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A NECESSARY AND SUFFICIENT CONDITION FOR POSITIVITY OF LINEAR MAPS ON M_4 CONSTRUCTED FROM PERMUTATION PAIRS

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Abstract. A necessary and sufficient condition for a D-type map Φ_{π_1,π_2} on 4×4 matrices constructed from a pair of arbitrary permutations $\{\pi_1,\pi_2\}$ to be positive is obtained.

1. Introduction

Denote by $M_n = M_n(\mathbb{C})$ the algebra of all $n\times n$ complex matrices and M_n^+ the set of all positive semi-definite matrices in M_n. A map $L: M_n \to M_n$ is positive if $L(M_n^+) \subseteq M_n^+$. The positive maps are important objects both in mathematics and quantum information theory, see [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 15].

Suppose $\Phi_D: M_n \to M_n$ is a linear map of the form

$$(a_{ij}) \longmapsto \mathrm{diag}(f_1, f_2, ..., f_n) - (a_{ij}) \tag{1.1}$$

with $(f_1, f_2, ..., f_n) = (a_{11}, a_{22}, ..., a_{nn})D$ for an $n\times n$ nonnegative matrix $D = (d_{ij})$ (i.e., $d_{ij} \geqslant 0$ for all i, j). The map Φ_D of the form Eq. (1.1) defined by a nonnegative matrix D is called a D-type map [9]. The question of when a D-type map is positive was studied intensively by many authors and applied in quantum information theory to detect entangled states and construct entanglement witnesses (ref., for instance, [9, 14] and the references therein).

A very interesting class of D-type maps is the class of maps constructed from permutations.

Assume that π is a permutation of $(1,2,\ldots,n)$. Recall that the permutation matrix $P_\pi = (p_{ij})$ of π is a $n\times n$ matrix determined by

$$p_{ij} = \begin{cases} 1 & \text{if } i = \pi(j), \\ 0 & \text{if } i \neq \pi(j). \end{cases}$$

The well-known Choi map $\Psi: M_3 \to M_3$ defined by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}+a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22}+a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33}+a_{22} \end{pmatrix}$$

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is clearly a D-type map induced from the permutation $(1,2,3) \to (2,3,1)$.

Recall also that a subset $(i_1,\ldots,i_l) \subseteq \{1,2,\ldots,n\}$ is an l-cycle of the permutation π if $\pi(i_j) = i_{j+1}$ for $j = 1,\ldots,l-1$ and $\pi(i_l) = i_1$. Note that every permutation π of $(1,\ldots,n)$ has a disjoint cycle decomposition $\pi = (\pi_1)(\pi_2)\cdots(\pi_r)$, that is, there exists a set $\{F_s\}_{s=1}^r$ of disjoint cycles of π with $\cup_{s=1}^r F_s = \{1,2,\ldots,n\}$ such that $\pi_s = \pi|_{F_s}$ and $\pi(i) = \pi_s(i)$ whenever $i \in F_s$. Let π be a permutation of $(1,2,\ldots,n)$ with disjoint cycle decomposition $(\pi_1)\cdots(\pi_r)$ such that the maximum length of π_i is equal to $l > 1$ and $P_\pi = (\delta_{i\pi(j)})$ is the permutation matrix associated with π. For $t \geqslant 0$, let $\Phi_{t,\pi} : M_n \to M_n$ be the D-type map of the form in Eq. (1.1) with $D = (n-t)I_n + tP_\pi$. It is shown in [9] that $\Phi_{t,\pi}$ is positive if and only if $0 \leqslant t \leqslant \frac{n}{l}$. Thus Φ_D with $D = (n-2)I_n + P_\pi + P_\pi$ is not positive if $\frac{n}{l} < 2$. This fact reveals that, in general, a D-type map with $D = (n-2)I_n + P_{\pi_1} + P_{\pi_2}$ is not a positive map.

Motivated by the above result, it was discussed in [16] the D-type maps constructed from a pair of permutations, that is,

$$\Phi_{n,\pi_1,\pi_2} = \Phi_{D_{\pi_1,\pi_2}} \text{ with } D_{\pi_1,\pi_2} = (n-2)I_n + P_{\pi_1} + P_{\pi_2}, \tag{1.2}$$

and the question that under what conditions that Φ_{n,π_1,π_2} of the form Eq. (1.2) are positive. A notion of the property (C) for pairs of permutations was introduced in [16] (see Definition 3.2 below), and it was proved that, if $\{\pi_1,\pi_2\}$ has property (C), then the D-type map $\Phi_{n,\pi_1,\pi_2} : M_n \to M_n$ with $n \geqslant 3$ is positive. The property (C) is characterized for $\{\pi_1,\pi_2\}$, and a criterion is given for the case that $\pi_1 = \pi^p$ and $\pi_2 = \pi^q$, where π is the permutation defined by $\pi(i) = i+1 \bmod n$ and $1 \leqslant p < q \leqslant n$. The results in [16] allow us to construct many new positive maps. However, the property (C) is only a sufficient condition for Φ_{n,π_1,π_2} to be positive. So, it is natural and interesting to ask the following.

PROBLEM 1.1. What is the necessary and sufficient condition for Φ_{n,π_1,π_2} to be positive?

The purpose of this paper is to give an answer to the above problem for low dimension cases, that is, the case $n \in \{3,4\}$. Since the results in [9], we always assume in this paper that $\pi_1 \neq \pi_2$ and, neither π_1 nor π_2 is the identity permutation. Furthermore, we denote by $l(\pi_1,\pi_2)$ the length of the pair $\{\pi_1,\pi_2\}$ of permutations defined by

$$l(\pi_1,\pi_2) = \max\{\#F : F \text{ is a minimal common invariant subset of } \pi_1,\pi_2\}.$$

In other words, $l(\pi_1,\pi_2)$ is the cardinality of the minimal common invariant subset of π_1 and π_2 which has the largest number of elements.

The following are the main results.

THEOREM 1.2. *Let π_1 and π_2 be two distinct permutations of $(1,2,3,4)$ that are not the identity, and let $\Phi_{\pi_1,\pi_2} : M_4(\mathbb{C}) \to M_4(\mathbb{C})$ be the D-type map defined by $D = 2I_4 + P_{\pi_1} + P_{\pi_2}$. Then Φ_{π_1,π_2} is positive if and only if either*

(i) *$l(\pi_1,\pi_2) = 2$; or*

(ii) *$l(\pi_1,\pi_2) \geqslant 3$ and the following two conditions hold:*

(1) if i is not the fixed point of both π_1 and π_2, then $\pi_1(i) \neq \pi_2(i)$;

(2) if π_1 and π_2 have no common fixed points and if there exist distinct i,j such that $\{\pi_1(i),\pi_2(i)\} = \{\pi_1(j),\pi_2(j)\}$, then neither π_1 nor π_2 has fixed points.

THEOREM 1.3. *Let π_1 and π_2 be two distinct permutations of $(1,2,3)$ that are not the identity, and let $\Phi_{\pi_1,\pi_2}: M_3(\mathbb{C}) \to M_3(\mathbb{C})$ be the D-type map defined by $D = I_3 + P_{\pi_1} + P_{\pi_2}$. Then Φ_{π_1,π_2} is positive if and only if $\pi_1(i) \neq \pi_2(i)$ holds for any i.*

The paper is organized as follows. In Section 2 we recall some preliminary inequalities from [16] that are needed in the remain part of the paper. Section 3 deals with the case that $n = 4$, $\{\pi_1, \pi_2\}$ has the property (C). A easy characterization of $\{\pi_1, \pi_2\}$ to have the property (C) is given and, based on this, in Section 4, for any pair of permutations of $(1,2,3,4)$, some criteria for $\Phi_{\pi_1,\pi_2}: M_4 \to M_4$ to be positive are established. The final section completes the proofs of Theorems 1.2 and 1.3.

2. Preliminary inequalities

In this section, we first recall some inequalities proved in [16].

LEMMA 2.1. [16, Lemma 2.1] *Let s, M be positive numbers and $f(u_1,u_2,...,u_m)$ be a function in m-variable defined by*

$$f(u_1,u_2,...,u_m) = \frac{1}{s+u_1} + \frac{1}{s+u_2} + ... + \frac{1}{s+u_m}$$

on the region $u_i > 0$ with $u_1u_2...u_m = M^m$, $i = 1,2,...,m$. Then we have

(1) *f has extremum values $\frac{rs^{\frac{m}{2r-m}} + (m-r)M^{\frac{m}{2r-m}}}{s(s^{\frac{m}{2r-m}} + M^{\frac{m}{2r-m}})}$ with $\frac{m}{2} < r \leqslant m$ at the points that r of u_is are $(\frac{M^m}{s^{2m-2r}})^{\frac{1}{2r-m}}$ and others are $(\frac{s^{2r}}{M^m})^{\frac{1}{2r-m}}$;*

(2) *f may also achieve the extremum $\frac{m}{s+M}$ when m is even, at points $\frac{m}{2}$ of u_is are u and others are $\frac{s^2}{u}$, in this case we must have $s = M$;*

(3) $\sup f(u_1,u_2,...,u_m) = \max\{\frac{m-1}{s}, \frac{m}{s+M}\}$.

