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A SURVEY ON THE BÖTTCHER–WENZEL CONJECTURE AND RELATED PROBLEMS

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Abstract. A fundamental fact in matrix theory is that the matrix multiplication is not commutative, i.e., there are square matrices X and Y such that $XY \neq YX$. The difference $XY - YX$ is called the commutator (or Lie product) of X and Y. The commutator plays an important role in diverse areas in mathematics, for instance, Lie group and Lie algebra theory, perturbation analysis, and matrix manifold computation. Böttcher and Wenzel proposed the following conjecture in 2005 : for any real $n \times n$ matrices X and Y,

$$\|XY - YX\|_F \leqslant \sqrt{2}\,\|X\|_F \|Y\|_F,$$

where $\|\cdot\|_F$ is the Frobenius norm. This survey is concerned with the proofs of this conjecture and the study of its related problems.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively. Let $\mathbf{i} \equiv \sqrt{-1}$ and $\mathrm{Re}(z)$ denote the real part of $z \in \mathbb{C}$. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $M_n(\mathbb{F})$ denote the vector space of all $n \times n$ matrices with entries in $\mathbb{F}$. Let $E_{jk} \in M_n(\mathbb{F})$ denote the matrix whose (j,k)-th entry equals 1 and all other entries equal 0. Let I_n denote the identity matrix of order n and $\mathbf{0}$ denote the zero matrix of appropriate order. For $X \in M_n(\mathbb{F})$, the conjugate, transpose and conjugate transpose of X are denoted by $\overline{X}$, X^T and X^* respectively. We also use the following notation

$$[X,Y] \equiv XY - YX$$

to denote the commutator of $X, Y \in M_n(\mathbb{F})$. Let $\lambda_1(X) \geqslant \cdots \geqslant \lambda_n(X)$ denote the eigenvalues of X when they are all real, and let $s_1(X) \geqslant \cdots \geqslant s_n(X)$ denote the singular values of X. The vector space $M_n(\mathbb{F})$ is equipped with the usual inner product $\langle X, Y\rangle = \mathrm{tr}\,(Y^*X)$, where $\mathrm{tr}\,(X)$ denotes the trace of X, and the Frobenius norm is given by $\|X\|_F = \langle X, X\rangle^{\frac{1}{2}}$. For $X \in M_n(\mathbb{F})$, the (p,k)-norm, where $1 \leqslant k \leqslant n$ and $1 \leqslant p \leqslant \infty$, is defined by

$$\|X\|_{(k),p} = \left(s_1^p(X) + \cdots + s_k^p(X)\right)^{\frac{1}{p}}.$$

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When $k=n$, $\|\cdot\|_{(n),p}$ is the Schatten p-norm and is denoted by $\|\cdot\|_p$. In particular,

$$\|\cdot\|_{(n),2} = \|\cdot\|_2 = \|\cdot\|_F.$$

When $p=\infty$, $\|X\|_{(k),p} = s_1(X)$ for $k=1,\ldots,n$.

The commutator plays an important role in diverse areas in mathematics, for instance, Lie group and Lie algebra theory, perturbation analysis, operator theory, and matrix manifold computation [1, 3, 4, 13, 14, 25]. In 2005, Böttcher and Wenzel proposed the following conjecture in [5]: the upper bound of the Frobenius norm of the commutator of any $X,Y \in M_n(\mathbb{R})$ is given by

$$\|XY-YX\|_F \leqslant \sqrt{2}\,\|X\|_F\|Y\|_F. \tag{1.1}$$

Note that the constant $\sqrt{2}$ is the best possible as shown by the following simple example:

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Böttcher and Wenzel proved this inequality for the case of 2-by-2 matrices [5]. Later, László proved the conjecture for the case of 3-by-3 matrices [15]. The conjecture was first proved for general n-by-n real matrices in 2007 by Vong and Jin, and their paper [21] was published in 2008. A few months later, Lu [18] independently gave a different proof and the result is now included in [17]. Böttcher and Wenzel [6] extended the result to complex matrices. Some other alternative proofs of the conjecture can be found in [2, 19]

In mathematics, a solution of a conjecture usually leads to new subsequent problems. After the affirmation of the Böttcher-Wenzel conjecture, there were several subsequent problems considered, namely,

(I) the maximal pairs of the inequality;

(II) the use of other norms in the inequality;

(III) the sharpening of the inequality;

(IV) the extension to other products similar to the commutator.

In fact, the first two considerations were raised by Böttcher and Wenzel [6] (see also [22]).

In this paper, we survey results concerned with the Böttcher-Wenzel conjecture and related problems. Some recent development and open problems will also be mentioned.

