{s,\mathbf{x}} \in \mathscr{M}$ with $\left\|B_{s,\mathbf{x}}\right\| \leqslant 2\|\mathbf{x}\|$. $\square$

We note in this connection that if $\mathbf{x} \in \ell^2_*(S, \mathscr{A}) \setminus \mathscr{X}$, then $B^{\#}_{s,\mathbf{x}}\mathbf{e}_s = \mathbf{x} \notin \mathscr{X}$. Thus $B^{\#}_{s,\mathbf{x}} \notin \mathscr{M}$ and hence $\mathscr{M}$ is not a C^*-algebra with the most natural adjoint operation $^{\#}$.

PROPOSITION 13.

(i) *For each $f \in \mathscr{K}^{\#}$, there is a unique function $\widetilde{f} :\ S \times\ S\ \to\ \mathscr{A}^{\#}$ such that*

$$f(A) = \sum_{s \in S} \sum_{t \in S} \widetilde{f}(s,t)(A(s,t)) \quad \text{for all } A \in \mathscr{K}. \tag{3}$$

Furthermore,

$$\widehat{f}(A) := \sum_{s\in S}\sum_{t\in S}\widetilde{f}(s,t)(A(s,t)) \quad \textit{converges for all } A \in \mathscr{M},$$

and $\widehat{f}$ *is a bounded linear functional on* $\mathscr{M}$ *with* $\left\|\widehat{f}\right\|_{\mathscr{M}^{\#}} = \|f\|_{\mathscr{K}^{\#}}$.

(ii) *Conversely if* $g : S\times S \to \mathscr{A}^{\#}$ *has the property that*

$$\sum_{s\in S}\sum_{t\in S} g(s,t)(A(s,t)) \quad \textit{converges for all } A \in \mathscr{K},$$

then the double sum defines a bounded linear functional on $\mathscr{K}$ *(and hence on* $\mathscr{M}$*).*

Proof. (i) For $(s,t)\in S\times S$ and $a\in\mathscr{A}$, let $E_{(s,t)}(a)$ be the function on $S\times S$ defined by

$$[E_{(s,t)}(a)](u,v) = \begin{cases} a & \text{if } (u,v) = (s,t) \\ 0 & \text{if } (u,v) \neq (s,t) \end{cases}$$

Then a straightforward calculation shows that

$$E_{(s,t)}(a) \in \mathscr{K} \quad \text{and} \quad \left\|E_{(s,t)}(a)\right\| = \|a\|.$$

Thus, for each $f\in\mathscr{K}^{\#}$ and each $(s,t)\in S\times S$,

$$(\widetilde{f}(s,t))(a) = f(E_{(s,t)}(a)) \qquad (a\in\mathscr{A})$$

is a well defined functional on $\mathscr{A}$. Since

$$\left|(\widetilde{f}(s,t))(a)\right| \leqslant \|f\|\left\|E_{(s,t)}(a)\right\| \leqslant \|f\|\,\|a\| \quad \forall\ a\in\mathscr{A},$$

$\widetilde{f}(s,t)\in\mathscr{A}^{\#}$, and $\widetilde{f}$ is a map from $S\times S$ to $\mathscr{A}^{\#}$.

To see the convergence of the inner sum, we first show that it converges for "rows" associated with functions from $\mathscr{X}$ as in Lemma 12. For each $\mathbf{x}\in\mathscr{X} = \ell^2_{*u}(S,\mathscr{A})$ and each $s\in S$, let $B_{s,\mathbf{x}}$ be as in Lemma 12. Then $B_{s,\mathbf{x}}\in\mathscr{M}$, and

$$\begin{aligned}\lim_{F\in\mathscr{F}}\left\|B_{s,\mathbf{x}} - (B_{s,\mathbf{x}})_{F\rfloor}\right\| &= \lim_{F\in\mathscr{F}}\left\|B_{s,\mathbf{x}} - B_{s,\mathbf{x}_F}\right\| = \lim_{F\in\mathscr{F}}\left\|B_{s,(\mathbf{x}-\mathbf{x}_F)}\right\| \\ &\leqslant \lim_{F\in\mathscr{F}} 2\left\|\mathbf{x}-\mathbf{x}_F\right\| = 0.\end{aligned}$$

Thus $B_{s,\mathbf{x}}\in\mathscr{K}$, and hence $f(B_{s,\mathbf{x}})$ exists. From the continuity of f,

$$\begin{aligned}f(B_{s,\mathbf{x}}) &= \lim_{F\in\mathscr{F}} f(B_{s,(\mathbf{x}_F)}) = \lim_{F\in\mathscr{F}} f\left(\sum_{t\in F} B_{s,(\mathbf{x}_{\{t\}})}\right) = \lim_{t\in F} f\left(\sum_{t\in F} E_{(s,t)}(\mathbf{x}^*(t))\right) \\ &= \lim_{F\in\mathscr{F}}\sum_{t\in F}[\widetilde{f}(s,t)]((\mathbf{x}(t))^*) = \sum_{t\in S}[\widetilde{f}(s,t)](\mathbf{x}^*(t)) = \sum_{t\in S}[\widetilde{f}(s,t)](B_{s,\mathbf{x}}(s,t)).\end{aligned}$$

That is, for each $s \in S$,

$$\sum_{t \in S} [\widetilde{f}(s,t)](\mathbf{x}^*(t)) \text{ converges for all } \mathbf{x} \in \mathscr{X}.$$

Notice that for each $A \in \mathscr{K}$ and each $s \in S$, the function $(A(s,\cdot))^* \in \mathscr{X}$, and hence

$$f(A_{\underline{\{s\}}}) = \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t)) \qquad \text{for each } s \in S.$$

By linearity the same is true for each $F \in \mathscr{F}(S)$ in place of the singleton set $\{s\}$. Continuity of f and the fact that

$$\|A - A_{\underline{F}}\| = \left\| \left[A - A_{F_\lrcorner}\right]_{\underline{(S \setminus F)}} \right\| \leqslant \left\|A - A_{F_\lrcorner}\right\|$$

imply that $f(A)$ is given by the double sum in (3) for each $A \in \mathscr{K}$.