Consequently, we have

COROLLARY 2.2. *Let s, M be positive numbers and $f(u_1,u_2,u_3,u_4)$ be a function in 4-variable defined by*

$$f(u_1,...,u_4) = \frac{1}{s+u_1} + \frac{1}{s+u_2} + \frac{1}{s+u_3} + \frac{1}{s+u_4}$$

on the region $u_i > 0$ with $u_1u_2u_3u_4 = M^m$. Then,

$$\sup f(u_1,u_2,u_3,u_m) = \max\{\frac{3}{s}, \frac{4}{s+M}\}.$$

Moreover, all possible extremum values of f is bounded by $\max\{\frac{4}{s+M}, \frac{2s}{(s^2+M^2)} + \frac{1}{s}\}$.

LEMMA 2.3. [16, Lemma 2.2] *Let s be a positive number, n,k be positive integers with $s > k$. Then for any nk positive real numbers $\{x_{hi}, h = 1,2,...,k; i = 1,2,...,n,\}$ satisfying $x_{h1}x_{h2}...x_{hn} = 1$ for each h with $1 \leqslant h \leqslant k$, we have*

$$\begin{aligned} f(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{k1},\ldots,x_{kn}) &= \textstyle\sum_{i=1}^{n} \frac{1}{s-k+x_{1i}+x_{2i}+...+x_{ki}} \\ &\leqslant \max\{\frac{n-1}{s-k}, \frac{n}{s}\}. \end{aligned}$$

Moreover, the extremum values of f *are*

$$\begin{array}{ll}\delta_r = \frac{r(s-k)^{\frac{n}{2r-n}}+(n-r)k^{\frac{n}{2r-n}}}{(s-k)((s-k)^{\frac{n}{2r-n}}+k^{\frac{n}{2r-n}})}, & [\frac{n}{2}]+1 \leqslant r \leqslant n; \\ \delta_{\frac{n}{2}} = \frac{n}{s} & \text{if } n \text{ is even,}\end{array}$$

where $[t]$ *stands for the integer part of real number* t.

The following corollary is immediate.

COROLLARY 2.4. *Let*

$$f(x_{11},\ldots,x_{14},x_{21},\ldots,x_{24}) = \sum_{i=1}^{4} \frac{1}{2+x_{1i}+x_{2i}}.$$

Then, $\sup f = \frac{3}{2}$ *and all extremum values of* f *is 1 on the region of* $x_{hi} > 0$, $i = 1,2,3,4$ *and* $h = 1,2$ *with* $x_{h1}x_{h2}x_{h3}x_{h4} = 1$.

3. Positivity of Φ_{π_1,π_2} on M_4 with $\{\pi_1,\pi_2\}$ having property (C)

For any two permutations π_1 and π_2 of $(1,2,3,4)$, let $\Phi_{\pi_1,\pi_2} : M_4(\mathbb{C}) \to M_4(\mathbb{C})$ be the D-type map of the form

$$(a_{ij}) \longmapsto \mathrm{diag}(f_1,f_2,f_3,f_4) - (a_{ij}), \tag{3.1}$$

where $(f_1,f_2,f_3,f_4) = (a_{11},a_{22},a_{33},a_{44})D$ and $D = 2I_4 + P_{\pi_1} + P_{\pi_2}$ with P_{π_h} the permutation matrix of π_h, $h = 1,2$.

The main purpose of this section is to show the following result.

PROPOSITION 3.1. *Let* $\Phi_{\pi_1,\pi_2} : M_4(\mathbb{C}) \to M_4(\mathbb{C})$ *be a* D*-type map defined by a pair of permutations* $\{\pi_1,\pi_2\}$ *as in Eq. (3.1). Then* Φ_{π_1,π_2} *is positive if any one of the following condition satisfied.*

(i) π_1,π_2 *have two common fixed points.*

(ii) π_1,π_2 *have one common fixed point* i*, and* $\pi_1(j) \neq \pi_2(j)$ *for any* $j \neq i$.

(iii) $\pi_1(i) \neq \pi_2(i)$ *for any* i *and* $\{\pi_1(k),\pi_2(k)\} = \{\pi_1(j),\pi_2(j)\} = \{k,j\}$ *for some distinct* k,j.

(iv) *For any* i, $\pi_1(i) \neq \pi_2(i)$ *and, for any distinct* k,j, $\{\pi_1(k),\pi_2(k)\} \neq \{\pi_1(j), \pi_2(j)\}$.

The following conception was introduced in [16].

DEFINITION 3.2. [16, Definition 3.2] A pair $\{\pi_1,\pi_2\}$ of permutations of $(1,2,\ldots,n)$ is said to have property (C) if, for any given $i \in \{1,2,\ldots,n\}$ and for any $j \neq i$, there exists $\pi_{h_j}(j) \in \{\pi_1(j),\pi_2(j)\}$ such that $\{\pi_{h_j}(j) : j = 1,2,\ldots,i-1,i+1,\ldots,n\} = \{1,2,\ldots,i-1,i+1,\ldots,n\}$, that is, $(\pi_{h_1}(1),\ldots,\pi_{h_{i-1}}(i-1),\pi_{h_{i+1}}(i+1),\ldots,\pi_{h_n}(n))$ is a permutation of $(1,2,\ldots,i-1,i+1,\ldots,n)$.

To make the meaning of the property (C) clear, let us see some examples before going ahead. Let π_1 and π_2 be the permutations $(1,2,3,4) \to (2,3,4,1)$ and $(3,4,1,2)$, respectively; then $\{\pi_1,\pi_2\}$ has the property (C). However the pair $\{\rho_1,\rho_2\}=\{(2,3,4,1),(4,1,2,3)\}$ of permutations of $(1,2,3,4)$ does not have the property (C). To see this, take $i=1$. One can not pick $\rho_{h_2}(2)\in\{\rho_1(2),\rho_2(2)\}=\{3,1\}$, $\rho_{h_3}(3)\in\{4,2\}$ and $\rho_{h_4}(4)\in\{1,3\}$ so that $\{\rho_{h_2}(2),\rho_{h_3}(3),\rho_{h_4}(4)\}=\{2,3,4\}$.

It was shown that for any $n\geqslant 3$ and any pair $\{\pi_1,\pi_2\}$ of permutations of $(1,2,\ldots,n)$, the D-type map Φ_{n,π_1,π_2} is positive if $\{\pi_1,\pi_2\}$ has the property (C). Thus particularly we have

PROPOSITION 3.3. *Let $\Phi_{\pi_1,\pi_2}:M_4(\mathbb{C})\to M_4(\mathbb{C})$ be a D-type map defined by a pair of permutations $\{\pi_1,\pi_2\}$ as in Eq. (3.1). If $\{\pi_1,\pi_2\}$ has the property (C), then Φ_{π_1,π_2} is positive.*

Since the case one of π_1 and π_2 is the identity permutation reduces to the situation had dealt with in [9], we may always assume in the sequel that $\pi_1\neq\mathrm{id}$ and $\pi_2\neq\mathrm{id}$.

By Proposition 3.3, to detect the positivity of a D-type map Φ_{π_1,π_2} on M_4, it is important to determine whether or not the pair $\{\pi_1,\pi_2\}$ of permutations has the property (C).

Let $\{\pi_1,\pi_2\}$ be a pair of permutations on $(1,2,\ldots,n)$. It is clear that the smaller n is the easier to check the property (C) of $\{\pi_1,\pi_2\}$. This motivates us to decompose the permutations into small ones. For a nonempty proper subset F of $\{1,2,\ldots,n\}$, if $\pi_h(F)=F$ holds for all $h=1,2$, we say that F is an invariant subset of $\{\pi_1,\pi_2\}$, or, F is a common invariant subset of π_1 and π_2. Obvious, there exist disjoint minimal invariant subsets $F_1,F_2,\ldots,F_r$ $(r<n)$, of $\{\pi_1,\pi_2\}$ such that $\sum_{s=1}^r\#F_s=n$ (i.e., $\cup_{s=1}^r F_s=\{1,2,\ldots,n\}$). We say $\{F_1,F_2,\ldots,F_l\}$ is the complete set of minimal invariant subsets of $\{\pi_1,\pi_2\}$. Thus one can reduce the pair $\{\pi_1,\pi_2\}$ of permutations into r pairs $\{\pi_{1s},\pi_{2s}\}_{s=1}^r$ of small ones, where $\pi_{hs}=\pi_h|_{F_s}$. It is easily checked that $\{\pi_1,\pi_2\}$ has the property (C) if and only if each of its sub-pairs $\{\pi_{1s},\pi_{2s}\}$ has the property (C).

Now let us come back to the case of $n=4$. Let $\{F_s\}_{s=1}^r$, $1\leqslant r\leqslant 4$, be the complete set of minimal invariant subsets of $\{\pi_1,\pi_2\}$. Assume that $\#F_1\leqslant\#F_2\leqslant\ldots\leqslant\#F_r$. It is clear that,
if $r=3$, then $\#F_1=\#F_2=1$, $\#F_3=2$, and hence $\{\pi_1,\pi_2\}$ has property (C) if and only if one of π_i is the identity;
if $r=2$, $\#F_1=1$ and $\#F_2=3$, then $\{\pi_1,\pi_2\}$ has property (C) if and only if for each $i\in F_2$, $\pi_1(i)\neq\pi_2(i)$;
if $r=2$, $\#F_1=\#F_2=2$, then $\{\pi_1,\pi_2\}$ has property (C) if and only if $\pi_1|_{F_s}\neq\pi_2|_{F_s}$, $s=1,2$, and equivalently, $\pi_1(i)\neq\pi_2(i)$ for any i and $\{\pi_1(k),\pi_2(k)\}=\{\pi_1(j),\pi_2(j)\}=\{k,j\}$ for some distinct k,j.

