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CLOSEST SOUTHEAST SUBMATRIX THAT MAKES MULTIPLE A DEFECTIVE EIGENVALUE OF THE NORTHWEST ONE

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Abstract. Given three complex matrices $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{m\times n}$, and given a defective eigenvalue z_0 of A, we study when the set $\mathcal{S}$ of matrices $X \in \mathbb{C}^{m\times m}$ such that z_0 is a multiple eigenvalue of the matrix

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}.$$

is nonempty. Moreover, when $\mathcal{S} \neq \emptyset$, given a fourth matrix $D \in \mathbb{C}^{m\times m}$ we find a matrix $X_0 \in \mathcal{S}$ such that

$$\|X_0 - D\| = \min\{\|X - D\| : X \in \mathcal{S}\}.$$

1. Introduction

Let us denote by $\|\cdot\|$ the spectral matrix norm. We write $\Lambda(M)$ for the spectrum of a square complex matrix M. If $\lambda_0 \in \Lambda(M)$ we denote by $\mathrm{m}(\lambda_0, M)$ the algebraic multiplicity of λ_0. We say that λ_0 is a *defective* eigenvalue of M if its algebraic multiplicity is greater than its geometric multiplicity; or, equivalently, λ_0 is defective if there exists a Jordan block of order $\geqslant 2$ associated to λ_0 in the Jordan canonical form of M. An eigenvalue α_0 of M is said to be *semisimple* if all the Jordan blocks associated to α_0 are of order one. So, an eigenvalue is defective if and only if is nonsemisimple. Let L_{nm} denote the Cartesian product $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{m\times n}$. Let $\Lambda_2(M)$ denote the set of multiple eigenvalues of M. For any matrix $N \in \mathbb{C}^{p\times q}$ we denote by $\nu(N)$ the nullity of N. That is, $\nu(N) = \dim \operatorname{Ker} N$. We denote by $\sigma_1(N) \geqslant \sigma_2(N) \geqslant \cdots \geqslant \sigma_{\min(p,q)}(N)$ the singular values of N. Two unitary column vectors u, v are a pair of singular vectors (left and right) of the matrix N associated with the singular value σ if $Nv = \sigma u$ and $N^*u = \sigma v$, where N^* denotes the conjugate transpose matrix of N. Finally, $N^\dagger$ denotes the Moore–Penrose inverse of N.

In [5] and [6] the second and third authors solved the following problems:

PROBLEM 1. Let $\alpha := (A, B, C) \in L_{nm}$ be a triple of matrices, and let us suppose that z_0 is a complex number such that: (1) either $z_0 \notin \Lambda(A)$; (2) or z_0 is a semisimple

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eigenvalue of A. Characterize the cases where the set $\mathcal{M}_2(z_0,\alpha)$ of matrices $X \in \mathbb{C}^{m\times m}$ such that z_0 is a multiple eigenvalue of

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}$$

is nonempty. The second and third authors gave solutions to this problem: in [5] when $z_0 \notin \Lambda(A)$; and in [6] when z_0 is a semisimple eigenvalue of A.

PROBLEM 2. Let $\alpha := (A,B,C) \in L_{nm}$ be a triple of matrices, and let us suppose that z_0 is a complex number such that: (1) either $z_0 \notin \Lambda(A)$; (2) or z_0 is a semisimple eigenvalue of A. In case of $\mathcal{M}_2(z_0,\alpha) \neq \emptyset$, given a fourth matrix $D \in \mathbb{C}^{m\times m}$, find a matrix $X_0 \in \mathcal{M}_2(z_0,\alpha)$ such that

$$\|X_0 - D\| = \min_{X\in\mathcal{M}_2(z_0,\alpha)} \|X - D\|. \tag{1}$$

The second and third authors gave solutions to this problem: in [5] when $z_0 \notin \Lambda(A)$; and in [6] when z_0 a semisimple eigenvalue of A.

In this paper we address these two problems when z_0 is a nonsemisimple eigenvalue of A. One more detailed motivation for this class of structured matrix problems can be seen in the introduction of paper [6]. To shorten notation, for a triple of matrices $\alpha := (A,B,C) \in L_{nm}$ and a matrix $X \in \mathbb{C}^{m\times m}$ we write $M(\alpha,X)$ instead of

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}.$$

To simplify Problems 1 and 2 there is no loss of generality in assuming that $z_0 = 0$. In fact, let $\alpha' = (A - z_0 I_n, B, C)$; then for $X \in \mathbb{C}^{m\times m}$, $\mathrm{m}(z_0, M(\alpha,X)) \geqslant 2$ if and only if $\mathrm{m}(0, M(\alpha', X - z_0 I_m)) \geqslant 2$. So, the set $\mathcal{M}_2(z_0,\alpha)$ is nonempty if and only if $\mathcal{M}_2(0,\alpha')$ is nonempty. In that case, given a matrix $D \in \mathbb{C}^{m\times m}$,

$$\min_{X\in\mathcal{M}_2(z_0,\alpha)} \|X - D\| = \min_{Y\in\mathcal{M}_2(0,\alpha')} \|Y - (D - z_0 I_m)\|.$$

Thus, from here on we suppose that $z_0 = 0$. We will denote the zero matrices by O and the row and column vectors by 0, disregarding their sizes. Note that when $B = O$ or $C = O$, as 0 is supposed to be a nonsemisimple eigenvalue of A, then 0 is a multiple eigenvalue of

$$\begin{pmatrix} A & O \\ C & X \end{pmatrix} \text{ or } \begin{pmatrix} A & B \\ O & X \end{pmatrix}$$

for every $X \in \mathbb{C}^{m\times m}$; so $\mathcal{M}_2(0,\alpha) = \mathbb{C}^{m\times m}$ and

$$\min_{X\in\mathcal{M}_2(0,\alpha)} \|X - D\| = \|D - D\| = 0.$$

Therefore, in what follows we will assume that B and C are nonzero matrices.

The organization of this paper is the following one. We will try to solve simultaneously the problems of emptiness of $\mathcal{M}_2(0,\alpha)$ and the minimization of $\|X - D\|$

subject to $X \in \mathcal{M}_2(0,\alpha)$. In Section 2 we will recall results in the literature about the nearest X to D that lowers the rank of $\left(\begin{smallmatrix} A & B \\ C & X \end{smallmatrix}\right)$ to a preassigned value less than the rank of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. We will also reformulate the surjective mapping theorem about functions of several variables. In Section 3 we will reduce the matrices A, B and C by means of unitary matrices to a simplified form that makes less difficult the solution of the Problems. Thus, they are reduced to five cases, whose analyses are made in Sections 4 and 5.

