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CONDITIONAL EXPECTATIONS OF RANDOM LINEAR OPERATORS IN BANACH SPACES

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Abstract. The paper generalizes in various ways factorization property, known till now only for real-valued random variables, to conditional expectations of random linear continuous transformations in Banach spaces measurable with respect to sub-σ-fields. We investigate compositions of random linear operations formed by means of outer and inner maps measurable with respect to sub-σ-fields and prove that under suitable assumptions such operations can be pulled out of the conditional expectation operator.

1. Introduction

Let f, g be real random variables (r.v.'s) on a probability space $(\Omega,\mathscr{F},P)$ and let $\mathscr{A}\subset\mathscr{F}$ be a sub-σ-field of $\mathscr{F}$. The following assertion is fairly well-known: if $E|g|<\infty$ and $E|f\cdot g|<\infty$, where f is $\mathscr{A}$-measurable, then (a) $E[f\cdot g|\mathscr{A}]=f\cdot E[g|\mathscr{A}]$ a.s. More generally, if ξ is a random element (r.e.) in a separable Banach space $\mathscr{X}$ such that $E\|\xi\|_{\mathscr{X}}<\infty$, and in addition $E\|f\cdot\xi\|_{\mathscr{X}}<\infty$, where f is a real-valued $\mathscr{A}$-measurable r.v., then (A) $E[f\cdot\xi|\mathscr{A}]=f\cdot E[\xi|\mathscr{A}]$ a.s. On the other hand, for an arbitrary linear bounded (nonrandom) operator $T:\mathscr{X}\to\mathscr{Y}$ acting on a separable Banach space $\mathscr{X}$ into a Banach space $\mathscr{Y}$, and any r.e. ξ in $\mathscr{X}$ with $E\|\xi\|_{\mathscr{X}}<\infty$, we have (B) $E[T(\xi)|\mathscr{A}]=T(E[\xi|\mathscr{A}])$ a.s., see e.g. [4], Ch. II, §4.1, (d), (f), p. 127. However, the analogous statement as (a) for random linear operators acting in Banach spaces is not known. The observations above motivate our study of random linear continuous operators. The aim of this article is to generalize (a), (A) and (B) to Banach space valued r.e.'s, assuming either $\mathscr{A}$-measurability of the r.e. ξ, or $\mathscr{A}$-measurability of random linear transformations taking values in the Banach space $\mathscr{L}(\mathscr{X},\mathscr{Y})=\{T:\mathscr{X}\to\mathscr{Y};\ T - \text{linear, continuous}\}$. Although our investigations are devoted mainly to some specific problems of probability theory, for the proof of our results we use some methods which may be interesting also from the point of view of functional analysis.

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2. Measurability of random compositions

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and let $\mathscr{X}$ be a real Banach space. Recall that the mapping $\xi : \Omega \to \mathscr{X}$ is called separably valued, if $\xi(\Omega)$ lies in a separable subspace $\mathscr{X}_0 \subseteq \mathscr{X}$, and ξ is said to be essentially separable (ess.sep.), if there exists a set $\Omega_1 \in \mathscr{F}$, $P[\Omega_1] = 1$, such that $\xi(\Omega_1)$ is a separable subset of $\mathscr{X}$. Let $\widehat{\mathscr{C}}(\mathscr{X})$ denote the cylindrical σ-field generated by all continuous linear functionals $x^* \in \mathscr{X}^*$, and let $\mathscr{B}(\mathscr{X})$ be the Borel σ-field in $\mathscr{X}$. The mapping $\xi : \Omega \to \mathscr{X}$ is called a random element (r.e.), if (i) $\xi^{-1}(\widehat{\mathscr{C}}(\mathscr{X})) \subseteq \mathscr{F}$, and it is called the Borel r.e. in $\mathscr{X}$, if (ii) $\xi^{-1}(\mathscr{B}(\mathscr{X})) \subseteq \mathscr{F}$. For separably valued maps the last two conditions are equivalent, cf. [4], Ch. II, §1.2, Th. 1.1, p. 89. Moreover, ξ is called P-measurable, if (iii) $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = 0$ a.s., where $\{\xi_n\}$ is a sequence of simple mappings, and ξ is called weakly P-measurable, if (iv) $x^*\xi$ is P-measurable for each $x^* \in \mathfrak{X}^*$. A P-measurable mapping $\xi : \Omega \to \mathscr{X}$, although it is ess.sep., need not be Borel measurable nor a r.e. in $\mathscr{X}$ unless $\mathscr{F}$ is P-complete, but if ξ is an ess.sep. map defined on the complete probability space, then all the above measurability conditions (i), (ii), (iii) and (iv) are equivalent, cf. [3], Ch. II, §1, Th. 2, p. 42, and [4], Ch. I, §1.4, Prop. 1.9, p. 12. Therefore, without loss of generality, we assume that all the σ-fields of subsets of the space Ω appearing below are P-complete. Such an assumption is not a severe restriction, for it is always possible to complete a given σ-field; otherwise a phrase "ess.sep. r.e." everywhere in what follows should be replaced by "P-measurable mapping".

