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MULTIVARIABLE BESSEL GABOR TRANSFORM AND APPLICATIONS

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Abstract. In this paper we consider multivariable Bessel operator. We define and study the multivariable Bessel Gabor transform. We prove a Plancherel formula, an inversion formula and a weak uncertainty principle for it. As applications, an analog of Heisenberg's inequality is obtained. At the end, we give an application of the theory of reproducing kernels to the Tikhonov regularization on the generalized Sobolev spaces associated with the multivariable Bessel operator.

1. Introduction

The multivariable Bessel operator $\mathscr{L}_\alpha$ on $\Omega_d = (0,\infty)^d$ is defined by:

$$\mathscr{L}_\alpha = \ell_{\alpha_1} \otimes \ldots \otimes \ell_{\alpha_d},$$

where $\alpha = (\alpha_1,\ldots,\alpha_d) \in (-1/2,\infty)^d$ and ℓ_{α_i} is the Bessel operator on Ω_1 given by

$$\ell_{\alpha_i} = \frac{d^2}{dx_i^2} + \frac{2\alpha_i+1}{x_i}\frac{d}{dx_i}, \quad i = 1,\ldots,d.$$

The operator $\mathscr{L}_\alpha$ is introduced by Chettaoui and Othmani in [2], in which the authors have studied the harmonic analysis associated with this operator. In particular they have defined the multivariable Bessel transform $\mathscr{F}_B$ on Ω_d, for a regular function f, by

$$\forall\, \lambda \in \mathbb{R}^d,\ \mathscr{F}_B f(\lambda) = \int_{\Omega_d} f(x)\Lambda_\alpha(\lambda,x)d\mu_\alpha(x) \tag{1.1}$$

where $\Lambda_\alpha(\lambda,x)$ represents the multivariable Bessel kernel on $\mathbb{C}^d \times \mathbb{R}^d$ and $d\mu_\alpha$ the measure given by

$$d\mu_\alpha(x) = \prod_{i=1}^{d} \frac{|x_i|^{2\alpha_i+1}}{2^{\alpha_i}\Gamma(\alpha_i+1)}dx_i.$$

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Moreover, they have proved real Paley-Wiener theorems for the transform $\mathscr{F}_B$.

In this work we are interested to the study of a generalized Gabor transform associated with the multivariable Bessel operator. More precisely, we establish generalized Plancherel and L^2 inversion formulas and we give some applications. In the classical case the Gabor transform is very fundamental and has many applications to Mathematical Sciences. In fact, Dennis Gabor [3] was the first to introduce the Gabor transform, in which he uses translations and modulations of a single Gaussian to represent one dimensional signal. Other names for this transform used in literature are: short time Fourier transform, Weyl-Heisenberg transform and windowed Fourier transform. In [6], Trimèche defined and studied the windowed Fourier transform for Gelfand pairs.

The paper is organized as follows. In §2, we recall the main results about the harmonic analysis related to the multivariable Bessel operator. In §3, we introduce the analog of the continuous Gabor transform associated with the multivariable Bessel operator and we give some harmonic properties for it (Plancheral formula, L^2_α inverse formula). The §4 is devoted to prove the analogue of Heisenberg's inequality for the generalized continuous Gabor transform. In §5 using the reproducing kernel theory given by Saitoh [5] we study the problem of approximative concentration. In the last section we give an application of the Tikhonov regularization method on the generalized Sobolev spaces associated with the multivariable Bessel operator, and we study the extremal function for the multivariable Bessel Gabor transform.

2. Preliminaries

For $\beta=(\beta_1,\dots,\beta_d)\in\mathbb{N}^d$, we denote by $\mathscr{L}_\alpha^\beta$ the operator $\mathscr{L}_\alpha^\beta=\ell_{\alpha_1}^{\beta_1}\otimes\dots\otimes\ell_{\alpha_d}^{\beta_d}$ and $\Delta_\alpha=\sum_{i=1}^{d}\ell_{\alpha_i}$. Let j_{α_i} be the normalized Bessel function defined for $\lambda_i\in\mathbb{C}$ and $x_i\in\mathbb{R}$ by

$$j_{\alpha_i}(\lambda_i x_i)=\Gamma(\alpha_i+1)\sum_{n=0}^{\infty}\frac{(-1)^d(\lambda_i x_i)^{2n}}{2^{2n}n!\Gamma(n+\alpha_i+1)}.$$

The multivariable Bessel kernel $\Lambda_\alpha(\lambda,x)$ defined by

$$\Lambda_\alpha(\lambda,x)=\prod_{i=1}^{d}j_{\alpha_i}(\lambda_i x_i),$$

where $\lambda=(\lambda_1,\dots,\lambda_d)\in\mathbb{C}^d$, $x=(x_1,\dots,x_d)\in\Omega_d$, is a solution of the equation

$$\begin{cases}\mathscr{L}_\alpha^\beta u(x)=(-1)^{|\beta|}\lambda^{2\beta}u(x)\\[2mm] u(0)=1,\ \dfrac{\partial}{\partial x_i}u(x)=0,\ \ i=1,\dots,d.\end{cases}$$

From the properties of the function j_α (cf. [6]), we deduce that the function Λ_α satisfies the following properties

i) For all $\lambda\in\mathbb{C}^d$, the function $(x_1,\dots,x_d)\mapsto\Lambda_\alpha(\lambda,(x_1,\dots,x_d))$ is of class $\mathscr{C}^\infty$ on $\mathbb{R}^d$ and even with respect to each variable.

ii) For all $x \in \mathbb{R}^d$, the function $(\lambda_1,\ldots,\lambda_d) \mapsto \Lambda_\alpha((\lambda_1,\ldots,\lambda_d),x)$ is entire on $\mathbb{C}^d$ and even with respect to each variable.

iii) For all $\lambda \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$, the function Λ_α admits the following integral representation

$$\Lambda_\alpha(\lambda,x) = \prod_{i=1}^{d} \frac{2\Gamma(\alpha_i+1)}{\sqrt{\pi}\Gamma(\alpha_i+1/2)} \int_0^1 (1-t^2)^{\alpha_i-1/2} \cos(\lambda_i x_i t)\,dt.$$

iv) For all $\nu \in \mathbb{N}^d$, $x \in \mathbb{R}^d$, $\lambda \in \mathbb{C}^d$ we have

$$|D_\lambda^\nu \Lambda_\alpha(\lambda,x)| \leqslant \|x\|^{|\nu|} \exp(\|x\|.\|Im(\lambda)\|). \tag{2.2}$$

In particular for all $\lambda \in \mathbb{R}^d$ we have $|\Lambda_\alpha(\lambda,x)| \leqslant 1$.

2.1. Multivariable Bessel transform on Ω_d

In this subsection we recall some basic results related to the multivariable Bessel transform on Ω_d.

NOTATIONS. We denote by
$x^\alpha := x_1^{\alpha_1} \ldots x_1^{\alpha_d}$, for all $x \in \mathbb{R}^d$ and for all $\alpha := (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d$.
$|\alpha| := \alpha_1 + \ldots + \alpha_d$, for all $\alpha := (\alpha_1,\ldots,\alpha_d) \in \mathbb{R}^d$.
$\mathscr{C}_*(\Omega_d)$ the space of continuous functions on Ω_d and even with respect to each variable.
$\mathscr{C}_*(\mathbb{R}^d)$ (resp $\mathscr{C}_{*,c}(\mathbb{R}^d)$) the space of continuous functions on $\mathbb{R}^d$ (resp with compact support), even with respect to each variable.
$S_*(\mathbb{R}^d)$ the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^d$, rapidly decreasing together with their derivatives which are even with respect to each variable.
$D_*(\mathbb{R}^d)$ the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^d$, with compact support and even with respect to each variable.
$L_\alpha^p(\Omega_d), 1 \leqslant p \leqslant \infty$, the space of measurable functions f on Ω_d, such that

$$\|f\|_{L_\alpha^p(\Omega_d)} = \left(\int_{\Omega_d} |f(x)|^p d\mu_\alpha(x)\right)^{1/r} < \infty, \ 1 \leqslant p < \infty$$

$$\|f\|_{L_\alpha^\infty(\Omega_d)} = ess \sup_{x\in\Omega_d} |f(x)| < \infty, \quad p = \infty.$$