2. Preliminary results

The following statement is a reformulation of results in [4, Theorem 1.1], [8, Theorem 19, (8.1), (8.2) and (8.6)], [3, Theorem 3], [11, Theorem 2.1] and Theorem 6.3.7 of the page 102 in the book [2].

THEOREM 1. *Let* $\alpha = (A,B,C) \in L_{nm}$ *be a triple of matrices and let* $D \in \mathbb{C}^{m\times m}$. *Let*

$$\rho := \operatorname{rank}(A,B) + \operatorname{rank}\begin{pmatrix} A \\ C \end{pmatrix} - \operatorname{rank} A,$$

and

$$M := (I - AA^\dagger)B, \quad N := C(I - A^\dagger A).$$

Then for $X \in \mathbb{C}^{m\times m}$,

$$\operatorname{rank} M(\alpha, X) = \rho + \operatorname{rank} S(X),$$

where

$$S(X) := (I - NN^\dagger)(X - CA^\dagger B)(I - M^\dagger M).$$

Furthermore, for each integer r *such that* $\rho \leqslant r < \operatorname{rank} M(\alpha, D)$, *there exits a matrix* X_0 *such that* $\operatorname{rank} M(\alpha, X_0) \leqslant r$ *and*

$$\|X_0 - D\| = \min_{\substack{X\in\mathbb{C}^{m\times m} \\ \operatorname{rank} M(\alpha,X)\leqslant r}} \|X - D\| = \sigma_{p+1}(S(D)),$$

where $p = r - \rho$. *In addition, if* $U, V \in \mathbb{C}^{m\times m}$ *are the unitary matrices which appear in the singular value decomposition of the matrix* $S(D)$, *i.e.*

$$U^* S(D) V = \operatorname{diag}(\sigma_1(S(D)), \ldots, \sigma_m(S(D))),$$

we can choose

$$X_0 = D - U \operatorname{diag}(0, \ldots, 0, \sigma_{p+1}(S(D)), \ldots, \sigma_m(S(D)))\, V^*.$$

Let $f : \Omega \to \mathbb{C}^m$ be a differentiable map defined on an open subset Ω of $\mathbb{C}^n$. For $z = (z_1, \ldots, z_n) \in \Omega$ write $f(z) = (f_1(z_1, \ldots, z_n), \ldots, f_m(z_1, \ldots, z_n))$. We will denote by

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(z_1, \ldots, z_n)}(z)$$

the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1}(z) & \cdots & \frac{\partial f_1}{\partial z_n}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1}(z) & \cdots & \frac{\partial f_m}{\partial z_n}(z) \end{pmatrix}.$$

We say that f belongs to class C^1 on Ω if it has continuous partial derivatives $\partial f_i/\partial z_j$, for $i=1,\ldots,m,\ j=1,\ldots,n$.

Let us suppose that $f:\Omega\to\mathbb{C}^{m\times p}$ is a map from Ω into $\mathbb{C}^{m\times p}$ with Ω an open subset of $\mathbb{C}^{n\times q}$. For each matrix $X=(x_{ij})\in\Omega,\ f(X)=(f_{ij}(X))$ is a $m\times p$ matrix. If f is differentiable on Ω, we define its Jacobian matrix at X in the following manner

$$\frac{\partial f}{\partial X}(X) := \frac{\partial(f_{11},\ldots,f_{1p},\ldots,f_{m1},\ldots,f_{mp})}{\partial(x_{11},\ldots,x_{1q},\ldots,x_{n1},\ldots,x_{nq})}(X).$$

This matrix has size $mp\times nq$. The symbol $\otimes$ denotes the Kronecker product of matrices and $^{\mathrm{T}}$ stands for the transpose matrix. With these notations, one has the following result ([9], Examples 3(b), p. 71; [7], p. 175).

LEMMA 2. *Let* $A\in\mathbb{C}^{m\times n},\ X\in\mathbb{C}^{n\times p},\ Z\in\mathbb{C}^{q\times m}$. *Then,*

(a) $\dfrac{\partial(AX)}{\partial X}=A\otimes I_p,$

(b) $\dfrac{\partial(ZA)}{\partial Z}=I_q\otimes A^{\mathrm{T}}.$

For a family of sets $S_1,\ldots,S_r$ we will denote the Cartesian product $S_1\times\cdots\times S_r$ by $\prod_{i=1}^r S_i$. Let us suppose that $g:\Omega\to\mathbb{C}^{m\times p}$ is a map from Ω into $\mathbb{C}^{m\times p}$ with Ω an open subset of $\prod_{i=1}^r\mathbb{C}^{n_i\times q_i}$. For each r-tuple of matrices $(X_1,\ldots,X_r)\in\Omega, X_k=\left(x_{ij}^{(k)}\right)$, $k=1,\ldots,r,\ g(X_1,\ldots,X_r)=(g_{ij}(X_1,\ldots,X_r))$ is a $m\times p$ matrix. If g is differentiable on Ω, we define its partial Jacobian matrix with respect to X_k at $(X_1,\ldots,X_r)$ in the following manner

$$\frac{\partial g}{\partial X_k}(X_1,\ldots,X_r) := \frac{\partial(g_{11},\ldots,g_{1p},\ldots,g_{m1},\ldots,g_{mp})}{\partial\left(x_{11}^{(k)},\ldots,x_{1q_k}^{(k)},\ldots,x_{n_k1}^{(k)},\ldots,x_{n_kq_k}^{(k)}\right)}(X_1,\ldots,X_r).$$

This matrix has size $mp\times n_kq_k$. A consequence of the Surjective Mapping Theorem ([1], Theorem 41.6, p. 378; [10], Lemma 12.4–1, p. 230) is the following lemma. Before its statement, we need some notations. For $1\leqslant i\leqslant n, 1\leqslant j\leqslant p$ and $1\leqslant k\leqslant s$, we are going to consider the vector spaces of matrices $\mathbb{C}^{n_i\times n_i'}$, $\mathbb{C}^{p_j\times p_j'}$ and $\mathbb{C}^{m_k\times m_k'}$. Let us denote

$$P:=\sum_{j=1}^{p}p_jp_j',\quad M:=\sum_{k=1}^{m}m_km_k'.$$

LEMMA 3. *Let* Ω *be an open subset of*

$$\left(\prod_{i=1}^{n}\mathbb{C}^{n_i\times n_i'}\right)\times\left(\prod_{j=1}^{p}\mathbb{C}^{p_j\times p_j'}\right).$$