So, to detect whether or not $\{\pi_1,\pi_2\}$ has the property (C), the only case left is $r=1$, that is, $\{\pi_1,\pi_2\}$ has no proper invariant subsets. This will be done in the next proposition.

PROPOSITION 3.4. *Let π_1,π_2 be two permutations of $(1,2,3,4)$ having no proper common invariant subsets. Then $\{\pi_1,\pi_2\}$ has property (C) if and only if the following conditions are satisfied:*

(1) *For any i, $\pi_1(i)\neq\pi_2(i)$.*

(2) *For any distinct* i, j, $\{\pi_1(i), \pi_2(i)\} \neq \{\pi_1(j), \pi_2(j)\}$.

Proof. Assume that $\{\pi_1, \pi_2\}$ satisfy the conditions (1)-(2). For any i, we have to show that we can choose one element in $\pi_{h_j}(j) \in \{\pi_1(j), \pi_2(j)\}$ for each $j \neq i$ so that $\{\pi_{h_j}(j), j \neq i\} = \{1,2,3,4\} \setminus \{i\}$.

Case (i). $i \in \{\pi_1(i), \pi_2(i)\}$. Say $\pi_1(i) = i$; then obviously the choice $\{\pi_1(j) : j \neq i\} = \{1,2,3,4\} \setminus \{i\}$.

Case (ii). $i \notin \{\pi_1(i), \pi_2(i)\}$.

Let j_1, j_2 such that $\pi_1(j_2) = i = \pi_2(j_1)$; then $i \notin \{j_1, j_2\}$. By the condition (2), $\pi_1(j_1) \neq \pi_2(j_2)$.

If $\{\pi_1(j_1), \pi_2(j_2)\} = \{\pi_1(i), \pi_2(i)\}$, then $\{\pi_1(j_1), \pi_2(j_2)\} \cup \{\pi_1(j) : j \notin \{i, j_1, j_2\}\} = \{1,2,3,4\} \setminus \{i\}$ and we finish the proof.

In the sequel, assume that $\{\pi_1(j_1), \pi_2(j_2)\} \neq \{\pi_1(i), \pi_2(i)\}$.

If $\pi_1(j_1) = \pi_2(i)$ or $\pi_2(j_2) = \pi_1(i)$, saying $\pi_2(j_2) = \pi_1(i)$, then we have $\{\pi_2(j_2)\} \cup \{\pi_1(j) : j \notin \{i, j_2\}\} = \{\pi_1(j) : j \neq j_2\} = \{1,2,3,4\} \setminus \{i\}$, and then the proof is finished.

Thus we may assume that $\{\pi_1(j_1), \pi_2(j_2)\} \cap \{\pi_1(i), \pi_2(i)\} = \emptyset$. Take j_3 so that $\pi_2(j_3) = \pi_1(j_1)$. As $\pi_1(j_1) \neq \pi_2(j_2)$, we have $j_3 \neq j_2$. Since $\pi_1(j_1) \neq \pi_2(i), \pi_1(j_1) \neq \pi_2(j_1) = i$, we have $j_3 \notin \{i, j_1, j_2\}$. We claim that $\pi_1(j_3) \neq \pi_2(j_2)$. In fact, if $\pi_1(j_3) = \pi_2(j_2)$, then one gets $\{\pi_1(j_1), \pi_2(j_2)\} = \{\pi_1(j_3), \pi_2(j_3)\}$. It is clear that $\{i, \pi_1(j_1), \pi_2(j_2)\}$ has three distinct elements, $\{\pi_1(j_1), \pi_2(j_2)\} = \{\pi_1(j_3), \pi_2(j_3)\}$ implies that $\pi_1(i) = \pi_2(i) \in \{1,2,3,4\} \setminus \{i, \pi_1(j_1), \pi_2(j_2)\}$, which contradicts to the condition (1). Thus we get a set $\{\pi_1(j_1), \pi_2(j_2), \pi_1(j_3)\}$ of distinct elements, and hence $\{\pi_1(j_1), \pi_2(j_2), \pi_1(j_3)\} = \{1,2,3,4\} \setminus \{i\}$. So the conditions (1) and (2) imply that $\{\pi_1, \pi_2\}$ has the property (C).

Conversely, if any one of the conditions (1) and (2) is broken, then it is easily checked that $\{\pi_1, \pi_2\}$ cannot have the property (C). For instance, if (1) is broken, then there is i such that $\pi_1(i) = \pi_2(i) = j$. As π_1 and π_2 have no proper common invariant subset, we must have $j \neq i$. It follows that $j \notin \{\pi_1(h), \pi_2(h); h \neq i\}$ and hence $\{\pi_1, \pi_2\}$ does not have the property (C). If the condition (2) is broken, then $\{\pi_1(i), \pi_2(i)\} = \{\pi_1(j), \pi_2(j)\}$ for some $i \neq j$. If $i \in \{\pi_1(i), \pi_2(i)\}$, saying $\pi_1(i) = i$, then $\pi_2(i) = \pi_1(j) \neq j$ as π_1 and π_2 have no proper common invariant subset $\{i, j\}$. This implies that $j \in \{\pi_1(h), \pi_2(h)\}$ for each $h \in \{1,2,3,4\} \setminus \{i, j\} = \{h_1, h_2\}$ and $\{\pi_1(h_1), \pi_2(h_2)\} = \{\pi_1(h_2), \pi_2(h_2)\}$. Now it is clear that there exists no choice of $\pi'(t) \in \{\pi_1(t), \pi_2(t)\}$ so that $\{\pi'(t) : t \neq j\} = \{1,2,3,4\} \setminus \{j\}$. If $t \notin \{\pi_1(t), \pi_2(t)\}$ for each $t \in \{i, j\}$, then for any choice of $\pi'(t) \in \{\pi_1(t), \pi_2(t)\}$, at least one of $\{\pi_1(i), \pi_2(i)\}$ does not belong to $\{\pi'(t) : t \neq j\}$. Hence $\{\pi_1, \pi_2\}$ has no the property (C) if (2) is broken. $\square$

Proof of Proposition 3.1. Obvious by Proposition 3.3, Proposition 3.4 and the discussion before it. $\square$

Before ending the section we list the following lemma which comes from [9] and will be used frequently in Section 4.

LEMMA 3.5. *Suppose $\Phi_D : M_n \to M_n$ is a D-type linear map of the form*

$$(a_{ij}) \longmapsto \mathrm{diag}(f_1, f_2, ..., f_n) - (a_{ij}) \tag{3.2}$$

with $(f_1, f_2, ..., f_n) = (a_{11}, a_{22}, ..., a_{nn})D$ for an $n \times n$ nonnegative matrix $D = (d_{ij})$. Then, Φ_D is positive if and only if, for any unit vector $u = (u_1, u_2, ..., u_n)^t \in \mathbb{C}^n$, we have $f_j(u) = \sum_{i=1}^n d_{ij}|u_i|^2 \neq 0$ whenever $u_j \neq 0$, and $\sum_{u_j \neq 0} \frac{|u_j|^2}{f_j(u)} \leqslant 1$.

4. Positivity of Φ_{π_1,π_2} on M_4 with arbitrary $\{\pi_1, \pi_2\}$

By Proposition 3.3, a D-type map Φ_{π_1,π_2} on 4×4 matrices constructed from a pair of permutations $\{\pi_1, \pi_2\}$ is positive if $\{\pi_1, \pi_2\}$ has the property (C). However, the property (C) is not a necessary condition. There are many examples that Φ_{π_1,π_2} is positive but $\{\pi_1, \pi_2\}$ doesn't have the property (C).

EXAMPLE 4.1. Let π_1, π_2 be permutations defined by $\pi_1(1) = 2$, $\pi_1(2) = 1$, $\pi_1(3) = 3$, $\pi_1(4) = 4$; and $\pi_2(1) = 2$, $\pi_2(2) = 1$, $\pi_2(3) = 4$, $\pi_2(4) = 3$. Clearly, $\{\pi_1, \pi_2\}$ does not have the property (C), but the D-type map $\Phi_{\pi_1,\pi_2} : M_4 \to M_4$ defined by Eq. (3.2) is positive (See Proposition 4.2).

The purpose of this section is to discuss the positivity of Φ_{π_1,π_2} for pair of arbitrary permutations, which are basic to our proof of the main result Theorem 1.2.

Let $\{F_s\}_{s=1}^r$ be the set of all minimal common invariant subsets of $\{\pi_1, \pi_2\}$. As $\pi_1 \neq \pi_2$, we have $r \leqslant 3$; also, if $r = 3$, by the discussion before Proposition 3.5, $\{\pi_1, \pi_2\}$ must have property (C).