Let $A \in \mathscr{M}$. For each $s \in S$, since $A_{\underline{\{s\}}} \in \mathscr{M}$, and, as a function on S, $A_{\underline{\{s\}}}(s,\cdot)$ has the property that

$$\left\langle\!\left\langle \mathbf{x},\ (A_{\underline{\{s\}}}(s,\cdot))^* \right\rangle\!\right\rangle = (A\mathbf{x})(s) \qquad \text{converges in } \mathscr{A} \text{ for each } \mathbf{x} \in \mathscr{X},$$

thus $(A_{\underline{\{s\}}}(s,\cdot))^* \in \ell^2_*(S,\mathscr{A})$ by Theorem 4 (ii). It then follows from Corollary 3 that the inner sums all converge for each $A \in \mathscr{M}$; i.e.,

$$\sum_{t \in S} (\widetilde{f}(s,t))(A(s,t)) \qquad \text{converges for all } A \in \mathscr{M} \text{ and all } s \in S.$$

Suppose the outer sum for $\widehat{f}$ does not converge for some $A \in \mathscr{M}$. Then by Cauchy criterion, there are an $\varepsilon > 0$ and a sequence $\{F_n\}_{n \in \mathbb{N}}$ of pairwise disjoint (finite) sets in $\mathscr{F}(S)$ such that

$$\left| \sum_{s \in F_n} \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t)) \right| \geqslant 2\varepsilon \qquad \text{for all } n \in \mathbb{N}.$$

For each n, the finiteness of F_n gives rise to a finite $G_n \in \mathscr{F}(S)$ such that

$$\left| \sum_{s \in F_n} \sum_{t \in G_n} [\widetilde{f}(s,t)](A(s,t)) \right| \geqslant \varepsilon.$$

Let α_n be the sum in the last expression without absolute value, and $\beta_n = \frac{\operatorname{sgn}(\alpha_n)}{n}$. Define

$$B(s,t) = \begin{cases} \beta_n A(s,t) & \text{if } (s,t) \in F_n \times G_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } (s,t) \in (S \times S) \setminus \left[\bigcup_{k=1}^{\infty} (F_k \times G_k) \right]. \end{cases}$$

Note that $B = \sum_{n=1}^{\infty} \beta_n \left(A_{\underline{F_n}}\right)_{G_n|}$, and each $\left(A_{\underline{F_n}}\right)_{G_n|}$ is adjointable with

$$\left\| \left[\left(A_{\underline{F_n}}\right)_{G_n|} \right]^{\#} \right\| = \left\| \left(A_{\underline{F_n}}\right)_{G_n|} \right\| \leqslant \|A\|.$$

Since $\left[\left(A_{\underline{F_n}}\right)_{G_n|} \right]^{\#} = \left([A^{\#}]_{F_n|} \right)_{\underline{G_n}}$, and the sequence $\{F_n\}$ is pairwise disjoint in $\mathscr{F}(S)$, $B \in \mathscr{K}$ by Proposition 11.

On the other hand, since $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$, for each M, there is a $\kappa \in \mathbb{N}$ such that $\sum_{k=1}^{n} \frac{1}{k} > \frac{M}{\varepsilon}$ for all $n \geqslant \kappa$. Let $F = \bigcup_{k=1}^{\kappa} F_k$. Then

$$\begin{aligned}
\sum_{s \in F} \left[\sum_{t \in S} [\widetilde{f}(s,t)](B(s,t)) \right] &= \sum_{k=1}^{\kappa} \sum_{s \in F_k} \sum_{t \in G_k} [\widetilde{f}(s,t)](\beta_k A(s,t)) \\
&= \sum_{k=1}^{\kappa} \frac{1}{k} \left| \sum_{s \in F_k} \sum_{t \in G_k} [\widetilde{f}(s,t)](A(s,t)) \right| > M.
\end{aligned}$$

That is

$$f(B) = \sum_{s \in S} \sum_{t \in S} [\widetilde{f}(s,t)](B(s,t)) \qquad \text{diverges.}$$

This contradicts $B \in \mathscr{K}$. Therefore the double sum must converge for all $A \in \mathscr{M}$.

Now we use a uniform boundedness argument to show that $\widehat{f} \in \mathscr{M}^{\#}$. For each fixed $s \in S$, and each $F \in \mathscr{F}(S)$, define

$$g_{s,F}(A) = \sum_{t \in F} \widetilde{f}(s,t)(A(s,t)) \qquad \text{for all } A \in \mathscr{M}.$$

Then

$$\begin{aligned}
\left| g_{s,F}(A) \right| &\leqslant \sum_{t \in F} \left| \widetilde{f}(s,t)(A(s,t)) \right| \leqslant \sum_{t \in F} \left\| \widetilde{f}(s,t) \right\| \|A(s,t)\| \\
&= \sum_{t \in F} \left\| \widetilde{f}(s,t) \right\| \left\| E_{(s,t)}(A(s,t)) \right\| \leqslant \sum_{t \in F} \|f\| \left\| (A_{\underline{\{s\}}})_{\{t\}|} \right\| \leqslant \sum_{t \in F} \|f\| \cdot \|A\|;
\end{aligned}$$

and hence $g_{s,F} \in \mathscr{M}^{\#}$ with $\left\| g_{s,F} \right\| \leqslant (\mathrm{Card\ F}) \cdot \|\mathrm{f}\|$. Since, for a fixed $s \in S$,

$$\sum_{t \in S} \widetilde{f}(s,t)(A(s,t)) \qquad \text{converges for all } A \in \mathscr{M},$$

the net of finite partial sums $\left\{ g_{s,F}(A) \right\}_{F \in \mathscr{F}(S)}$ is bounded and thus there is an M_A such that $\left| g_{s,F}(A) \right| \leqslant M_A$ for all $F \in \mathscr{F}(S)$. Uniform boundedness principle implies that there is an M such that $\|g_{s,F}\| \leqslant M$ for all $F \in \mathscr{F}(S)$. Thus the functional g_s defined by

$$g_s(A) = \sum_{t \in S} \widetilde{f}(s,t)(A(s,t)) \qquad (A \in \mathscr{M})$$

is bounded:

$$\|g_s(A)\| = \lim_{F \in \mathscr{F}(S)} |g_{s,F}(A)| \leqslant \limsup_{F \in \mathscr{F}(S)} \|g_{s,F}\| \, \|A\| \leqslant M \|A\| \qquad \forall \; A \in \mathscr{M}.$$

Using the convergence of the outer sum for $\widehat{f}$, a similar uniform boundedness argument shows that $\widehat{f}(A) = \sum_{s \in S} g_s(A)$ is bounded on $\mathscr{M}$.

