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HAHN–BANACH TYPE EXTENSION THEOREMS ON p–OPERATOR SPACES

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Abstract. Let $V \subseteq W$ be two operator spaces. Arveson-Wittstock-Hahn-Banach theorem asserts that every completely contractive map $\varphi : V \to \mathscr{B}(H)$ has a completely contractive extension $\tilde{\varphi} : W \to \mathscr{B}(H)$, where $\mathscr{B}(H)$ denotes the space of all bounded operators from a Hilbert space H to itself. In this paper, we show that this is not in general true for p-operator spaces, that is, we show that there are p-operator spaces $V \subseteq W$, an SQ_p space E, and a p-completely contractive map $\varphi : V \to \mathscr{B}(E)$ such that φ does not extend to a p-completely contractive map on W. Restricting E to L_p spaces, we also consider a condition on W under which every completely contractive map $\varphi : V \to \mathscr{B}(L_p(\mu))$ has a completely contractive extension $\tilde{\varphi} : W \to \mathscr{B}(L_p(\mu))$.

1. Introduction to p-operator spaces

Throughout this paper, we assume $1 < p, p' < \infty$ with $1/p + 1/p' = 1$, unless stated otherwise. For a Banach space X, we denote by $\mathbb{M}_{m,n}(X)$ the linear space of all $m \times n$ matrices with entries in X. By $\mathbb{M}_n(X)$, we will denote $\mathbb{M}_{n,n}(X)$. When $X = \mathbb{C}$, we will simply use $\mathbb{M}_{m,n}$ (respectively, $\mathbb{M}_n$) for $\mathbb{M}_{m,n}(\mathbb{C})$ (respectively, $\mathbb{M}_n(\mathbb{C})$). For Banach spaces X and Y, we will denote by $\mathscr{B}(X,Y)$ the space of all bounded linear operators from X to Y. We will also use $\mathscr{B}(X)$ for $\mathscr{B}(X,X)$. The ℓ_p direct sum of n copies of X will be denoted by $\ell_p^n(X)$.

DEFINITION 1.1. Let SQ_p denote the collection of subspaces of quotients of L_p spaces. A Banach space X is called a *concrete p-operator space* if X is a closed subspace of $\mathscr{B}(E)$ for some $E \in SQ_p$.

Let $E \in SQ_p$. For a concrete p-operator space $X \subseteq \mathscr{B}(E)$ and for each $n \in \mathbb{N}$, define a norm $\|\cdot\|_n$ on $\mathbb{M}_n(X)$ by identifying $\mathbb{M}_n(X)$ as a subspace of $\mathscr{B}(\ell_p^n(E))$, and let $M_n(X)$ denote the corresponding normed space. The norms $\|\cdot\|_n$ then satisfy

$\mathscr{D}_\infty$ for $u \in M_n(X)$ and $v \in M_m(X)$, we have $\|u \oplus v\|_{M_{n+m}(X)} = \max\{\|u\|_n, \|v\|_m\}$.

$\mathscr{M}_p$ for $u \in M_m(X)$, $\alpha \in \mathbb{M}_{n,m}$, and $\beta \in \mathbb{M}_{m,n}$, we have $\|\alpha u \beta\|_n \leqslant \|\alpha\|\|u\|_m\|\beta\|$, where $\|\alpha\|$ is the norm of α as a member of $\mathscr{B}(\ell_p^m, \ell_p^n)$, and similarly for β.

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When $p=2$, these are Ruan's axioms and 2-operator spaces are simply operator spaces because the SQ_2 spaces are exactly the same as Hilbert spaces.

As in operator spaces, we can also define abstract p-operator spaces.

DEFINITION 1.2. An *abstract p-operator space* is a Banach space X together with a sequence of norms $\|\cdot\|_n$ defined on $\mathbb{M}_n(X)$ satisfying the conditions $\mathscr{D}_\infty$ and $\mathscr{M}_p$ above.

Thanks to Ruan's representation theorem [8], we do not distinguish between concrete and abstract operator spaces. Le Merdy showed that this remains true for p-operator spaces.

THEOREM 1.3. [6, Theorem 4.1] *An abstract p-operator space X can be isometrically embedded in $\mathscr{B}(E)$ for some $E \in SQ_p$ in such a way that the canonical norms on $\mathbb{M}_n(X)$ arising from this embedding agree with the given norms.*

EXAMPLE 1.4.

a. Suppose E and F are SQ_p spaces and let $L = E \oplus_p F$, the ℓ_p direct sum of E and F. Then L is also an SQ_p space [4, Proposition 5] and the mapping

$$x \mapsto \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$$

is an isometric embedding of $\mathscr{B}(E,F)$ into $\mathscr{B}(L)$. Using this we can view $\mathscr{B}(E,F)$ as a p-operator space. Note that $M_n(\mathscr{B}(E,F))$ is isometrically isomorphic to $\mathscr{B}(\ell_p^n(E), \ell_p^n(F))$.

b. The identification $L_p(\mu) = \mathscr{B}(\mathbb{C}, L_p(\mu)) \subseteq \mathscr{B}(\mathbb{C} \oplus_p L_p(\mu))$ gives a p-operator space structure on $L_p(\mu)$ called the *column p-operator space structure* of $L_p(\mu)$, which we denote by $L_p^c(\mu)$. Similarly, the identification $L_{p'}(\mu) = \mathscr{B}(L_p(\mu), \mathbb{C})$ gives rise to p-operator space structure on $L_{p'}(\mu)$ which we denote by $L_{p'}^r(\mu)$ and call the *row p-operator space structure* of $L_{p'}(\mu)$. In general, we can define E^c and $(E')^r$ for any $E \in SQ_p$, where E' is the Banach dual space of E.