2. The proofs of the Böttcher-Wenzel conjecture

In this section, we introduce different proofs of (1.1). Following the notation used in [6], let $X=USV$ be the singular value decomposition of X where $U,V \in M_n(\mathbb{C})$ are unitary and $S=\mathrm{diag}(s_1,\ldots,s_n)$ with $s_i = s_i(X)$ for $i=1,\ldots,n$. Put

$$C=[c_{jk}] = VYV^*, \qquad D=[d_{jk}] = U^*YU.$$

Then

$$\|XY-YX\|_F^2 = \|SC-DS\|_F^2 = \sum_{j,k=1}^{n} |s_j c_{jk} - d_{jk} s_k|^2 \tag{2.1}$$

$$\leqslant \sum_{j\neq k} \left(s_j^2|c_{jk}|^2+s_k^2|c_{jk}|^2+s_j^2|d_{jk}|^2+s_k^2|d_{jk}|^2\right) + \sum_{j=1}^{n} s_j^2|c_{jj}-d_{jj}|^2 \tag{2.2}$$

$$= \sum_{j=1}^{n} s_j^2 \Delta_j, \tag{2.3}$$

where

$$\Delta_j = |c_{jj}-d_{jj}|^2 + \sum_{k\neq j} \left(|c_{jk}|^2+|c_{kj}|^2+|d_{jk}|^2+|d_{kj}|^2\right).$$

The main results in Vong and Jin [21] and in Böttcher and Wenzel [6] are of showing that

$$\Delta_j \leqslant 2\|Y\|_F^2, \quad j=1,\dots,n. \tag{2.4}$$

Then (1.1) follows. Vong and Jin used an inequality approach. The main tools are the Cauchy-Schwarz inequality and the Lagrange identity, and the computation is highly technical. Böttcher and Wenzel's proof is more matrix-oriented.

Lu [17] adopted an operator approach to prove (1.1) (for real matrices). Suppose X is fixed and $\|X\|_F = 1$. Then

$$\|XY-YX\|_F^2 = \langle XY-YX, XY-YX\rangle = \langle T_X(Y), Y\rangle \leqslant \lambda_1(T_X)\|Y\|_F^2,$$

where $T_X : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is the positive semidefinte operator given by

$$T_X(Y) = [X^T, [X,Y]]. \tag{2.5}$$

The problem is then to find an upper bound of $\lambda_1(T_X)$. In his proof, a very special property is used: the geometric multiplicity of $\lambda_1(T_X)$ is at least two. This is a special property of the commutator and can be deduced as follows: suppose Y is an eigenvector of T_X with respect to $\lambda_1(T_X)$, i.e.,

$$T_X(Y) = \lambda_1(T_X)Y.$$

Then, by direct verification,

$$T_X\left([X^T, Y^T]\right) = \lambda_1(T_X)[X^T, Y^T]$$

and that Y and $[X^T, Y^T]$ are linearly independent. This observation allows an eigenvector to be chosen such that its (1,1)-th entry is zero, which is crucial in the proof.

Now, we state the Böttcher-Wenzel conjecture as a theorem below and include a complete proof provided in [2]. This proof is more elementary and easier to be understood than those proofs provided in [6, 17, 21]. We begin with a lemma.

LEMMA 2.1. *Let $p_j \geqslant 0$ for $j=1,2,\ldots,n$ with $\sum_{j=1}^{n} p_j = 1$, and let $z_j \in \mathbb{R}$ for $j=1,2,\ldots,n$. Then*

$$\sum_{j=1}^{n} p_j z_j^2 - \Big(\sum_{j=1}^{n} p_j z_j\Big)^2 \leqslant \sum_{j=1}^{n} \frac{z_j^2}{2}.$$

Proof. From direct calculations, we have

$$\sum_{j=1}^{n} p_j z_j^2 - \Big(\sum_{j=1}^{n} p_j z_j\Big)^2 = \sum_{j=1}^{n} p_j \Big[z_j - \big(\sum_{k=1}^{n} p_k z_k\big)\Big]^2.$$

Assuming that $z_1 \geqslant z_2 \geqslant \cdots \geqslant z_n$ and denoting $d = \frac{1}{2}(z_1+z_n) - \sum_{j=1}^{n} p_j z_j$, we have

$$\begin{aligned}
\sum_{j=1}^{n} p_j z_j^2 - \Big(\sum_{j=1}^{n} p_j z_j\Big)^2 &\leqslant \sum_{j=1}^{n} p_j \Big[z_j - \big(\sum_{k=1}^{n} p_k z_k\big)\Big]^2 + d^2 \\
&= \sum_{j=1}^{n} p_j \Big[z_j - \big(\sum_{k=1}^{n} p_k z_k\big)\Big]^2 - 2d \sum_{j=1}^{n} p_j \Big(z_j - \sum_{k=1}^{n} p_k z_k\Big) + d^2 \\
&= \sum_{j=1}^{n} p_j \Big[z_j - \big(\sum_{k=1}^{n} p_k z_k\big) - d\Big]^2 = \sum_{j=1}^{n} p_j \Big(\frac{z_j - z_n}{2} - \frac{z_1 - z_j}{2}\Big)^2 \\
&\leqslant \sum_{j=1}^{n} p_j \Big(\frac{z_1 - z_n}{2}\Big)^2 \leqslant \frac{1}{4}(2z_1^2 + 2z_n^2) \leqslant \sum_{j=1}^{n} \frac{z_j^2}{2}. \quad \square
\end{aligned}$$