Let $F \in \mathscr{F}(S)$ and $A \in \mathscr{M}$. For each $G \in \mathscr{F}(S)$, $\left\|A_{\underline{G}|}\right\| \leqslant \|A\|$ by Lemma 9, and $A_{\underline{G}|} \in \mathscr{K}$,

$$\begin{aligned} \left|\widehat{f}(A_{\underline{F}})\right| &= \left|\sum_{s \in F} \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t))\right| = \lim_{G \in \mathscr{F}(S)} \left|\sum_{s \in F} \sum_{t \in G} [\widetilde{f}(s,t)](A(s,t))\right| \\ &= \lim_{G \in \mathscr{F}(S)} \left|f([A_{\underline{F}}]_{\underline{G}})\right| = \left|f(A_{\underline{F}})\right| \leqslant \|f\| \left\|A_{\underline{F}}\right\| \leqslant \|f\|_{\mathscr{K}^{\#}} \|A\|. \end{aligned}$$

Therefore

$$\left|\widehat{f}(A)\right| = \lim_{F \in \mathscr{F}(S)} \left|\sum_{s \in F} g_s(A)\right| = \lim_{F \in \mathscr{F}(S)} \left|\widehat{f}(A_{\underline{F}})\right| \leqslant \|f\|_{\mathscr{K}^{\#}} \|A\| \qquad \text{for all } \; A \in \mathscr{M}.$$

That is $\left\|\widehat{f}\right\|_{\mathscr{M}^{\#}} \leqslant \|f\|_{\mathscr{K}^{\#}}$. But since $\widehat{f} = f$ on $\mathscr{K}$, we see that $\left\|\widehat{f}\right\|_{\mathscr{M}^{\#}} = \|f\|_{\mathscr{K}^{\#}}$ must hold. Uniqueness of the function $\widetilde{f} : S \times S \to \mathscr{A}^{\#}$ is clear from the construction.

(ii) This follows from a uniform boundedness argument similar to the one used in the preceding proof, and is omitted. $\square$

An immediate consequence of this proposition is that we may, and will, just treat $\mathscr{K}^{\#}$ as a subspace of $\mathscr{M}^{\#}$.

The trace formula, trace $AB =$ trace BA, for a trace class operator A and a bounded operator B on a Hilbert space has the following generalization.

PROPOSITION 14. *Let $\xi : S \times S \to \mathscr{A}^{\#}$ be a function such that*

$$g(A) = \sum_{s \in S} \sum_{t \in S} [\xi(s,t)](A(s,t)) \qquad \textit{converges for all } A \in \mathscr{K}.$$

Then

$$\sum_{s \in S} \sum_{t \in S} [\xi(s,t)](A(s,t)) = \sum_{t \in S} \sum_{s \in S} [\xi(s,t)](A(s,t)) \qquad \textit{for all } A \in \mathscr{M}.$$

Proof. Uniform boundedness arguments similar to that used in the proof of Proposition 13 (i) show that g defines a bounded linear functional on $\mathscr{K}$. Note that for each $A \in \mathscr{M}$ and each $t \in S$, $A_{\{t\}|} \in \mathscr{K}$. By Lemma 10,

$$\sum_{s \in S} [\xi(s,t)](A(s,t)) \quad \text{converges for each } A \in \mathscr{M} \text{ and each fixed } t \in S.$$

For each $G \in \mathscr{F}$, define

$$h_G(A) = \sum_{t \in G} \sum_{s \in S} [\xi(s,t)](A(s,t)) = \sum_{s \in S} \sum_{t \in G} [\xi(s,t)](A(s,t)) \qquad (A \in \mathscr{M}).$$

Again a uniform boundedness argument can be used to show that $h_G \in \mathscr{M}^{\#}$. We claim that $\{h_G\}_{G \in \mathscr{F}}$ is a Cauchy net in $\mathscr{M}^{\#}$. For otherwise, by the Cauchy criterion, there is an $\varepsilon > 0$ such that for all $G \in \mathscr{F}$, there are $H_G, K_G \in \mathscr{F}$ such that $G \subseteq H := H_G$, $G \subseteq K := K_G$, and $\left\|h_H - h_K\right\| \geqslant 2\varepsilon$. Thus there is an $A := A^G \in [\mathscr{M}]_1$ (the closed unit ball of $\mathscr{M}$) such that

$$\left|h_H(A) - h_K(A)\right| = \left|h_{[H \setminus K]}(A) - h_{[K \setminus H]}(A)\right| > \varepsilon.$$

Denote by

$$\alpha = \operatorname{sgn}\left[h_{[H \setminus K]}(A) - h_{[K \setminus H]}(A)\right]; \quad \text{and let} \quad B = \alpha[A_{(H \setminus K)|} - A_{(K \setminus H)|}].$$

Then, since $(H \setminus K) \cap (K \setminus H) = \emptyset$, we see that

$$\begin{aligned} \|B\| &\leqslant \left\|(B)_{(H \setminus K)|}\right\| + \left\|(B)_{(K \setminus H)|}\right\| = \left\|A_{(H \setminus K)|}\right\| + \left\|A_{(K \setminus H)|}\right\| \\ &\leqslant 2\|A\| \leqslant 2. \end{aligned}$$

Note that, a straightforward calculation reveals that $h_{F_1}(C_{F_2|}) = h_{F_1 \cap F_2}(C_{(F_1 \cap F_2)|})$ for all $C \in \mathscr{M}$ and all $F_1, F_2 \in \mathscr{F}$; consequently,

$$\begin{aligned} h_{[H \triangle K]}(B) &= h_{[H \triangle K]}(\alpha[A_{(H \setminus K)|} - A_{(K \setminus H)|}]) \\ &= \left|h_{(H \setminus K)}(A) - h_{(K \setminus H)}(A)\right| > \varepsilon. \end{aligned}$$

Since $H, K \supseteq G$, $H \triangle K = (H \setminus K) \cup (K \setminus H) \subseteq S \setminus G$. Note also that the sum in the previous expression involves only $A(s,t)$ with $t \in H \triangle K$, and that $h_{H \triangle K}(A) = h_{H \triangle K}(A_{(H \triangle K)|}) = g(A)$ if $A = A_{(H \triangle K)|}$.