For $1\leqslant k\leqslant s$ *consider the matrix functions*

$$f_k:\Omega\to\mathbb{C}^{m_k\times m_k'}$$

of class C^1 *on* Ω. *Let*

$$Z_0:=(X_1^0,X_2^0,\ldots,X_n^0,Y_1^0,Y_2^0,\ldots,Y_p^0)=(X^0,Y^0)\in\Omega,$$

with

$$X_i^0\in\mathbb{C}^{n_i\times n_i'}\quad 1\leqslant i\leqslant n,$$
$$Y_j^0\in\mathbb{C}^{p_j\times p_j'}\quad 1\leqslant j\leqslant p,$$

be a point that satisfies

$$\begin{cases} f_1(X^0,Y^0)=O,\\ f_2(X^0,Y^0)=O,\\ \qquad\vdots\\ f_s(X^0,Y^0)=O.\end{cases}$$

Assume $M\leqslant P$ *and that the rank of the partial Jacobian matrix*

$$\frac{\partial(f_1,f_2,\ldots,f_s)}{\partial(Y_1,Y_2,\ldots,Y_p)}(Z_0):=\begin{pmatrix}\frac{\partial f_1}{\partial Y_1}(Z_0) & \frac{\partial f_1}{\partial Y_2}(Z_0) & \cdots & \frac{\partial f_1}{\partial Y_p}(Z_0)\\ \frac{\partial f_2}{\partial Y_1}(Z_0) & \frac{\partial f_2}{\partial Y_2}(Z_0) & \cdots & \frac{\partial f_2}{\partial Y_p}(Z_0)\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_s}{\partial Y_1}(Z_0) & \frac{\partial f_s}{\partial Y_2}(Z_0) & \cdots & \frac{\partial f_s}{\partial Y_p}(Z_0)\end{pmatrix}$$

is equal to M. *Then, for every sequence*

$$\{X^q\}_{q=1}^{\infty}=\{(X_1^q,X_2^q,\ldots,X_n^q)\}_{q=1}^{\infty}$$

in $\prod_{i=1}^{n}\mathbb{C}^{n_i\times n_i'}$ *that converges to* X^0 *when* $q\to\infty$, *there exists at least a sequence*

$$\{Y^q\}_{q=1}^{\infty}=\{(Y_1^q,Y_2^q,\ldots,Y_p^q)\}_{q=1}^{\infty}$$

in $\prod_{j=1}^{p}\mathbb{C}^{p_j\times p_j'}$ *that converges to* Y^0 *when* $q\to\infty$ *and such that for* $q\geqslant 1$,

$$\begin{cases} f_1(X^q,Y^q)=O,\\ f_2(X^q,Y^q)=O,\\ \qquad\vdots\\ f_s(X^q,Y^q)=O.\end{cases}$$

3. A reduction of the problems

For a simplification of the Problems we make the following remarks. Given a triple of matrices $\alpha = (A,B,C) \in L_{nm}$, let us define

$$\alpha' = (A',B',C') = (PAP^*, PBQ^*, QCP^*),$$

with P,Q unitary matrices. Then, one readily sees that $\mathcal{M}_2(0,\alpha)$ is nonempty if and only if $\mathcal{M}_2(0,\alpha')$ is nonempty. In that case, let $D \in \mathbb{C}^{m\times m}$, and let $D' = QDQ^*$, then

$$\min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X-D\| = \min_{\substack{Y\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha',Y))\geqslant 2}} \|Y-D'\|.$$

REMARK 4. To find the minimum in (1) there is no loss of generality in considering another triple $\alpha' = (A',B',C') \in L_{nm}$ and another matrix $D' \in \mathbb{C}^{m\times m}$ such that

$$(A',B',C',D') = (PAP^*, PBQ^*, QCP^*, QDQ^*),$$

with unitary matrices P,Q, instead of α and D, respectively.

We say that two matrices $N_1, N_2 \in \mathbb{C}^{(n+m)\times(n+m)}$ are (n,m) *block-diagonal unitarily similar* if there exist two unitary matrices $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$ that satisfy

$$N_1 = \begin{pmatrix} U & O \\ O & V \end{pmatrix} N_2 \begin{pmatrix} U & O \\ O & V \end{pmatrix}^*.$$

From this definition we get the following lemma, showed in [5, Lemma 11].

LEMMA 5. *Let $\alpha := (A,B,C) \in L_{nm}$. Assume that B and C are nonzero matrices. Then, the matrix $M(\alpha,O)$ is (n,m) block-diagonal unitarily similar to a matrix in the reduced form:*

(a) *either*

$$\left(\begin{array}{cccc|c} A_{11} & O & O & O & O \\ A_{21} & A_{22} & O & O & O \\ A_{31} & A_{32} & A_{33} & A_{34} & B_3 \\ A_{41} & A_{42} & O & A_{44} & B_4 \\ \hline C_1 & O & O & C_4 & O \end{array}\right) = \left(\begin{array}{c|c} A_r & B_r \\ \hline C_r & O \end{array}\right), \tag{2}$$

with controllable pairs

$$\left(\begin{array}{cc|c} A_{33} & A_{34} & B_3 \\ O & A_{44} & B_4 \end{array}\right), \qquad (A_{44}, B_4)$$

and observable pairs

$$(C_1, A_{11}), \quad (C_4, A_{44});$$

(b) *or*

$$\left(\begin{array}{ccc|c} \hat{A}_{11} & O & O & O \\ \hat{A}_{21} & \hat{A}_{22} & O & O \\ \hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33} & \hat{B}_3 \\ \hline \hat{C}_1 & O & O & O \end{array}\right) = \left(\begin{array}{c|c} \hat{A}_r & \hat{B}_r \\ \hline \hat{C}_r & O \end{array}\right), \tag{3}$$

with $(\hat{A}_{33}, \hat{B}_3)$ and $(\hat{C}_1, \hat{A}_{11})$ controllable and observable pairs, respectively.

REMARK 6. Concerning the submatrices in (2) and (3) we notice that: the sum of the numbers of columns of the matrices A_{11}, A_{22}, A_{33} and A_{44} is n; the matrices B_3 and B_4 have m columns; the matrices C_1 and C_4 have m rows; the sum of the numbers of columns of the matrices $\hat{A}_{11}$, $\hat{A}_{22}$ and $\hat{A}_{33}$ is n; the matrix $\hat{B}_3$ has m columns; and the matrix $\hat{C}_1$ has m rows.

According to Remark 4 in addressing the Problems there is no loss of generality in assuming that the matrix $M(\alpha, O)$ has the reduced form (a) or (b). That is, there is no loss of generality in considering the triples $\alpha_r := (A_r, B_r, C_r)$ or $\hat{\alpha}_r := (\hat{A}_r, \hat{B}_r, \hat{C}_r)$, respectively, instead of the triple $\alpha = (A, B, C)$.