PROPOSITION 1. *i) For all f in $L_\alpha^1(\Omega_d)$, the function $\mathscr{F}_B(f)$ given by (1.1), is continuous on $\mathbb{R}^d$, goes to zero at infinity and we have*

$$\|\mathscr{F}_B(f)\|_{L_\alpha^\infty(\Omega_d)} \leqslant \|f\|_{L_\alpha^1(\Omega_d)}. \tag{2.3}$$

ii) For all f in $D_*(\mathbb{R}^d)$, we have

$$\forall\, \lambda \in \mathbb{R}^d, \ \mathscr{F}_B(\mathscr{L}_\alpha^\beta f)(x) = (-1)^{|\beta|} \lambda^{2\beta} \mathscr{F}_B(f)(\lambda) \tag{2.4}$$

$$\forall\,\lambda \in \mathbb{R}^d,\ \mathscr{L}_\alpha^\beta(\mathscr{F}_B(f))(\lambda) = (-1)^{|\beta|}\mathscr{F}_B(x^{2\beta}f)(\lambda). \tag{2.5}$$

iii) For all f *in* $L^1_\alpha(\Omega_d)$, *if* $\mathscr{F}_B(f)$ *belongs to* $L^1_\alpha(\Omega_d)$, *then*

$$f(y) = \int_{\Omega_d} \mathscr{F}_B(f)(x)\Lambda(x,y)d\mu_\alpha(x), \quad a.e. \tag{2.6}$$

PROPOSITION 2. *Let* f *be in* $D_*(\mathbb{R}^d)$, *then we have the inversion formula*

$$\forall\, x \in \Omega_d,\ f(x) = \int_{\Omega_d} \mathscr{F}_B(f)(\lambda)\Lambda_\alpha(\lambda,x)d\mu_\alpha(\lambda). \tag{2.7}$$

PROPOSITION 3. *The multivariable Bessel transform* $\mathscr{F}_B$ *is a topological isomorphism from* $S_*(\mathbb{R}^d)$ *onto itself.*

THEOREM 1. *(i)* (Plancherel formula). *For all* f,g *in* $S_*(\mathbb{R}^d)$. *We have*

$$\int_{\Omega_d} |f(x)|^2 d\mu_\alpha(x) = \int_{\Omega_d} |\mathscr{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda). \tag{2.8}$$

(ii) The transform $\mathscr{F}_B$ *can be extended to an isometric isomorphism of* $L^2_\alpha(\Omega_d)$ *onto itself.*

2.2. Generalized convolution product associated with the Bessel operator on Ω_d

DEFINITION 1. The generalized translation operators $\tau_x, x \in \mathbb{R}^d$, associated with the multivariable Bessel operator on $\mathbb{R}^d$, are defined for f in $\mathscr{C}_*(\mathbb{R}^d)$ by

$$\tau_x f(y) = c_\alpha \int_{[0,\pi]^d} f\Big(\sqrt{x_1^2+y_1^2-2x_1y_1\cos\theta_1},\dots,\sqrt{x_d^2+y_d^2-2x_dy_d\cos\theta_d}\Big)$$

$$\times (\sin\theta_1)^{2\alpha_1}\dots(\sin\theta_d)^{2\alpha_d}d\theta_1\dots d\theta_d, \tag{2.9}$$

where $x=(x_1,\dots,x_d)$, $y=(y_1,\dots,y_d)$ and $c_\alpha = \displaystyle\prod_{i=1}^{d} \frac{\Gamma(\alpha_i+1)}{\sqrt{\pi}\,\Gamma(\alpha_i+1/2)}$.

By using the multivariable Bessel kernel, we can also define a generalized translation. For a function $f \in L^2_\alpha(\Omega_d)$ and $y \in \Omega_d$, the generalized translation $\tau_y f$ is defined by the following relation:

$$\mathscr{F}_B(\tau_y f)(x) = \Lambda(x,y)\mathscr{F}_B(f)(x). \tag{2.10}$$

DEFINITION 2. The generalized convolution product associated with the multivariable Bessel operator in Ω_d of f and g in $\mathscr{C}_{*,c}(\mathbb{R}^d)$ is defined by

$$\forall\, x \in \mathbb{R}^d,\ f *_B g(x) = \int_{\Omega_d} \tau_x f(y)g(y)d\mu_\alpha(y). \tag{2.11}$$

PROPOSITION 4. *i) Let f be in $L^1_\alpha(\Omega_d)$. Then for all $x \in \Omega_d$, we have*

$$\forall \lambda \in \mathbb{R}^d, \ \mathscr{F}_B(\tau_x f)(\lambda) = \Lambda_\alpha(\lambda, x)\mathscr{F}_B(f)(\lambda). \tag{2.12}$$

ii) Let $f \in L^1_\alpha(\Omega_d)$ and $g \in L^2_\alpha(\Omega_d)$ then $f *_B g$ is defined almost every where, belongs to $L^2_\alpha(\Omega_d)$ and we have

$$\mathscr{F}_B(f *_B g) = \mathscr{F}_B(f)\mathscr{F}_B(g). \tag{2.13}$$

PROPOSITION 5. *i) Let f be in $L^1_\alpha(\Omega_d)$ and g in $L^\infty_\alpha(\Omega_d)$. Then we have*

$$\|f *_B g\|_{L^\infty_\alpha(\Omega_d)} \leqslant \|f\|_{L^1_\alpha(\Omega_d)} \|g\|_{L^\infty_\alpha(\Omega_d)}. \tag{2.14}$$

ii) Let g be in $L^1_\alpha(\Omega_d)$ and f in $L^p_\alpha(\Omega_d)$ $1 \leqslant p \leqslant \infty$, then

$$\|f *_B g\|_{L^p_\alpha(\Omega_d)} \leqslant \|f\|_{L^p_\alpha(\Omega_d)} \|g\|_{L^1_\alpha(\Omega_d)}. \tag{2.15}$$

iii) Let $p,q,r \in [1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If f is in $L^p_\alpha(\Omega_d)$, g in $L^q_\alpha(\Omega_d)$. Then $f *_B g \in L^r_\alpha(\Omega_d)$ and we have

$$\|f *_B g\|_{L^r_\alpha(\Omega_d)} \leqslant \|f\|_{L^p_\alpha(\Omega_d)} \|g\|_{L^q_\alpha(\Omega_d)}. \tag{2.16}$$

3. The multivariable Bessel Gabor transform

NOTATIONS. We denote by:

X^p_α, $p \in [1,\infty]$ the space of measurable functions f on $\Omega_d \times \Omega_d$ with respect to the measure $d\omega_\alpha(x,y) = d\mu_\alpha(x)d\mu_\alpha(y)$ such that

$$\begin{aligned} \|f\|_{p,\mu_\alpha} &:= \left(\int_{\Omega_d \times \Omega_d} |f(x,y)|^p d\omega_\alpha(x,y) \right)^{\frac{1}{p}} < \infty, \quad 1 \leqslant p < \infty \\ \|f\|_{\infty,\mu_\alpha} &:= \operatorname*{ess\,sup}_{x,y \in \Omega_d} |f(x,y)| < \infty. \end{aligned}$$

DEFINITION 3. For any function g in $L^2_\alpha(\Omega_d)$ and any $\nu \in \Omega_d$, we define the modulation of g by ν as :

$$\mathscr{M}_\nu g := g_\nu := \mathscr{F}_B\left(\sqrt{\tau_\nu(|g|^2)}\right), \tag{3.17}$$

where τ_y, $y \in \Omega_d$, are the generalized translation operators given by (2.10).

REMARK 1. For g in $L^2_\alpha(\Omega_d)$, we have

$$\|g_\nu\|_{L^2_\alpha(\Omega_d)} = \|g\|_{L^2_\alpha(\Omega_d)}.$$

For $\nu, y \in \Omega_d$, we denote by $g_{\nu,y}$ the function defined on Ω_d by

$$g_{\nu,y} = \tau_y g_\nu$$

DEFINITION 4. Let g be in $L^2_\alpha(\Omega_d)$. For a function f in $L^2_\alpha(\Omega_d)$ we define its continuous generalized Gabor transform by

$$\mathscr{G}_g f(y,\nu) := \int_{\Omega_d} f(x) g_{\nu,y}(x) d\mu_\alpha(x), \tag{3.18}$$

which can also be written in the form

$$\mathscr{G}_g f(y,\nu) := f *_B g_\nu(y). \tag{3.19}$$

PROPOSITION 6. *Let $f,g \in L^2_\alpha(\Omega_d)$. Then $f *_B g \in L^2_\alpha(\Omega_d)$ if and only if $\mathscr{F}_B(f)\mathscr{F}_B(g)$ belongs to $L^2_\alpha(\Omega_d)$, and in this case we have*

$$\mathscr{F}_B(f *_B g) = \mathscr{F}_B(f)\mathscr{F}_B(g).$$

The proof of this proposition is a consequence of the following two lemmas.