Note that a linear map $u : X \to Y$ between p-operator spaces X and Y induces a map $u_n : M_n(X) \to M_n(Y)$ by applying u entrywise. We say that u is *p-completely bounded* if $\|u\|_{pcb} := \sup_n \|u_n\| < \infty$. Similarly, we define *$p$-completely contractive*, *p-completely isometric*, and *p-completely quotient* maps. We write $\mathscr{CB}_p(X,Y)$ for the space of all p-completely bounded maps from X into Y.

To turn the mapping space $\mathscr{CB}_p(X,Y)$ between two p-operator spaces X and Y into a p-operator space, we define a norm on $\mathbb{M}_n(\mathscr{CB}_p(X,Y))$ by identifying this space with $\mathscr{CB}_p(X, M_n(Y))$. Using Le Merdy's theorem, one can show that $\mathscr{CB}_p(X,Y)$ itself is a p-operator space. In particular, the *p-operator dual space* of X is defined to be $\mathscr{CB}_p(X,\mathbb{C})$. The next lemma by Daws shows that we may identify the Banach dual space X' of X with the p-operator dual space $\mathscr{CB}_p(X,\mathbb{C})$ of X.

LEMMA 1.5. [1, Lemma 4.2] *Let X be a p-operator space, and let $\varphi \in X'$, the Banach dual of X. Then φ is p-completely bounded as a map to $\mathbb{C}$. Moreover, $\|\varphi\|_{pcb} = \|\varphi\|$.*

If $E = L_p(\mu)$ for some measure μ and $X \subseteq \mathscr{B}(E) = \mathscr{B}(L_p(\mu))$, then we say that X is a *p-operator space on L_p space*. These p-operator spaces are often easier to work with. For example, let $\kappa_X : X \to X''$ denote the canonical inclusion from a p-operator space X into its second dual. Contrary to operator spaces, κ_X is *not* always p-completely isometric. Thanks to the following theorem by Daws, however, we can easily characterize those p-operator spaces with the property that the canonical inclusion is p-completely isometric.

PROPOSITION 1.6. [1, Proposition 4.4] *Let X be a p-operator space. Then κ_X is a p-complete contraction. Moreover, κ_X is a p-complete isometry if and only if $X \subseteq \mathscr{B}(L_p(\mu))$ p-completely isometrically for some measure μ.*

2. Non-existence of p-Arveson-Wittstock-Hahn-Banach theorem

Let $V \subseteq W$ be two operator spaces. Arveson-Wittstock-Hahn-Banach theorem asserts that every completely bounded map $\varphi : V \to \mathscr{B}(H)$ has a completely bounded extension $\tilde{\varphi} : W \to \mathscr{B}(H)$, where H is a Hilbert space. For p-operator spaces, the following question naturally arises.

QUESTION 2.1. Let $V \subseteq W$ be p-operator spaces and E an SQ_p space. Does every p-completely bounded map $\varphi : V \to \mathscr{B}(E)$ have a p-completely bounded extension $\tilde{\varphi} : W \to \mathscr{B}(E)$?

To show that this question has a negative answer, let $p \neq 2$, and let E and $L_p(\Omega)$ such that E is a Hilbert space embedding to $L_p(\Omega)$. The existence of such E and $L_p(\Omega)$ is guaranteed by, for example, [2, Proposition 8.7]. Let $J : E \hookrightarrow L_p(\Omega)$ denote the isometric embedding, then we can view E as a subspace of $L_p(\Omega)$.

LEMMA 2.2. *Let J be as above. With p-operator space structure E^c and $L_p(\Omega)^c$, J becomes a p-complete isometry.*

Proof. From Example 1.4, we note that $M_n(E^c) \subseteq M_n(\mathscr{B}(\mathbb{C}, E)) = \mathscr{B}(\ell_p^n, \ell_p^n(E))$. For $[\xi_{ij}] \in M_n(E^c)$, the norm is given by

$$\|[\xi_{ij}]\|^p = \sup\left\{ \sum_{i=1}^n \left\| \sum_{j=1}^n \lambda_j \xi_{ij} \right\|_E^p : \lambda_j \in \mathbb{C},\ \sum_{j=1}^n |\lambda_j|^p \leqslant 1 \right\}.$$