In case (a) we write

$$\tilde{A} := \operatorname{diag}(A_{11}, A_{22}, A_{33}), \qquad A_4 := A_{44} \in \mathbb{C}^{n_4 \times n_4} \tag{4}$$

for short. Given $X \in \mathbb{C}^{m \times m}$, if $M(\alpha, O)$ is (n,m) block-diagonal unitarily similar to (2), then using the notations in (4) we immediately obtain

$$\det\left(\lambda I_{n+m} - \begin{pmatrix} A & B \\ C & X \end{pmatrix}\right) = \det(\lambda I_{n-n_4} - \tilde{A}) \det\left(\lambda I_{n_4+m} - \begin{pmatrix} A_4 & B_4 \\ C_4 & X \end{pmatrix}\right). \tag{5}$$

On the other hand, if $M(\alpha, O)$ is (n,m) block-diagonal unitarily similar to (3), then we have

$$\det\left(\lambda I_{n+m} - \begin{pmatrix} A & B \\ C & X \end{pmatrix}\right) = \det(\lambda I_n - A) \det(\lambda I_m - X). \tag{6}$$

According to the disjunctive (a) or (b) and $\tilde{A}$ being the matrix defined in (4), the analyses of the Problems can be reduced to the consideration of the cases:

(a) $M(\alpha, O)$ is (n,m) block-diagonal unitarily similar to (2), with the following subcases:

$$\begin{cases} \text{(a-1) } 0 \in \Lambda_2(\tilde{A}), \\ \text{(a-2) } 0 \in \Lambda(\tilde{A}) \setminus \Lambda_2(\tilde{A}), \\ \text{(a-3) } 0 \notin \Lambda(\tilde{A}) \text{ and } m = 1, \\ \text{(a-4) } 0 \notin \Lambda(\tilde{A}) \text{ and } m > 1. \end{cases}$$

(b) $M(\alpha, O)$ is (n,m) block-diagonal unitarily similar to (3).

REMARK 7. Let us note that as 0 is a multiple eigenvalue of A, then in the subcases (a-3) and (a-4) it follows that 0 is a multiple eigenvalue of A_4. Therefore in these subcases we see that $n_4 > 1$.

In section 4 we will analyze all the cases, except for the subcase (a-4), which will be studied in Section 5.

4. Cases: (b), (a-1), (a-2) and (a-3)

4.1. Cases: (b) and (a-1)

We have the next theorem.

THEOREM 8. *In the cases* (b) *and* (a-1) *with the notations in* (4)*, if, either* $M(\alpha,O)$ *is* (n,m) *block-diagonal unitarily similar to* (2) *and* 0 *is a multiple eigenvalue of* $\tilde{A}$*, or* $M(\alpha,O)$ *is* (n,m) *block-diagonal unitarily similar to* (3)*, then* $\mathcal{M}_2(0,\alpha)\neq\emptyset$ *and*

$$\min_{X\in\mathcal{M}_2(0,\alpha)}\|X-D\|=0.$$

Proof. It is a consequence of (6) and (5). □

4.2. Subcase (a-2)

Since $M(\alpha,O)$ is (n,m) block-diagonal unitarily similar to (2) and $0\in\Lambda(\tilde{A})\setminus\Lambda_2(\tilde{A})$, fixing $X\in\mathbb{C}^{m\times m}$, from (5),

$$0\in\Lambda_2\begin{pmatrix}A & B\\ C & X\end{pmatrix}\Longleftrightarrow 0\in\Lambda\begin{pmatrix}A_4 & B_4\\ C_4 & X\end{pmatrix}.$$

Therefore, denoting $\alpha_4=(A_4,B_4,C_4)$, where $A_4\in\mathbb{C}^{n_4\times n_4}$, we have

$$\min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}}\|X-D\|=\min_{\substack{X\in\mathbb{C}^{m\times m}\\ \operatorname{rank}M(\alpha_4,X)<n_4+m}}\|X-D\|.$$

With these considerations, for this case we are going to prove the next result.

THEOREM 9. *In the subcase* (a-2)*, with the hypotheses and notations above, let*

$$p:=m-\nu(A_4)-1.$$

(i) *If* $p\geqslant 0$*, then* $\mathcal{M}_2(0,\alpha)\neq\emptyset$ *and the equality*

$$\min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}}\|X-D\|=\sigma_{p+1}(S(D))$$

holds, where

$$S(D):=(I-NN^{\dagger})(D-C_4A_4^{\dagger}B_4)(I-M^{\dagger}M),$$

with

$$M:=(I-A_4A_4^{\dagger})B_4,\quad N:=C_4(I-A_4^{\dagger}A_4).$$

In addition, if $U,V\in\mathbb{C}^{m\times m}$ *are the unitary matrices which satisfy* $U^*S(D)V=\operatorname{diag}(\sigma_1(S(D)),\dots,\sigma_m(S(D)))$ *and* $p\geqslant 0$*, then defining*

$$X_0:=D-U\operatorname{diag}(0,\dots,0,\sigma_{p+1}(S(D)),\dots,\sigma_m(S(D)))V^*,$$

we have $\mathrm{m}(0,M(\alpha,X_0))\geqslant 2$ *i.e.* $\operatorname{rank}M(\alpha_4,X_0)<n_4+m$*, and* $\|X_0-D\|=\sigma_{p+1}(S(D))$*.*

(ii) *If* $p<0$*, then* $\mathcal{M}_2(0,\alpha)=\emptyset$*.*

Proof. We are going to apply Theorem 1. First, since (A_4,B_4) is controllable and (C_4,A_4) is observable,

$$\rho = \operatorname{rank}(A_4,B_4) + \operatorname{rank}\begin{pmatrix} A_4 \\ C_4 \end{pmatrix} - \operatorname{rank} A_4 = n_4 + n_4 - \operatorname{rank} A_4 = n_4 + \nu(A_4).$$

Setting $r = n_4 + m - 1$, it follows that

$$\rho \leqslant r \Leftrightarrow \nu(A_4) + 1 \leqslant m \Leftrightarrow p \geqslant 0.$$

Suppose that $p \geqslant 0$. If $\operatorname{rank} M(\alpha_4,D) < n_4 + m$, i.e. $r \geqslant \operatorname{rank} M(\alpha_4,D)$, then 0 is an eigenvalue of the matrix $M(\alpha_4,D)$ and

$$\min_{\substack{X\in\mathbb{C}^{m\times m} \\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X-D\| = 0.$$

But, by Theorem 1,

$$n_4 + m > \operatorname{rank} M(\alpha_4,D) = \rho + \operatorname{rank} S(D),$$

which implies

$$\operatorname{rank} S(D) < m - \nu(A_4) = p + 1.$$

Therefore $\sigma_{p+1}(S(D)) = 0$ and the theorem has been proved in this case.