Since we are going to consider r.e.'s in Banach spaces $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ consisting of linear continuous operators $T : \mathscr{X} \to \mathscr{Y}$, where $\mathscr{X}, \mathscr{Y}$ are both infinite dimensional Banach spaces, it should be pointed out here that an example of an infinite dimensional (possibly separable) Banach space $\mathscr{X}$ for which $\mathscr{L}(\mathscr{X}, \mathscr{X})$ is also a separable Banach space is not yet known. Furthermore, it seems that the question whether $\mathscr{L}(\mathscr{X}, \mathscr{X})$ is non-separable for every infinite dimensional Banach space $\mathscr{X}$ is still an open, difficult problem, cf. [1]. In this context our approach based on consideration of ess.sep. r.e.'s in non-separable Banach spaces is thereby the most reasonable and well-founded.

First we have to elucidate the problem concerning measurability of compound random mappings.

LEMMA 1. *Let $\xi : \Omega \to \mathscr{X}$ be an ess.sep. r.e. in a Banach space $\mathscr{X}$ and let $V : \Omega \to \mathscr{L}(\mathscr{X}, \mathscr{Y})$ be an ess.sep. r.e. in the Banach space $\mathscr{L}(\mathscr{X}, \mathscr{Y})$. Then $V(\xi) : \Omega \to \mathscr{Y}$ is an ess.sep. r.e. in the Banach space $\mathscr{Y}$.*

Operator version of the above lemma is the following result.

LEMMA 2. *Let $V : \Omega \to \mathscr{L}(\mathscr{X}, \mathscr{Y})$ and $U : \Omega \to \mathscr{L}(\mathscr{Y}, \mathscr{Z})$ be ess.sep. r.e.'s in Banach spaces $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ and $\mathscr{L}(\mathscr{Y}, \mathscr{Z})$ resp., where $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ are Banach spaces. Then $U(V) : \Omega \to \mathscr{L}(\mathscr{X}, \mathscr{Z})$ is also ess.sep. r.e. taking values in the Banach space $\mathscr{L}(\mathscr{X}, \mathscr{Z})$.*

The proof of Lemmas 1 and 2 is based on standard methods, i.e. verification of the statement for elementary r.e.'s taking at most countably many values and next an extension of it to the general case.

3. Conditional expectations of random linear transformations

According to the standard terminology, an ess.sep. $\mathscr{X}$-valued r.e. ξ is said to be of strong order 1, if $E\|\xi\|_{\mathscr{X}} < \infty$. The last condition is equivalent to the existence of Bochner integral, cf. [3], Ch. II, §2, Th. 2, p. 45. Let $\mathscr{A}$ be a (P-complete) sub-σ-field of the σ-field $\mathscr{F}$. Construction of conditional expectation $E(\xi|\mathscr{A})$ for separably valued r.e. ξ satisfying condition $E\|\xi\|_{\mathscr{X}} < \infty$ in a Banach space $\mathscr{X}$ can be found in [4]. If ξ is an ess.sep. r.e. of strong order 1 in $\mathscr{X}$, then the conditional expectation $E(\xi|\mathscr{A})$ is also well-defined, cf. [3], Ch. V, §1, Th. 4, p. 123, and the main features of conditional expectations in Banach spaces given for separably valued r.e.'s in [4], p. 127, remain valid also for ess. sep. r.e.'s.

Now we are able to generalize properties (A) and (B) of conditional expectations in Banach spaces (and at the same time, equation (a) valid only for real-valued r.v.'s).