If $r \leqslant 2$, then we have two cases: $\#F_1 = \#F_2 = 2$ and $\#F_1 = 1$, $\#F_2 = 3$. We deal with these two cases in Proposition 4.2 and Proposition 4.3 respectively.

PROPOSITION 4.2. *Let π_1, π_2 be two permutations of $(1,2,3,4)$ with $\{F_1, F_2\}$ the set of minimal common invariant subsets. If $\#F_1 = \#F_2 = 2$, then Φ_{π_1,π_2} is positive.*

Proof. Let $F_1 = \{i_1, i_2\}$ and $F_2 = \{i_3, i_4\}$. By Proposition 3.3, we may assume that $\{\pi_1, \pi_2\}$ has no property (C). Thus, by Proposition 3.1 and the discussion before Proposition 3.4, with no loss of generality, we may assume that

$$\begin{aligned} &\pi_1(i_1) = i_2,\ \pi_1(i_2) = i_1,\ \pi_1(i_3) = i_3,\ \pi_1(i_4) = i_4; \\ &\pi_2(i_1) = i_2,\ \pi_2(i_2) = i_1,\ \pi_1(i_3) = i_4,\ \pi_1(i_4) = i_3, \end{aligned}$$

where $\{i_1, i_2, i_3, i_4\} = \{1,2,3,4\}$. By Lemma 3.5, Φ_{π_1,π_2} is positive if

$$\begin{aligned} f(x_1,x_2,x_3,x_4) &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i + x_{\pi_1(i)} + x_{\pi_2(i)}} \\ &= \textstyle\sum_{i_h \in F_1} \frac{x_{i_h}}{2x_{i_h} + x_{\pi_1(i_h)} + x_{\pi_2(i_h)}} + \sum_{i_h \in F_2} \frac{x_{i_h}}{2x_{i_h} + x_{\pi_1(i_h)} + x_{\pi_2(i_h)}} \\ &= \frac{x_{i_1}}{2x_{i_1} + x_{\pi_1(i_1)} + x_{\pi_2(i_1)}} + \frac{x_{i_2}}{2x_{i_2} + x_{\pi_1(i_2)} + x_{\pi_2(i_2)}} \\ &\quad + \frac{x_{i_3}}{2x_{i_3} + x_{\pi_1(i_3)} + x_{\pi_2(i_3)}} + \frac{x_{i_4}}{2x_{i_4} + x_{\pi_1(i_4)} + x_{\pi_2(i_4)}} \\ &= \frac{x_{i_1}}{2x_{i_1} + 2x_{i_2}} + \frac{x_{i_2}}{2x_{i_2} + 2x_{i_1}} + \frac{x_{i_3}}{2x_{i_3} + x_{i_3} + x_{i_4}} + \frac{x_{i_4}}{2x_{i_4} + x_{i_4} + x_{i_3}} \leqslant 1 \end{aligned} \tag{4.1}$$

holds for any point (x_1,x_2,x_3,x_4) with $x_i \geqslant 0$ and $x_1+x_2+x_3+x_4=1$.

By Corollary 2.4, all possible extremum values of f are bounded above by 1. So, the inequality (4.1) holds if f is also bounded above by 1 at the points that some x_is are zero. Clearly, if there are at least two of x_is are 0, then, $f(x_1,\dots,x_4)<1$. So, we need check the case that only one of x_is is 0.

If $x_{i_1}=0$, or, if $x_{i_2}=0$, we get

$$\begin{aligned} f(x_1,x_2,x_3,x_4) &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \tfrac{1}{2}+\frac{1}{3+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{3+\frac{x_{i_3}}{x_{i_4}}} \leqslant 1 \end{aligned}$$

by Lemma 2.1.

If $x_{i_3}=0$, or, if $x_{i_4}=0$, we have

$$\begin{aligned} f(x_1,x_2,x_3,x_4) &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+2\frac{x_{i_2}}{x_{i_1}}}+\frac{1}{2+2\frac{x_{i_1}}{x_{i_2}}}+\tfrac{1}{3}=\tfrac{5}{6}<1. \end{aligned}$$

Therefore $f(x_1,x_2,x_3,x_4)\leqslant 1$ holds for all non-negative $x_1,\dots,x_4\in\mathbb{R}$ with $x_1+\cdots+x_4=1$, and consequently, Φ_{π_1,π_2} is positive. $\square$

PROPOSITION 4.3. *Let π_1,π_2 be two permutations of $(1,2,3,4)$ with $\{F_1,F_2\}$ the set of minimal common invariant subsets. If $\#F_1=1$ and $\#F_2=3$, then Φ_{π_1,π_2} is positive if and only if $\{\pi_1,\pi_2\}$ has the property (C), that is, for any $i\in F_2$, $\pi_1(i)\neq\pi_2(i)$.*

Proof. Let $F_1=\{i_1\}$ and $F_2=\{i_2,i_3,i_4\}$. By Proposition 3.3, we may assume that $\{\pi_1,\pi_2\}$ does not have the property (C) and show that Φ_{π_1,π_2} is not positive. Thus, by Proposition 3.1 or the discussion before Proposition 3.4, there is at least one $i\in F_2$ so that $\pi_1(i)=\pi_2(i)\neq i$. As $\pi_1\neq\pi_2$, we may assume further that $\pi_1(i_2)=\pi_2(i_2)=i_3$. So we have

$$\pi_1(i_1)=i_1,\ \pi_1(i_2)=i_3,\ \pi_1(i_3)=i_4,\ \pi_1(i_4)=i_2$$

and

$$\pi_2(i_1)=i_1,\ \pi_2(i_2)=i_3,\ \pi_2(i_3)=i_2,\ \pi_2(i_4)=i_4;$$

Now it is clear by Lemma 3.5 that Φ_{π_1,π_2} is not positive whenever

$$\begin{aligned} f(x_1,x_2,x_3,x_4) &= \textstyle\sum_{k=1}^4 \frac{x_{i_k}}{2x_{i_k}+x_{\pi_1(i_k)}+x_{\pi_2(i_k)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+x_{i_1}+x_{i_1}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_3}+x_{i_3}} \\ &\quad+\frac{x_{i_3}}{2x_{i_3}+x_{i_4}+x_{i_2}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_2}+x_{i_4}} \\ &= \tfrac{1}{4}+\frac{x_{i_2}}{2x_{i_2}+2x_{i_3}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_4}+x_{i_2}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_2}+x_{i_4}}>1 \end{aligned}$$

for some points (x_1,x_2,x_3,x_4) with $x_i\geqslant 0$ and $x_1+x_2+x_3+x_4=1$. This is true because, if $x_{i_2}=0$, then we have

$$f=\frac{1}{4}+\frac{1}{2+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{3},$$

which is not bounded above by 1. For instance, taking $x_{i_1} = \frac{89999}{10000}$, $x_{i_2} = 0$, $x_{i_3} = \frac{1}{10}$ and $x_{i_4} = \frac{1}{10000}$, then $f = \frac{1}{4} + \frac{1}{2+\frac{1}{1000}} + \frac{1}{3} > 1.083 > 1$. $\square$

For the case $r = 1$, that is, $l(\pi_1, \pi_2) = 4$, we have

PROPOSITION 4.4. *Assume the permutation pair* $\{\pi_1, \pi_2\}$ *of* $(1,2,3,4)$ *has no proper common invariant subsets. Then* $\Phi_{\pi_1,\pi_2} : M_4 \to M_4$ *is positive if and only if the following conditions are satisfied.*

(1) $\pi_1(i) \neq \pi_2(i)$ *for any* i*;*

(2) *if there are distinct* i, j*, such that* $\{\pi_1(i), \pi_2(i)\} = \{\pi_1(j), \pi_2(j)\}$*, then neither* π_1 *nor* π_2 *has fixed point.*

Proof. Note that, if $\{\pi_1, \pi_2\}$ has the property (C), then (1) is satisfied.

Firstly, let us prove that if π_1, π_2 satisfy the conditions (1) and (2), then Φ_{π_1,π_2} is positive.