When $\operatorname{rank} M(\alpha_4,D) = n_4 + m$, i.e. $r < \operatorname{rank} M(\alpha_4,D)$, the theorem immediately follows from Theorem 1. This ends the proof of (i).

Now we will prove (ii). Let us observe in first place that if $p < 0$ then $\nu(A_4) \geqslant m$. As (A_4,B_4) is controllable, then $\nu(A_4) \leqslant m$. Hence $\nu(A_4) = m$, i.e. $\rho = n_4 + m$. By Theorem 1, for $X \in \mathbb{C}^{m\times m}$, we deduce that $\operatorname{rank} M(\alpha_4,X) \geqslant \rho = n_4 + m$. Thus, there is no matrix $X \in \mathbb{C}^{m\times m}$ such that $\operatorname{rank} M(\alpha_4,X) < n_4 + m$. $\square$

4.3. Subcase (a-3)

THEOREM 10. *In the subcase* (a-3)*, there is no matrix* $X_0 \in \mathbb{C}^{1\times 1}$ *such that* $\mathrm{m}(0,M(\alpha,X_0)) \geqslant 2$.

Proof. First, let us observe that in the proof of Theorem 9 we have proved $\rho = n_4 + \nu(A_4)$. Now then, by Theorem 1, for any $X \in \mathbb{C}^{1\times 1}$ we conclude that $\operatorname{rank} M(\alpha_4,X) \geqslant \rho = n_4 + 1$. In consequence, as $0 \notin \Lambda(\tilde{A})$, we infer that there is no matrix $X_0 \in \mathbb{C}^{1\times 1}$ such that $\mathrm{m}(0,M(\alpha,X_0)) \geqslant 2$. $\square$

5. Subcase (a-4)

Let $\alpha_4 = (A_4,B_4,C_4)$. Since 0 is not an eigenvalue of $\tilde{A}$, from (5) we deduce the following assertion: *Given a matrix* $X \in \mathbb{C}^{m\times m}$*, then* 0 *is a multiple eigenvalue of* $M(\alpha,X)$ *if and only if* 0 *is a multiple eigenvalue of* $M(\alpha_4,X)$*. For this reason* $\mathcal{M}_2(0,\alpha) = \mathcal{M}_2(0,\alpha_4)$*, and if this set is nonempty,*

$$\min_{X \in \mathcal{M}_2(0,\alpha)} \|X - D\| = \min_{X \in \mathcal{M}_2(0,\alpha_4)} \|X - D\|.$$

The pairs (A_4, B_4) and (C_4, A_4) are controllable and observable, respectively, and 0 is an eigenvalue of A_4. Therefore, a solution to the Problems is given by means of the forthcoming Theorem 14. To ease the meaning of this theorem we need the following three results.

PROPOSITION 11. *Let any $\alpha = (A,B,C) \in L_{nm}$ with $m > 1$. Then for every $z_0 \in \mathbb{C} \setminus \Lambda(A)$, the set $\mathcal{M}_2(z_0, \alpha)$ is nonempty.*

Proof. As

$$\begin{pmatrix} I_n & (A - z_0 I_n)^{-1} B \\ O & I_m \end{pmatrix} \begin{pmatrix} A & B \\ C & z_0 I_m + C(A - z_0 I_n)^{-1} B \end{pmatrix} \begin{pmatrix} I_n & -(A - z_0 I_n)^{-1} B \\ O & I_m \end{pmatrix} = \begin{pmatrix} A + (A - z_0 I_n)^{-1} BC & O \\ C & z_0 I_m \end{pmatrix}$$

and $m > 1$, it follows that z_0 is a multiple eigenvalue of the matrix

$$M(\alpha, z_0 I_m + C(A - z_0 I_n)^{-1} B). \quad \square$$

COROLLARY 12. *Let $\alpha = (A,B,C) \in L_{nm}$ where A is invertible and $m > 1$. Then $\mathcal{M}_2(0, \alpha) \neq \emptyset$.*

Let us remind the following theorem about the minimum distance from a given matrix D to the matrices X in the set $\mathcal{M}_2(0, \alpha)$, which the second and third authors showed in [5, Theorem 25, page 1205].

THEOREM 13. *Let $\alpha = (A,B,C) \in L_{nm}$ where A is invertible and $m > 1$. Let $D \in \mathbb{C}^{m \times m}$. Then*

$$\sup_{t \in \mathbb{R}} \sigma_{2m-1} \begin{pmatrix} D - CA^{-1}B & t(I_m + CA^{-2}B) \\ O & D - CA^{-1}B \end{pmatrix} = \min_{X \in \mathcal{M}_2(0,\alpha)} \|X - D\|.$$

Now we are prepared to establish the main result in this paper.

THEOREM 14. *Let any triple $\alpha = (A,B,C) \in L_{nm}$ with $m > 1$. Let us assume that the pair (A,B) is controllable and the pair (C,A) is observable. Let $\{\alpha_q = (A_q, B_q, C_q)\}_{q=1}^{\infty}$ be a sequence of triples of matrices in L_{nm} that converges to α when $q \to \infty$, and where for every q the matrix A_q is invertible. Then there exists the limit*

$$\lim_{q \to \infty} \min_{X \in \mathcal{M}_2(0,\alpha_q)} \|X\|,$$

finite $(\ell \in \mathbb{R})$ *or infinite* (∞). *Also,*

$$\lim_{q\to\infty} \min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X\| = \begin{cases} \ell \in \mathbb{R} & \Longleftrightarrow \quad \mathcal{M}_2(0,\alpha) \neq \emptyset, \\ \infty & \Longleftrightarrow \quad \mathcal{M}_2(0,\alpha) = \emptyset. \end{cases}$$

Moreover, when this limit is $\ell < \infty$ *then*

$$\min_{X\in\mathcal{M}_2(0,\alpha)} \|X\| = \ell.$$

REMARK 15. Let us make the following observations about the statement of this theorem:

1. The matrix A can be invertible or not.
2. The convergence of $\min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X\|$ to a real number ℓ (to ∞, respectively), and this limit, is independent of the choice of the sequence $\{\alpha_q\}_{q=1}^{\infty}$ converging to α.
3. The invertibility of the matrices A_q guarantees the existence of the minimum $\min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X\|$ and the computation of its value.
4. The sequence of nonnegative numbers

$$\left\{ \min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X\| \right\}_{q=1}^{\infty}$$

does not oscillate; more precisely,

$$\liminf_{q\to\infty} \min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X\| = \limsup_{q\to\infty} \min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X\|.$$