THEOREM 2.2. *For any matrices $X, Y \in M_n(\mathbb{C})$, we have*

$$\|XY - YX\|_F^2 \leqslant 2\, \|X\|_F^2 \|Y\|_F^2. \tag{2.6}$$

Proof. If $X = \mathbf{0}$, then (2.6) holds obviously. Now suppose $X \neq \mathbf{0}$ and thus $\|X\|_F > 0$. For any $V, W \in M_n(\mathbb{C})$ in the following, we use the property $\mathrm{tr}(VW) = \mathrm{tr}(WV)$ repeatedly. We have

$$\begin{aligned}
\|XY - YX\|_F^2 &= \mathrm{tr}[(XY - YX)(XY - YX)^*] \\
&= \mathrm{tr}(XYY^*X^* - XYX^*Y^* - YXY^*X^* + YXX^*Y^*) \\
&= \mathrm{tr}(X^*XYY^* - XYX^*Y^* - YXY^*X^* + XX^*Y^*Y),
\end{aligned}$$

and similarly

$$\|X^*Y + YX^*\|_F^2 = \mathrm{tr}(XX^*YY^* + YXY^*X^* + XYX^*Y^* + X^*XY^*Y).$$

Thus,

$$\begin{aligned}
\|XY - YX\|_F^2 + \|X^*Y + YX^*\|_F^2 &= \mathrm{tr}(X^*XYY^* + XX^*Y^*Y + XX^*YY^* + X^*XY^*Y) \\
&= \mathrm{tr}[(X^*X + XX^*)(Y^*Y + YY^*)]. \qquad (2.7)
\end{aligned}$$

By using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|\mathrm{tr}[Y(X^*X+XX^*)]| &= |\mathrm{tr}[(X^*Y+YX^*)X]| = |\langle X^*Y+YX^*, X^*\rangle| \\ &\leqslant \|X^*Y+YX^*\|_F\|X\|_F.\end{aligned}$$

Thus,

$$\|X^*Y+YX^*\|_F^2 \geqslant |\mathrm{tr}[Y(X^*X+XX^*)]|^2/\|X\|_F^2. \tag{2.8}$$

Combining (2.7) and (2.8) then gives

$$\|XY-YX\|_F^2 \leqslant \mathrm{tr}[(X^*X+XX^*)(Y^*Y+YY^*)] - |\mathrm{tr}[Y(X^*X+XX^*)]|^2/\|X\|_F^2. \tag{2.9}$$

Let

$$D \equiv (X^*X+XX^*)/(2\|X\|_F^2).$$

We can simplify (2.9) by using D as follows:

$$\|XY-YX\|_F^2 \leqslant 4\|X\|_F^2\left\{\mathrm{tr}[D(Y^*Y+YY^*)/2] - |\mathrm{tr}(DY)|^2\right\}. \tag{2.10}$$

Note that D is positive semidefinite with $\mathrm{tr}(D)=1$. Such a matrix is called a density matrix.

Now we consider the Cartesian decomposition $Y = A+\mathbf{i}B$, where A,B are Hermitian. Obviously,

$$\frac{1}{2}(Y^*Y+YY^*) = A^2+B^2,$$

and then

$$\|Y\|_F^2 = \|A\|_F^2 + \|B\|_F^2. \tag{2.11}$$

Using the fact that the trace of the product of two Hermitian matrices is a real number, we therefore have

$$|\mathrm{tr}(DY)|^2 = |\mathrm{tr}(DA)+\mathbf{i}\mathrm{tr}(DB)|^2 = [\mathrm{tr}(DA)]^2 + [\mathrm{tr}(DB)]^2.$$

Hence,

$$\begin{aligned}\mathrm{tr}[D(Y^*Y+YY^*)/2] - |\mathrm{tr}(DY)|^2 &= \mathrm{tr}[D(A^2+B^2)] - [\mathrm{tr}(DA)]^2 - [\mathrm{tr}(DB)]^2 \\ &= \left(\mathrm{tr}(DA^2) - [\mathrm{tr}(DA)]^2\right) + \left(\mathrm{tr}(DB^2) - [\mathrm{tr}(DB)]^2\right).\end{aligned} \tag{2.12}$$

Combining (2.10) and (2.12), we have

$$\|XY-YX\|_F^2 \leqslant 4\|X\|_F^2\left\{\left(\mathrm{tr}(DA^2) - [\mathrm{tr}(DA)]^2\right) + \left(\mathrm{tr}(DB^2) - [\mathrm{tr}(DB)]^2\right)\right\}. \tag{2.13}$$

Next we want to show that for any Hermitian matrix $H \in M_n(\mathbb{C})$,

$$\mathrm{tr}(DH^2) - [\mathrm{tr}(DH)]^2 \leqslant \frac{\|H\|_F^2}{2}. \tag{2.14}$$

By the spectral decomposition theorem of Hermitian matrices [12, p. 171], we have $H = U\Lambda U^*$, where U is unitary and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$.