Since J is an isometry,

$$\left\| J\left(\sum_{j=1}^n \lambda_j \xi_{ij} \right) \right\|_{L_p(\Omega)} = \left\| \sum_{j=1}^n \lambda_j \xi_{ij} \right\|_E$$

and it follows that

$$\begin{aligned}
\|J_n([\xi_{ij}])\|^p &= \sup\left\{\sum_{i=1}^n \left\|\sum_{j=1}^n \lambda_j J(\xi_{ij})\right\|_{L_p(\Omega)}^p : \lambda_j \in \mathbb{C},\ \sum_{j=1}^n |\lambda_j|^p \leqslant 1\right\} \\
&= \sup\left\{\sum_{i=1}^n \left\|J\left(\sum_{j=1}^n \lambda_j \xi_{ij}\right)\right\|_{L_p(\Omega)}^p : \lambda_j \in \mathbb{C},\ \sum_{j=1}^n |\lambda_j|^p \leqslant 1\right\} \\
&= \|[\xi_{ij}]\|^p. \quad \square
\end{aligned}$$

Let $\tilde{E} = \mathbb{C} \oplus_p E$. Let $\pi : \tilde{E} \to E$ denote the projection from $\tilde{E}$ onto E and define $\varphi : E^c \to \mathscr{B}(\tilde{E})$ and $\psi : \mathscr{B}(\tilde{E}) \to E^c$ by

$$\varphi(\xi) = T_\xi, \quad T_\xi(\lambda \oplus_p e) = 0 \oplus_p \lambda\xi, \quad \lambda \in \mathbb{C}, \quad e \in E$$

and

$$\psi(T) = \pi T(1 \oplus_p 0), \quad T \in \mathscr{B}(\tilde{E})$$

(see the diagram below).

$$\begin{tikzcd}
L_p(\Omega)^c \arrow[dr, bend left, "\tilde{\varphi}"] & \\
E^c \arrow[u, "J"] \arrow[r, shift left, "\varphi"] & \mathscr{B}(\tilde{E}) \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

It is then easy to check that φ and ψ are p-complete contractions with $\psi \circ \varphi = id_{E^c}$. Suppose that $\varphi : E^c \to \mathscr{B}(\tilde{E})$ extends to $\tilde{\varphi} : L_p(\Omega)^c \to \mathscr{B}(\tilde{E})$. Define $P : L_p(\Omega)^c \to E^c$ by $P = \psi \circ \tilde{\varphi}$, then it follows that P is a p-completely contractive projection onto E^c, meaning that E must be a 1-complemented subspace of $L_p(\Omega)$. This is, however, impossible, because it would imply that a Hilbert space E is isometrically isomorphic to some L_p space with $p \neq 2$.

3. A predual of $\mathscr{CB}_p(V, M_n)$

In this section, we define a normed space structure on $\mathbb{M}_n(V)$ whose Banach dual is isometrically isomorphic to $\mathscr{CB}_p(V, M_n)$.

LEMMA 3.1. *Let $1 < p, p' < \infty$ with $1/p + 1/p' = 1$. Let $\lambda = \{\lambda_j\}_{1 \leqslant j \leqslant n}$ be a finite sequence in $\mathbb{C}$. Then*

$$\|\lambda\|_{\ell_p^n} \leqslant n^{|1/p - 1/p'|} \cdot \|\lambda\|_{\ell_{p'}^n}.$$

Proof. There is nothing to prove if $p = p' = 2$. If $p > p'$, then $\|\lambda\|_{\ell_p^n} \leqslant \|\lambda\|_{\ell_{p'}^n} \leqslant n^{|1/p-1/p'|} \cdot \|\lambda\|_{\ell_{p'}^n}$ since $n^{|1/p-1/p'|} \geqslant 1$. Finally, assume $1 < p < p'$ and let $q = \frac{p'}{p} > 1$ and let q' be the conjugate exponent to q. By Hölder's inequality,

$$\|\lambda\|_{\ell_p^n}^p \leqslant \left(\sum_{j=1}^n |\lambda_j|^{pq}\right)^{1/q} \cdot n^{1/q'} = \left(\sum_{j=1}^n |\lambda_j|^{p'}\right)^{p/p'} \cdot n^{1-p/p'}$$

and hence $\|\lambda\|_{\ell_p^n} \leqslant n^{|1/p-1/p'|} \cdot \|\lambda\|_{\ell_{p'}^n}$. □

LEMMA 3.2. *Let $\alpha = [\alpha_{ij}] \in \mathbb{M}_{n,r}$ and $\beta = [\beta_{kl}] \in \mathbb{M}_{r,n}$. Let $1 < p, p' < \infty$ with $1/p + 1/p' = 1$. Then we have*

$$\|\alpha\|_{\mathscr{B}(\ell_p^r, \ell_p^n)} \leqslant \|\alpha\|_{p'} \cdot n^{|1/p-1/p'|} \quad \text{and} \quad \|\beta\|_{\mathscr{B}(\ell_p^n, \ell_p^r)} \leqslant \|\beta\|_p \cdot n^{|1/p-1/p'|},$$

where

$$\|\alpha\|_{p'} = \left(\sum_{i=1}^n \sum_{j=1}^r |\alpha_{ij}|^{p'}\right)^{1/p'} \quad \text{and} \quad \|\beta\|_p = \left(\sum_{k=1}^r \sum_{l=1}^n |\beta_{kl}|^p\right)^{1/p}.$$