This shows that, under the assumption that $\{h_G\}_{G \in \mathscr{F}}$ is not a Cauchy net, there are an $\varepsilon > 0$, and (with the set G above in each step taken to be the set in the previous step) a pairwise disjoint sequence $\{H_n\}_{n \in \mathbb{N}} \subseteq \mathscr{F}$ (in place of the $H \triangle K$ above), and a sequence $\{B_n\}_{n \in \mathbb{N}} \subseteq \mathscr{M}$ bounded by 2 in norm, such that

$$B_n = (B_n)_{(H_n)|} \quad \text{and} \quad g(B_n) = h_{H_n}(B_n) > \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

By Proposition 11, $B := \sum_{n=1}^{\infty} \frac{1}{n} B_n$ converges in $\mathscr{K}$. On the other hand, since $g(B_n) = h_{H_n}(B_n) > \varepsilon$ for each $n \in \mathbb{N}$, we have

$$g(B) = \sum_{n=1}^{\infty} \frac{g(B_n)}{n} > \sum_{n=1}^{\infty} \frac{\varepsilon}{n} = \infty.$$

This contradicts the fact that $g(A)$ converges for each $A \in \mathscr{K}$.

Therefore $\{h_F\}_{F \in \mathscr{F}}$ is a Cauchy net in $\mathscr{M}^{\#}$, and hence there is an $h \in \mathscr{M}^{\#}$ such that $\lim_{F \in \mathscr{F}} \|h_F - h\|_{\mathscr{M}^{\#}} = 0$.

Let $A \in \mathscr{K}$. For each $G \in \mathscr{F}$, (since $A_{\underline{G}} - (A_{\underline{G}})_{F_{\lrcorner}} = (A - A_{F_{\lrcorner}})_{\underline{G}}$ for all $F \in \mathscr{F}$, and $\left\|B_{\underline{G}}\right\| \leqslant \|B\|$ for all $B \in \mathscr{M}$) we have $A_{\underline{G}} \in \mathscr{K}$ and hence

$$\begin{aligned} g(A_{\underline{G}}) &= \sum_{s \in G} \sum_{t \in S} [\xi(s,t)](A(s,t)) = \lim_{F \in \mathscr{F}} \left[\sum_{t \in F} \sum_{s \in G} [\xi(s,t)](A(s,t)) \right] \\ &= \lim_{F \in \mathscr{F}} h_F(A_{\underline{G}}) = h(A_{\underline{G}}). \end{aligned}$$

For $A \in \mathscr{K}$, since

$$\begin{aligned} \left\|(A - A_{\underline{G}})\mathbf{x}\right\| &= \|A\mathbf{x} - (A\mathbf{x})_G\| = \left\|(A\mathbf{x})_{S \setminus G}\right\| = \left\|[(A - A_{G_{\lrcorner}})\mathbf{x}]_{S \setminus G}\right\| \leqslant \left\|(A - A_{G_{\lrcorner}})\mathbf{x}\right\| \\ &\leqslant \left\|A - A_{G_{\lrcorner}}\right\| \|\mathbf{x}\| \quad \text{for all } \mathbf{x} \in \mathscr{X}, \end{aligned}$$

$\lim_{G \in \mathscr{F}} \left\|A_{\underline{G}} - A\right\| \leqslant \lim_{G \in \mathscr{F}} \left\|A - A_{G_{\lrcorner}}\right\| = 0$. By the continuity of g and h,

$$g(A) = \lim_{G \in \mathscr{F}} g(A_{\underline{G}}) = \lim_{G \in \mathscr{F}} h(A_{\underline{G}}) = h(A).$$

That is $h\big|_{\mathscr{K}} = g$ on $\mathscr{K}$. From the convergence $\|h_F - h\|_{\mathscr{M}^{\#}} \to 0$, we also have, for each $A \in \mathscr{M}$,

$$h(A) = \lim_{F \in \mathscr{F}} h_F(A) = \lim_{F \in \mathscr{F}} \sum_{t \in F} \sum_{s \in S} (\xi(s,t))(A(s,t)) = \sum_{t \in S} \sum_{s \in S} (\xi(s,t))(A(s,t)).$$

Let $\widehat{g}$ be the unique extension of g to all of $\mathscr{M}$, as in Proposition 13. Then

$$\widehat{g}(A) = \sum_{s \in S} \sum_{t \in S} (\xi(s,t))(A(s,t)) \qquad \text{for all } A \in \mathscr{M}.$$

For $A \in \mathscr{M}$ and $G \in \mathscr{F}(S)$,

$$\begin{aligned} (\widehat{g} - h_G)(A) &= \widehat{g}(A) - h_G(A) = \sum_{s \in S} \sum_{t \in S} (\xi(s,t))(A(s,t)) - \sum_{s \in S} \sum_{t \in G} (\xi(s,t))(A(s,t)) \\ &= \sum_{s \in S} \sum_{t \in S \setminus G} (\xi(s,t))(A(s,t)) = \sum_{s \in S} \sum_{t \in S} (\widetilde{\xi}(s,t))(A(s,t)), \end{aligned}$$

where $\widetilde{\xi}(s,t) = \xi(s,t)$ if $(s,t) \in S \times (S \setminus G)$ and $\widetilde{\xi}(s,t) = 0$ otherwise. By Proposition 13 again, we have

$$\lim_{G \in \mathscr{F}(S)} \left\| \widehat{g} - h_G \right\|_{\mathscr{M}^\#} = \lim_{G \in \mathscr{F}(S)} \left\| [g - (h_G)|_{\mathscr{K}}]\widehat{\ } \right\|_{\mathscr{K}^\#} = \lim_{G \in \mathscr{F}(S)} \left\| g - (h_G)|_{\mathscr{K}} \right\|_{\mathscr{K}^\#} = 0.$$

Thus $\widehat{g} = \lim_{G \in \mathscr{F}(S)} h_G = h$. Hence, for all $A \in \mathscr{M}$, we have as asserted,

$$\sum_{s \in S} \sum_{t \in S} [\xi(s,t)](A(s,t)) = \widehat{g}(A) = h(A) = \sum_{t \in S} \sum_{s \in S} [\xi(s,t)](A(s,t)). \quad \square$$

6. The Hilbert C^*-module $\mathscr{X}$ and adjointable matrix operators

In this section we will use the fact that $\mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$ is a Hilbert C^*-module to establish a bound for the norm of block diagonal matrix operators, which will be used in the Dixmier decomposition theorem (Theorem 19). In a Hilbert space we have $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ for orthogonal vectors x and y; in particular for $x,\ y \in \ell^2(s)$ with disjoint supports; i.e., $x(s)y(s) = 0$ for all $s \in S$. However, this is not true for functions in $\mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$ or $\ell^2_*(S, \mathscr{A})$, as the following example shows.