Assume (1) and (2); then, for any i, we have $\pi_1(i) \neq \pi_2(i)$ and $i \notin \{\pi_1(i), \pi_2(i)\}$. By Proposition 3.3 we may assume that $\{\pi_1, \pi_2\}$ does not possesses the property (C). Thus it follows that, there are $i_1, i_2, i_3, i_4 \in \{1,2,3,4\}$ with $\{i_1, i_2, i_3, i_4\} = \{1,2,3,4\}$ such that

$$\begin{array}{l} \{\pi_1(i_1), \pi_2(i_1)\} = \{\pi_1(i_2), \pi_2(i_2)\} = \{i_3, i_4\} \\ \{\pi_1(i_3), \pi_2(i_3)\} = \{\pi_1(i_4), \pi_2(i_4)\} = \{i_1, i_2\}. \end{array} \tag{4.2}$$

By Lemma 3.5, the D-type map Φ_{π_1,π_2} is positive if and only if

$$f(x_1,x_2,x_3,x_4) = \sum_{i=1}^{4} \frac{x_i}{2x_i + x_{\pi_1(i)} + x_{\pi_2(i)}} \leqslant 1$$

holds for all non-negative $x_1, x_2, x_3, x_4 \in \mathbb{R}$ with $x_1 + \cdots + x_4 = 1$. By Eq. (4.2), we have

$$\begin{aligned} & f(x_1,x_2,x_3,x_4) = \textstyle\sum_{i=1}^{4} \frac{x_i}{2x_i + x_{\pi_1(i)} + x_{\pi_2(i)}} \\ = & \frac{x_{i_1}}{2x_{i_1} + x_{\pi_1(i_1)} + x_{\pi_2(i_1)}} + \frac{x_{i_2}}{2x_{i_2} + x_{\pi_1(i_2)} + x_{\pi_2(i_2)}} + \frac{x_{i_3}}{2x_{i_3} + x_{\pi_1(i_3)} + x_{\pi_2(i_3)}} + \frac{x_{i_4}}{2x_{i_4} + x_{\pi_1(i_4)} + x_{\pi_2(i_4)}} \\ = & \frac{x_{i_1}}{2x_{i_1} + x_{i_3} + x_{i_4}} + \frac{x_{i_2}}{2x_{i_2} + x_{i_3} + x_{i_4}} + \frac{x_{i_3}}{2x_{i_3} + x_{i_1} + x_{i_2}} + \frac{x_{i_4}}{2x_{i_4} + x_{i_1} + x_{i_2}}. \end{aligned}$$

By Corollary 2.4, it is easily seen that all extremum values of f are bounded above by 1. For the values of f at points on the boundary of the region $\{(x_1, x_2, x_3, x_4) : x_i \geqslant 0, x_1 + x_2 + x_3 + x_4 = 1\}$, if at least two of x_i s are 0, then obviously $f(x_1, x_2, x_3, x_4) < 1$. Assume that only one of x_i s is 0.

Consider the function

$$g(s,t) = \frac{1}{2+s+t} + \frac{1}{2+\frac{1}{s}} + \frac{1}{2+\frac{1}{t}} = \frac{1}{2+s+t} + \frac{s}{2s+1} + \frac{t}{2t+1},$$

where $s > 0$ and $t > 0$. As

$$\begin{aligned} & (2s+1)(2t+1) + s(s+t+2)(2t+1) + t(s+t+2)(2s+1) \\ = & 4s^2t + 4st^2 + 14st + s^2 + t^2 + 4s + 4t + 1, \end{aligned}$$

$$(2s+1)(2t+1)(s+t+2) = 4s^2t+4st^2+12st+2s^2+2t^2+5s+5t+2$$

and $2st < s^2+t^2+s+t+1$, it is easily checked that

$$g(s,t) = 1 - \frac{(s-t)^2+s+t+1}{(2s+1)(2t+1)(s+t+2)} < 1$$

holds for any $t > 0$ and $s > 0$. Applying the above inequality, we see that if $x_{i_1} = 0$, then

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+\frac{x_{i_3}}{x_{i_2}}+\frac{x_{i_4}}{x_{i_2}}} + \frac{1}{2+\frac{x_{i_2}}{x_{i_3}}} + \frac{1}{2+\frac{x_{i_2}}{x_{i_4}}} < 1; \end{aligned}$$

if $x_{i_2} = 0$, we get

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+\frac{x_{i_3}}{x_{i_1}}+\frac{x_{i_4}}{x_{i_1}}} + \frac{1}{2+\frac{x_{i_1}}{x_{i_3}}} + \frac{1}{2+\frac{x_{i_1}}{x_{i_4}}} < 1; \end{aligned}$$

if $x_{i_3} = 0$, we get

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+\frac{x_{i_4}}{x_{i_1}}} + \frac{1}{2+\frac{x_{i_4}}{x_{i_2}}} + \frac{1}{2+\frac{x_{i_1}}{x_{i_4}}+\frac{x_{i_2}}{x_{i_4}}} < 1; \end{aligned}$$

if $x_{i_4} = 0$, we get

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+\frac{x_{i_3}}{x_{i_1}}} + \frac{1}{2+\frac{x_{i_3}}{x_{i_2}}} + \frac{1}{2+\frac{x_{i_1}}{x_{i_3}}+\frac{x_{i_2}}{x_{i_3}}} < 1. \end{aligned}$$

So we have shown that $f(x_1,x_2,x_3,x_4) \leqslant 1$ holds for any $x_i \geqslant 0$, $i = 1,2,3,4$, with $x_1+x_2+x_3+x_4 = 1$. Therefore, Φ_{π_1,π_2} is positive.

Conversely, we show that $\Phi_{\pi_1,\pi_2} \geqslant 0$ implies both (1) and (2) hold. To do this, it suffices to show that any one of the following conditions (a) and (b) will imply that Φ_{π_1,π_2} is not positive:

(a) there is i such that $\pi_1(i) = \pi_2(i)$;

(b) if there are distinct i,j such that $\{\pi_1(i),\pi_2(i)\} = \{\pi_1(j),\pi_2(j)\}$, then π_1 or π_2 has fixed point.

Since the proof of "(a)$\Rightarrow$ Φ_{π_1,π_2} is not positive" is a little more complex, we first treat the case (b).

CLAIM 1. (b)$\Rightarrow$ Φ_{π_1,π_2} is not positive.

Suppose that (b) holds. Because of (a), we may assume that $\pi_1(k) \neq \pi_2(k)$ for any $k = 1,2,3,4$. With no loss of generality, say π_1 has fixed points.

Case (i). π_1 has two fixed points. In this case π_1 and π_2 have the forms

$$\begin{aligned} &\pi_1(i_1) = i_1, \quad \pi_1(i_2) = i_2, \quad \pi_1(i_3) = i_4, \quad \pi_1(i_4) = i_3; \\ &\pi_2(i_1) = i_3, \quad \pi_2(i_2) = i_4, \quad \pi_1(i_3) = i_1, \quad \pi_1(i_4) = i_2. \end{aligned}$$

Then we have

$$\{\pi_1(i_1),\pi_2(i_1)\}=\{\pi_1(i_4),\pi_2(i_4)\}$$

and thus, by Lemma 3.5, Φ_{π_1,π_2} is not positive if

$$\begin{aligned} &f(x_1,x_2,x_3,x_4)=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ &=\frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}+\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\ &=\frac{x_{i_1}}{2x_{i_1}+x_{i_1}+x_{i_3}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_2}+x_{i_4}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_4}+x_{i_2}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_3}+x_{i_1}} \end{aligned} \tag{4.3}$$

>1 for some point (x_1,x_2,x_3,x_4) with $x_i\geqslant 0$ and $x_1+x_2+x_3+x_4=1$.

Let $x_{i_3}=0$; then $x_{i_1}+x_{i_2}+x_{i_4}=1$ and, by Eq. (4.3),

$$\begin{aligned} f&=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ &=\frac{1}{3}+\frac{1}{3+\frac{x_{i_4}}{x_{i_2}}}+\frac{1}{2+\frac{x_{i_1}}{x_{i_4}}}. \end{aligned}$$

If we take $x_{i_1}=\frac{1}{10000}$, $x_{i_4}=\frac{1}{100}$ and $x_{i_2}=1-\frac{1}{10000}-\frac{1}{100}=\frac{9899}{10000}$, then

$$\begin{aligned} f&=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ &=\frac{1}{3}+\frac{1}{3+\frac{100}{9899}}+\frac{1}{2+\frac{1}{100}}\approx 1.1631>1. \end{aligned}$$

So, Φ_{π_1,π_2} is not positive.

Case (ii). π_1 has only one fixed point.

We check this case by considering six subcases.

Subcase (1). π_1,π_2 have respectively the forms

$$\begin{aligned} &\pi_1(i_1)=i_1,\quad \pi_1(i_2)=i_3,\quad \pi_1(i_3)=i_4,\quad \pi_1(i_4)=i_2;\\ &\pi_2(i_1)=i_3,\quad \pi_2(i_2)=i_1,\quad \pi_1(i_3)=i_2,\quad \pi_1(i_4)=i_4. \end{aligned}$$

Then

$$\begin{aligned} \{\pi_1(i_1),\pi_2(i_1)\}=\{\pi_1(i_2),\pi_2(i_2)\}=\{i_1,i_3\}\\ \{\pi_1(i_3),\pi_2(i_3)\}=\{\pi_1(i_4),\pi_2(i_4)\}=\{i_2,i_4\}. \end{aligned}$$

Thus Φ_{π_1,π_2} is not positive if

$$\begin{aligned} &f(x_1,x_2,x_3,x_4)=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ &=\frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}+\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\ &=\frac{x_{i_1}}{2x_{i_1}+x_{i_1}+x_{i_3}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_3}+x_{i_1}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_4}+x_{i_2}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_2}+x_{i_4}} \end{aligned} \tag{4.4}$$

greater than 1 at some point.

Let $x_{i_2}=0$, we get

$$\begin{aligned} f&=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ &=\frac{1}{3+\frac{x_{i_3}}{x_{i_1}}}+\frac{1}{2+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{3}. \end{aligned}$$

Now take $x_{i_4}=\frac{1}{10000}$, $x_{i_3}=\frac{1}{100}$, and $x_{i_1}=\frac{9899}{10000}$, we get $f\approx 1.1631>1$, as desired.