Proof. Suppose $\xi = \{\xi_j\}_{j=1}^r$ is a unit vector in ℓ_p^r. For each i, $1 \leqslant i \leqslant n$, let $\eta_i = \left|\sum_{j=1}^r \alpha_{ij}\xi_j\right|$, then by Hölder's inequality, $\eta_i \leqslant \left(\sum_{j=1}^r |\alpha_{ij}|^{p'}\right)^{1/p'}$ and by Lemma 3.1,

$$\left(\sum_{i=1}^n \eta_i^p\right)^{1/p} \leqslant n^{|1/p-1/p'|} \cdot \left(\sum_{i=1}^n \eta_i^{p'}\right)^{1/p'} \leqslant n^{|1/p-1/p'|} \cdot \|\alpha\|_{p'}$$

and hence we get $\|\alpha\|_{\mathscr{B}(\ell_p^r, \ell_p^n)} \leqslant n^{|1/p-1/p'|} \cdot \|\alpha\|_{p'}$. To prove the second inequality, let $\gamma = \beta^T \in \mathbb{M}_{n,r}$, the transpose of β. Then by the argument above we have

$$\|\gamma\|_{\mathscr{B}(\ell_{p'}^r, \ell_{p'}^n)} \leqslant \|\gamma\|_p \cdot n^{|1/p-1/p'|}.$$

Since $\|\gamma\|_{\mathscr{B}(\ell_{p'}^r, \ell_{p'}^n)} = \|\beta\|_{\mathscr{B}(\ell_p^n, \ell_p^r)}$ and $\|\gamma\|_p = \|\beta\|_p$, we get the desired inequality. □

Let V be a p-operator space. Fix $n \in \mathbb{N}$ and define $\|\cdot\|_{1,n} : \mathbb{M}_n(V) \to [0, \infty)$ by

$$\|v\|_{1,n} = \inf\{\|\alpha\|_{p'}\|w\|\|\beta\|_p : r \in \mathbb{N},\ v = \alpha w \beta,\ \alpha \in \mathbb{M}_{n,r},\ \beta \in \mathbb{M}_{r,n},\ w \in M_r(V)\}, \tag{3.1}$$

where $\|\cdot\|_{p'}$ and $\|\cdot\|_p$ as in Lemma 3.2.

PROPOSITION 3.3. *Suppose that V is a p-operator space and $n \in \mathbb{N}$. Then $\|\cdot\|_{1,n}$ defines a norm on $\mathbb{M}_n(V)$.*

Proof. Suppose $v_1, v_2 \in \mathbb{M}_n(V)$. Let $\varepsilon > 0$. For $i = 1,2$, we can find α_i, β_i, and w_i such that $v_i = \alpha_i w_i \beta_i$ with $\|w_i\| \leqslant 1$ and

$$\|\alpha_i\|_{p'} < (\|v_i\|_{1,n} + \varepsilon)^{1/p'}, \quad \|\beta_i\|_p < (\|v_i\|_{1,n} + \varepsilon)^{1/p}. \tag{3.2}$$

Let

$$\alpha = [\alpha_1 \ \alpha_2], \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \text{and} \quad w = \begin{bmatrix} w_1 & \\ & w_2 \end{bmatrix},$$

then $\|\alpha\|_{p'}^{p'} = \|\alpha_1\|_{p'}^{p'} + \|\alpha_2\|_{p'}^{p'}$, $\|\beta\|_p^p = \|\beta_1\|_p^p + \|\beta_2\|_p^p$, and $\|w\| \leqslant 1$. Since $v_1 + v_2 = \alpha w \beta$, it follows that

$$\begin{aligned} \|v_1 + v_2\|_{1,n} &\leqslant \|\alpha\|_{p'} \|\beta\|_p \\ (\text{Young's inequality}) &\leqslant \frac{\|\alpha\|_{p'}^{p'}}{p'} + \frac{\|\beta\|_p^p}{p} \\ &= \frac{\|\alpha_1\|_{p'}^{p'} + \|\alpha_2\|_{p'}^{p'}}{p'} + \frac{\|\beta_1\|_p^p + \|\beta_2\|_p^p}{p} \\ (\text{by } (3.2)) &< \frac{\|v_1\|_{1,n} + \|v_2\|_{1,n} + 2\varepsilon}{p'} + \frac{\|v_1\|_{1,n} + \|v_2\|_{1,n} + 2\varepsilon}{p} \\ &= \|v_1\|_{1,n} + \|v_2\|_{1,n} + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, we get $\|v_1 + v_2\|_{1,n} \leqslant \|v_1\|_{1,n} + \|v_2\|_{1,n}$.

For any $c \in \mathbb{C}$, if $v = \alpha w \beta$, then we have $cv = \alpha(cw)\beta$ and hence $\|cv\|_{1,n} \leqslant \|\alpha\|_{p'} |c| \|w\| \|\beta\|_p$. Taking the infimum, we get

$$\|cv\|_{1,n} \leqslant |c| \|v\|_{1,n}. \tag{3.3}$$

When $c \neq 0$, replacing c by $1/c$ and v by cv in (3.3) gives

$$|c| \|v\|_{1,n} \leqslant \|cv\|_{1,n}, \tag{3.4}$$

so (3.3) together with (3.4) gives $\|cv\|_{1,n} = |c| \|v\|_{1,n}$, which is obviously true when $c = 0$.