EXAMPLE 15. With $\mathscr{A} = C[0,1]$ and $S = \mathbb{N}$, there are $\mathbf{x}, \mathbf{y} \in \mathscr{X}$ with disjoint supports (i.e., for all $n \in \mathbb{N}$, $\mathbf{x}(n) = 0$ or $\mathbf{y}(n) = 0$) such that $\|\mathbf{x}+\mathbf{y}\|^2 < \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Proof. Define $\mathbf{x}, \mathbf{y} : \mathbb{N} \to \mathscr{A}$ by

$$(\mathbf{x}(1))(t) = \begin{cases} 1 - 2t & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < t \leqslant 1; \end{cases} \qquad (\mathbf{y}(2))(t) = \begin{cases} 0 & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ 2t - 1 & \text{for } \frac{1}{2} < t \leqslant 1; \end{cases}$$

and $\mathbf{x}(n) = 0$ for all $n \neq 1$ and $\mathbf{y}(n) = 0$ for $n \neq 2$. Then, since $\mathbf{x}(1)$ and $\mathbf{y}(2)$ are self-adjoint elements in $\mathscr{A}$,

$$\begin{aligned} \|\mathbf{x}+y\|^2 &= \sup_{\varphi \in s(\mathscr{A})} [\varphi((\mathbf{x}(1))^2) + \varphi((\mathbf{y}(2))^2)] = \sup_{\varphi \in s(\mathscr{A})} \varphi((\mathbf{x}(1))^2 + (\mathbf{y}(2))^2) \\ &= \left\| (\mathbf{x}(1))^2 + (\mathbf{y}(2))^2 \right\| = 1 < 2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2. \quad \square \end{aligned}$$

The Pythagorean property implies that the norm of a block diagonal matrix operator is the maximum of the norms of the blocks. The result remains true for operator matrices on $\mathscr{X}$. The following is a proof of this fact by using properties of the Hilbert C^*-module $\mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$ [6, p. 4].

LEMMA 16.

(i) The space $\mathscr{X} = \ell^2_{*u}(S, \mathscr{A})$ (with the $\mathscr{A}$-valued inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$) is a Hilbert C^*-module over $\mathscr{A}$.

(ii) Each adjointable matrix operator A on $\mathscr{X}$ is right $\mathscr{A}$-linear.

(To have the $\mathscr{A}$-valued inner product linear in the second argument as in [6], just define $\langle\!\langle \mathbf{x},\, \mathbf{y}\rangle\!\rangle_1 = \langle\!\langle \mathbf{y},\, \mathbf{x}\rangle\!\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathscr{X}$.)

Proof. (i) Let $\mathbf{x} \in \mathscr{X}$ and $a \in \mathscr{A}$. Let $\varepsilon > 0$. Since $\sum_{s\in S} \mathbf{x}^*(s)\mathbf{x}(s)$ converges in $\mathscr{A}$, there is an $F_\varepsilon \in \mathscr{F}(S)$ such that

$$\left\| \sum_{s\in F} \mathbf{x}^*(s)\mathbf{x}(s) \right\| < \frac{\varepsilon}{\|a\|^2 + 1} \qquad \text{for all } F_\varepsilon \subseteq F \in \mathscr{F}(S).$$

If $F_\varepsilon \subseteq F \in \mathscr{F}(S)$, then

$$\begin{aligned}\left\| \sum_{s\in F} a^* \mathbf{x}^*(s)\mathbf{x}(s)a \right\| &= \left\| a^* \left[\sum_{s\in F} \mathbf{x}^*(s)\mathbf{x}(s) \right] a \right\| \leqslant \left\| a^* \right\| \left\| \sum_{s\in F} \mathbf{x}^*(s)\mathbf{x}(s) \right\| \|a\| \\ &= \|a\|^2 \left\| \sum_{s\in F} \mathbf{x}^*(s)\mathbf{x}(s) \right\| < \varepsilon.\end{aligned}$$

Thus

$$\sum_{s\in S} a^* \mathbf{x}^*(s)\mathbf{x}(s)a \quad \text{ converges in } \mathscr{A}, \text{ and hence } \mathbf{x}a \in \mathscr{X}.$$

That $\langle\!\langle \cdot,\, \cdot \rangle\!\rangle$ is an $\mathscr{A}$-valued inner product on $\mathscr{X}$ is routine to check. Therefore $\mathscr{X}$ is an $\mathscr{A}$-module (this is in fact an example in [6]).

(ii) This follows from the distributive property of the multiplication on $\mathscr{A}$. For if $\mathbf{x} \in \mathscr{X}$ an $a \in \mathscr{A}$, we have, for each $s \in S$,

$$[A(\mathbf{x}a)](s) = \sum_{t\in S} [A(s,t)((\mathbf{x}(t))a)] = \sum_{t\in S} (A(s,t)\mathbf{x}(t))\,a = ((A\mathbf{x})(s))a. \quad \square$$

Denote by $\mathfrak{L}(\mathscr{X})$ the set of all adjointable $\mathscr{A}$-linear bounded operators on $\mathscr{X}$. Then $\mathfrak{L}(\mathscr{X})$ is a C^*-algebra with the operator norm [6, p. 8]. A routine verification reveals that the adjoint operation on the adjointable $\mathscr{A}$-matrix operators coincides with the $^\#$ operation here. For convenience of reference we state the following lemma in the form that is more suitable in our situation.

LEMMA 17. [6, Lemma 4.1 (p. 32)] *Let T be an $\mathscr{A}$-linear bounded operator on $\mathscr{X}$. Then T is positive element of $\mathfrak{L}(\mathscr{X})$ iff $\langle\!\langle T\mathbf{x},\, \mathbf{x}\rangle\!\rangle \geqslant 0$ for all $\mathbf{x} \in \mathscr{X}$.*

LEMMA 18. *Let $A \in \mathscr{M}$. For each $\mathbf{x} \in \mathscr{X}$,*

$$\langle\!\langle A\mathbf{x},\, A\mathbf{x}\rangle\!\rangle \leqslant \|A\|^2 \langle\!\langle \mathbf{x},\, \mathbf{x}\rangle\!\rangle \quad \textit{in } \mathscr{A}.$$