Set $E \equiv U^*DU = [p_{ij}]$. Then E is also a positive semidefinite matrix with $\mathrm{tr}(E) = 1$. Since

$$p_{jj} \geqslant 0, \quad j = 1,2,\ldots,n; \qquad \sum_{j=1}^{n} p_{jj} = 1,$$

we have by Lemma 2.1,

$$\mathrm{tr}(DH^2) - [\mathrm{tr}(DH)]^2 = \sum_{j=1}^{n} p_{jj}\lambda_j^2 - \Big(\sum_{j=1}^{n} p_{jj}\lambda_j\Big)^2 \leqslant \sum_{j=1}^{n} \frac{\lambda_j^2}{2} = \frac{\|H\|_F^2}{2}.$$

Then (2.14) holds. Applying (2.14) and then (2.11) to (2.13), we finally obtain

$$\|XY - YX\|_F^2 \leqslant 4\,\|X\|_F^2 \frac{\|A\|_F^2 + \|B\|_F^2}{2} = 2\,\|X\|_F^2\|Y\|_F^2. \quad \square$$

3. Related problems

In this section, we review the study of the four problems mentioned in section 1.

3.1. The maximal pairs of the inequality

Problem (I) concerns with maximal pairs. By maximal pairs, it means those matrices X and Y such that

$$\|XY - YX\|_F = \sqrt{2}\,\|X\|_F\|Y\|_F, \tag{3.1}$$

i.e., those matrices X and Y such that (1.1) holds in equality. When [6] was published, there were already three different proofs for (1.1). However, the maximal pairs could not be derived from the proofs. The following necessary conditions were derived in [6, Corollary 4.2].

LEMMA 3.1. *Let $n > 1$ and $X, Y \in M_n(\mathbb{C})$ be nonzero. If (X,Y) is a maximal pair, then*

(i) $\mathrm{rank}\,(X) \leqslant 2, \quad \mathrm{rank}\,(Y) \leqslant 2$;

(ii) $\mathrm{tr}\,(X) = \mathrm{tr}\,(Y) = 0$;

(iii) $\langle X, Y^m\rangle = 0, \quad \langle X^m, Y\rangle = 0, \quad m = 1,2,\ldots$.

One can easily check that the conditions in Lemma 3.1 are not sufficient. For example, when

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

conditions (i)-(iii) in the above lemma hold but $\|XY - YX\|_F^2 = 1 \neq 4 = 2\|X\|_F^2\|Y\|_F^2$. Nevertheless, it was also found that, for matrices in $M_2(\mathbb{C})$, these conditions are both necessary and sufficient. In this particular situation, if we further assume that X is normal, one can easily see that these conditions are equivalent to $X = UX_0U^*$ and $Y = UY_0U^*$ for some unitary matrix U, where

$$X_0 = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}, \qquad Y_0 = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, \tag{3.2}$$

with $\lambda \neq 0$ and $|a|^2 + |b|^2 > 0$. This fact has shown to be true even for matrices of larger sizes. More precisely, if $X \in M_n(\mathbb{C})$ is normal, the pair (X,Y) is maximal if and only if there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that

$$X = U(X_0 \oplus \mathbf{0})U^* \quad \text{and} \quad Y = U(Y_0 \oplus \mathbf{0})U^*,$$

where X_0 and Y_0 are given by (3.2), and $\mathbf{0}$ is the $(n-2)\times(n-2)$ zero matrix.

Inspired by these facts, Cheng et al. [9] gave a complete characterization of the maximal pairs.

THEOREM 3.2. *Let $n > 1$ and $X, Y \in M_n(\mathbb{C})$ be nonzero. Then (X,Y) is maximal if and only if*

(i) *X and Y are simultaneously unitarily similar to matrices in $M_2(\mathbb{C}) \oplus \mathbf{0}$;*

(ii) $\operatorname{tr}(X) = \operatorname{tr}(Y) = 0$*;*

(iii) $\langle X, Y \rangle = 0$.

Their results were cited very recently in a book titled " *Matrix Theory* " [25] in 2013. The characterization obtained in [9] is essential in the study of the maximal pairs of other commutator bounds (see [23]). We remark that the proof in [9] was deduced by heavy calculations and an alternative proof of the characterization of the maximal pairs was obtained by Cheng et al. [8] again based on the proof in [2] (i.e., the proof of Theorem 2.2 in the above section).

3.2. The use of some other norms in the inequality

Problem (II) concerns with unitarily invariant norms. A norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is called a unitarily invariant norm if, for any $A, U, V \in M_n(\mathbb{C})$ with U and V unitary,

$$\|A\| = \|UAV\|.$$

If further that $\|E_{11}\| = 1$, then it is called a normalized unitarily invariant norm. The Frobenius norm $\|\cdot\|_F$ is an example of a normalized unitarily invariant norm.