LEMMA 1. *Let $f \in L^\infty_\alpha(\Omega_d), g \in L^1_\alpha(\Omega_d)$ and assume that for all $\chi \in L^1_\alpha(\Omega_d) \cap L^2_\alpha(\Omega_d)$ we have*

$$\int_{\Omega_d} f(y)\overline{\chi(y)} d\mu_\alpha(y) = \int_{\Omega_d} g(\xi)\overline{\mathscr{F}_B(\chi)(\xi)} d\mu_\alpha(\xi).$$

Then $f \in L^2_\alpha(\Omega_d)$ if and only if $g \in L^2_\alpha(\Omega_d)$, and in this case we have

$$\mathscr{F}_B(f) = g \quad a.e.$$

Proof. This follows from an easy application of the Plancherel formula. $\square$

LEMMA 2. *Let $f,g \in L^2_\alpha(\Omega_d), \chi \in L^1_\alpha(\Omega_d) \cap L^2_\alpha(\Omega_d)$ we have*

$$\int_{\Omega_d} f *_B g(y)\overline{\chi(y)} d\mu_\alpha(y) = \int_{\Omega_d} \mathscr{F}_B(f)(\xi)\mathscr{F}_B(g)(\xi)\overline{\mathscr{F}_B(\chi)(\xi)} d\mu_\alpha(\xi).$$

Proof. First note the following general fact: if $f \in L^1_\alpha(\Omega_d) \cap L^2_\alpha(\Omega_d)$ and $g \in L^2_\alpha(\Omega_d)$ then

$$\mathscr{F}_B(f *_B g) = \mathscr{F}_B(f)\mathscr{F}_B(g) \quad a.e.$$

This follows from the analogue fact for $L^1_\alpha(\Omega_d)$ functions and the possibility to approximate g in $L^2_\alpha(\Omega_d)$ with functions in $L^1_\alpha(\Omega_d) \cap L^2_\alpha(\Omega_d)$.

Next fix $g \in L^2_\alpha(\Omega_d)$ and define on $L^1_\alpha(\Omega_d) \cap L^2_\alpha(\Omega_d)$ the two functionals

$$S_1(f) := \int_{\Omega_d} f *_B g(y)\overline{\chi(y)} d\mu_\alpha(y),$$
$$S_2(f) := \int_{\Omega_d} \mathscr{F}_B(f)(\xi)\mathscr{F}_B(g)(\xi)\overline{\mathscr{F}_B(\chi)(\xi)} d\mu_\alpha(\xi).$$

By the previous fact and Plancherel's identity, S_1 and S_2 coincide on $L^1_\alpha(\Omega_d) \cap L^2_\alpha(\Omega_d)$. It is easy to show that both functionals are bounded with respect to the L^2_α norm, and therefore can be extended to the whole space $L^2_\alpha(\Omega_d)$, where they still coincide. $\square$

An immediate consequence of Proposition 6 and the Plancherel formula that will be used in the next section is the following.

PROPOSITION 7. *Let f and g be in $L^2_\alpha(\Omega_d)$. Then, we have*

$$\int_{\Omega_d} |f *_B g(x)|^2 d\mu_\alpha(x) = \int_{\Omega_d} |\mathscr{F}_B(f)(\xi)|^2 |\mathscr{F}_B(g)(\xi)|^2 d\mu_\alpha(\xi) \tag{3.20}$$

where both sides are finite or infinite.

THEOREM 2. (L^2_α inversion formula) *Let g be in $(L^2_\alpha(\Omega_d)\cap L^\infty_\alpha(\Omega_d))\setminus\{0\}$ such that $||g||_{L^2_\alpha(\Omega_d)} = 1$. Then, for any function f in $L^2_\alpha(\Omega_d)$, we have*

$$f_n(x) = \int_{B^d_+(0,n)} \int_{\Omega_d} \mathscr{G}_g(f)(y,\nu)\tau_y g_\nu(x) d\omega_\alpha(\nu,y) \tag{3.21}$$

in $L^2_\alpha(\Omega_d)$ and satisfies

$$\lim_{n\to\infty} ||f - f_n||_{L^2_\alpha(\Omega_d)} = 0,$$

where

$$B^d_+(0,n) = \{x \in \Omega_d : \ ||x|| \leqslant n\}.$$

To prove this theorem we need the following Lemmas.

LEMMA 3. *Let g be as above. For any positive integer n define the two functions*

$$G_n(x) := \int_{B^d_+(0,n)} \int_{\Omega_d} \Lambda(\xi,x) |\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\nu) d\mu_\alpha(\xi), \ x \in \Omega_d$$

and

$$H_n(\xi) := \int_{B^d_+(0,n)} |\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\nu), \ x \in \Omega_d.$$

Then

$$G_n \in L^2_\alpha(\Omega_d), \quad H_n \in L^1_\alpha(\Omega_d)\cap L^\infty_\alpha(\Omega_d) \quad \textit{and } \mathscr{F}_B(G_n) = H_n.$$

Proof. Using the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}|G_n(x)|^2 &\leqslant \Big(\int_{B^d_+(0,n)} d\mu_\alpha(\nu)\Big) \int_{B^d_+(0,n)} \Big| \int_{\Omega_d} \Lambda(\xi,x)|\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\xi)\Big|^2 d\mu_\alpha(\nu) \\ &\leqslant C \int_{B^d_+(0,n)} \Big| \int_{\Omega_d} \Lambda(\xi,x)|\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\xi)\Big|^2 d\mu_\alpha(\nu).\end{aligned}$$

Therefore by Fubini theorem, the inversion theorem, the Plancherel formula we get

$$\begin{aligned}\int_{\Omega_d} |G_n(x)|^2 d\mu_\alpha(x) &\leqslant C \int_{B^d_+(0,n)} \int_{\Omega_d} \Big| \int_{\Omega_d} \Lambda(\xi,x)|\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\xi)\Big|^2 d\mu_\alpha(\nu) d\mu_\alpha(x) \\ &\leqslant C \int_{B^d_+(0,n)} \int_{\Omega_d} |\mathscr{F}_B^{-1}(|\mathscr{F}_B(g_\nu)|^2)(x)|^2 d\mu_\alpha(x) d\mu_\alpha(\nu) \\ &\leqslant C \int_{B^d_+(0,n)} \int_{\Omega_d} |\tau_\nu |g|^2(\xi)|^2 d\mu_\alpha(\nu) d\mu_\alpha(\xi) \\ &\leqslant C \int_{B^d_+(0,n)} \Big|\Big|\tau_\nu|g|^2\Big|\Big|_{L^1_\alpha(\Omega_d)} \Big|\Big|\tau_\nu|g|^2\Big|\Big|_{L^\infty_\alpha(\Omega_d)} d\mu_\alpha(\nu) \\ &\leqslant C \int_{B^d_+(0,n)} ||\tau_\nu|g|^2||_{L^\infty_\alpha(\Omega_d)} d\mu_\alpha(\nu) < \infty.\end{aligned}$$

On the other hand, one can easily see that $H_n \in L^1_\alpha(\Omega_d) \cap L^\infty_\alpha(\Omega_d)$ and using Fubini's theorem we deduce

$$\begin{aligned}\mathscr{F}_B^{-1}(H_n)(y) &= \int_{\Omega_d} H_n(\xi)\Lambda(\xi,y)d\mu_\alpha(\xi)\\ &= \int_{\Omega_d} \Lambda(\xi,y)\int_{B^d_+(0,n)} |\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\nu)d\mu_\alpha(\xi)\\ &= \int_{B^d_+(0,n)}\int_{\Omega_d} \Lambda(\xi,y)|\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\nu)d\mu_\alpha(\xi) = G_n(y). \quad \square\end{aligned}$$

LEMMA 4. *Let g be as above. For any positive integer n the function*

$$G_n(x) := \int_{B^d_+(0,n)}\int_{\Omega_d} \Lambda(\xi,x)|\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\xi)d\mu_\alpha(\nu),$$

can be written

$$G_n(x) = \int_{B^d_+(0,n)} g_\nu *_B g_\nu(x)d\mu_\alpha(\nu).$$

Proof. From Proposition 6 we have

$$\begin{aligned}G_n(x) &= \int_{B^d_+(0,n)}\int_{\Omega_d} \Lambda(\xi,x)|\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\nu)d\mu_\alpha(\xi)\\ &= \int_{B^d_+(0,n)} \mathscr{F}_B^{-1}(|\mathscr{F}_B(g_\nu)|^2)(x)d\mu_\alpha(\nu)\\ &= \int_{B^d_+(0,n)} \Big(g_\nu *_B g_\nu(x)\Big)d\mu_\alpha(\nu). \quad \square\end{aligned}$$

LEMMA 5. *Let g be in $(L^2_\alpha(\Omega_d)\cap L^\infty_\alpha(\Omega_d))\setminus\{0\}$. Then, for any function f in $L^2_\alpha(\Omega_d)$, we have*

$$f_n = G_n *_B f. \tag{3.22}$$

Proof. We have

$$\begin{aligned}f_n(x) &= \int_{B^d_+(0,n)}\int_{\Omega_d} \mathscr{G}_g(f)(y,\nu)\tau_y g_\nu(x)d\omega_\alpha(\nu,y)\\ &= \int_{B^d_+(0,n)} \Big(\mathscr{G}_g(f)(.,\nu) *_B g_\nu\Big)(x)d\mu_\alpha(\nu)\\ &= \int_{B^d_+(0,n)} f *_B g_\nu *_B g_\nu(x)d\mu_\alpha(\nu)\\ &= \int_{B^d_+(0,n)}\int_{\Omega_d} \tau_x f(y)(g_\nu *_B g_\nu)(y)d\omega_\alpha(\nu,y)\\ &= \int_{\Omega_d} \tau_x f(y)\Big(\int_{B^d_+(0,n)} (g_\nu *_B g_\nu)(y)d\mu_\alpha(\nu)\Big)d\mu_\alpha(y)\\ &= \int_{\Omega_d} \tau_x f(y)G_n(y)d\mu_\alpha(y)\\ &= f *_B G_n(x).\end{aligned}$$