Proof. By Lemma 16, $\mathscr{X} = \ell^2_{*u}(S,\mathscr{A})$ is a two-sided Hilbert C^*-module. For each $F \in \mathscr{F}(S)$, $A_{F|}$ is adjointable with adjoint $(A_{F|})^{\#}$ (though A may not be adjointable), and hence $(A_{F|})^{\#}(A_{F|})$ is adjointable. For each $\mathbf{x} \in \mathscr{X}$,

$$\left\langle\!\!\left\langle (A_{F|})^{\#}(A_{F|})\mathbf{x},\, \mathbf{x} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle (A_{F|})\mathbf{x},\, (A_{F|})\mathbf{x} \right\rangle\!\!\right\rangle \geqslant 0 \quad \text{in } \mathscr{A}.$$

Thus $(A_{F|})^{\#}A_{F|}$ is positive in the C^*-algebra $\mathfrak{L}(\mathscr{X}])$ by Lemma 17. Since

$$\left\|(A_{F|})\mathbf{x}\right\| = \|A(\mathbf{x}_F)\| \leqslant \|A\|\,\|\mathbf{x}_F\| \leqslant \|A\|\,\|\mathbf{x}\| \qquad \forall\ \mathbf{x} \in \mathscr{X},$$

$\|A\|^2 - (A_{F|})^{\#}(A_{F|})$ is a positive element in the C^*-algebra $\mathfrak{L}(\mathscr{X})$. Applying Lemma 17 again, with the opposite implication, we have also, for each $\mathbf{x} \in \mathscr{X}$,

$$\begin{aligned} 0 &\leqslant \left\langle\!\!\left\langle [\|A\|^2 - (A_{F|})^{\#}(A_{F|})]\mathbf{x},\, \mathbf{x} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \|A\|^2\mathbf{x},\, \mathbf{x} \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle (A_{F|})^{\#}(A_{F|})\mathbf{x},\, \mathbf{x} \right\rangle\!\!\right\rangle \\ &= \|A\|^2\langle\!\langle \mathbf{x},\, \mathbf{x}\rangle\!\rangle - \left\langle\!\!\left\langle (A_{F|})\mathbf{x},\, (A_{F|})\mathbf{x} \right\rangle\!\!\right\rangle = \|A\|^2\langle\!\langle \mathbf{x},\, \mathbf{x}\rangle\!\rangle - \langle\!\langle A(\mathbf{x}_F),\, A(\mathbf{x}_F)\rangle\!\rangle \end{aligned}$$

That is $\langle\!\langle A(\mathbf{x}_F),\, A(\mathbf{x}_F)\rangle\!\rangle \leqslant \|A\|^2\langle\!\langle \mathbf{x},\, \mathbf{x}\rangle\!\rangle$ for all $F \in \mathscr{F}(S)$ and all $\mathbf{x} \in X$. Since

$$\lim_{F\in\mathscr{F}(S)} \|\mathbf{x}_F - \mathbf{x}\| = 0 \ \text{ for all } \mathbf{x} \in \mathscr{X}, \quad \text{and} \quad \langle\!\langle \cdot,\, \cdot\rangle\!\rangle \text{ is continuous in both variables,}$$

we have

$$\langle\!\langle A\mathbf{x},\, A\mathbf{x}\rangle\!\rangle = \lim_{F\in\mathscr{F}(S)} \langle\!\langle A(\mathbf{x}_F),\, A(\mathbf{x}_F)\rangle\!\rangle \leqslant \lim_{F\in\mathscr{F}(S)} \|A\|^2\langle\!\langle \mathbf{x},\, \mathbf{x}\rangle\!\rangle = \|A\|^2\langle\!\langle \mathbf{x},\, \mathbf{x}\rangle\!\rangle \quad \square$$

7. A decomposition theorem for $\mathscr{M}^{\#}$

Now we are ready to prove a decomposition theorem analogous to the Dixmier decomposition theorem for the pair $\mathscr{K}$ and $\mathscr{M}$; i.e., $\mathscr{K}$ is an M-ideal in $\mathscr{M}$. As a subspace of the set of adjointable matrix operators $\mathscr{M}_0 (= \mathfrak{L}(\mathscr{X}) \cap \mathscr{M})$, $\mathscr{K}$ is an M-ideal, by a theorem of Smith and Ward [7], simply because the space $\mathscr{M}_0$ is a C^*-algebra and $\mathscr{K}$ is an ideal in $\mathscr{M}_0$. However, $\mathscr{M}$ properly contains $\mathscr{M}_0$ and $\mathscr{M}$ is not a C^*-algebra, as noted following the proof of Lemma 12. It is not hard to show that if J is an M-ideal in a Banach space X, and X is contained in a Banach space Y, then J may not, in general, be an M-ideal in Y. However, in this case, we will show that $\mathscr{K}$ is an M-ideal in $\mathscr{M}$.

THEOREM 19. *Each $g \in \mathscr{K}^{\#}$ has a unique Hahn-Banach extension, also denoted by g, to all of $\mathscr{M}$ with $\|g\|_{\mathscr{K}^{\#}} = \|g\|_{\mathscr{M}^{\#}}$. For each $f \in \mathscr{M}^{\#}$, there are unique $g \in \mathscr{K}^{\#}$ (as a subspace of $\mathscr{M}^{\#}$, via the uniqueness of extensions) and $h \in \mathscr{K}^{\perp}$ such that $f = g + h$ and $\|f\| = \|g\| + \|h\|$.*

Proof. Uniqueness of Hahn-Banach extension of $g \in \mathscr{K}^{\#}$ is immediate from Proposition 13. Let $f \in [\mathscr{M}]^{\#}$. Then by Proposition 13, there is a map $\widetilde{f} : S \times S \to \mathscr{A}^{\#}$ such that

$$g(A) = \sum_{s \in S} \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t))$$

converges for all $A \in \mathscr{M}$, and $g = f$ on $\mathscr{K}$. Let $h = f - g$. Then $h = 0$ on $\mathscr{K}$ and $f = g + h$. Uniqueness is clear from the construction: for if another function $f' : S \times S \to \mathscr{A}^{\#}$ satisfies

$$g'(A) = \sum_{s \in S} \sum_{t \in S} [f'(s,t)](A(s,t))$$

converges for all $A \in \mathscr{M}$ and $g' = f$ on $\mathscr{K}$, then, for each $(s,t) \in S \times S$,

$$[f'(s,t)](a) = g'(E_{(s,t)}(a)) = f(E_{(s,t)}(a)) = g(E_{(s,t)}(a)) = [\widetilde{f}(s,t)](a),$$

for all $a \in \mathscr{A}$, and hence $f' = \widetilde{f}$.