In [6], it was shown that if $\|\cdot\|$ is a normalized unitarily invariant norm, then

$$\sup\left\{ \frac{\|XY - YX\|}{\|X\|\|Y\|} : X, Y \in M_n(\mathbb{C}) \setminus \{\mathbf{0}\} \right\} \geqslant \sqrt{2}.$$

Together with (1.1), it means

$$\min_{\|\cdot\|} \sup\left\{\frac{\|XY-YX\|}{\|X\|\|Y\|} : X,Y \in M_n(\mathbb{C}) \setminus \{\mathbf{0}\}\right\} = \sqrt{2},$$

where the minimum is taken over all normalized unitarily invariant norms $\|\cdot\|$, and the minimum is attained for the Frobenius norm $\|\cdot\|_F$. It was also proved that the Frobenius norm is the only normalized unitarily invariant norm $\|\cdot\|$ on $M_2(\mathbb{C})$ such that the inequality

$$\|XY-YX\| \leqslant \sqrt{2}\,\|X\|\|Y\| \tag{3.3}$$

is true for all $X,Y \in M_2(\mathbb{C})$. On the other hand, an example of a non-unitarily invariant norm such that (3.3) holds was constructed in [22]. An immediate question raised was that whether the Frobenius norm is the only normalized unitarily invariant norm $\|\cdot\|$ on $M_n(\mathbb{C})$ such that inequality (3.3) is true for all $X,Y \in M_n(\mathbb{C})$ with $n \geqslant 3$.

Fong et al. [10] answered the question. They proved that the dual norm of the (2,2)-norm, $\|\cdot\|^D_{(2),2}$, given by

$$\begin{aligned}\|X\|^D_{(2),2} &= \max\{|\mathrm{tr}\,(Y^*X)| : \|Y\|_{(2),2} = 1, Y \in M_n(\mathbb{C})\}\\ &= \begin{cases}\sqrt{s_1^2(X) + \left(\sum\limits_{j=2}^n s_j(X)\right)^2}, & \text{if } s_1(X) \geqslant \sum\limits_{j=2}^n s_j(X),\\ \dfrac{1}{\sqrt{2}}\sum\limits_{j=1}^n s_j(X), & \text{if } s_1(X) < \sum\limits_{j=2}^n s_j(X),\end{cases}\end{aligned}$$

is another normalized unitarily invariant norm on $M_n(\mathbb{C})$ such that (3.3) is true for all $X,Y \in M_n(\mathbb{C})$ with $n \geqslant 3$. In the proof, the inequality (3.7) below is crucial.

The Frobenius norm is a member of the class of the Schatten p-norm $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$, and it is natural to consider similar inequalities using Schatten p-norm. The problem was first studied for only one p-norm [6, 22], but finally it comes to a general problem using three Schatten norms as follows (see [23]): find the best possible constant $C_{p,q,r}$ such that

$$\|XY-YX\|_p \leqslant C_{p,q,r}\,\|X\|_q\|Y\|_r, \tag{3.4}$$

for all $X,Y \in M_n(\mathbb{C})$. The main tool used in [23] is the Riesz-Thorin theorem which, in matrix form, is

THEOREM 3.3. *Let $1 \leqslant p_0 \leqslant p_1 \leqslant \infty$ and $1 \leqslant q_0 \leqslant q_1 \leqslant \infty$ be given. Suppose there exist constants c_0 and c_1 such that for the linear operator $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$,*

$$\|T(X)\|_{p_0} \leqslant c_0\|X\|_{q_0} \quad \text{and} \quad \|T(X)\|_{p_1} \leqslant c_1\|X\|_{q_1}, \quad \forall X \in M_n(\mathbb{C}).$$

Then for any $\theta \in [0,1]$,

$$\|T(X)\|_p \leqslant c_0^{1-\theta}c_1^{\theta}\|X\|_q, \quad \forall X \in M_n(\mathbb{C}),$$

where $p \in [p_0,p_1]$ and $q \in [q_0,q_1]$ are defined by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let us illustrate how Theorem 3.3 works. For any fixed $X \in M_n(\mathbb{C})$ with $\|X\|_2 = 1$, define $T_X : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $T_X(Y) = XY - YX$. Clearly, from (1.1), we have

$$\|T_X(Y)\|_2 \leqslant \sqrt{2}\,\|Y\|_2.$$

On the other hand, we can easily get

$$\|T_X(Y)\|_2 \leqslant 2\,\|Y\|_\infty.$$

By Theorem 3.3, with $p_0 = p_1 = q_0 = 2$ and $q_1 = \infty$, we get, with $\theta \in (0,1)$,

$$\frac{1}{2} = \frac{1-\theta}{2} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{\infty},$$

i.e.,

$$\theta = 1 - \frac{2}{r}.$$

Thus, we get $r > 2$ and $\|T_X(Y)\|_2 \leqslant \sqrt{2}^{1-\theta} 2^{\theta} \|Y\|_r$. That is,

$$\|XY - YX\|_2 \leqslant 2^{1-1/r} \|X\|_2 \|Y\|_r,$$

and hence $C_{2,2,r} \leqslant 2^{1-1/r}$. On the other hand, the example (taken from [23])

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad XY - YX = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},$$

yields

$$\|XY - YX\|_2 = 2^{1-1/r} \|X\|_2 \|Y\|_r.$$

As a result, $C_{2,2,r} = 2^{1-1/r}$ for $r > 2$. From this example, we know that some constants are more important, for example $C_{2,2,2} = \sqrt{2}$ and $C_{2,2,\infty} = 2$, as we use them to interpolate.