Finally, suppose $\|v\|_{1,n} = 0$. To show that $v = 0$, it suffices to show that

$$\|v\| \leqslant n^{2|1/p - 1/p'|} \cdot \|v\|_{1,n}. \tag{3.5}$$

Indeed, if $v = \alpha w \beta$ with $\alpha \in \mathbb{M}_{n,r}$, $\beta \in \mathbb{M}_{r,n}$, and $w \in M_r(v)$, then

$$\begin{aligned} \|v\| &\leqslant \|\alpha\| \|w\| \|\beta\| \\ (\text{by Lemma 3.2}) &\leqslant \|\alpha\|_{p'} \cdot n^{|1/p - 1/p'|} \cdot \|w\| \cdot \|\beta\|_p \cdot n^{|1/p - 1/p'|} \\ &= n^{2|1/p - 1/p'|} \cdot \|\alpha\|_{p'} \cdot \|w\| \cdot \|\beta\|_p. \end{aligned}$$

Taking the infimum, (3.5) follows. $\square$

For a p-operator space V, let $\mathscr{T}_n(V)$ denote the normed space $(\mathbb{M}_n(V), \|\cdot\|_{1,n})$.

LEMMA 3.4. *For a p-operator space V, $\mathcal{T}_n(V)' = M_n(V') = \mathcal{CB}_p(V,M_n)$ isometrically.*

Proof. The second isometric isomorphism comes from the definition of the p-operator space structure on V'. We follow the idea as in [3, §4.1]. Let $f = [f_{ij}] \in M_n(V') = \mathcal{CB}_p(V,M_n)$. Note that

$$\|f\| = \sup\{\|\langle\langle f,\tilde{v}\rangle\rangle\| : r \in \mathbb{N},\ \tilde{v} = [\tilde{v}_{kl}] \in M_r(V),\ \|\tilde{v}\| \leqslant 1\}.$$

Let $D^p_{n\times r}$ denote the closed unit ball of $\ell_p^{n\times r}$, then

$$\begin{aligned}
\|f\| &= \sup\{|\langle\langle\langle f,\tilde{v}\rangle\rangle\eta,\xi\ \rangle| : r \in \mathbb{N},\ \tilde{v} = [\tilde{v}_{kl}] \in M_r(V),\ \|\tilde{v}\| \leqslant 1,\ \eta \in D^p_{n\times r},\ \xi \in D^{p'}_{n\times r}\} \\
&= \sup\left\{\left|\sum_{i,j,k,l} f_{ij}(\tilde{v}_{kl})\eta_{(j,l)}\xi_{(i,k)}\right| : r \in \mathbb{N},\ \tilde{v} = [\tilde{v}_{kl}] \in M_r(V),\ \|\tilde{v}\| \leqslant 1,\right. \\
&\qquad \left.\eta \in D^p_{n\times r},\ \xi \in D^{p'}_{n\times r}\right\} \\
&= \sup\left\{\left|\sum_{i,j=1}^{n}\left\langle f_{ij}, \sum_{k,l=1}^{r}\xi_{(i,k)}\tilde{v}_{kl}\eta_{(j,l)}\right\rangle\right| : r \in \mathbb{N},\ \tilde{v} = [\tilde{v}_{kl}] \in M_r(V),\ \|\tilde{v}\| \leqslant 1,\right. \\
&\qquad \left.\eta \in D^p_{n\times r},\ \xi \in D^{p'}_{n\times r}\right\}.
\end{aligned}$$

Note that $\sum_{k,l=1}^r \xi_{(i,k)}\tilde{v}_{kl}\eta_{(j,l)}$ is the (i,j)-entry of the matrix product $\alpha\tilde{v}\beta$, where

$$\alpha = \begin{bmatrix} \xi_{(1,1)} & \cdots & \xi_{(1,r)} \\ \vdots & \ddots & \vdots \\ \xi_{(n,1)} & \cdots & \xi_{(n,r)} \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} \eta_{(1,1)} & \cdots & \eta_{(n,1)} \\ \vdots & \ddots & \vdots \\ \beta_{(1,r)} & \cdots & \eta_{(n,r)} \end{bmatrix},$$

so

$$\begin{aligned}
\|f\| &= \sup\left\{\left|\sum_{i,j=1}^{n}\langle f_{ij},(\alpha\tilde{v}\beta)_{ij}\rangle\right| : \|\tilde{v}\| \leqslant 1,\ \|\alpha\|_{p'} \leqslant 1,\ \|\beta\|_p \leqslant 1\right\} \\
&= \sup\{|\langle f,v\rangle| : v = \alpha\tilde{v}\beta,\ \|\tilde{v}\| \leqslant 1,\ \|\alpha\|_{p'} \leqslant 1,\ \|\beta\|_p \leqslant 1\} \\
&= \sup\{|\langle f,v\rangle| : \|v\|_{1,n} \leqslant 1\}. \qquad (3.6)
\end{aligned}$$

Define the scalar pairing $\Phi : M_n(V') \to \mathcal{T}_n(V)'$ by $f \mapsto \langle f,\cdot\rangle$, then from (3.6) it follows that Φ is an isometric isomorphism. $\square$