Since $\|f\| \leqslant \|g\| + \|h\|$, it suffices to establish the nontrivial opposite inequality. To that end, let $\varepsilon > 0$. There are $A, B \in \mathscr{M}$ such that

$$\|A\| = \|B\| = 1, \quad g(A) > \|g\| - \frac{\varepsilon}{6}, \quad \text{and} \quad h(B) > \|h\| - \frac{\varepsilon}{6}. \tag{4}$$

From the convergence of $g(A)$ to a positive number, there is an $F_1 \in \mathscr{F}$ such that

$$\Re\left[\sum_{s \in F} \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t))\right] > g(A) - \frac{\varepsilon}{6} > \|g\| - \frac{\varepsilon}{3} \quad \forall\ F \in \mathscr{F},\ F \supseteq F_1.$$

From the finiteness of F_1 and the convergence of the inner sums in the last expression, there is a $G_1 \in \mathscr{F}$ such that

$$\Re\left[\sum_{s \in F_1} \sum_{t \in G_1} [\widetilde{f}(s,t)](A(s,t))\right] > \Re\left[\sum_{s \in F_1} \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t))\right] - \frac{\varepsilon}{6} > \|g\| - \frac{2\varepsilon}{3}. \tag{5}$$

From the convergence of $g(B)$, there is a finite subset (of S) $F_2 \supseteq F_1$ such that

$$\left|\sum_{s \in S \setminus F_2} \sum_{t \in S} [\widetilde{f}(s,t)](B(s,t))\right| < \frac{\varepsilon}{6}.$$

Since $B - B_{\underline{F_2}} \in \mathscr{M}$,

$$\sum_{s \in S \setminus F_2} \sum_{t \in S} [\widetilde{f}(s,t)](B(s,t)) = \sum_{t \in S} \sum_{s \in S \setminus F_2} [\widetilde{f}(s,t)](B(s,t)),$$

by Proposition 14, hence there is a finite subset (of S) $G_2 \supseteq G_1$ such that

$$\left|\sum_{t\in S\setminus G_2}\ \sum_{s\in S\setminus F_2}[\widetilde{f}(s,t)](B(s,t))\right| = \left|\sum_{s\in S\setminus F_2}\ \sum_{t\in S\setminus G_2}[\widetilde{f}(s,t)](B(s,t))\right| < \frac{\varepsilon}{6}. \tag{6}$$

Let

$$A_0 = (A_{\underline{F_1}})_{G_1|}, \qquad B_0 = (B - B_{\underline{F_2}}) - (B - B_{\underline{F_2}})_{G_2|}, \qquad \text{and} \qquad C = A_0 + B_0.$$

Then inequalities (5) and (6) are, respectively,

$$\Re(g(A_0)) > \|g\| - \frac{2\varepsilon}{3}, \qquad \text{and} \qquad |g(B_0)| < \frac{\varepsilon}{6}.$$

For each $\mathbf{x} \in \mathscr{X}$, since $G_1 \subseteq G_2$, we have

$$\left\langle\!\!\left\langle \left[A_0(\mathbf{x}_{G_1})\right]_{F_1},\ \left[B_0(\mathbf{x}_{S\setminus G_2})\right]_{S\setminus F_2} \right\rangle\!\!\right\rangle = 0 \qquad \text{and}$$

$$\left\langle\!\!\left\langle \left[B_0(\mathbf{x}_{S\setminus G_2})\right]_{S\setminus F_2},\ \left[A_0(\mathbf{x}_{G_1})\right]_{F_1} \right\rangle\!\!\right\rangle = 0,$$

since each pair of functions have disjoint supports. Thus, from Lemma 18,

$$\begin{aligned}
&\langle\!\langle C\mathbf{x},\ C\mathbf{x}\rangle\!\rangle = \left\langle (A_0+B_0)\mathbf{x},\ (A_0+B_0)\mathbf{x}\right\rangle \\
&= \left\langle A_0\mathbf{x},\ A_0\mathbf{x}\right\rangle + \left\langle A_0\mathbf{x},\ B_0\mathbf{x}\right\rangle + \left\langle B_0\mathbf{x},\ A_0\mathbf{x}\right\rangle + \left\langle B_0\mathbf{x},\ B_0\mathbf{x}\right\rangle \\
&= \left\langle\!\left\langle A_0(\mathbf{x}_{G_1}),\ A_0(\mathbf{x}_{G_1})\right\rangle\!\right\rangle + \left\langle\!\left\langle A_0(\mathbf{x}_{G_1}),\ B_0(\mathbf{x}_{S\setminus G_2})\right\rangle\!\right\rangle \\
&\quad + \left\langle\!\left\langle B_0(\mathbf{x}_{S\setminus G_2}),\ A_0(\mathbf{x}_{G_1})\right\rangle\!\right\rangle + \left\langle\!\left\langle B_0(\mathbf{x}_{S\setminus G_2}),\ B_0(\mathbf{x}_{S\setminus G_2})\right\rangle\!\right\rangle \\
&= \left\langle\!\!\left\langle \left[A_0(\mathbf{x}_{G_1})\right]_{F_1},\ \left[A_0(\mathbf{x}_{G_1})\right]_{F_1}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \left[A_0(\mathbf{x}_{G_1})\right]_{F_1},\ \left[B_0(\mathbf{x}_{S\setminus G_2})\right]_{S\setminus F_2}\right\rangle\!\!\right\rangle \\
&\quad + \left\langle\!\!\left\langle \left[B_0(\mathbf{x}_{S\setminus G_2})\right]_{S\setminus F_2},\ \left[A_0(\mathbf{x}_{G_1})\right]_{F_1}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \left[B_0(\mathbf{x}_{S\setminus G_2})\right]_{S\setminus F_2},\ \left[B_0(\mathbf{x}_{S\setminus G_2})\right]_{S\setminus F_2}\right\rangle\!\!\right\rangle \\
&= \left\langle\!\!\left\langle \left[A(\mathbf{x}_{G_1})\right]_{F_1},\ \left[A(\mathbf{x}_{G_1})\right]_{F_1}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \left[B(\mathbf{x}_{S\setminus G_2})\right]_{S\setminus F_2},\ \left[B(\mathbf{x}_{S\setminus G_2})\right]_{S\setminus F_2}\right\rangle\!\!\right\rangle \\
&\leqslant \left\langle\!\left\langle A(\mathbf{x}_{G_1}),\ A(\mathbf{x}_{G_1})\right\rangle\!\right\rangle + \left\langle\!\left\langle B(\mathbf{x}_{S\setminus G_2}),\ B(\mathbf{x}_{S\setminus G_2})\right\rangle\!\right\rangle \\
&\leqslant \left\langle\!\left\langle \mathbf{x}_{G_1},\ \mathbf{x}_{G_1}\right\rangle\!\right\rangle + \left\langle\!\left\langle \mathbf{x}_{S\setminus G_2},\ \mathbf{x}_{S\setminus G_2}\right\rangle\!\right\rangle \leqslant \left\langle\!\left\langle \mathbf{x}_{G_1},\ \mathbf{x}_{G_1}\right\rangle\!\right\rangle + \left\langle\!\left\langle \mathbf{x}_{S\setminus G_1},\ \mathbf{x}_{S\setminus G_1}\right\rangle\!\right\rangle \\
&= \langle\!\langle \mathbf{x},\ \mathbf{x}\rangle\!\rangle
\end{aligned}$$