On the follow we justify the use of Fubini's theorem in the last sequence of equalities observe that

$$\Big|\int_{B_+^d(0,n)}\int_{\Omega_d}\tau_x f(y)(g_\nu *_B g_\nu)(y)d\omega_\alpha(\nu,y)\Big| \leqslant \int_{B_+^d(0,n)}\Big|f *_B g_\nu *_B g_\nu(x)\Big|d\mu_\alpha(\nu).$$

Now, using Proposition 6 and hypothesis on g we see that $g_\nu *_B g_\nu \in L^2_\alpha(\Omega_d)$. Next using Young's inequality and Parseval theorem we obtain

$$||f *_B g_\nu *_B g_\nu||_{L^\infty_\alpha(\Omega_d)} \leqslant ||f||_{L^2_\alpha(\Omega_d)}||g_\nu *_B g_\nu||_{L^2_\alpha(\Omega_d)} \leqslant ||f||_{L^2_\alpha(\Omega_d)}||g||_{L^2_\alpha(\Omega_d)}||g||_{L^\infty_\alpha(\Omega_d)}$$

and

$$\int_{B_+^d(0,n)}\Big|f *_B g_\nu *_B g_\nu(x)\Big|d\mu_\alpha(\nu) \leqslant \Big(\int_{B_+^d(0,n)} d\mu_\alpha(\nu)\Big)||f||_{L^2_\alpha(\Omega_d)}||g||_{L^2_\alpha(\Omega_d)}||g||_{L^\infty_\alpha(\Omega_d)}.$$

The proof is complete. □

Proof of Theorem 2. It follows from Proposition 6 and Lemma 3 that $f_n \in L^2_\alpha(\Omega_d)$ and

$$\mathscr{F}_B(f_n) = H_n\mathscr{F}_B(f).$$

By this, the Plancherel formula, the fact that $H_n \to 1$ pointwise as $n \to \infty$, and the dominated convergence theorem, it follows that

$$\begin{aligned}||f-f_n||^2_{L^2_\alpha(\Omega_d)} &= \int_{\Omega_d}|\mathscr{F}_B(f)(\xi) - H_n(\xi)\mathscr{F}_B(f)(\xi)|^2 d\mu_\alpha(\xi)\\ &= \int_{\Omega_d}|\mathscr{F}_B(f)(\xi)(1-H_n(\xi))|^2 d\mu_\alpha(\xi) \to 0\end{aligned}$$

as $n \to \infty$ which achieves the proof. □

PROPOSITION 8. *For f in $L^2_\alpha(\Omega_d)$ and g in $L^2_\alpha(\Omega_d)$ we have*

$$||\mathscr{G}_g f||_{\infty,\mu_\alpha} \leqslant ||f||_{L^2_\alpha(\Omega_d)}||g||_{L^2_\alpha(\Omega_d)}. \tag{3.23}$$

PROPOSITION 9. (Plancherel formula) *Let g be in $L^2_\alpha(\Omega_d)$. Then, for all f in $L^2_\alpha(\Omega_d)$, we have*

$$||\mathscr{G}_g f||_{2,\mu_\alpha} = ||g||_{L^2_\alpha(\Omega_d)}||f||_{L^2_\alpha(\Omega_d)}. \tag{3.24}$$

Proof. Using relation (3.20), Fubini's theorem and Plancherel's formula for the multivariable Bessel transform, we have

$$\begin{aligned}&\int_{\Omega_d}\int_{\Omega_d}|f *_B g_\nu|^2(y)d\mu_\alpha(y)d\mu_\alpha(\nu)\\ &= \int_{\Omega_d}\int_{\Omega_d}|\mathscr{F}_B f(\xi)|^2|\mathscr{F}_B(g_\nu)(\xi)|^2 d\mu_\alpha(\xi)d\mu_\alpha(\nu)\\ &= \int_{\Omega_d}\int_{\Omega_d}|\mathscr{F}_B f(\xi)|^2\tau_\nu(|g|^2)(\xi)d\mu_\alpha(\xi)d\mu_\alpha(\nu)\\ &= \int_{\Omega_d}\int_{\Omega_d}|\mathscr{F}_B f(\xi)|^2\tau_\xi(|g|^2)(\nu)d\mu_\alpha(\nu)d\mu_\alpha(\xi)\\ &= ||f||^2_{L^2_\alpha(\Omega_d)}||g||^2_{L^2_\alpha(\Omega_d)}. \quad\square\end{aligned}$$

As in the classical case, the multivariable Bessel Gabor transform preserves the orthogonality relation. However, we have the following result.

COROLLARY 1. *Let g be in $L^2_\alpha(\Omega_d)$. Then, for all f,h in $L^2_\alpha(\Omega_d)$, we have*

$$\int_{\Omega_d}\int_{\Omega_d} \mathscr{G}_g f(y,\nu)\overline{\mathscr{G}_g h(y,\nu)}d\mu_\alpha(y)d\mu_\alpha(\nu) = \|g\|^2_{L^2_\alpha(\Omega_d)} \int_{\Omega_d} f(x)\overline{h(x)}d\mu_\alpha(x). \tag{3.25}$$

4. Uncertainty principles of Heisenberg type

In this section we will prove the Heisenberg inequality for the multivariable Bessel Gabor transform.

PROPOSITION 10. (Uncertainty principle of Heisenberg type for $\mathscr{F}_B$) *Let f be in $L^2_\alpha(\Omega_d)$, the following inequality holds*

$$\left(\int_{\Omega_d} ||x||^2 |\mathscr{F}_B(f)(x)|^2 d\mu_\alpha(x)\right)^{\frac{1}{2}} \left(\int_{\Omega_d} ||y||^2 |f(y)|^2 d\mu_\alpha(y)\right)^{\frac{1}{2}} \geqslant (|\alpha|+d)\|f\|^2_{L^2_\alpha(\Omega_d)}.$$

Proof. By a similar arguments as in [4] and using the Heisenberg inequality for the Fourier-Bessel transform on Ω_1, we obtain the result. □

THEOREM 3. (Uncertainty principles of Heisenberg Type for $\mathscr{G}_g$) *Let g be in $L^2_\alpha(\Omega_d)$. Then, for all f in $L^2_\alpha(\Omega_d)$, the following inequality holds*

$$\begin{aligned} &\left(\int_{\Omega_d} ||x||^2 |\mathscr{F}_B f(x)|^2 d\mu_\alpha(x)\right)^{\frac{1}{2}} \left(\int_{\Omega_d}\int_{\Omega_d} ||y||^2 |\mathscr{G}_g f(y,\nu)|^2 d\omega_\alpha(y,\nu)\right)^{\frac{1}{2}} \\ &\geqslant (|\alpha|+d)\|g\|_{L^2_\alpha(\Omega_d)}\|f\|^2_{L^2_\alpha(\Omega_d)}. \end{aligned} \tag{4.26}$$

Proof. Let us assume the non-trivial case that both integrals on the left hand side of (4.26) are finite. Fixing ν arbitrary, Heisenberg's inequality for the multivariable Bessel transform gives that

$$\begin{aligned} &\left(\int_{\Omega_d} ||y||^2 |\mathscr{F}_B(\mathscr{G}_g f(.,\nu))(y)|^2 d\mu_\alpha(y)\right)^{\frac{1}{2}} \left(\int_{\Omega_d} ||y||^2 |\mathscr{G}_g f(y,\nu)|^2 d\mu_\alpha(y)\right)^{\frac{1}{2}} \\ &\geqslant (|\alpha|+d)\int_{\Omega_d} |\mathscr{G}_g f(y,\nu)|^2 d\mu_\alpha(y). \end{aligned}$$

Integrating over ν and using Cauchy Schwartz inequality we obtain

$$\begin{aligned} &\left(\int_{\Omega_d}\int_{\Omega_d} ||y||^2 |\mathscr{F}_B(\mathscr{G}_g f(.,\nu))(y)|^2 d\mu_\alpha(y)d\mu_\alpha(\nu)\right)^{\frac{1}{2}} \\ &\times\left(\int_{\Omega_d}\int_{\Omega_d} ||y||^2 |\mathscr{G}_g f(y,\nu)|^2 d\omega_\alpha(y,\nu)\right)^{\frac{1}{2}} \\ &\geqslant (|\alpha|+d)\int_{\Omega_d}\int_{\Omega_d} |\mathscr{G}_g f(y,\nu)|^2 d\omega_\alpha(y,\nu). \end{aligned}$$