PROPOSITION 3.5. *Let $V \subseteq W$ be p-operator spaces such that the inclusion $\mathcal{T}_n(V) \hookrightarrow \mathcal{T}_n(W)$ is isometric. Then every p-completely contractive map $\varphi : V \to \mathcal{B}(L_p(\Omega))$ has a completely contractive extension $\tilde{\varphi} : W \to \mathcal{B}(L_p(\Omega))$.*

Proof. Following [3, Corollay 4.1.4, Theorem 4.1.5], it suffices to assume that $\mathcal{B}(L_p(\Omega)) = \mathcal{B}(\ell_p^n) = M_n$. If the inclusion $i : \mathcal{T}_n(V) \hookrightarrow \mathcal{T}_n(W)$ is isometric, then by Lemma 3.4, the adjoint $i' : \mathcal{CB}_p(W,M_n) \to \mathcal{CB}_p(V,M_n)$, which is a restriction mapping, is an exact quotient mapping. $\square$

4. ℓ_p-polar decomposition

Let $V \subseteq W$ be p-operator spaces. By Proposition 3.5, if the inclusion $\mathscr{T}_n(V) \hookrightarrow \mathscr{T}_n(W)$ is isometric, then every p-completely contractive map $\varphi : V \to \mathscr{B}(L_p(\Omega))$ has a completely contractive extension $\tilde{\varphi} : W \to \mathscr{B}(L_p(\Omega))$. In this section, we consider a condition on W under which the inclusion $\mathscr{T}_n(V) \hookrightarrow \mathscr{T}_n(W)$ is isometric. Recall that the vector p-norm of $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

If we identify $\mathbb{M}_{r,n}$ with $\mathscr{B}(\ell_2^n, \ell_2^r)$, the space of all bounded linear operators from ℓ_2^n to ℓ_2^r, it is well known that every $\beta \in \mathbb{M}_{r,n}$ with $r \geqslant n$ has a *polar decomposition*, that is, β can be written as $\beta = \tau\beta_0$, where $\tau \in \mathbb{M}_{r,n}$ has orthonormal columns, that is, τ is an isometry, and $\beta_0 \in \mathbb{M}_n$ is positive semidefinite [5, §7.3]. For $p \neq 2$ and $r \geqslant n$, regarding $\mathbb{M}_{r,n}$ as $\mathscr{B}(\ell_p^n, \ell_p^r)$, the space of all bounded linear operators from ℓ_p^n to ℓ_p^r, we ask if there is an ℓ_p-analogue of the polar decomposition. First of all, we need to define what we should mean by polar decomposition when $p \neq 2$, because, for example, if $T : \ell_p^n \to \ell_p^n$, then the adjoint T' is from $\ell_{p'}^n$ to $\ell_{p'}^n$, where $1/p + 1/p' = 1$, and therefore $T'T$ is not defined, which in turn means we lose the concept of positive (semi)definiteness. We use the definition below as a natural p-analogue of the polar decomposition.

DEFINITION 4.1. Let $r \geqslant n$. We say that $\beta \in \mathbb{M}_{r,n} = \mathscr{B}(\ell_p^n, \ell_p^r)$ is ℓ_p*-polar decomposible* if there is an isometry $\tau \in \mathbb{M}_{r,n}$ and an operator $\beta_0 \in \mathbb{M}_n$ such that $\beta = \tau\beta_0$. In this case, we say that $\beta = \tau\beta_0$ is an ℓ_p*-polar decomposition* of β. The set of all full rank ℓ_p-polar decomposible $r \times n$ matrices is denoted by $\mathbb{M}_{r,n}^{(p)}$.

REMARK 4.2.

a. If $r < n$, then there is no isometry in $\mathbb{M}_{r,n} = \mathscr{B}(\ell_p^n, \ell_p^r)$ and hence we only consider the case $r \geqslant n$ in Definition 4.1.

b. It is well known [5, §0.4] that $\operatorname{rank} AB \leqslant \min\{\operatorname{rank} A, \operatorname{rank} B\}$ whenever AB is defined for matrices A and B, so if $\beta = \tau\beta_0$ is an ℓ_p-polar decomposition of a full rank $r \times n$ matrix β, then

$$n = \operatorname{rank} \beta \leqslant \min\{\operatorname{rank} \tau, \operatorname{rank} \beta_0\} \leqslant n$$

and it follows that $\operatorname{rank} \tau = \operatorname{rank} \beta_0 = n$. In particular, β_0 is nonsingular.

c. If $\beta = \tau\beta_0$ is an ℓ_p-polar decomposition of β, then $\|\beta\|_p = \|\beta_0\|_p$, where $\|\cdot\|_p$ is as in Lemma 3.2.

To give a characterization of ℓ_p-polar decomposible matrices, we begin with a characterization of isometries from ℓ_p^n to ℓ_p^r. Recall that for a vector $x = (x_1, \ldots, x_m)$, we define $\operatorname{supp} x$, the *support* of x, by $\operatorname{supp} x = \{i : 1 \leqslant i \leqslant m, \quad x_i \neq 0\}$.

LEMMA 4.3. *Let $1<p<\infty$, $p\neq 2$, and $r\geqslant n$. Then $\tau:\ell_p^n\to\ell_p^r$ is an isometry if and only if the columns of τ have mutually disjoint supports with each vector p-norm equal to* 1.