THEOREM 1. *Let $\mathscr{X},\mathscr{Y}$ be arbitrary real Banach spaces and let $\xi:\Omega\to\mathscr{X}$ and $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y})$ be ess.sep. r.e.'s such that ξ and $V(\xi)$ are Bochner integrable. Moreover, let V be measurable with respect to the σ-field $\mathscr{A}\subset\mathscr{F}$. Then*

$$E[V(\xi)|\mathscr{A}] = V(E[\xi|\mathscr{A}]) \quad P|_{\mathscr{A}} - a.s.$$

Proof. As in the proof of Lemma I.1.2 of [4], p. 10, one can construct a sequence of $(\mathscr{B}(\mathscr{L}(\mathscr{X},\mathscr{Y})),\mathscr{A})$-measurable r.e.'s $V_n = \sum_{j=1}^{\infty} T_j^{(n)}\cdot \mathbb{1}_{A_j^{(n)}} + 0\cdot \mathbb{1}_{\Omega\setminus\bigcup_{j=1}^{\infty}A_j^{(n)}}$, such that

$$\|V(\omega)-V_n(\omega)\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})} < 1/n^2, \quad \omega\in\Omega_1 = \bigcup_{j=1}^{\infty} A_j^{(n)},$$

where $T_j^{(n)}\in\mathscr{L}(\mathscr{X},\mathscr{Y})$, the sets $A_j^{(n)}\in\mathscr{A}$, $j\geqslant 1$, are disjoint for all $n\geqslant 1$ and $P[\Omega_1]=1$. Clearly, $(V-V_n)(\xi)$ and $V_n(\xi)$ are then ess.sep. r.e.'s of strong order 1. Observe next that the norm $\|\cdot\|_{\mathscr{Y}}$ is a convex function in $\mathscr{Y}$, thus in view of Jensen's inequality for conditional expectations in Banach spaces given in [5], cf. also [3], Ch. V, §1, Th. 4, p. 123, we have

$$\begin{aligned} &E\|E[V(\xi)|\mathscr{A}] - E[V_n(\xi)|\mathscr{A}]\|_{\mathscr{Y}} \leqslant E\{E[\|(V-V_n)(\xi)\|_{\mathscr{Y}}|\mathscr{A}]\} \\ &= E\|(V-V_n)(\xi)\|_{\mathscr{Y}} \leqslant E\|V-V_n\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}\cdot\|\xi\|_{\mathscr{X}} \leqslant \frac{1}{n^2}\cdot E\|\xi\|_{\mathscr{X}} \to 0, \end{aligned}$$

as $n\to\infty$. Hence, for each $\varepsilon>0$,

$$P\left[\bigcup_{k\geqslant n}\{\|E(V-V_k)(\xi)|\mathscr{A}\|_{\mathscr{Y}} > \varepsilon\}\right] \leqslant \frac{E\|\xi\|_{\mathscr{X}}}{\varepsilon}\sum_{k=n}^{\infty}\frac{1}{k^2} \to 0, \quad n\to\infty,$$

therefore $\|E[V(\xi)|\mathscr{A}]-E[V_n(\xi)|\mathscr{A}]\|_{\mathscr{Y}} \to 0$ a.s. and in $L^1(R)$. Furthermore, let $V_n^m(\omega)=\sum_{j=1}^m T_j^{(n)}\cdot \mathbb{1}_{A_j^{(n)}}(\omega)$, $m \geqslant 1$. Since for all $m \geqslant 1$,

$$\|V_n(\xi)(\omega)\|_{\mathscr{Y}} = \sum_{j=1}^{m} \left\|T_j^{(n)}(\xi(\omega))\right\|_{\mathscr{Y}} \cdot \mathbb{1}_{A_j^{(n)}}(\omega) + \left\|\sum_{j=m+1}^{\infty} T_j^{(n)}(\xi(\omega))\cdot \mathbb{1}_{A_j^{(n)}}(\omega)\right\|_{\mathscr{Y}},$$

and *a fortiori*, for each $\omega\in\Omega$,

$$\|V_n(\xi)(\omega)-V_n^m(\xi)(\omega)\|_{\mathscr{Y}} = \left\|\sum_{j=m+1}^{\infty} T_j^{(n)}(\xi(\omega))\cdot \mathbb{1}_{A_j^{(n)}}(\omega)\right\|_{\mathscr{Y}} \leqslant \|V_n(\xi)(\omega)\|_{\mathscr{Y}},$$

we conclude that both sides above are P-integrable, because

$$\|V_n(\xi)\|_{\mathscr{Y}} \leqslant \|(V_n-V)(\xi)\|_{\mathscr{Y}} + \|V(\xi)\|_{\mathscr{Y}} \leqslant \frac{1}{n^2}\cdot\|\xi\|_{\mathscr{X}} + \|V(\xi)\|_{\mathscr{Y}} \quad a.s.$$