Before the proof of this theorem we are going to prove a proposition and a lemma. With the hypotheses of Theorem 14 for the triple $\alpha = (A,B,C) \in L_{nm}$, let us assume that there exists a matrix $X_0 \in \mathbb{C}^{m\times m}$ such that $\mathrm{m}(0, M(\alpha, X_0)) \geqslant 2$. Therefore, there exist vectors $u_1, v_1 \in \mathbb{C}^{n\times 1}$, $u_2, v_2 \in \mathbb{C}^{m\times 1}$ and a complex number β such that

$$\operatorname{rank} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = 2, \tag{7}$$

and

$$\begin{pmatrix} A & B \\ C & X_0 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}. \tag{8}$$

That is

$$Au_1 + Bu_2 = 0, \tag{9a}$$

$$Cu_1 + X_0 u_2 = 0, \tag{9b}$$

$$Av_1 + Bv_2 - u_1\beta = 0, \tag{9c}$$

$$Cv_1 + X_0 v_2 - u_2\beta = 0. \tag{9d}$$

We have the following result.

PROPOSITION 16. $u_2 \neq 0$.

Proof. Suppose, contrary to our claim, that $u_2 = 0$. Then, by (9a) and (9b), $Au_1 = 0$ and $Cu_1 = 0$. Since (C,A) is an observable pair, then $u_1 = 0$. Hence $\binom{u_1}{u_2} = 0$. This contradicts (7). □

LEMMA 17. *Let $\alpha = (A,B,C) \in L_{nm}$ be any triple of matrices, with $m > 1$. Let us assume that (A,B) is controllable and (C,A) is observable. Let us suppose that there is a matrix $X_0 \in \mathbb{C}^{m\times m}$ such that 0 is a multiple eigenvalue of $M(\alpha, X_0)$. Let $\{\alpha_q\}_{q=1}^{\infty}$ be a sequence in L_{nm} that converges to α when $q \to \infty$. Then there exist a sequence of matrices $\{X_q\}_{q=1}^{\infty}$ converging to X_0 when $q \to \infty$, such that 0 is a multiple eigenvalue of $M(\alpha_q, X_q)$, for each q.*

Proof. Since 0 is a multiple eigenvalue of $M(\alpha, X_0)$, there exist vectors $u_1, v_1 \in \mathbb{C}^{n\times 1}$, $u_2, v_2 \in \mathbb{C}^{m\times 1}$ and a complex number β such that (7) and (8) are satisfied. Let $\alpha_q := (A+\Delta_1^q, B+\Delta_2^q, C+\Delta_3^q)$.

The proof of this lemma will be ended once we have proved the existence of sequences of matrices $\{\Delta_4^q\}_{q=1}^{\infty}$ and sequences of vectors $\{s_i^q\}_{q=1}^{\infty}$, $i = 1,2,3,4$, of adequate sizes, converging to O and 0 when $q \to \infty$, such that for each q,

$$\begin{pmatrix} A+\Delta_1^q & B+\Delta_2^q \\ C+\Delta_3^q & X_0+\Delta_4^q \end{pmatrix} \begin{pmatrix} u_1+s_1^q & v_1+s_2^q \\ u_2+s_3^q & v_2+s_4^q \end{pmatrix} = \begin{pmatrix} u_1+s_1^q & v_1+s_2^q \\ u_2+s_3^q & v_2+s_4^q \end{pmatrix} \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}. \tag{10}$$

Case 1. We assume that u_2 and v_2 are linearly independent. Operating by blocks in (10), our problem is reduced to find sequences $\{\Delta_4^q\}_{q=1}^{\infty}$ and $\{s_i^q\}_{q=1}^{\infty}$ converging to O and 0 when $q \to \infty$, such that for each q,

$$\begin{cases} (A+\Delta_1^q)(u_1+s_1^q) + (B+\Delta_2^q)(u_2+s_3^q) = 0, \\ (C+\Delta_3^q)(u_1+s_1^q) + (X_0+\Delta_4^q)(u_2+s_3^q) = 0, \\ (A+\Delta_1^q)(v_1+s_2^q) + (B+\Delta_2^q)(v_2+s_4^q) - (u_1+s_1^q)\beta = 0, \\ (C+\Delta_3^q)(v_1+s_2^q) + (X_0+\Delta_4^q)(v_2+s_4^q) - (u_2+s_3^q)\beta = 0. \end{cases} \tag{11}$$

To solve this question, we are going to take into account Lemma 3. Let $P_{n,m}$ be the product space

$$\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{m\times n} \times \mathbb{C}^{n\times 1} \times \mathbb{C}^{n\times 1} \times \mathbb{C}^{n\times 1} \times \mathbb{C}^{n\times 1} \times \mathbb{C}^{m\times m}.$$

Let $f_1 \colon P_{n,m} \to \mathbb{C}^{n\times 1}$, $f_2 \colon P_{n,m} \to \mathbb{C}^{m\times 1}$, $f_3 \colon P_{n,m} \to \mathbb{C}^{n\times 1}$, $f_4 \colon P_{n,m} \to \mathbb{C}^{m\times 1}$ be the maps defined by

$$f_1(\Delta_1,\Delta_2,\Delta_3,s_1,s_2,s_3,s_4,\Delta_4) := (A+\Delta_1)(u_1+s_1) + (B+\Delta_2)(u_2+s_3),$$
$$f_2(\Delta_1,\Delta_2,\Delta_3,s_1,s_2,s_3,s_4,\Delta_4) := (C+\Delta_3)(u_1+s_1) + (X_0+\Delta_4)(u_2+s_3),$$
$$f_3(\Delta_1,\Delta_2,\Delta_3,s_1,s_2,s_3,s_4,\Delta_4) := (A+\Delta_1)(v_1+s_2) + (B+\Delta_2)(v_2+s_4) - (u_1+s_1)\beta,$$
$$f_4(\Delta_1,\Delta_2,\Delta_3,s_1,s_2,s_3,s_4,\Delta_4) := (C+\Delta_3)(v_1+s_2) + (X_0+\Delta_4)(v_2+s_4) - (u_2+s_3)\beta,$$

for

$$(\Delta_1, \Delta_2, \Delta_3, s_1, s_2, s_3, s_4, \Delta_4) \in P_{n,m}.$$

First, by (9) we deduce that

$$f_i(O,O,O,0,0,0,0,O) = 0$$

for $i = 1,2,3,4$. Second, due to Lemma 2, the partial Jacobian matrix

$$\frac{\partial(f_1, f_2, f_3, f_4)}{\partial(s_1, s_2, s_3, s_4, \Delta_4)}$$

evaluated at the point $(O,O,O,0,0,0,0,O) \in P_{n,m}$, is the matrix

$$\mathscr{J} = \begin{pmatrix} A & O & B & O & O \\ C & O & X_0 & O & I_m \otimes u_2^{\mathrm{T}} \\ -\beta I_n & A & O & B & O \\ O & C & -\beta I_m & X_0 & I_m \otimes v_2^{\mathrm{T}} \end{pmatrix}.$$