Thus, using the fact that

$$\int_{\Omega_d}\int_{\Omega_d}||y||^2|\mathscr{F}_B(\mathscr{G}_g f(.,\nu))(y)|^2 d\mu_\alpha(y)d\mu_\alpha(\nu)=\|g\|^2_{L^2_\alpha(\Omega_d)}\int_{\Omega_d}||y||^2|\mathscr{F}_B f(y)|^2 d\mu_\alpha(y),$$

we obtain

$$\|g\|_{L^2_\alpha(\Omega_d)}\left(\int_{\Omega_d}||x||^2|\mathscr{F}_B f(x)|^2 d\mu_\alpha(x)\right)^{\frac{1}{2}}\left(\int_{\Omega_d}\int_{\Omega_d}||y||^2|\mathscr{G}_g f(y,\nu)|^2 d\omega_\alpha(y,\nu)\right)^{\frac{1}{2}}$$
$$\geqslant (|\alpha|+d)\int_{\Omega_d}\int_{\Omega_d}|\mathscr{G}_g f(y,\nu)|^2 d\omega_\alpha(y,\nu)=(|\alpha|+d)\|g\|^2_{L^2_\alpha(\mathbb{R})}\|f\|^2_{L^2_\alpha(\mathbb{R})}.$$

This proves the result. □

5. Reproducing kernel

COROLLARY 2. (Reproducing kernel) *Let g be in $(L^2_\alpha(\Omega_d)\cap L^\infty_\alpha(\Omega_d))\setminus\{0\}$. Then, $\mathscr{G}_g(L^2_\alpha(\Omega_d))$ is a reproducing kernel Hilbert space in X^2_α with kernel function*

$$\mathscr{W}_g(y',\nu';y,\nu):=\frac{1}{\|g\|^2_{L^2_\alpha(\Omega_d)}}\int_{\Omega_d}\tau_y g_\nu(x)\tau_{y'}g_{\nu'}(x)d\mu_\alpha(x)$$

$$:=\frac{1}{\|g\|^2_{L^2_\alpha(\Omega_d)}}\tau_y g_\nu *_B g_{\nu'}(y'). \tag{5.27}$$

The kernel is pointwise bounded:

$$|\mathscr{W}_g(y',\nu';y,\nu)|\leqslant 1;\qquad \text{for all}\quad (y',\nu'),(y,\nu)\in\Omega_d\times\Omega_d. \tag{5.28}$$

Proof. We have

$$\mathscr{G}_g f(y,\nu)=\int_{\Omega_d}f(x)\overline{g_{\nu,y}(x)}d\mu_\alpha(x).$$

Using the relation (3.25), we obtain

$$\mathscr{G}_g f(y,\nu)=\frac{1}{\|g\|^2_{L^2_\alpha(\Omega_d)}}\int_{\Omega_d}\int_{\Omega_d}\mathscr{G}_g f(x,\nu')\overline{\mathscr{G}_g(g_{\nu,y})(x',\nu')}d\mu_\alpha(x)d\mu_\alpha(\nu).$$

On the other hand using Proposition 6, one can easily see that for every $y,\nu,\nu'\in\Omega_d$, the function

$$x\longmapsto\frac{1}{\|g\|^2_{L^2_\alpha(\Omega_d)}}\overline{\mathscr{G}_g(g_{\nu,y})(x,\nu')}=\frac{1}{\|g\|^2_{L^2_\alpha(\Omega_d)}}\tau_y g_\nu *_B g_{\nu'}(x)$$

belongs to $L^2_\alpha(\Omega_d)$. Therefore, the result is obtained. □

In the following theorem, we will show that the portion of the multivariable Bessel Gabor transform lying outside some sufficiently small set of finite measure cannot be arbitrarily too small. Then, in order to prove a concentration result of the multivariable Bessel Gabor transform, we need the following notations:

$P_g : X^2_\alpha \longrightarrow X^2_\alpha$ the orthogonal projection from X^2_α onto $\mathscr{G}_g(L^2_\alpha(\Omega_d))$.

$P_U : X^2_\alpha \longrightarrow X^2_\alpha$ the orthogonal projection from X^2_α onto the subspace of function supported in the subset $U \subset \Omega_d \times \Omega_d$ with $\omega_\alpha(U) < \infty$.

We put

$$\|P_U P_g\| = \sup\Big\{\|P_U P_g v\|_{2,\mu_\alpha},\ v \in X^2_\alpha;\ \|v\|_{2,\mu_\alpha} = 1\Big\}. \tag{5.29}$$

The essential result of this section is the following.

THEOREM 4. (Concentration of $\mathscr{G}_g f$ in small sets) *Let g be in $(L^2_\alpha(\Omega_d)\cap L^\infty_\alpha(\Omega_d)) \setminus \{0\}$ and $U \subset \Omega_d \times \Omega_d$ with $w_\alpha(U) < 1$. Then, for all f in $L^2_\alpha(\Omega_d)$ we have*

$$\|\mathscr{G}_g f - \chi_U \mathscr{G}_g f\|_{2,\mu_\alpha} \geqslant \Big(1 - \sqrt{w_\alpha(U)}\Big)\|g\|_{L^2_\alpha(\Omega_d)}\|f\|_{L^2_\alpha(\Omega_d)}. \tag{5.30}$$

Proof. From the definition of P_U and P_g we have

$$\|\mathscr{G}_g f - \chi_U \mathscr{G}_g f\|_{2,\mu_\alpha} = \|(I - P_U P_g)\mathscr{G}_g f\|_{2,\mu_\alpha}.$$

Then, using the Proposition 9 we get

$$\|\mathscr{G}_g f - \chi_U \mathscr{G}_g f\|_{2,\mu_\alpha} \geqslant \|\mathscr{G}_g f\|_{2,\mu_\alpha}(1 - \|P_U P_g\|) \tag{5.31}$$

$$\geqslant \|g\|_{L^2_\alpha(\Omega_d)}\|f\|_{L^2_\alpha(\Omega_d)}(1 - \|P_U P_g\|). \tag{5.32}$$

As P_g is a projection onto a reproducing kernel Hilbert space, then, from Saitoh [5], P_g can be represented by

$$P_g F(y,\nu) = \int_{\Omega_d\times\Omega_d} F(y',\nu')\mathscr{W}_g(y',\nu';y,\nu)dw_\alpha(y',\nu'),$$

with $\mathscr{W}_g$ defined by (5.27). Hence, for $F \in X^2_\alpha$ arbitrary, we have

$$P_U P_g F(y,\nu) = \int_{\Omega_d\times\Omega_d} \chi_U(y,\nu)F(y',\nu')\mathscr{W}_g(y',\nu';y,\nu)d\omega_\alpha(y',\nu')$$

and its Hilbert-Schmidt norm

$$\|P_U P_g\|_{HS} = \Big(\int_{\Omega_d^2\times\Omega_d^2} |\chi_U(y,\nu)|^2|\mathscr{W}_g(y',\nu';y,\nu)|^2 d\omega_\alpha(y',\nu')d\omega_\alpha(y,\nu)\Big)^{\frac{1}{2}}.$$

By the Cauchy-Schwartz inequality we see that

$$\|P_U P_g\|_{HS} \geqslant \|P_U P_g\|. \tag{5.33}$$

On the other hand, from (5.27) and Fubini's theorem, it is easy to see that

$$\|P_U P_g\|_{HS} \leqslant \sqrt{w_\alpha(U)}. \tag{5.34}$$

Thus, from the relations (5.32),(5.33) and (5.34) we obtain the result. $\square$

6. Extremal functions on the generalized Sobolev spaces

6.1. Reproducing kernel

In this subsection we give reproducing kernel for the generalized Sobolev spaces associated with the multivariable Bessel operator.

Let $s\in\mathbb{R}$. We define the space $H^s_\alpha(\Omega_d)$ by

$$H^s_\alpha(\Omega_d) := \Big\{f\in L^2_\alpha(\Omega_d)\,:\,(1+||\xi||^2)^{s/2}\mathscr{F}_B(f)\in L^2_\alpha(\Omega_d)\Big\}.$$

The space $H^s_\alpha(\Omega_d)$ provided with the inner product

$$\langle f,g\rangle_{H^s_\alpha(\Omega_d)} = \int_{\Omega_d}(1+||\xi||^2)^s\mathscr{F}_B(f)(\xi)\overline{\mathscr{F}_B(g)}(\xi)d\mu_\alpha(\xi), \tag{6.35}$$

and the norm $\|f\|^2_{H^s_\alpha(\Omega_d)} = \langle f,f\rangle_{H^s_\alpha(\Omega_d)}$, is a Hilbert space.

PROPOSITION 11. *For $s>\dfrac{d}{2}$, the Hilbert space $H^s_\alpha(\Omega_d)$ admits the following reproducing kernel:*

$$K_s(x,y) = \int_{\Omega_d}\frac{\Lambda(\xi,x)\Lambda(\xi,y)d\mu_\alpha(\xi)}{(1+||\xi||^2)^s}$$

that is

(i) For all $y\in\Omega_d$, the function $x\mapsto K_s(x,y)$ belongs to $H^s_\alpha(\Omega_d)$.