The following subcases (2)-(6) are dealt with similarly.

Subcase (2). π_1,π_2 have respectively the forms

$$\begin{array}{llll}\pi_1(i_1)=i_1, & \pi_1(i_2)=i_3, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_2;\\ \pi_2(i_1)=i_4, & \pi_2(i_2)=i_2, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_3.\end{array}$$

Subcase (3).

$$\begin{array}{llll}\pi_1(i_1)=i_1, & \pi_1(i_2)=i_3, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_2;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_4, & \pi_1(i_3)=i_3, & \pi_1(i_4)=i_1.\end{array}$$

Subcase (4).

$$\begin{array}{llll}\pi_1(i_1)=i_1, & \pi_1(i_2)=i_4, & \pi_1(i_3)=i_2, & \pi_1(i_4)=i_3;\\ \pi_2(i_1)=i_4, & \pi_2(i_2)=i_1, & \pi_1(i_3)=i_3, & \pi_1(i_4)=i_2.\end{array}$$

Subcase (5).

$$\begin{array}{llll}\pi_1(i_1)=i_1, & \pi_1(i_2)=i_4, & \pi_1(i_3)=i_2, & \pi_1(i_4)=i_3;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_3, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_4.\end{array}$$

Subcase (6).

$$\begin{array}{llll}\pi_1(i_1)=i_1, & \pi_1(i_2)=i_4, & \pi_1(i_3)=i_2, & \pi_1(i_4)=i_3;\\ \pi_2(i_1)=i_3, & \pi_2(i_2)=i_2, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_1.\end{array}$$

Therefore Claim 1 is true.

CLAIM 2. (a)$\Rightarrow \Phi_{\pi_1,\pi_2}$ is not positive.

As π_1,π_2 have no proper common invariant subsets, if there exists i such that $\pi_1(i)=\pi_2(i)$, then $\pi_h(i)\neq i$, $h=1,2$.

Case (i). There are i_1,i_2 such that $\pi_1(i_1)=\pi_2(i_1)$ and $\pi_1(i_2)=\pi_2(i_2)$. We have six different situations.

Subcase (1).

$$\begin{array}{llll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_3, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_4;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_3, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_1.\end{array}$$

In this situation,

$$\begin{aligned}&f(x_1,x_2,x_3,x_4)=\textstyle\sum_{i=1}^4\frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ =&\frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}+\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\ =&\frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+2x_{i_3}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_1}+x_{i_4}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_4}+x_{i_1}}.\end{aligned}\tag{4.5}$$

Let $x_{i_1}=0$, we have

$$\begin{aligned}f&=\textstyle\sum_{i=1}^4\frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ &=\frac{1}{2+2\frac{x_{i_3}}{x_{i_2}}}+\frac{1}{2+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{3}.\end{aligned}$$

Taking $x_{i_4}=\frac{1}{10000}$, $x_{i_3}=\frac{1}{100}$ and $x_{i_2}=\frac{9899}{10000}$ gives

$$\begin{aligned} f &= \textstyle\sum_{i=1}^{4}\frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+2\frac{100}{9899}}+\frac{1}{2+\frac{1}{100}}+\frac{1}{3}\approx 1.3258>1. \end{aligned}$$

Then, by the Lemma 3.5, Φ_{π_1,π_2} is not positive.

Subcase (2).

$$\begin{array}{llll} \pi_1(i_1)=i_2, & \pi_1(i_2)=i_4, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_3; \\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_4, & \pi_1(i_3)=i_3, & \pi_1(i_4)=i_1. \end{array}$$

By Lemma 3.5, Φ_D is not positive if

$$\begin{aligned} &f(x_1,x_2,x_3,x_4)=\textstyle\sum_{i=1}^{4}\frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}+\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+2x_{i_4}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_1}+x_{i_3}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_3}+x_{i_1}}. \end{aligned} \tag{4.6}$$

>1 at some points (x_1,x_2,x_3,x_4) with $x_i\geqslant 0$ and $x_1+x_2+x_3+x_4=1$. Let $x_{i_1}=0$, we get

$$\begin{aligned} f &= \textstyle\sum_{i=1}^{4}\frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+2\frac{x_{i_4}}{x_{i_2}}}+\frac{1}{3}+\frac{1}{2+\frac{x_{i_3}}{x_{i_4}}}. \end{aligned}$$

Taking $x_{i_3}=\frac{1}{10000}$, $x_{i_4}=\frac{1}{100}$ and $x_{i_2}=\frac{9899}{10000}$ gives

$$\begin{aligned} f &= \textstyle\sum_{i=1}^{4}\frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+2\frac{100}{9899}}+\frac{1}{3}+\frac{1}{2+\frac{1}{100}}\approx 1.3258>1. \end{aligned}$$

Subcase (3).

$$\begin{array}{llll} \pi_1(i_1)=i_3, & \pi_1(i_2)=i_4, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_2; \\ \pi_2(i_1)=i_3, & \pi_2(i_2)=i_4, & \pi_1(i_3)=i_2, & \pi_1(i_4)=i_1. \end{array}$$

In this subcase, Φ_{π_1,π_2} is not positive if

$$\begin{aligned} &f(x_1,x_2,x_3,x_4)=\textstyle\sum_{i=1}^{4}\frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}+\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+2x_{i_3}}+\frac{x_{i_2}}{2x_{i_2}+2x_{i_4}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_1}+x_{i_2}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_2}+x_{i_1}} \end{aligned} \tag{4.7}$$

>1 at some points (x_1,x_2,x_3,x_4) with $x_i\geqslant 0$ and $x_1+x_2+x_3+x_4=1$.

Let $x_{i_3}=0$, we get

$$\begin{aligned} f &= \textstyle\sum_{i=1}^{4}\frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2}+\frac{1}{2+2\frac{x_{i_4}}{x_{i_2}}}+\frac{1}{2+\frac{x_{i_1}}{x_{i_4}}+\frac{x_{i_2}}{x_{i_4}}}. \end{aligned}$$

Take $x_{i_4}=\frac{9}{10}$, $x_{i_1}=\frac{1}{100}$ and $x_{i_2}=\frac{9}{100}$. Then

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2}+\frac{1}{2+20}+\frac{9}{19}\approx 1.019>1. \end{aligned}$$

Subcase (4).

$$\begin{array}{llll} \pi_1(i_1)=i_3, & \pi_1(i_2)=i_1, & \pi_1(i_3)=i_2, & \pi_1(i_4)=i_4; \\ \pi_2(i_1)=i_3, & \pi_2(i_2)=i_1, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_2. \end{array}$$

In this subcase we have to check

$$\begin{aligned} & f(x_1,x_2,x_3,x_4)=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}+\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+2x_{i_3}}+\frac{x_{i_2}}{2x_{i_2}+2x_{i_1}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_2}+x_{i_4}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_4}+x_{i_2}} \end{aligned} \tag{4.8}$$

>1 at some points (x_1,x_2,x_3,x_4) with $x_i\geqslant 0$ and $x_1+x_2+x_3+x_4=1$.

Letting $x_{i_2}=0$, we get

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+2\frac{x_{i_3}}{x_{i_1}}}+\frac{1}{2+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{3}. \end{aligned}$$

If $x_{i_4}=\frac{1}{10000}$, $x_{i_3}=\frac{1}{100}$, and $x_{i_1}=\frac{9899}{10000}$, then

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+2\frac{100}{9899}}+\frac{1}{2+\frac{1}{100}}+\frac{1}{3}\approx 1.3258>1. \end{aligned}$$

Subcase (5).

$$\begin{array}{llll} \pi_1(i_1)=i_4, & \pi_1(i_2)=i_1, & \pi_1(i_3)=i_2, & \pi_1(i_4)=i_3; \\ \pi_2(i_1)=i_4, & \pi_2(i_2)=i_1, & \pi_1(i_3)=i_3, & \pi_1(i_4)=i_2. \end{array}$$

Φ_{π_1,π_2} is not positive because the function

$$\begin{aligned} & \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}+\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+2x_{i_4}}+\frac{x_{i_2}}{2x_{i_2}+2x_{i_1}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_2}+x_{i_3}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_3}+x_{i_2}} \end{aligned} \tag{4.9}$$

has value $\approx 1.3258>1$ at the point of $x_{i_1}=\frac{9899}{10000}$, $x_{i_2}=0$, $x_{i_3}=\frac{1}{10000}$ and $x_{i_4}=\frac{1}{100}$.

Subcase (6).

$$\begin{array}{llll} \pi_1(i_1)=i_4, & \pi_1(i_2)=i_3, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_2; \\ \pi_2(i_1)=i_4, & \pi_2(i_2)=i_3, & \pi_1(i_3)=i_2, & \pi_1(i_4)=i_1. \end{array}$$

Φ_{π_1,π_2} is not positive in this subcase because

$$\begin{aligned}&\sum_{i=1}^4 \tfrac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} = \tfrac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}\\&+\tfrac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\tfrac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\tfrac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\=&\tfrac{x_{i_1}}{2x_{i_1}+2x_{i_4}}+\tfrac{x_{i_2}}{2x_{i_2}+2x_{i_3}}+\tfrac{x_{i_3}}{2x_{i_3}+x_{i_1}+x_{i_2}}+\tfrac{x_{i_4}}{2x_{i_4}+x_{i_2}+x_{i_1}}\end{aligned} \tag{4.10}$$

achieves its value $\approx 1.019 > 1$ at the point of $x_{i_1}=\frac{9}{100}$, $x_{i_2}=\frac{1}{100}$, $x_{i_3}=0$ and $x_{i_4}=\frac{9}{10}$.