To finish the proof, it suffices to see that the $(2n+2m) \times (2n+3m)$ matrix $\mathscr{J}$ has rank $2n+2m$. Note that

$$\operatorname{rank} \mathscr{J} = 2m + \operatorname{rank} \begin{pmatrix} A & B & O & O \\ -\beta I_n & O & A & B \end{pmatrix},$$

because u_2 and v_2 are linearly independent. Finally, since (A,B) is a controllable pair we conclude that

$$\operatorname{rank} \begin{pmatrix} A & B & O & O \\ -\beta I_n & O & A & B \end{pmatrix} = 2n.$$

Thus, $\operatorname{rank} \mathscr{J} = 2m + 2n$.

Case 2. We assume that u_2 and v_2 are linearly dependent. Then, by Proposition 16, since $u_2 \neq 0$ we see that $v_2 = \lambda u_2$ for some $\lambda \in \mathbb{C}$. From (8), we deduce that

$$\begin{pmatrix} A & B \\ C & X_0 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix};$$

that is

$$\begin{pmatrix} A & B \\ C & X_0 \end{pmatrix} \begin{pmatrix} u_1 & v_1 - \lambda u_1 \\ u_2 & 0 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 - \lambda u_1 \\ u_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}.$$

Given that u_2 and v_2 are linearly dependent, there is no loss of generality in considering that $v_2 = 0$. Then, by (8),

$$\begin{pmatrix} A & B \\ C & X_0 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & 0 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \tag{12}$$

where u_2 and v_1 are nonzero vectors. Let $\{T_p\}_{p=0}^{\infty}$ be a sequence of matrices in $\mathbb{C}^{m\times n}$ such that for each p $\operatorname{rank}(u_2 + T_p u_1, T_p v_1) = 2$, $(A - BT_p, B)$ is a controllable pair,

$(C+T_pA-X_0T_p-T_pBT_p, A-BT_p)$ is observable, and $\|T_p\|<1/p$. By (12) we see that

$$\begin{pmatrix} I_n & O \\ T_p & I_m \end{pmatrix}\begin{pmatrix} A & B \\ C & X_0 \end{pmatrix}\begin{pmatrix} I_n & O \\ -T_p & I_m \end{pmatrix}\begin{pmatrix} I_n & O \\ T_p & I_m \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2 & 0 \end{pmatrix}=\begin{pmatrix} I_n & O \\ T_p & I_m \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix},$$

that is

$$\begin{pmatrix} A-BT_p & B \\ C+T_pA-X_0T_p-T_pBT_p & X_0+T_pB \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2+T_pu_1 & T_pv_1 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2+T_pu_1 & T_pv_1 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}.$$

Since $\operatorname{rank}(u_2+T_pu_1, T_pv_1)=2$, $(A-BT_p, B)$ is a controllable pair, and $(C+T_pA-X_0T_p-T_pBT_p, A-BT_p)$ is observable, by the already proved in Case 1 and given that the sequence of $\{(\Omega_1^{p,q},\Omega_2^{p,q},\Omega_3^{p,q})\}_{q=1}^{\infty}$ converges to $O\in L_{nm}$ when $q\to\infty$, we infer that there exist sequences $\{E_q^p\}_{q=1}^{\infty}$, $\{s_i^{p,q}\}_{q=1}^{\infty}$ of adequate sizes converging to 0, such that for each q,

$$\begin{pmatrix} A-BT_p+\Omega_1^{p,q} & B+\Omega_2^{p,q} \\ C+T_pA-X_0T_p-T_pBT_p+\Omega_3^{p,q} & X_0+T_pB+E_q^P \end{pmatrix} \times \begin{pmatrix} u_1+s_1^{p,q} & v_1+s_2^{p,q} \\ u_2+T_pu_1+s_3^{p,q} & T_pv_1+s_4^{p,q} \end{pmatrix} = \begin{pmatrix} u_1+s_1^{p,q} & v_1+s_2^{p,q} \\ u_2+T_pu_1+s_3^{p,q} & T_pv_1+s_4^{p,q} \end{pmatrix}\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}. \quad (13)$$

Defining

$$\Omega_1^{q,q} := \Delta_1^q+BT_q,\ \Omega_2^{q,q} := \Delta_2^q,\ \Omega_3^{q,q} := \Delta_3^q-T_qA+X_0T_q+T_qBT_q,$$

$s_3^q := T_qs_3^{q,q}$ and $s_i^q := s_i^{q,q}$ $i=\{1,2,4\}$, from (13) we conclude the proof in this case. Observe that $\Delta_4^q = T_qB+E_q^q \to O$. □

We are in a position to prove Theorem 14.

Proof of Theorem 14. Let us consider an arbitrary sequence of triples of matrices $\{\alpha_q=(A_q,B_q,C_q)\}_{q=1}^{\infty}$ converging to α, such that for each q, A_q is invertible. Since A_q is invertible, from Corollary 12 and Theorem 13 we see that there exists a sequence of matrices $\{Y_q\}_{q=1}^{\infty}$ such that for each $q=1,2,\ldots$,

$$\mu_q := \min_{\substack{X\in\mathbb{C}^{m\times m} \\ \mathrm{m}(0,M(\alpha_q,X))\geqslant 2}} \|X\| = \|Y_q\|, \quad (14)$$

where $\mathrm{m}(0,M(\alpha_q,Y_q))\geqslant 2$.