(ii) The reproducing property: for all $f\in H^s_\alpha(\Omega_d)$ and $y\in\Omega_d$,

$$f(y) = \langle f,K_s(x,y)\rangle_{H^s_\alpha(\Omega_d)}.$$

Proof. It can be easily observed, from the properties of the normalized Bessel function j_α, that the function

$$(\xi,x)\longmapsto\left[\prod_{i=1}^d(\xi_ix_i)^{\alpha_i+\frac12}\right]\Lambda(\xi,x),\ (\xi,x)\in\Omega_d\times\Omega_d, \tag{6.36}$$

is bounded.

i) It is clear that, for $y\in\Omega_d$, the function

$$\xi\longmapsto\frac{\Lambda(\xi,y)}{(1+||\xi||^2)^s} \tag{6.37}$$

belongs to $L^1_\alpha(\Omega_d)\cap L^2_\alpha(\Omega_d)$. Then the function $K_s(.,y)$ is well defined and by (2.6) we have

$$K_s(.,y) = \mathscr{F}_B^{-1}\Big(\frac{\Lambda(.,y)}{(1+||.||^2)^s}\Big).$$

From Theorem 1 (ii), it follows that $K_s(.,y)$ belongs to $L^2_\alpha(\Omega_d)$, and we have

$$\mathscr{F}_B\Big(K_s(.,y)\Big) = \frac{\Lambda(.,y)}{(1+||.||^2)^s}. \tag{6.38}$$

Therefore, by using relations (6.36) and (6.38), we obtain

$$||K_s(.,y)||^2_{H^s_\alpha(\Omega_d)} \leqslant C^2\left[\prod_{i=1}^d y_i^{-2\alpha_i-1}\right]\int_{\Omega_d}\frac{d\xi}{(1+||\xi||^2)^s} < \infty.$$

This proves that for all $y\in\Omega_d$ the function $K_s(.,y)$ belongs to $H^s_\alpha(\Omega_d)$.

(ii) Let f be in $H^s_\alpha(\Omega_d)$ and y in Ω_d. Then by (6.35) and (6.38) we get

$$\langle f, K_s(.,y)\rangle_{H^s_\alpha(\Omega_d)} = \int_{\Omega_d}\mathscr{F}_B(f)(\xi)\Lambda(\xi,y)d\mu_\alpha(\xi), \tag{6.39}$$

and from inversion formula, we obtain the reproducing property

$$f(y) = \langle f, K_s(x,y)\rangle_{H^s_\alpha(\Omega_d)}.$$

This completes the proof of the theorem. □

COROLLARY 3. *For $s > \frac{d}{2}$, the Hilbert space $H^s_\alpha(\Omega_d)$ is embedded in $C_*(\Omega_d)$.*

REMARK 2. From the proof of Proposition 11 and using inequality (2.2), it can be observed that for all $s > \frac{d}{2}$ there exists a positive constant C_s such that

$$||K_s(.,y)||^2_{H^s_\alpha(\Omega_d)} \leqslant C^2_s(y) = \begin{cases} C_s\left[\prod_{i=1}^d y_i^{-2\alpha_i-1}\right] & \text{for } \frac{d}{2} < s \leqslant |\alpha|+d \\ C_s & \text{for } |\alpha|+d < s. \end{cases}$$

PROPOSITION 12. *Let g be a function in $L^2_\alpha(\Omega_d)\bigcap L^\infty_\alpha(\Omega_d)$, and $\nu\in\Omega_d$. The integral transform $\mathscr{G}_g(.,\nu)$, is a bounded linear operator from $H^s_\alpha(\Omega_d)$, s in $\mathbb{R}_+$, into $L^2_\alpha(\Omega_d)$, and we have*

$$||\mathscr{G}_g f(.,\nu)||_{L^2_\alpha(\Omega_d)} \leqslant ||g||_{L^\infty_\alpha(\Omega_d)}||f||_{H^s_\alpha(\Omega_d)}.$$

Proof. Let f be in $H^s_\alpha(\Omega_d)$. Using Theorem 1 (ii) we have

$$||\mathscr{G}_g f(.,\nu)||^2_{L^2_\alpha(\Omega_d)} = ||\mathscr{F}_B(\mathscr{G}_g f(.,\nu))||^2_{L^2_\alpha(\Omega_d)}.$$

Invoking the relation ships (3.19) and (3.20) we can write

$$||\mathscr{G}_g f(.,\nu)||^2_{L^2_\alpha(\Omega_d)} = \int_{\Omega_d}|\mathscr{F}_B(f)(\xi)|^2\tau_\nu(|g|^2)(\xi)d\mu_\alpha(\xi).$$

Therefore

$$||\mathscr{G}_g f(.,\nu)||_{L^2_\alpha(\Omega_d)} \leqslant ||g||_{L^\infty_\alpha(\Omega_d)}||f||_{H^s_\alpha(\Omega_d)}. \quad \square$$

DEFINITION 5. Let g be a function in $L^2_\alpha(\Omega_d)\bigcap L^\infty_\alpha(\Omega_d)$. Let $r>0$, $\nu\in\Omega_d$ and $s\in\mathbb{R}_+$. We define the Hilbert space $H^{r,s}_\alpha(\Omega_d)$ as the subspace of $H^s_\alpha(\Omega_d)$ with the inner product:

$$\langle f,h\rangle_{H^{r,s}_\alpha}=r\langle f,h\rangle_{H^s_\alpha(\Omega_d)}+\langle \mathscr{G}_g f(.,\nu),\mathscr{G}_g h(.,\nu)\rangle_{L^2_\alpha(\Omega_d)},\quad f,h\in H^s_\alpha(\Omega_d).$$

The norm associated to the inner product is defined by:

$$\|f\|^2_{H^{r,s}_\alpha}:=r\|f\|^2_{H^s_\alpha(\Omega_d)}+\|\mathscr{G}_g f(.,\nu)\|^2_{L^2_\alpha(\Omega_d)}.$$

PROPOSITION 13. *Let g be a function in $L^2_\alpha(\Omega_d)\bigcap L^\infty_\alpha(\Omega_d)$. For $s>\frac{d}{2}$, the Hilbert space $H^{r,s}_\alpha(\Omega_d)$ admits the following reproducing kernel:*

$$\mathscr{K}_{g,r}(x,y)=\int_{\Omega_d}\frac{\Lambda(\xi,x)\Lambda(\xi,y)d\mu_\alpha(\xi)}{r(1+||\xi||^2)^s+\tau_\nu(|g|^2)(\xi)}.$$

Proof. i) Let y be in Ω_d. Using inequality (6.36) and Theorem 1 (ii) it can be deduced, as in Proposition 11, that there exists a function $x\mapsto\mathscr{K}_{g,r}(x,y)$ belongs to $L^2_\alpha(\Omega_d)$ such that we have

$$\mathscr{F}_B\Big(\mathscr{K}_{g,r}(.,y)\Big)=\frac{\Lambda(.,y)}{r(1+||.||^2)^s+\tau_\nu(|g|^2)}.\tag{6.40}$$

On the other hand we have

$$\mathscr{F}_B\Big(\mathscr{G}_g(\mathscr{K}_{g,r}(.,y))(.,\nu)\Big)=\sqrt{\tau_\nu(|g|^2)}\mathscr{F}_B\Big(\mathscr{K}_{g,r}(.,y)\Big)\text{ in }L^2_\alpha(\Omega_d).\tag{6.41}$$

Hence from Theorem 1 (ii), we obtain

$$\begin{aligned}\|\mathscr{G}_g(\mathscr{K}_{g,r}(.,y))(.,\nu)\|^2_{L^2_\alpha(\Omega_d)}&=\int_{\Omega_d}\tau_\nu(|g|^2)(\xi)|\mathscr{F}_B\Big(\mathscr{K}_{g,r}(.,y)\Big)(\xi)|^2d\mu_\alpha(\xi)\\&\leqslant\frac{C}{r^2}\int_{\Omega_d}\frac{\tau_\nu(|g|^2)(\xi)|\Lambda(\xi,y)|^2}{(1+||\xi||^2)^{2s}}d\mu_\alpha(\xi)<\infty.\end{aligned}$$

Therefore we conclude that $\|\mathscr{K}_{g,r}(.,y)\|^2_{H^{r,s}_\alpha}<\infty$.

ii) Let f be in $H^{r,s}_\alpha(\Omega_d)$ and y in Ω_d. Then

$$\langle f,\mathscr{K}_{g,r}(.,y)\rangle_{H^{r,s}_\alpha}=rI_1+I_2,\tag{6.42}$$

where

$$I_1=\langle f,\mathscr{K}_{g,r}(.,y)\rangle_{H^s_\alpha(\Omega_d)}\quad\text{and}\quad I_2=\langle\mathscr{G}_g f(.,\nu),\mathscr{G}_g(\mathscr{K}_{g,r}(.,y))(.,\nu)\rangle_{L^2_\alpha(\Omega_d)}.$$