Many results are obtained for different values of p, q and r. Nevertheless, there are cases that are not solved yet. We will come to this again in section 4. We note that the constant may or may not involve the order n. For example, $C_{p,p,r} = \max\{2^{1/p}, 2^{1-1/p}, 2^{1-1/r}\}$ [23, Theorem 2] whereas

$$C_{1,\infty,\infty} = \begin{cases} n\sqrt{2 + 2\cos(\pi/n)}, & \text{if } n \text{ is odd,} \\ 2n, & \text{if } n \text{ is even,} \end{cases}$$

[23, Theorem 5].

Maximal pairs were also considered in [23]. It turns out that the maximal pairs of (1.1) (see Theorem 3.2) play a significant role in the characterization.

3.3. The sharpening of the inequality

An immediate improvement of (1.1) was given in [6] is as follows:

THEOREM 3.4. *Let $X,Y \in M_n(\mathbb{C})$. Then*

$$\|XY - YX\|_F \leqslant \sqrt{2}\left[\|X\|_F^2\|Y\|_F^2 - |\mathrm{tr}\,(Y^*X)|^2\right]. \tag{3.5}$$

Based on this, a further improvement can be found in [24]. Though the proof there is for real matrices, the idea works well for complex matrices. For any given $X,Y \in M_n(\mathbb{C})$, let $\hat{Y} = Y + \alpha Z$ where α is real and Z is chosen such that X,Z are linearly independent and commute. Then, by (3.5),

$$\|XY - YX\|_F^2 = \|X\hat{Y} - \hat{Y}X\|_F^2 \leqslant 2\left[\|X\|_F^2\|Y + \alpha Z\|_F^2 - |\mathrm{tr}\,[(Y+\alpha Z)^*X]|^2\right]. \tag{3.6}$$

The right-hand side in (3.6) is a quadratic polynomial in α and consequently $\|XY - YX\|_F^2$ is not larger than the minimum of this quadratic polynomial. By direct calculation, we have

$$\begin{aligned}\|XY - YX\|_F^2 &\leqslant \min_{\alpha\in\mathbb{R}}\left\{2\left[\|X\|_F^2\|Y+\alpha Z\|_F^2 - |\mathrm{tr}\,[(Y+\alpha Z)^*X]|^2\right]\right\}\\ &= 2\|X\|_F^2\|Y\|_F^2 - 2|\mathrm{tr}\,(Y^*X)|^2 - \frac{\left[\mathrm{Re}\,[\|X\|_F^2\mathrm{tr}\,(Y^*Z) - \mathrm{tr}\,(Y^*X)\mathrm{tr}\,(Z^*X)]\right]^2}{\|X\|_F^2\|Z\|_F^2 - |\mathrm{tr}\,(Z^*X)|^2}.\end{aligned}$$

Of course, a further step may be done by replacing Z by θZ with $|\theta| = 1$ and having

$$\min_{|\theta|=1}\left[\mathrm{Re}\,[\theta\|X\|_F^2\mathrm{tr}\,(Y^*Z) - \overline{\theta}\mathrm{tr}\,(Y^*X)\mathrm{tr}\,(Z^*X)]\right]^2.$$

The same argument can be applied on Y. We can assume X is not a scalar multiple of I_n, otherwise $XY - YX = \mathbf{0}$. Then, an obvious possible choice of Z is I_n. The improvement may be significant. The following example is given in [24]. Let

$$X = \begin{bmatrix} -1 & 4 \\ 0 & 1 \end{bmatrix}, \qquad Y = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then $\|XY - YX\|_F^2 = 72$, $2\|X\|_F^2\|Y\|_F^2 - 2|\mathrm{tr}\,(Y^*X)|^2 = 234$, and with $Z = I_2$, the improved bound is 153.

In [2], with the introduction of the variance of a matrix, Audenaert improved (1.1) to

$$\|XY - YX\|_F \leqslant \sqrt{2}\,\|X\|_F\|Y\|_{(2),2}. \tag{3.7}$$

Though there are now two different norms on the right-hand side, it is evident that (3.7) is equivalent to

$$\|XY - YX\|_F \leqslant \sqrt{2}\,\|X\|_{(2),2}\|Y\|_F. \tag{3.8}$$

Note that for $n > 2$,

$$\|Y\|_{(2),2} = \sqrt{s_1^2(Y) + s_2^2(Y)} \leqslant \sqrt{s_1^2(Y) + s_2^2(Y) + \cdots + s_n^2(Y)} = \|Y\|_F$$

and hence the upper bound in (3.7) is sharper than that in (1.1). The variance (with respect to a density matrix D) of a matrix Y is defined as

$$\mathrm{Var}_D(Y) = \mathrm{tr}[D(Y^*Y + YY^*)/2] - |\mathrm{tr}(DY)|^2.$$

Among many other estimates, it is proved in [2] that

$$\max\{\mathrm{Var}_D(Y) : D \in M_n(\mathbb{C}) \text{ is a density matrix}\} \leqslant \frac{1}{2}\|Y\|_{(2),2}^2. \tag{3.9}$$

Thus (3.7) follows from (2.10). To date, this is still the leading result.