Moreover, if $\omega\in\Omega_1$, then $\omega\in A_{m_0}^{(n)}$ for some $m_0(\omega)\geqslant 1$, and so $\omega\notin\bigcup_{j=m+1}^{\infty}A_j^{(n)}$ when $m\geqslant m_0(\omega)$; consequently,

$$\|V_n(\xi)(\omega)-V_n^m(\xi)(\omega)\|_{\mathscr{Y}} = \left\|\sum_{j=m+1}^{\infty} T_j^{(n)}(\xi(\omega))\cdot \mathbb{1}_{A_j^{(n)}}(\omega)\right\|_{\mathscr{Y}} = 0$$

whenever $m\geqslant m_0(\omega)$. In other words, $\|V_n(\xi)(\omega)-V_n^m(\xi)(\omega)\|_{\mathscr{Y}}\to 0$ a.s. for every fixed $n\geqslant 1$ as $m\to\infty$. Applying the Lebesgue dominated convergence theorem we obtain

$$E\,\|(V_n-V_n^m)(\xi)\|_{\mathscr{Y}} \to 0 \quad \text{as} \quad m\to\infty.$$

Choosing an appropriate subsequence $m_n \nearrow \infty$ satisfying conditions

$$P\left[\bigcup_{j=m_n}^{\infty} A_j^{(n)}\right] < \frac{1}{n^2} \quad \text{and} \quad E\,\|(V_n-V_n^m)(\xi)\|_{\mathscr{Y}} < \frac{1}{n^2} \quad \text{for} \quad m\geqslant m_n,\ n\geqslant 1,$$

we have for each $\varepsilon>0$,

$$P\left[\bigcup_{k\geqslant n}\left\{\|V_k-V_k^{m_k}\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}>\varepsilon\right\}\right] \leqslant \sum_{k\geqslant n} P\left[\bigcup_{j=m_k}^{\infty} A_j^{(k)}\right] \leqslant \sum_{k\geqslant n}\frac{1}{k^2}\to 0,$$

so that $\|V_n-V_n^{m_n}\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}\to 0$ a.s. Arguing as before we also conclude that

$$\|E[V_n(\xi)|\mathscr{A}]-E[V_n^{m_n}(\xi)|\mathscr{A}]\|_{\mathscr{Y}} \to 0 \quad a.s. \text{ and in } L^1(R).$$

Hence, it follows that for suitably chosen simple r.e.'s $\{V_n^{m_n}, n\geqslant 1\}$,

$$\|E[V(\xi)|\mathscr{A}]-E[V_n^{m_n}(\xi)|\mathscr{A}]\|_{\mathscr{Y}} \to 0 \quad a.s. \text{ and in } L^1(R). \tag{1}$$

However, equations (A), (B) ensure that

$$E[V_n^{m_n}(\xi)|\mathscr{A}] = \sum_{j=1}^{m_n} T_j^{(n)}(E[\xi|\mathscr{A}]) \cdot \mathbb{1}_{A_j^{(n)}} = V_n^{m_n}(E(\xi|\mathscr{A})) \quad P|_{\mathscr{A}} - a.s., \tag{2}$$

and moreover

$$\|V(E(\xi|\mathscr{A})) - V_n^{m_n}(E(\xi|\mathscr{A}))\|_{\mathscr{Y}} \leqslant \tag{3}$$

$$\leqslant \left(\|V - V_n\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})} + \|V_n - V_n^{m_n}\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}\right) \cdot \|E(\xi|\mathscr{A})\|_{\mathscr{X}} \to 0 \quad P|_{\mathscr{A}} - a.s.$$

Finally, from (1), (2), (3) we infer that $E[V(\xi)|\mathscr{A}] = V(E[\xi|\mathscr{A}])$ $P|_{\mathscr{A}} - a.s.$ From the last equation and our assumption $E\|V(\xi)\|_{\mathscr{Y}} < \infty$ integrability of $\|V(E[\xi|\mathscr{A}])\|_{\mathscr{Y}}$ follows immediately. □

Now we discuss the case of composition $V(\xi)$, where the inner r.e. ξ is measurable with respect to a sub-σ-field. To obtain analogous statement as above we need an auxiliary result.

Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces, and let $\mathscr{X}' = \mathscr{L}(\mathscr{X},\mathscr{Y})$. Then each element $x \in \mathscr{X}$ can be considered as a function $x : \mathscr{X}' \to \mathscr{Y}$, given by $x(T) = T(x)$ for $T \in \mathscr{L}(\mathscr{X},\mathscr{Y}) = \mathscr{X}'$. On the basis of the Hahn-Banach theorem, see e.g. [2], Ch. III, Th. 6.2 and Corollary 6.8, p. 78-79, we can deduce the following result.

LEMMA 3. *The above defined mapping $x : \mathscr{X}' \to \mathscr{Y}$ is linear and continuous operator, with the norm $\|x\|_{\mathscr{L}(\mathscr{X}',\mathscr{Y})} = \|x\|_{\mathscr{X}}$. In consequence, $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ can be treated as a closed linear subspace of $(\mathscr{L}(\mathscr{X}',\mathscr{Y}), \|\cdot\|_{\mathscr{L}(\mathscr{X}',\mathscr{Y})})$.*

We are now in a position to extend slightly the properties (A), (B) of conditional expectations in Banach spaces quoted at the beginning.

LEMMA A_1. *Let $V : \Omega \to \mathscr{L}(\mathscr{X},\mathscr{Y})$ be an ess. sep. Bochner integrable r.e. in the Banach space $\mathscr{L}(\mathscr{X},\mathscr{Y})$, and let $f : \Omega \to R$ be $\mathscr{A}$-measurable r.v. such that $E\|f \cdot V\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})} < \infty$. Then*

$$E[f \cdot V|\mathscr{A}] = f \cdot E[V|\mathscr{A}] \quad P|_{\mathscr{A}} - a.s.;$$

in particular, for an arbitrary $x \in \mathscr{X}$,

$$E[V(fx)|\mathscr{A}] = f \cdot E[V(x)|\mathscr{A}] \quad P|_{\mathscr{A}} - a.s.$$

Proof. The first statement is a special case of (A). Moreover,

$$E\|V(fx)\|_{\mathscr{Y}} \leqslant E\|f \cdot V\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})} \cdot \|x\|_{\mathscr{X}} < \infty,$$

and if $V(\Omega_1)$ is a separable subset of $\mathscr{L}(\mathscr{X},\mathscr{Y})$, where $\Omega_1 \in \mathscr{F}$, $P[\Omega_1] = 1$, then $V(x)(\Omega_1)$ is a separable subset of $\mathscr{Y}$. Hence the second assertion follows. □

In the situation described by Lemma 3, we are able to obtain also an analogue of property (B), which may be of independent interest. The result below follows easily from Lemma 3 and needs no proof.

LEMMA B$_1$. *Let $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y})$ be an ess.sep. Bochner integrable r.e. in the Banach space $\mathscr{L}(\mathscr{X},\mathscr{Y})$, and let $x\in\mathscr{X}$ be a fixed point. Then*

$$E[V(x)|\mathscr{A}]=E[x(V)|\mathscr{A}]=x(E[V|\mathscr{A}])=E[V|\mathscr{A}](x) \quad P|_{\mathscr{A}}-a.s.$$

Lemma B$_1$ enables us to prove in a direct way the next result, but we prefer to use Lemma 3 and Theorem 1 for the proof of it.