Case 1. Let us assume that $\mathcal{M}_2(0,\alpha)\neq\emptyset$. Let X_0 be such that $\mathrm{m}(0,M(\alpha,X_0))\geqslant 2$ and

$$\mu_0 := \|X_0\| = \min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X\|. \tag{15}$$

Since $\{\alpha_q\}_{q=1}^{\infty}$ converges to α, by Lemma 17 there exists a sequence $\{X_q\}_{q=1}^{\infty}$ converging to X_0, such that for each q, 0 is a multiple eigenvalue of $M(\alpha_q,X_q)$. Let

$$\hat{\mu}_q := \|X_q\|. \tag{16}$$

Then

$$\lim_{q\to\infty}\hat{\mu}_q = \mu_0. \tag{17}$$

Since $\mu_q \leqslant \hat{\mu}_q$, by (17)

$$\limsup_{q\to\infty}\mu_q \leqslant \limsup_{q\to\infty}\hat{\mu}_q = \mu_0. \tag{18}$$

Let $\{\mu_{q_k}\}_{k=1}^{\infty}$ be a subsequence of $\{\mu_q\}_{q=1}^{\infty}$ such that

$$\liminf_{q\to\infty}\mu_q = \lim_{k\to\infty}\mu_{q_k}. \tag{19}$$

Since $\{Y_{q_k}\}_{k=1}^{\infty}$ is bounded, there exists a subsequence $\{Y_{q_{k_i}}\}_{i=1}^{\infty}$ that converges to a matrix $\hat{Y}_0$. As 0 is a multiple eigenvalue of $M(\alpha_{q_{k_i}},Y_{q_{k_i}})$, 0 is a multiple eigenvalue of $M(\alpha,\hat{Y}_0)$. By (19), (14) and (15), we see that

$$\liminf_{q\to\infty}\mu_q = \lim_{i\to\infty}\mu_{q_{k_i}} = \lim_{i\to\infty}\|Y_{q_{k_i}}\| = \|\hat{Y}_0\| \geqslant \mu_0. \tag{20}$$

Combining inequalities (20) and (18) we conclude that

$$\mu_0 \leqslant \liminf_{q\to\infty}\mu_q \leqslant \limsup_{q\to\infty}\mu_q \leqslant \mu_0,$$

that is

$$\lim_{q\to\infty}\mu_q = \mu_0.$$

Case 2. Let us suppose that $\mathcal{M}_2(0,\alpha)=\emptyset$. We are going to prove that $\lim_{q\to\infty}\mu_q = \infty$. Let us assume the opposite. Then, it follows from (14) that there exist convergent subsequences $\{\mu_{q_j}\}_{j=1}^{\infty}$ and $\{Y_{q_j}\}_{j=1}^{\infty}$ of $\{\mu_q\}_{q=1}^{\infty}$ and $\{Y_q\}_{q=1}^{\infty}$, respectively. Let us call

$$\hat{Z}_0 := \lim_{j\to\infty} Y_{q_j}.$$

As 0 is a multiple eigenvalue of $M(\alpha_{q_j},Y_{q_j})$ for each j, then 0 is a multiple eigenvalue of $M(\alpha,\hat{Z}_0)$; a contradiction. $\square$

REMARK 18. A careful analysis of the proof of this theorem let us see that the following assertions are true:

1. If the limit
$$\lim_{q\to\infty} \min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X\|$$
is finite (infinite, respectively), the same holds for the limit
$$\lim_{q\to\infty} \min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X-D\|$$
whatever the matrix $D\in\mathbb{C}^{m\times m}$ is.

2. The value of the limit
$$\lim_{q\to\infty} \min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X-D\|$$
depends only on D, but it does not depend on the sequence $\{\alpha_q\}_{q=1}^{\infty}$ converging to α.

COROLLARY 19. *Let any triple $\alpha=(A,B,C)\in L_{nm}$ with $m>1$, controllable (A,B) and observable (C,A). Let $\{\alpha_q=(A_q,B_q,C_q)\}_{q=1}^{\infty}$ be a sequence of triples of matrices in L_{nm} that converges to α when $q\to\infty$, and where for every q the matrix A_q is invertible. Then*

- $\mathcal{M}_2(0,\alpha)\neq\emptyset \Longleftrightarrow \lim_{q\to\infty}\min_{X\in\mathcal{M}_2(0,\alpha_q)}\|X\|$ *is finite.*
- $\mathcal{M}_2(0,\alpha)=\emptyset \Longleftrightarrow \lim_{q\to\infty}\min_{X\in\mathcal{M}_2(0,\alpha_q)}\|X\|$ *is infinite.*

COROLLARY 20. *Let any triple $\alpha=(A,B,C)\in L_{nm}$ with $m>1$, controllable (A,B) and observable (C,A). Let $\{\alpha_q=(A_q,B_q,C_q)\}_{q=1}^{\infty}$ be a sequence of triples of matrices in L_{nm} that converges to α when $q\to\infty$, and where for every q the matrix A_q is invertible. In case of $\mathcal{M}_2(0,\alpha)\neq\emptyset$, then for any $D\in\mathbb{C}^{m\times m}$,*
$$\min_{X\in\mathcal{M}_2(0,\alpha)}\|X-D\| = \lim_{q\to\infty} \min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X-D\|.$$

Concluding remarks

1. Let $\alpha=(A,B,C)\in L_{nm}$ and $z_0\in\mathbb{C}$ such that z_0 is a defective eigenvalue of A. The set $\mathcal{M}_2(z_0,\alpha)$ of matrices $X\in\mathbb{C}^{m\times m}$ such that z_0 is a multiple eigenvalue of the matrix
$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}$$
can be empty. Let α' denote $(A-z_0I_n,B,C)$. Reducing the matrix $M(\alpha',O)$, instead of $M(\alpha,O)$, to the form (2), and using the same notations as in Lemma 5 for A_4,B_4,C_4 and $\tilde{A}$, where (A_4,B_4) is controllable and (C_4,A_4) is observable, we deduce that $\mathcal{M}_2(z_0,\alpha)=\emptyset$ just in the cases when
   - 0 is a simple eigenvalue of $\tilde{A}$ and $m<\nu(A_4)+1$ (special case of (a-2));
   - $0\notin\Lambda(\tilde{A})$ and $m=1$ (case (a-3));

- $0 \notin \Lambda(\tilde{A}), m > 1$ and

$$\lim_{q\to\infty} \min_{X\in\mathcal{M}_2(0,\alpha_q)} \|X\| = \infty,$$

where $\{\alpha_q = (A_q, B_q, C_q)\}_{q=1}^{\infty}$ is any sequence of triples of matrices of adequate sizes, with invertible A_q for every q, converging to (A_4, B_4, C_4) (special case of (a-4)).

In the two first items the small value of m restricts the number of entries of $\left(\begin{smallmatrix} A & B \\ C & X \end{smallmatrix}\right)$ we may choose to do multiple the eigenvalue z_0.

2. Moreover, let $D \in \mathbb{C}^{m\times m}$ be a fourth matrix. With this paper we complete a solution of the problems of feasibility and finding the minimum distance

$$\min_{X\in\mathcal{M}_2(z_0,\alpha)} \|X - D\|,$$

whatever the complex number z_0 is related to the spectrum of A, which the second and third authors began in [5] and [6].

Summing up, when

- z_0 is not an eigenvalue of A, see [5];
- z_0 is a semisimple eigenvalue of A, see [6];
- z_0 is a nonsemisimple eigenvalue of A, see the current paper.

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