Alternative proofs of (3.7) can be found in Lu [19] and Cheng et al. [8]. Lu again used an operator approach. A refinement of the operator defined in [17] is used and the proof in [19] is more conceptual and less computational. The proof of Cheng et al. is based on the proof in [6], and the additional argument is elementary, as follows. We refer to (2.2). Regarding s_1 as a real variable while taking $s_2,\ldots,s_n$ as constants, define a function $g : [s_2,\infty) \to \mathbb{R}$ as follows:

$$g(s_1) = \sum_{j\neq k} \left(s_j^2|c_{jk}|^2 + s_k^2|c_{jk}|^2 + s_j^2|d_{jk}|^2 + s_k^2|d_{jk}|^2\right) + \sum_{j=1}^{n} s_j^2|c_{jj} - d_{jj}|^2.$$

Also, define a function $h : [s_2,\infty) \to \mathbb{R}$ by

$$h(s_1) = 2(s_1^2 + s_2^2)\|Y\|_F^2.$$

Then, to prove (3.8), it suffices to show that

$$g(s_1) \leqslant h(s_1), \quad \forall s_1 \in [s_2,\infty).$$

Note that

$$g'(s_1) = 2s_1\left[|c_{11} - d_{11}|^2 + \sum_{k=2}^{n}(|c_{1k}|^2 + |c_{k1}|^2 + |d_{1k}|^2 + |d_{k1}|^2)\right] = 2s_1\Delta_1,$$

and

$$h'(s_1) = 4s_1\|Y\|_F^2.$$

From (2.4), we know that

$$g'(s_1) \leqslant h'(s_1), \quad \forall s_1 > s_2.$$

As g and h are continuous at $s_1 = s_2$, it remains to show that $g(s_2) \leqslant h(s_2)$. The verification is straightforward. With $s_1 = s_2$,

$$\begin{aligned} g(s_2) &\leqslant s_2^2 \sum_{j,k=1}^{n} (|c_{jk}|^2 + |c_{jk}|^2 + |d_{jk}|^2 + |d_{jk}|^2) \\ &= s_2^2(2\|C\|_F^2 + 2\|D\|_F^2) = 2(s_2^2 + s_2^2)\|Y\|_F^2 = h(s_2). \end{aligned}$$

Thus, (3.8) is true.

3.4. The extension to other products similar to the commutator

Problem (IV) is about the extension of (1.1) to other products similar to the commutator. Various products that are similar to the commutator are also of interest to many authors, e.g., see [16, 20]. Parallel to (1.1), let us consider a general question about the Frobenius norm of the products of the form

$$XY \pm Y^{\dagger}X^{\ddagger}$$

where $Y^{\dagger} \in \{Y, \overline{Y}, Y^T, Y^*\}$ and $X^{\ddagger} \in \{X, \overline{X}, X^T, X^*\}$. By the triangle inequality and the submultiplicative property of the Frobenius norm, one easily gets

$$\|XY \pm Y^{\dagger}X^{\ddagger}\|_F \leqslant 2\|X\|_F\|Y\|_F.$$

Besides the commutator, the products $XY - YX^T$ (equivalently $XY - Y^TX$) and $XY - Y^TX^T$, it is easy to check that the above inequality becomes tight when X and Y are suitably chosen from $\{E_{11}, E_{11}\mathbf{i}\}$. In these cases, the best possible constant in the right-hand side of the inequality is 2. Thus, we are left with $XY - YX^T$ and $XY - Y^TX^T$.

Let us first consider $XY - YX^T$. We ask if the inequality

$$\|XY - YX^T\|_F \leqslant \sqrt{2}\,\|X\|_F\|Y\|_F \tag{3.10}$$

is true or not, or further if an inequality similar to (3.7) or (3.8) can be established or not. Fong et al. showed in [11] that, parallel to (2.10), the norm $\|XY - YX^T\|_F$ can also be related to the variance of X as:

$$\|XY - YX^T\|_F^2 \leqslant 4\|Y\|_F^2\left\{\mathrm{tr}[D(X^*X + XX^*)/2] - |\mathrm{tr}(DX)|^2\right\} = 4\|Y\|_F^2\mathrm{Var}_D(X), \tag{3.11}$$

where $D = \frac{1}{2\|Y\|_F^2}(YY^* + Y^T\overline{Y})$ is a density matrix. Thus, by (3.9), the following inequality is true:

$$\|XY - YX^T\|_F \leqslant \sqrt{2}\,\|X\|_{(2),2}\|Y\|_F. \tag{3.12}$$

As a consequence, (3.10) is also true.