THEOREM 2. *Let $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y})$, $\xi:\Omega\to\mathscr{X}$ be ess.sep. r.e.'s in Banach spaces $\mathscr{L}(\mathscr{X},\mathscr{Y})$ and $\mathscr{X}$ resp., such that V and $V(\xi)$ are Bochner integrable. Moreover, suppose ξ is measurable with respect to the σ-field $\mathscr{A}\subset\mathscr{F}$. Then*

$$E[V(\xi)|\mathscr{A}]=(E[V|\mathscr{A}])(\xi) \quad P|_{\mathscr{A}}-a.s.$$

Proof. Denote $\mathscr{X}'=\mathscr{L}(\mathscr{X},\mathscr{Y})$ and observe that in view of Lemma 3, ξ can be treated as an ess.sep. r.e. taking values in the Banach space $\mathscr{L}(\mathscr{X}',\mathscr{Y})$. In such a case the pair (V,ξ) satisfies the assumptions of Theorem 1, provided the roles of ξ and V are interchanged. Essentially, in the proof of Theorem 1 the random linear transformation V need not take values in the whole space $\mathscr{L}(\mathscr{X},\mathscr{Y})$, but it may assume values in some closed linear subspace of $\mathscr{L}(\mathscr{X},\mathscr{Y})$, and in fact in our situation the random linear transformation ξ takes values in the space $\mathscr{X}\subset\mathscr{L}(\mathscr{X}',\mathscr{Y})$. Thus the conclusion $E[V(\xi)|\mathscr{A}]=E[\xi V|\mathscr{A}]=\xi(E[V|\mathscr{A}])=E[V|\mathscr{A}](\xi)$ a.s. of Theorem 2 follows easily from Lemma 3 and Theorem 1. □

4. Conditional expectations for compositions of random linear operators

In this section we consider conditional expectations of the form $E[U(V)|\mathscr{A}]$, where both U and V are random linear transformations. Although now the situation seems to be different than before, we show that the problem can be reduced to the case described already in Theorem 1.

Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be Banach spaces and let $T:\mathscr{X}\to\mathscr{Y}$ and $S:\mathscr{Y}\to\mathscr{Z}$ be (nonrandom) continuous linear operators, i.e. $T\in\mathscr{L}(\mathscr{X},\mathscr{Y})$, $S\in\mathscr{L}(\mathscr{Y},\mathscr{Z})$. Then in view of the inequality $\|S(T)\|_{\mathscr{L}(\mathscr{X},\mathscr{Z})}\leqslant\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\cdot\|T\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}$, the composition $S(T):\mathscr{X}\to\mathscr{Z}$ is obviously also linear and continuous, i.e. $S(T)\in\mathscr{L}(\mathscr{X},\mathscr{Z})$.

Denote $\mathscr{X}'=\mathscr{L}(\mathscr{X},\mathscr{Y})$, $\mathscr{Z}'=\mathscr{L}(\mathscr{X},\mathscr{Z})$ and observe that S may be viewed as a function $S:\mathscr{X}'=\mathscr{L}(\mathscr{X},\mathscr{Y})\to\mathscr{Z}'=\mathscr{L}(\mathscr{X},\mathscr{Z})$. Applying again the Hahn-Banach theorem already mentioned, cf. [2], Ch. III, Th. 6.2 and Corollary 6.8, p. 78-79, we are able to prove the following result.

LEMMA 4. *The mapping $S:\mathscr{X}'\to\mathscr{Z}'$ is a linear continuous operator with the norm $\|S\|_{\mathscr{L}(\mathscr{X}',\mathscr{Z}')}=\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}$. In consequence, $\left(\mathscr{L}(\mathscr{Y},\mathscr{Z}),\|\cdot\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\right)$ is a closed linear subspace of the Banach space $\left(\mathscr{L}(\mathscr{X}',\mathscr{Z}'),\|\cdot\|_{\mathscr{L}(\mathscr{X}',\mathscr{Z}')}\right)$.*

A version of Lemma B$_1$ for the composition of random linear transformations may be also of interest.

LEMMA B_2. *Let $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y})$ be an ess.sep. Bochner integrable r.e. in the Banach space $\mathscr{L}(\mathscr{X},\mathscr{Y})$, and let $S:\mathscr{Y}\to\mathscr{Z}$ be a fixed continuous linear operator. Then for each σ-field $\mathscr{A}\subset\mathscr{F}$,*

$$E[S(V)|\mathscr{A}]=S(E[V|\mathscr{A}])\quad P|_{\mathscr{A}}-a.s.$$

On the basis of these considerations we obtain in addition the following analogue of Theorem 1 for compositions of random linear operators.