From (6.35) and (6.40), we have

$$I_1=\int_{\Omega_d}\frac{(1+||\xi||^2)^s\mathscr{F}_B(f)(\xi)\Lambda(\xi,y)d\mu_\alpha(\xi)}{r(1+||\xi||^2)^s+\tau_\nu(|g|^2)(\xi)}.\tag{6.43}$$

From (6.40), (6.41) and Proposition 6, we have

$$I_2 = \int_{\Omega_d} \frac{\tau_v(|g|^2)(\xi)\mathscr{F}_B(f)(\xi)\Lambda(\xi,y)d\mu_\alpha(\xi)}{r(1+||\xi||^2)^s+\tau_v(|g|^2)(\xi)}. \tag{6.44}$$

Using relation (6.42) and combining (6.43) and (6.44), we deduce that

$$\langle f, \mathscr{K}_{g,r}(.,y)\rangle_{H_\alpha^{r,s}} = f(y), \ a.e. \quad \square$$

COROLLARY 4. *The kernel* $\mathscr{K}_{g,r}$ *satisfies the following properties:*
(i) $||\mathscr{K}_{g,r}(.,y)||_{H_\alpha^{r,s}(\Omega_d)} \leqslant \frac{C_s(y)}{r}$.
(ii) $||\mathscr{G}_g\mathscr{K}_{g,r}(.,y)||_{L_\alpha^2(\Omega_d)} \leqslant \frac{C_s(y)}{\sqrt{r}}$.
(iii) $||\mathscr{G}_g^*\mathscr{G}_g\mathscr{K}_{g,r}(.,y)||_{H_\alpha^{r,s}(\Omega_d)} \leqslant C_s(y)$,
where $\mathscr{G}_g^* : L_\alpha^2(\Omega_d) \longrightarrow H_\alpha^s(\Omega_d)$ *is the adjoint operator of* $\mathscr{G}_g$ *given by*

$$\langle \mathscr{G}_g f, h\rangle_{L_\alpha^2(\Omega_d)} = \langle f, \mathscr{G}_g^* h\rangle_{H_\alpha^s(\Omega_d)}, \quad f \in H_\alpha^s(\Omega_d),\ h \in L_\alpha^2(\Omega_d).$$

Proof. From the previous proposition, we have

$$\begin{aligned} f(y) &= \langle f, \mathscr{K}_{g,r}(.,y)\rangle_{H_\alpha^{r,s}}. \\ &= r\langle f, \mathscr{K}_{g,r}(.,y)\rangle_{H_\alpha^s(\Omega_d)} + \langle \mathscr{G}_g f, \mathscr{G}_g\mathscr{K}_{g,r}(.,y)\rangle_{L_\alpha^2(\Omega_d)} \\ &= \langle f, (rI+\mathscr{G}_g^*\mathscr{G}_g)\mathscr{K}_{g,r}(.,y)\rangle_{H_\alpha^s(\Omega_d)}. \end{aligned}$$

Thus,

$$(rI+\mathscr{G}_g^*\mathscr{G}_g)\mathscr{K}_{g,r}(.,y) = K_s(.,y). \tag{6.45}$$

Furthermore, the previous identity implies that

$$\begin{aligned} &r^2||\mathscr{K}_{g,r}(.,y)||^2_{H_\alpha^s(\Omega_d)} + 2r||\mathscr{G}_g\mathscr{K}_{g,r}(.,y)||^2_{L_\alpha^2(\Omega_d)} + ||\mathscr{G}_g^*\mathscr{G}_g\mathscr{K}_{g,r}(.,y)||^2_{H_\alpha^s(\Omega_d)} \\ &= ||K_s(.,y)||^2_{H_\alpha^s(\Omega_d)}. \end{aligned}$$

From this relation and using the fact that

$$||K_s(.,y)||_{H_\alpha^s(\Omega_d)} \leqslant C_s(y),$$

we obtain the properties (i), (ii) and (iii). $\square$

6.2. Tikhonov regularization on $H_\alpha^s(\Omega_d)$

In this subsection we shall give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the approximate solutions for a bounded linear operator equations on the Hilbert spaces $H_\alpha^s(\Omega_d)$.

More precisely, we prove for a given function g in $L^2_\alpha(\Omega_d)\bigcap L^\infty_\alpha(\Omega_d)$ that the infinitum of

$$\left\{r\|f\|^2_{H^s_\alpha(\Omega_d)}+\|h-\mathscr{G}_g f(.,\nu)\|^2_{L^2_\alpha(\Omega_d)}:\ f\in H^s_\alpha(\Omega_d)\right\}$$

is attained at some function denoted by $f^*_{r,h}$, which is unique, called the extremal function. We start it with the following fundamental theorem (cf. [5]).

THEOREM 5. *Let H_K be a Hilbert space admitting the reproducing kernel $K(p,q)$ on a set E and H a Hilbert space. Let $L: H_K\to H$ be a bounded linear operator on H_K into H. For $r>0$, we introduce the inner product in H_K and we call it H_{K_r} as*

$$\langle f_1,f_2\rangle_{H_{K_r}}=r\langle f_1,f_2\rangle_{H_K}+\langle Lf_1,Lf_2\rangle_H.$$

Then:

i) *H_{K_r} is a Hilbert space with the reproducing kernel $K_r(p,q)$ on E and satisfying the equation*

$$K(.,q)=(rI+L^*L)K_r(.,q),$$

where L^* is the adjoint operator of $L: H_K\to H$.

ii) *For any $r>0$ and for any h in H, the infimum*

$$\inf_{f\in H_K}\left\{r\|f\|^2_{H_K}+\|Lf-h\|^2_H\right\}$$

is attained by a unique function $f^*_{r,h}$ in H_K and this extremal function is given by

$$f^*_{r,h}(p)=\langle h,LK_r(.,p)\rangle_H. \tag{6.46}$$

We can now state the main result of this paragraph.

THEOREM 6. *Let g be a function in $L^2_\alpha(\Omega_d)\bigcap L^\infty_\alpha(\Omega_d)$. Let $s>\frac{d}{2}$. For any h in $L^2_\alpha(\Omega_d)$ and for any $r>0$,*

(i) the infimum

$$\inf_{f\in H^s_\alpha(\Omega_d)}\left\{r\|f\|^2_{H^s_\alpha(\Omega_d)}+\|h-\mathscr{G}_g f(.,\nu)\|^2_{L^2_\alpha(\Omega_d)}\right\} \tag{6.47}$$

is attained by a unique function $f^*_{r,h}$ given by

$$f^*_{r,h}(x)=\int_{\Omega_d}h(y)Q_r(x,y)d\mu_\alpha(y), \tag{6.48}$$

where

$$Q_r(x,y)=Q_{r,s}(x,y)=\int_{\Omega_d}\frac{\sqrt{\tau_\nu(|g|^2)(\xi)}\Lambda(\xi,x)\Lambda(\xi,y)}{r(1+||\xi||^2)^s+\tau_\nu(|g|^2)(\xi)}d\mu_\alpha(\xi). \tag{6.49}$$

(ii) The extremal function $f^*_{r,h}$ satisfies the following inequality:

$$|f^*_{r,h}(y)|\leqslant\frac{C_s(y)}{\sqrt{r}}||h||_{L^2_\alpha(\Omega_d)}.$$

Proof. (i) By Proposition 13 and Theorem 5 ii), the infimum given by (6.47) is attained by a unique function $f^*_{r,h}$, and the extremal function $f^*_{r,h}$ is represented by

$$f^*_{r,h}(y) = \langle h, \mathscr{G}_g(\mathscr{K}_{g,r}(.,y))(.,\nu)\rangle_{L^2_\alpha(\Omega_d)}, \quad y \in \Omega_d,$$

where $\mathscr{K}_{g,r}$ is the kernel given by Proposition 13.

On the other hand we have

$$\mathscr{G}_g f(x,\nu) = \int_{\Omega_d} \sqrt{\tau_\nu(|g|^2)(\xi)}\mathscr{F}_B(f)(\xi)\Lambda(\xi,x)d\mu_\alpha(\xi), \quad \text{for all } x \in \Omega_d.$$

Hence by (6.40), we obtain

$$\begin{aligned}\mathscr{G}_g\Big(\mathscr{K}_{g,r}(.,y)\Big)(.,\nu)(x) &= \int_{\Omega_d} \frac{\sqrt{\tau_\nu(|g|^2)(\xi)}\Lambda(\xi,x)\Lambda(\xi,y)}{r(1+||\xi||^2)^s + \tau_\nu(|g|^2)(\xi)} d\mu_\alpha(\xi) \\ &= Q_r(x,y).\end{aligned}$$

This gives (6.49).