A natural question is to ask if the (2,2)-norm $\|\cdot\|_{(2),2}$ on the right-hand side of (3.12) can be used on Y instead of X, i.e., whether the following inequality is true or not:

$$\|XY - YX^T\|_F \leqslant \sqrt{2}\,\|X\|_F\|Y\|_{(2),2}. \tag{3.13}$$

It is noted in [8] that the norm $\|XY - YX^T\|_F$, which can be bounded by the variance of X as in (3.11), cannot be similarly bounded by the variance of Y. The norm $\|XY - YX^T\|_F$ is invariant if X is replaced by $X + \alpha I_n$ for any $\alpha \in \mathbb{C}$. However, this is not true for Y. If X is not symmetric, we have $\|X(Y + \alpha I_n) - (Y + \alpha I_n)X^T\|_F \to \infty$ as $|\alpha| \to \infty$. On the other hand, the maximum of the variance of a matrix is also invariant under a translation of the identity matrix (see [2, Theorem 9]). This suggests a reason why, while (3.12) can be proved using the variance of a matrix, the method is not suitable for proving (3.13).

The other approach to prove (3.8) is an operator approach given by Lu in [17, 19]. As mentioned in section 2, a very special property of T_X (in (2.5)) is used: the geometric multiplicity of $\lambda_1(T_X)$ is at least two. When considering the product $XY - YX^T$, the corresponding operator $\hat{T}_Y$ does not have this property in general. An example (for any $n \geqslant 2$) is to take $Y = E_{12}$. Then X is an eigenvector corresponding to $\lambda_1(\hat{T}_{E_{12}})$ if and only if the maximum of $\|XE_{12} - E_{12}X^T\|_F^2/\|X\|_F^2$ is attained at X, and this happens if and only if X is a nonzero multiple of $E_{11} - E_{22}$. In other words, the geometric multiplicity of $\lambda_1(\hat{T}_{E_{12}})$ is only one. Again this approach is not applicable for proving (3.13).

Cheng et al. [8] found that a common approach can be used to prove (3.8), (3.12) and (3.13) at the same time. In fact, it was the attempt to prove (3.13) that this approach was found and the idea is illustrated above in proving (3.8): with (2.1)–(2.4), all we need are some elementary calculus arguments. Inequalities (3.12) and (3.13) can be proved similarly by first establishing inequalities similar to (2.1)–(2.4) and then applying the elementary calculus argument. This provides a new understanding about the similarities of these three inequalities. Moreover, this common approach has an advantage that the maximal pairs of all the three inequalities can be deduced from the proofs. Note that with only (2.1)–(2.4), the maximal pairs of (1.1) could not be determined. In particular it is found that, while there are two common analogous cases of maximal pairs for all the three inequalities, (3.13) admits an extra type of maximal pairs that neither (3.8) nor (3.12) has as a corresponding counterpart. This reveals a difference between (3.13) and the other two inequalities.

We finally consider $XY - Y^TX^T$. With $X = USV$ being the singular value decomposition of X, we have

$$\|XY - Y^TX^T\|_F = \|S(VY\overline{U}) - (VY\overline{U})^TS\|_F$$

and thus the corresponding results follow from (3.12) and (3.13).

4. Open problems

In this section, we discuss some open problems.

PROBLEM 1. Determine the best possible constant $C_{p,q,r}$ in inequality (3.4). This problem is still open when

(1) n is odd and p,q,r satisfy $\frac{1}{p} > \frac{1}{q} + \frac{1}{r}$ (except for $(p,q,r) = (1,\infty,\infty)$); and

(2) $p > 2$, $q < 2$ and $r < 2$ (except for $(p,q,r) = (\infty,1,1)$).

As discussed in [23], interpolation using the known results only produces upper bounds for $C_{p,q,r}$.

PROBLEM 2. One may consider an improvement of (3.4) by combining (3.4) and (3.7), i.e., to find the best possible constant $\tilde{C}_{p,q,r}$ such that

$$\|XY - YX\|_p \leqslant \tilde{C}_{p,q,r}\,\|X\|_q\|Y\|_{(2),r}.$$

Is it true that $\tilde{C}_{p,q,r} = C_{p,q,r}$ (where $C_{p,q,r}$ is the constant in (3.4)) ? Some work has been done in this direction. In particular, the Riesz-Thorin theorem is proved to be valid for $(r,2)$-norm (see [11]).

PROBLEM 3. Problem 2 may also be considered for the product $XY - YX^T$. Notice that one needs to consider two inequalities:

$$\|XY - YX^T\|_p \leqslant \bar{C}_{p,q,r}\ \|X\|_{(2),q}\|Y\|_r$$

and

$$\|XY - YX^T\|_p \leqslant \hat{C}_{p,q,r}\ \|X\|_q\|Y\|_{(2),r}.$$

Are the two constants the same? When $p = q = r = 2$, they are. Nevertheless, their maximal pairs are not the same.

In summary, since the appearance of the Böttcher-Wenzel conjecture in 2005, many scholars around the world have made great contributions to this important topic. Moreover, subsequent problems (I)–(IV) have been proposed and then answered. Nevertheless, there are still some open problems. After almost a decade, undoubtedly, the accomplishment in the research of this conjecture and related problems laid a milestone for the study of norm inequality for commutator.

Note added in proofs. Recently, the first author and Chunyu Lei solved (1) of Problem 1, and the result has been submitted for publication.

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