Proof. Let $\tau_j=\begin{bmatrix}\tau_{1j}\\ \vdots\\ \tau_{rj}\end{bmatrix}$ denote the j^{th} column of an $r\times n$ matrix τ. If $\tau_1,\dots,\tau_n$ have mutually disjoint supports with each p-norm equal to 1, then for any $x=(x_1,\dots,x_n)$ $\in\ell_p^n$, we get

$$\begin{aligned}\|\tau x\|_p^p&=\sum_{i=1}^{r}\left|\sum_{j=1}^{n}\tau_{ij}x_j\right|^p=\sum_{k=1}^{n}\sum_{i\in\operatorname{supp}\tau_k}\left|\sum_{j=1}^{n}\tau_{ij}x_j\right|^p\\&=\sum_{k=1}^{n}\sum_{i\in\operatorname{supp}\tau_k}|\tau_{ik}x_k|^p=\sum_{k=1}^{n}|x_k|^p\sum_{i\in\operatorname{supp}\tau_k}|\tau_{ik}|^p\\&=\|x\|_p^p.\end{aligned}$$

Conversely, suppose $\tau:\ell_p^n\to\ell_p^r$ is an isometry. Since $\tau_j=\tau e_j$ for each j, where e_j denotes the unit vector in ℓ_p^n whose only non-zero entry is 1 at the j^{th} place, it follows that τ_j is of norm 1. To show that columns of τ have mutually disjoint supports, let $j\neq k$ and consider $e_j\pm e_k$ in ℓ_p^n. Since $\|e_j\pm e_k\|_p=2^{1/p}$, we get $\|\tau_j\pm\tau_k\|_p^p=2$ and the result follows from [7, Lemma 15.7.23]. □

REMARK 4.4. The result above remains true when $p=1$.

Let V be a p-operator space. For $v\in\mathbb{M}_n(V)$, we define

$$\|v\|_{2,n}=\inf\{\|\alpha\|_{p'}\|w\|\|\beta\|_p : r\in\mathbb{N},\ v=\alpha w\beta,\ \alpha^T\in\mathbb{M}_{r,n}^{(p')},\ \beta\in\mathbb{M}_{r,n}^{(p)},\ w\in M_r(V)\},\tag{4.1}$$

where α^T denotes the transpose of α and

$$\|\alpha\|_{p'}=\left(\sum_{i=1}^{n}\sum_{j=1}^{r}|\alpha_{ij}|^{p'}\right)^{1/p'}\quad\text{and}\quad\|\beta\|_p=\left(\sum_{k=1}^{r}\sum_{l=1}^{n}|\beta_{kl}|^p\right)^{1/p}.$$

PROPOSITION 4.5. *Let $V\subseteq W$ be p-operator spaces. If $\|w\|_{2,n}=\|w\|_{1,n}$ for all $w\in\mathbb{M}_n(W)$, then the inclusion $\mathscr{T}_n(V)\hookrightarrow\mathscr{T}_n(W)$ is isometric.*

Proof. Let $v\in\mathbb{M}_n(V)$. It is clear that $\|v\|_{\mathscr{T}_n(W)}\leqslant\|v\|_{\mathscr{T}_n(V)}$. Suppose $\|v\|_{\mathscr{T}_n(W)}<1$, then by assumption, one can find $r\in\mathbb{N}$, $\alpha\in\mathbb{M}_{n,r}$, $\beta\in\mathbb{M}_{r,n}$, and $w\in M_r(W)$ such that $v=\alpha w\beta$, $\alpha^T\in\mathbb{M}_{r,n}^{(p')}$, $\beta\in\mathbb{M}_{r,n}^{(p)}$, $\|\alpha\|_{p'}<1$, $\|w\|<1$, and $\|\beta\|_p<1$. Let $\beta=\tau\beta_0$ (respectively, $\alpha^T=\sigma\alpha_0$) be ℓ_p-(respectively, ℓ_p'-) polar decomposition of β (respectively, α^T), and set $\tilde{w}=\sigma^T w\tau$, then $\|\tilde{w}\|_{M_n(W)}<1$. Moreover, by Remark 4.2, α_0 and β_0 are invertible and hence $\tilde{w}=(\alpha_0^T)^{-1}v\beta_0^{-1}\in M_n(V)$, giving that $\|\tilde{w}\|_{M_n(V)}<$

1. Since $v=\alpha_0^T\tilde{w}\beta_0$, $\|\alpha_0^T\|_{p'}=\|\alpha\|_{p'}<1$, and $\|\beta_0\|_p=\|\beta\|_p<1$ by Remark 4.2, it follows that $\|v\|_{\mathscr{T}_n(V)}<1$. $\square$

For any $v\in\mathbb{M}_n(V)$, it is clear that $\|v\|_{1,n}\leqslant\|v\|_{2,n}$ At this moment of writing, we do not know of any nontrivial example of p-operator space V such that $\|\cdot\|_{1,n}=\|\cdot\|_{2,n}$. It is not even clear whether $\|\cdot\|_{2,n}$ defines a norm on $\mathbb{M}_n(V)$ for some p-operator space V (see Remark 4.7). However, thanks to Lemma 4.3, we can give a characterization of ℓ_p-polar decomposible matrices which may lead to finding a nontrivial example of p-operator spaces V such that $\|v\|_{1,n}=\|v\|_{2,n}$ for all $v\in\mathbb{M}_n(V)$.