Case (ii). There is only one i such that $\pi_1(i)=\pi_2(i)$.

We have twelve subcases.

Subcase (1).

$$\begin{aligned}&\pi_1(i_1)=i_2,\quad \pi_1(i_2)=i_3,\quad \pi_1(i_3)=i_4,\quad \pi_1(i_4)=i_1;\\&\pi_2(i_1)=i_2,\quad \pi_2(i_2)=i_4,\quad \pi_1(i_3)=i_1,\quad \pi_1(i_4)=i_3.\end{aligned}$$

In this case

$$\begin{aligned}&f(x_1,x_2,x_3,x_4)=\sum_{i=1}^4 \tfrac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} = \tfrac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}\\&+\tfrac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\tfrac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\tfrac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\=&\tfrac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\tfrac{x_{i_2}}{2x_{i_2}+x_{i_3}+x_{i_4}}+\tfrac{x_{i_3}}{2x_{i_3}+x_{i_4}+x_{i_1}}+\tfrac{x_{i_4}}{2x_{i_4}+x_{i_1}+x_{i_3}}.\end{aligned} \tag{4.11}$$

If we let $x_{i_1}=0$, then

$$f=\sum_{i=1}^4 \tfrac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\tfrac{1}{2+\frac{x_{i_3}}{x_{i_2}}+\frac{x_{i_4}}{x_{i_2}}}+\tfrac{1}{2+\frac{x_{i_4}}{x_{i_3}}}+\tfrac{1}{2+\frac{x_{i_3}}{x_{i_4}}}.$$

Take $x_{i_3}=x_{i_4}=\frac{1}{100}$, $x_{i_2}=\frac{98}{100}$, we get

$$f=\sum_{i=1}^4 \tfrac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\tfrac{1}{1+\frac{100}{98}}+\tfrac{2}{3}\approx 1.16>1.$$

So Φ_{π_1,π_2} is not positive.

Subcase (2).

$$\begin{aligned}&\pi_1(i_1)=i_2,\quad \pi_1(i_2)=i_3,\quad \pi_1(i_3)=i_4,\quad \pi_1(i_4)=i_1;\\&\pi_2(i_1)=i_2,\quad \pi_2(i_2)=i_1,\quad \pi_1(i_3)=i_3,\quad \pi_1(i_4)=i_4.\end{aligned}$$

Then

$$\begin{aligned}&f(x_1,x_2,x_3,x_4)=\sum_{i=1}^4 \tfrac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} = \tfrac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}\\&+\tfrac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\tfrac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\tfrac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\=&\tfrac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\tfrac{x_{i_2}}{2x_{i_2}+x_{i_3}+x_{i_1}}+\tfrac{x_{i_3}}{2x_{i_3}+x_{i_4}+x_{i_3}}+\tfrac{x_{i_4}}{2x_{i_4}+x_{i_1}+x_{i_4}}.\end{aligned} \tag{4.12}$$

Let $x_{i_2}=0$, we get

$$f=\sum_{i=1}^4 \tfrac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\tfrac{1}{2}+\tfrac{1}{3+\frac{x_{i_4}}{x_{i_3}}}+\tfrac{1}{3+\frac{x_{i_1}}{x_{i_4}}}.$$

It is then clear that $x_{i_1}=\frac{1}{100}$, $x_{i_4}=\frac{1}{10}$ and $x_{i_3}=1-\frac{1}{100}-\frac{1}{10}=\frac{89}{100}$ gives

$$f=\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{1}{2}+\frac{1}{3+\frac{10}{89}}+\frac{1}{3+\frac{1}{10}}\approx 1.144>1.$$

Subcase (3).

$$\begin{array}{llll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_3, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_4;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_4, & \pi_1(i_3)=i_3, & \pi_1(i_4)=i_1.\end{array}$$

We have

$$\begin{aligned} f(x_1,x_2,x_3,x_4)&=\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}\\ &+\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\ &=\frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_3}+x_{i_4}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_1}+x_{i_3}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_4}+x_{i_1}}.\end{aligned}\tag{4.13}$$

Letting $x_{i_2}=0$ gives

$$f=\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{1}{2}+\frac{1}{3+\frac{x_{i_1}}{x_{i_3}}}+\frac{1}{3+\frac{x_{i_1}}{x_{i_4}}}.$$

So, taking $x_{i_1}=\frac{1}{100}$, $x_{i_4}=\frac{1}{10}$ and $x_{i_3}=1-\frac{1}{100}-\frac{1}{10}=\frac{89}{100}$, one gets

$$f=\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{1}{2}+\frac{1}{3+\frac{1}{89}}+\frac{1}{3+\frac{1}{10}}\approx 1.155>1.$$

Subcase (4).

$$\begin{array}{llll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_3, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_4;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_1, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_3.\end{array}$$

Then we have

$$\begin{aligned} f(x_1,x_2,x_3,x_4)&=\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}\\ &+\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\ &=\frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_3}+x_{i_1}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_1}+x_{i_4}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_4}+x_{i_3}}.\end{aligned}\tag{4.14}$$

Let $x_{i_2}=0$, we get

$$f=\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{1}{2}+\frac{1}{2+\frac{x_{i_1}}{x_{i_3}}+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{3+\frac{x_{i_3}}{x_{i_4}}}.$$

Then taking $x_{i_1}=\frac{1}{20}$, $x_{i_4}=\frac{1}{20}$ and $x_{i_3}=1-\frac{1}{20}-\frac{1}{20}=\frac{9}{10}$, we have

$$f=\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{1}{2}+\frac{9}{19}+\frac{1}{21}\approx 1.021>1.$$

Subcase (5).

$$\begin{array}{llll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_4, & \pi_1(i_3)=i_3, & \pi_1(i_4)=i_1;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_1, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_3.\end{array}$$

For this subcase we have

$$\begin{aligned} & f(x_1,x_2,x_3,x_4)=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}\\ & +\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\ =\ & \frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_4}+x_{i_1}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_3}+x_{i_4}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_1}+x_{i_3}}. \end{aligned} \tag{4.15}$$

It is clear that, if $x_{i_2}=0$, then

$$f=\sum_{i=1}^4\frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{1}{2}+\frac{1}{3+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{2+\frac{x_{i_1}}{x_{i_4}}+\frac{x_{i_3}}{x_{i_4}}}.$$

Thus if we take $x_{i_4}=\frac{9}{10}$, $x_{i_1}=\frac{1}{20}$ and $x_{i_3}=\frac{1}{20}$, we get

$$f=\frac{1}{2}+\frac{1}{21}+\frac{9}{19}\approx 1.021>1.$$

Subcase (6).

$$\begin{aligned} &\pi_1(i_1)=i_2, \quad \pi_1(i_2)=i_4, \quad \pi_1(i_3)=i_3, \quad \pi_1(i_4)=i_1;\\ &\pi_2(i_1)=i_2, \quad \pi_2(i_2)=i_3, \quad \pi_1(i_3)=i_1, \quad \pi_1(i_4)=i_4. \end{aligned}$$

In this subcase we have

$$\begin{aligned} & f(x_1,x_2,x_3,x_4)=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{x_{i_1}}{2x_{i_1}+x_{\pi_1(i_1)}+x_{\pi_2(i_1)}}\\ & +\frac{x_{i_2}}{2x_{i_2}+x_{\pi_1(i_2)}+x_{\pi_2(i_2)}}+\frac{x_{i_3}}{2x_{i_3}+x_{\pi_1(i_3)}+x_{\pi_2(i_3)}}+\frac{x_{i_4}}{2x_{i_4}+x_{\pi_1(i_4)}+x_{\pi_2(i_4)}}\\ =\ & \frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_4}+x_{i_3}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_3}+x_{i_1}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_4}+x_{i_1}}. \end{aligned} \tag{4.16}$$

Then letting $x_{i_1}=0$ and $x_{i_2}=\frac{3}{4}$ gives

$$f=\frac{1}{2+\frac{x_{i_3}}{x_{i_2}}+\frac{x_{i_4}}{x_{i_2}}}+\frac{1}{3}+\frac{1}{3}=\frac{2}{3}+\frac{1}{1+\frac{1}{x_{i_2}}}=\frac{2}{3}+\frac{3}{7}=\frac{23}{21}>1.$$

Subcase (7).

$$\begin{aligned} &\pi_1(i_1)=i_2, \quad \pi_1(i_2)=i_4, \quad \pi_1(i_3)=i_1, \quad \pi_1(i_4)=i_3;\\ &\pi_2(i_1)=i_2, \quad \pi_2(i_2)=i_1, \quad \pi_1(i_3)=i_3, \quad \pi_1(i_4)=i_4. \end{aligned}$$

In this subcase we have

$$\begin{aligned} & f(x_1,x_2,x_3,x_4)=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ =\ & \frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_4}+x_{i_1}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_3}+x_{i_1}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_4}+x_{i_1}}. \end{aligned} \tag{4.17}$$