For each $\varphi \in s(\mathscr{A})$, we have

$$\sum_{s\in S}\varphi\left((C\mathbf{x})^*(s)(C\mathbf{x})(s)\right) = \varphi\left(\sum_{s\in S}(C\mathbf{x})^*(s)(C\mathbf{x})(s)\right) = \varphi\left(\langle\!\langle C\mathbf{x},\, C\mathbf{x}\rangle\!\rangle\right)$$
$$\leqslant \varphi\left(\langle\!\langle \mathbf{x},\, \mathbf{x}\rangle\!\rangle\right) = \varphi\left(\sum_{s\in S}\mathbf{x}^*(s)\mathbf{x}(s)\right)$$
$$= \sum_{s\in S}\varphi\left(\mathbf{x}^*(s)\mathbf{x}(s)\right) \leqslant \|\mathbf{x}\|^2.$$

Thus

$$\|C\mathbf{x}\|^2 = \sup_{\varphi\in s(\mathscr{A})}\sum_{s\in S}\varphi\left((C\mathbf{x})^*(s)(C\mathbf{x})(s)\right) \leqslant \|\mathbf{x}\|^2,$$

and hence $\|C\mathbf{x}\| \leqslant \|\mathbf{x}\|$. Since $\mathbf{x} \in \mathscr{X}$ is arbitrary, we have $\|C\| \leqslant 1$.

Now, since $A_0 \in \mathscr{K}$, $h(A_0) = 0$. Since $B_{\underline{F_2}}$ and $(B - B_{\underline{F_2}})_{G_2|}$ are in $\mathscr{K}$, and h vanishes on $\mathscr{K}$,

$$h(B_0) = h(B - B_{\underline{F_2}}) - h\left((B - B_{\underline{F_2}})_{G_2|}\right) = h(B).$$

These together with the inequality (4) we have

$$\|f\| \geqslant |f(C)| = \left|g(A_0) + g(B_0) + h(A_0) + h(B_0)\right| \geqslant \left|g(A_0) + h(B_0)\right| - \left|g(B_0)\right|$$
$$> \Re\left(g(A_0) + h(B_0)\right) - \frac{\varepsilon}{6} = \Re(g(A_0)) + \Re(h(B)) - \frac{\varepsilon}{6}$$
$$> \|g\| - \frac{2\varepsilon}{3} + \|h\| - \frac{\varepsilon}{6} - \frac{\varepsilon}{6} = \|g\| + \|h\| - \varepsilon.$$

Since this argument holds for all $\varepsilon > 0$, we have $\|f\| \geqslant \|g\| + \|h\|$. Combining this with the triangle inequality, we have $\|f\| = \|g\| + \|h\|$ as asserted. $\square$

REFERENCES

[1] E. M. ALFSEN AND E. G. EFFROS, *Structure in real Banach spaces*, Ann. of Math. **96** (1972), 98–173.

[2] J. DIXMIER, *Les fonctionnelles lin'earies sur l'ensemble des op'erateurs born'es dans espace de Hilbert*, Ann. Math. **51** (1950), 387–408.

[3] PETER HARMAND, DIRK WERNER, AND WEND WERNER, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Mathematics, vol. 1547, Springer-Verlag, 1993.

[4] J. HENNEFELD, *A decomposition for $B(X)^*$ and unique Hahn-Banach extensions*, Pac. J Math. **46** (1973), 197–199.

[5] R. V. KADISON AND J. R. RINGROSE, *Fundamentals of the Theory of Operator Algebras*, vol. I, Academic Press, New York, 1983.

[6] E. C. LANCE, *Hilbert C^*-Modules – A toolkit for operator algebraists*, LMS Lecture Note Series, no. 210, Cambridge University Press, 1995.

[7] R. R. SMITH AND J. D. WARD, *M-ideal structure in Banach algebras*, J. Funct. Anal. **27** (1978), 337–349.

[8] T. WOOTIJIRATTIKAL AND S.-C. ONG, *Functional decomposition of state induced C^*-matrix spaces*, Banach J. Math. Anal. **5** (2011), no. 2, 106–121.

[9] T. WOOTIJIRATTIKAL, S.-C. ONG, AND J. RAKBUD, *Functional decompositions on vector-valued function spaces via operators*, J. Math. Anal. Appl. **389** (2012), 1173–1190.

[10] O. WOOTIJIRUTTIKAL, S.-C. ONG, P. CHAISURIYA, AND J. RAKBUD, *Banach spaces of functions taking values in a C^*-algebra*, Oper. Matrices **3** (2009), 373–396.

(Received February 28, 2012)

Titarii Wootijirattikal
Department of Mathematics, Statistics and Computer
Faculty of Science, Ubon Ratchathani University
Ubon Ratchathani 34190, Thailand
and
Centre of Excellence in Mathematics
CHE, Si Ayutthaya Rd., Bangkok 10400, Thailand
e-mail: sctitawo@ubu.ac.th, ma_pintto@yahoo.com

Sing-Cheong Ong
Department of Mathematics
Central Michigan University
Mount Pleasant, MI 48859, USA
e-mail: ong1s@cmich.edu, ong3pf@gmail.com

Jitti Rakbud
Department of Mathematics, Faculty of Science
Silpakorn University
Nakhon Pathom 73000, Thailand
e-mail: jitti@su.ac.th, jitrakbud@yahoo.com