(ii) From Corollary 4 (ii), we have

$$|f^*_{r,h}(y)| \leqslant ||h||_{L^2_\alpha(\Omega_d)} ||\mathscr{G}_g\Big(\mathscr{K}_{g,r}(.,y)\Big)||_{L^2_\alpha(\Omega_d)} \leqslant \frac{C_s(y)}{\sqrt{r}}||h||_{L^2_\alpha(\Omega_d)}.$$

Thus the theorem is proved. □

COROLLARY 5. *Let g be a function in $L^2_\alpha(\Omega_d)\bigcap L^\infty_\alpha(\Omega_d)$, $s > \frac{d}{2}$; $r,\delta > 0$ and h, h_δ in $L^2_\alpha(\Omega_d)$ such that*

$$||h - h_\delta||_{L^2_\alpha(\Omega_d)} \leqslant \delta.$$

Then

$$||f^*_{r,h} - f^*_{r,h_\delta}||_{H^s_\alpha(\Omega_d)} \leqslant \frac{\delta}{2\sqrt{r}}.$$

Proof. It is clear, from (6.49), that

$$Q_r(x,y) = \mathscr{F}_B\left(\frac{\sqrt{\tau_\nu(|g|^2)}\Lambda(.,x)}{r(1+||.||^2)^s + \tau_\nu(|g|^2)}\right)(y)$$

then, using the fact that

$$\int_{\Omega_d} h(y)\mathscr{F}_B(Q_r(x,.))(y)d\mu_\alpha(y) = \int_{\Omega_d} \mathscr{F}_B(h)(y)Q_r(x,y)d\mu_\alpha(y)$$

it follows, by (6.48), that

$$f^*_{r,h}(x) = \int_{\Omega_d} \frac{\sqrt{\tau_\nu(|g|^2)(\xi)}\mathscr{F}_B(h)(\xi)\Lambda(\xi,x)}{r(1+||\xi||^2)^s + \tau_\nu(|g|^2)(\xi)} d\mu_\alpha(\xi)$$

and so

$$\mathscr{F}_B(f^*_{r,h})(\xi) = \frac{\sqrt{\tau_\nu(|g|^2)(\xi)}\mathscr{F}_B(h)(\xi)}{r(1+||\xi||^2)^s + \tau_\nu(|g|^2)(\xi)}. \tag{6.50}$$

Hence

$$\mathscr{F}_B(f^*_{r,h} - f^*_{r,h_\delta})(\xi) = \frac{\sqrt{\tau_\nu(|g|^2)(\xi)}\mathscr{F}_B(h-h_\delta)(\xi)}{r(1+||\xi||^2)^s + \tau_\nu(|g|^2)(\xi)}.$$

Using the inequality $(x+y)^2 \geqslant 4xy$, we obtain

$$(1+||\xi||^2)^s \left|\mathscr{F}_B(f^*_{r,h} - f^*_{r,h_\delta})(\xi)\right|^2 \leqslant \frac{1}{4r}|\mathscr{F}_B(h-h_\delta)(\xi)|^2.$$

Thus, and from Theorem 1 (ii), we obtain

$$\|f^*_{r,h} - f^*_{r,h_\delta}\|^2_{H^s_\alpha(\Omega_d)} \leqslant \frac{1}{4r}\|\mathscr{F}_B(h-h_\delta)\|^2_{L^2_\alpha(\Omega_d)} = \frac{1}{4r}\|h-h_\delta\|^2_{L^2_\alpha(\Omega_d)},$$

which gives the desired result. □

COROLLARY 6. *Let g be a function in $L^2_\alpha(\Omega_d)\bigcap L^\infty_\alpha(\Omega_d)$. Let $s > \frac{d}{2}$ and $r > 0$. If f is in $H^s_\alpha(\Omega_d)$ and $h = \mathscr{G}_g f(.,\nu)$. Then*

$$\|f^*_{r,h} - f\|^2_{H^s_\alpha(\Omega_d)} \to 0 \quad \text{as } r \to 0.$$

Proof. From (6.49), we have

$$\mathscr{F}_B(f^*_{r,h})(\xi) = \frac{\sqrt{\tau_\nu(|g|^2)(\xi)}\mathscr{F}_B(h)(\xi)}{r(1+||\xi||^2)^s + \tau_\nu(|g|^2)(\xi)}.$$

Hence

$$\mathscr{F}_B(f^*_{r,h} - f)(\xi) = \frac{-r(1+||\xi||^2)^s\mathscr{F}_B(f)(\xi)}{r(1+||\xi||^2)^s + \tau_\nu(|g|^2)(\xi)}.$$

Then we obtain

$$\|f^*_{r,h} - f\|^2_{H^s_\alpha(\Omega_d)} = \int_{\mathbb{R}^d_+} h_{r,t,s}(\xi)|\mathscr{F}_B(f)(\xi)|^2 d\mu_\alpha(\xi),$$

with

$$h_{r,t,s}(\xi) = \frac{r^2(1+||\xi||^2)^{3s}}{\Big(r(1+||\xi||^2)^s + \tau_\nu(|g|^2)(\xi)\Big)^2}.$$

Since

$$\lim_{r\to 0} h_{r,t,s}(\xi) = 0$$

and

$$|h_{r,t,s}(\xi)| \leqslant (1+||\xi||^2)^s,$$

we obtain the result from the dominated convergence theorem. □

COROLLARY 7. *Let g be a function in $L^2_\alpha(\Omega_d)\bigcap L^\infty_\alpha(\Omega_d)$. Let $s>\frac{d}{2}$ and $r>0$. If f is in $H^s_\alpha(\Omega_d)$ and $h=\mathscr{G}_g f(.,\nu)$. Then*

(i) $f(y)=\lim_{r\to 0^+} f^*_{r,h}(y)$.

(ii) $|f(y)-f^*_{r,h}(y)|\leqslant C_s(y)||f||_{H^s_\alpha(\Omega_d)}$.

(iii) $|f^*_{r,h}(y)|\leqslant C_s(y)||f||_{H^s_\alpha(\Omega_d)}$.

Proof. Let f is in $H^s_\alpha(\Omega_d)$.

(i) Then

$$f^*_{r,h}(y)=\langle f,\mathscr{G}_g^*\mathscr{G}_g\mathscr{K}_{g,r}(.,y)\rangle_{H^s_\alpha(\Omega_d)}. \tag{6.51}$$

But from (6.45), we have

$$\lim_{r\to 0^+}\mathscr{G}_g^*\mathscr{G}_g\mathscr{K}_{g,r}(.,y)=K_s(.,y).$$

Thus

$$\lim_{r\to 0^+} f^*_{r,h}(y)=\langle f,K_s(.,y)\rangle_{H^s_\alpha(\Omega_d)}=f(y).$$

(ii) From (6.45) and (6.51), the extremal function $f^*_{r,h}$ satisfies

$$f^*_{r,h}(y)=f(y)-r\langle f,\mathscr{K}_{g,r}(.,y)\rangle_{H^s_\alpha(\Omega_d)}.$$

Thus and by Corollary 4 (i) we obtain

$$|f^*_{r,h}(y)-f(y)|\leqslant r||f||_{H^s_\alpha(\Omega_d)}||\mathscr{K}_{g,r}(.,y)||_{H^s_\alpha(\Omega_d)}\leqslant C_s(y)||f||_{H^s_\alpha(\Omega_d)}.$$

(iii) From (6.51) and Corollary 4 (iii), the extremal function $f^*_{r,h}$ satisfies

$$|f^*_{r,h}(y)|\leqslant ||f||_{H^s_\alpha(\Omega_d)}||\mathscr{G}_g^*\mathscr{G}_g\mathscr{K}_{g,r}(.,y)||_{H^s_\alpha(\Omega_d)}\leqslant C_s(y)||f||_{H^s_\alpha(\Omega_d)}. \quad \square$$

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REFERENCES

[1] L. P. CASTRO, S. SAITOH, Y. SAWANO, A. M. SIMOES, *General Inhomogeneous Discrete Linear Partial Differential Equations with Constant Coefficients on the Whole Spaces*, Complex Analysis and Operator Theory **05** (2012) 6 (1) 307–324.

[2] C. CHETTAOUI AND Y. OTHMANI, *Real Paley-Wiener theorems for the multivariable Bessel transform*, Int. J. Open Problems Complex Analysis, Vol. 6, No. 1, March (2014) 90–110.

[3] D. GABOR, *Theory of communication*, JIEE, 93(26), Part III, (1946), 429–457.

[4] M. RÖSLER AND M. VOIT, *An uncertainty principle for Hankel transforms*, Proc. Amer. Math. Soc. **127** (1999), 183–194.

[5] S. SAITOH, *Theory of reproducing kernels and its applications*, Longman Scientific Technical, Harlow, (1988).

[6] K. TRIMÈCHE, *Generalized harmonic analysis and wavelet packets*, Gordon and Breach Sciences Publishers, (2001).

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