Then, taking $x_{i_2}=0$, $x_{i_1}=\frac{1}{100}$, $x_{i_3}=\frac{1}{10}$ and $x_{i_4}=1-\frac{1}{100}-\frac{1}{10}=\frac{89}{100}$, we get

$$f=\frac{1}{2}+\frac{1}{3+\frac{x_{i_1}}{x_{i_3}}}+\frac{1}{3+\frac{x_{i_3}}{x_{i_4}}}=\frac{1}{2}+\frac{1}{3+\frac{1}{10}}+\frac{1}{3+\frac{10}{89}}\approx 1.144>1.$$

Subcase (8).

$$\begin{array}{llll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_4, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_3;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_3, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_1.\end{array}$$

Then we have

$$\begin{aligned} f(x_1,x_2,x_3,x_4) &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_4}+x_{i_3}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_1}+x_{i_4}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_1}+x_{i_3}}. \end{aligned} \tag{4.18}$$

If $x_{i_1}=0$, $x_{i_3}=x_{i_4}=\frac{1}{100}$ and $x_{i_2}=1-\frac{1}{100}-\frac{1}{100}=\frac{98}{100}$, we get

$$f=\frac{1}{2+\frac{x_{i_3}}{x_{i_2}}+\frac{x_{i_4}}{x_{i_2}}}+\frac{1}{2+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{2+\frac{x_{i_3}}{x_{i_4}}}=\frac{1}{1+\frac{100}{98}}+\frac{2}{3}\approx 1.16>1.$$

Subcase (9).

$$\begin{array}{llll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_1, & \pi_1(i_3)=i_3, & \pi_1(i_4)=i_4;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_3, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_1.\end{array}$$

Obviously, we have

$$\begin{aligned} f(x_1,x_2,x_3,x_4) &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_1}+x_{i_3}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_3}+x_{i_4}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_4}+x_{i_1}}. \end{aligned} \tag{4.19}$$

Now if we take $x_{i_1}=0$, $x_{i_3}=\frac{1}{100}$, $x_{i_4}=\frac{1}{10000}$, and $x_{i_2}=\frac{9899}{10000}$, then

$$f=\frac{1}{2+\frac{x_{i_3}}{x_{i_2}}}+\frac{1}{3+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{3}=\frac{1}{2+\frac{100}{9899}}+\frac{1}{3+\frac{1}{100}}+\frac{1}{3}\approx 1.163>1.$$

Subcase (10).

$$\begin{array}{llll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_1, & \pi_1(i_3)=i_3, & \pi_1(i_4)=i_4;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_4, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_3.\end{array}$$

Clearly,

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_1}+x_{i_4}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_3}+x_{i_1}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_4}+x_{i_3}}. \end{aligned} \tag{4.20}$$

If $x_{i_1}=0$, we get

$$f=\textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}=\frac{1}{2+\frac{x_{i_4}}{x_{i_2}}}+\frac{1}{3}+\frac{1}{3+\frac{x_{i_3}}{x_{i_4}}}.$$

Letting $x_{i_3}=\frac{1}{10000}$, $x_{i_4}=\frac{1}{100}$ and $x_{i_2}=\frac{9899}{10000}$, we get

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}} \\ &= \frac{1}{2+\frac{100}{9899}}+\frac{1}{3}+\frac{1}{3+\frac{1}{100}}\approx 1.163>1. \end{aligned}$$

Subcase (11).

$$\begin{array}{llll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_1, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_3;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_3, & \pi_1(i_3)=i_1, & \pi_1(i_4)=i_4.\end{array}$$

For this case we have

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ &= \frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_1}+x_{i_3}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_4}+x_{i_1}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_4}+x_{i_3}}.\end{aligned} \tag{4.21}$$

Thus, if $x_{i_2}=0$, $x_{i_1}=\frac{1}{20}$, $x_{i_4}=\frac{1}{20}$ and $x_{i_3}=\frac{9}{10}$, we get

$$f=\frac{1}{2}+\frac{1}{2+\frac{x_{i_1}}{x_{i_3}}+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{3+\frac{x_{i_3}}{x_{i_4}}}=\frac{1}{2}+\frac{9}{19}+\frac{1}{21}\approx 1.021>1.$$

Subcase (12).

$$\begin{array}{llll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_1, & \pi_1(i_3)=i_4, & \pi_1(i_4)=i_3;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_4, & \pi_1(i_3)=i_3, & \pi_1(i_4)=i_1.\end{array}$$

Then we have

$$\begin{aligned} f &= \textstyle\sum_{i=1}^4 \frac{x_i}{2x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ &= \frac{x_{i_1}}{2x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{2x_{i_2}+x_{i_1}+x_{i_4}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_3}+x_{i_4}}+\frac{x_{i_4}}{2x_{i_4}+x_{i_3}+x_{i_1}}.\end{aligned} \tag{4.22}$$

If $x_{i_2}=0$, $x_{i_1}=\frac{1}{20}$, $x_{i_3}=\frac{1}{20}$ and $x_{i_4}=\frac{9}{10}$, we get

$$f=\frac{1}{2}+\frac{1}{3+\frac{x_{i_4}}{x_{i_3}}}+\frac{1}{2+\frac{x_{i_3}}{x_{i_4}}+\frac{x_{i_1}}{x_{i_4}}}=\frac{1}{2}+\frac{1}{21}+\frac{9}{19}\approx 1.021>1.$$

By the Lemma 3.5, for subcases (1)–(12) of Case (ii), Φ_{π_1,π_2} is not positive, either.

Hence we have proved that if there exist i such that $\pi_1(i)=\pi_2(i)\neq i$, then Φ_{π_1,π_2} is not positive. So, Φ_{π_1,π_2} is positive implies that there is no i so that $\pi_1(i)=\pi_2(i)\neq i$, this finishes the proof of Proposition 4.4. □

5. Proofs of the main results

Now we are in a position to complete the proofs of Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. Note that, by the assumption, π_1 and π_2 are not the identity permutation and $\pi_1\neq\pi_2$. Still denote by $\{F_s\}_{s=1}^r$ the set of minimal common invariant subsets of π_1 and π_2 and denote by $l(\pi_1,\pi_2)$ the length of $\{\pi_1,\pi_2\}$, i.e., $l(\pi_1,\pi_2)=\max\{\#F_s\}_{s=1}^r$.

If $l(\pi_1,\pi_2)=2$, then Φ_{π_1,π_2} is always positive. In fact, $l(\pi)=2$ implies either $r=3$, in this situation one of π_1,π_2 is the identity; or $r=2$ with $\#F_1=\#F_2=2$, in this situation we apply Proposition 4.2.

If $l(\pi_1,\pi_2)=3$ with $\#F_1=1$, then by Proposition 4.3, Φ_{π_1,π_2} is positive if and only if for any $i\in F_2$ we have $\pi_1(i)\neq\pi_2(i)$, and in turn, if and only if the condition (1) in (ii) holds. Since π_1,π_2 has a common fixed point, the condition (2) in (ii) holds emptily. Hence the theorem is true for this case.

If $l(\pi_1,\pi_2)=4$, then π_1,π_2 have no common fixed point. By Proposition 4.4, it is obvious that Φ_{π_1,π_2} is positive if and only if (1) and (2) in (ii) hold. $\Box$

Proof of Theorem 1.3. As π_1,π_2 are not the identity, $\{\pi_1,\pi_2\}$ has the property (C) if and only if $\pi_1(i)\neq\pi_2(i)$ for any $i=1,2,3$. Thus by [16], $\pi_1(i)\neq\pi_2(i)$ for any $i=1,2,3$ implies that $\Phi_{\pi_1,\pi_2}:M_3\to M_3$ is positive. Conversely, if there is some $i_1\in\{1,2,3\}$ so that $\pi_1(i_1)=\pi_2(i_1)$, then $\pi_1(i_1)=\pi_2(i_1)=i_2\in\{i_2,i_3\}=\{1,2,3\}\setminus\{i_1\}$. Thus, with no loss of generality, we may assume that

$$\begin{array}{lll}\pi_1(i_1)=i_2, & \pi_1(i_2)=i_3, & \pi_1(i_3)=i_1;\\ \pi_2(i_1)=i_2, & \pi_2(i_2)=i_1, & \pi_2(i_3)=i_3.\end{array}$$

It follows from Lemma 3.5 that Φ_{π_1,π_2} is not positive if

$$\begin{aligned} f(x_1,x_2,x_3) &= \textstyle\sum_{i=1}^{3}\frac{x_i}{x_i+x_{\pi_1(i)}+x_{\pi_2(i)}}\\ &= \frac{x_{i_1}}{x_{i_1}+2x_{i_2}}+\frac{x_{i_2}}{x_{i_2}+x_{i_1}+x_{i_3}}+\frac{x_{i_3}}{2x_{i_3}+x_{i_1}}\end{aligned}$$

is greater than 1 at some point. This is the case because letting $x_{i_1}=0$ gives

$$f=\frac{x_{i_2}}{x_{i_2}+x_{i_3}}+\frac{1}{2}$$

which has supremum $\frac{3}{2}>1$. $\Box$

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