PROPOSITION 4.6. *Let $1<p<\infty$, $p\neq 2$, and $r\geqslant n$. Then* $\beta=\begin{bmatrix} — & u_1 & — \\ & \vdots & \\ — & u_r & — \end{bmatrix}\in$ $\mathbb{M}_{r,n}=\mathscr{B}(\ell_p^n,\ell_p^r)$ *is ℓ_p-polar decomposible if and only if there are $u_{j_1},u_{j_2},\dots,u_{j_n}$, not necessarily distinct, such that each u_i $(1\leqslant i\leqslant r)$ is a scalar multiple of u_{j_k} for some k, $1\leqslant k\leqslant n$.*

Proof. Let $\beta=\begin{bmatrix} — & u_1 & — \\ & \vdots & \\ — & u_r & — \end{bmatrix}\in\mathbb{M}_{r,n}=\mathscr{B}(\ell_p^n,\ell_p^r)$. Suppose that there are $u_{j_1},u_{j_2},\dots,u_{j_n}$ (not necessarily distinct) such that each u_i $(1\leqslant i\leqslant r)$ is a scalar multiple of u_{j_k} for some k, $1\leqslant k\leqslant n$. Rearranging rows of β with an appropriate permutation if necessary, we may assume that $1=j_1<j_2<j_3<\cdots<j_n\leqslant r$ and that for i with $j_k\leqslant i<j_{k+1}$, $u_i=c_iu_{j_k}$ for some scalar c_i. For each k, $1\leqslant k\leqslant n$, we define $\lambda_k=\left(\sum_{j_k\leqslant i<j_{k+1}}|c_i|^p\right)^{-p}$. Note that λ_k is well defined since $c_{j_k}=1$. Define $\tau\in\mathbb{M}_{r,n}$ and $\beta_0\in\mathbb{M}_n$ by

$$\tau=\begin{bmatrix} c_1\lambda_1 & 0 & 0\cdots & 0 \\ c_2\lambda_1 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots\ \ddots & \vdots \\ c_{j_2-1}\lambda_1 & 0 & 0\cdots & 0 \\ 0 & c_{j_2}\lambda_2 & 0\cdots & 0 \\ 0 & c_{j_2+1}\lambda_2 & 0\cdots & 0 \\ \vdots & \vdots & \vdots\ \ddots & \vdots \\ 0 & c_{j_3-1}\lambda_2 & 0\cdots & 0 \\ \vdots & \vdots & \vdots\ \ddots & \vdots \\ 0 & 0 & 0\cdots & c_{j_n}\lambda_n \\ 0 & 0 & 0\cdots & c_{j_n+1}\lambda_n \\ \vdots & \vdots & \vdots\ \ddots & \vdots \\ 0 & 0 & 0\cdots & c_r\lambda_n \end{bmatrix}\quad\text{and}\quad\beta_0=\begin{bmatrix} — & \frac{1}{\lambda_1}u_{j_1} & — \\ — & \frac{1}{\lambda_2}u_{j_2} & — \\ & \vdots & \\ — & \frac{1}{\lambda_n}u_{j_n} & — \end{bmatrix},$$

then by Lemma 4.3, it follows that $\beta=\tau\beta_0$ is an ℓ_p-polar decomposition of β.

Conversely, assume that $\beta = \tau\beta_0$ is a p-polar decomposition of β. To exclude triviality, we may assume that β contains no rows of only zeros. Let τ_k denote the k^{th} column of τ. By Lemma 4.3, $\operatorname{supp}\tau_k \neq \emptyset$ so we can pick $j_k \in \operatorname{supp}\tau_k$. Moreover, for each i, $1 \leqslant i \leqslant r$, there is exactly one $k(i)$ such that $i \in \operatorname{supp}\tau_{k(i)}$ and it follows that u_i is a constant multiple of $u_{j_{k(i)}}$. □

REMARK 4.7. Let $v_1 \in \mathbb{M}_n(V)$ and $v_2 \in \mathbb{M}_m(V)$ for some p-operator space V, then one can easily show that $\|cv_1\|_{2,n} = |c|\|v_1\|_{2,n}$. Moreover, the decomposition $v_1 = \alpha_1^T w_1 \beta_1$ and $v_2 = \alpha_2^T w_2 \beta_2$ gives

$$\begin{bmatrix} v_1 & \\ & v_2 \end{bmatrix} = \begin{bmatrix} \alpha_1^T & \\ & \alpha_2^T \end{bmatrix} \begin{bmatrix} w_1 & \\ & w_2 \end{bmatrix} \begin{bmatrix} \beta_1 & \\ & \beta_2 \end{bmatrix}, \tag{4.2}$$

which, combined with Proposition 4.6, shows that $\|v_1 \oplus v_2\|_{2,n+m} \leqslant \|v_1\|_{2,n} + \|v_2\|_{2,m}$.

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