THEOREM 3. *Let $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y})$ be an ess.sep. Bochner integrable r.e. with values in the Banach space $\mathscr{L}(\mathscr{X},\mathscr{Y})$. Moreover, let $U:\Omega\to\mathscr{L}(\mathscr{Y},\mathscr{Z})$ be an ess.sep. r.e. in the Banach space $\mathscr{L}(\mathscr{Y},\mathscr{Z})$, measurable with respect to the σ-field $\mathscr{A}\subset\mathscr{F}$, such that $U(V):\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Z})$ is also Bochner integrable. Then*

$$E[U(V)|\mathscr{A}]=U(E[V|\mathscr{A}])\quad P|_{\mathscr{A}}-a.s.$$

Proof. Denote $\mathscr{X}'=\mathscr{L}(\mathscr{X},\mathscr{Y})$, $\mathscr{Z}'=\mathscr{L}(\mathscr{X},\mathscr{Z})$, and consider U as a random linear transformation taking values in the Banach space $\mathscr{L}(\mathscr{X}',\mathscr{Z}')$. Then, by Lemma 4, the pair of mappings (U,V), playing the same role as before (V,ξ), satisfies the assumptions of Theorem 1. Hence the conclusion follows. □

As before, in order to obtain an analogous statement in the case when inner operation is measurable with respect to a sub-σ-field, we interchange the roles played by considered transformations.

Let $T:\mathscr{X}\to\mathscr{Y}$, $T\in\mathscr{L}(\mathscr{X},\mathscr{Y})$ and $S:\mathscr{Y}\to\mathscr{Z}$, $S\in\mathscr{L}(\mathscr{Y},\mathscr{Z})$ be continuous linear operators acting in Banach spaces $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$. Then $S(T):\mathscr{X}\to\mathscr{Z}$ is also a continuous linear operator, i.e. $S(T)\in\mathscr{L}(\mathscr{X},\mathscr{Z})$. Consider now $T\in\mathscr{L}(\mathscr{X},\mathscr{Y})$ as a map $T:\mathscr{L}(\mathscr{Y},\mathscr{Z})\to\mathscr{L}(\mathscr{X},\mathscr{Z})$, given by

$$\mathscr{L}(\mathscr{Y},\mathscr{Z})\ni S\mapsto T(S)=S(T)\in\mathscr{L}(\mathscr{X},\mathscr{Z}).$$

LEMMA 5. *Let $T:\mathscr{L}(\mathscr{Y},\mathscr{Z})\to\mathscr{L}(\mathscr{X},\mathscr{Z})$ be the mapping defined by the above formula. Then T is a continuous linear operator with the norm*

$$\|T\|_{\mathscr{L}(\mathscr{L}(\mathscr{Y},\mathscr{Z}),\mathscr{L}(\mathscr{X},\mathscr{Z}))}=\|T\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}.$$

Consequently, $\left(\mathscr{L}(\mathscr{X},\mathscr{Y}),\|\cdot\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}\right)$ can be treated as a closed linear subspace of the Banach space $\left(\mathscr{L}(\mathscr{L}(\mathscr{Y},\mathscr{Z}),\mathscr{L}(\mathscr{X},\mathscr{Z})),\|\cdot\|_{\mathscr{L}(\mathscr{L}(\mathscr{Y},\mathscr{Z}),\mathscr{L}(\mathscr{X},\mathscr{Z}))}\right)$.

Proof. Evidently, the operator T defined above is linear. To simplify the writing, put $\mathscr{Y}'=\mathscr{L}(\mathscr{Y},\mathscr{Z})$, $\mathscr{Z}'=\mathscr{L}(\mathscr{X},\mathscr{Z})$. Then

$$\begin{aligned}\|T\|_{\mathscr{L}(\mathscr{Y}',\mathscr{Z}')}&=\sup_{\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\leqslant 1}\|S(T)\|_{\mathscr{L}(\mathscr{X},\mathscr{Z})}=\sup_{\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\leqslant 1}\left(\sup_{\|x\|_{\mathscr{X}}\leqslant 1}\|S(T)(x)\|_{\mathscr{Z}}\right)\\&\leqslant\sup_{\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\leqslant 1}\left(\sup_{\|x\|_{\mathscr{X}}\leqslant 1}\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\cdot\|T(x)\|_{\mathscr{Y}}\right)=1\cdot\|T\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})},\end{aligned}$$

thus $T:\mathscr{Y}'\to\mathscr{Z}'$ is continuous. Furthermore, from the above inequality it follows that $\|T\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}=0$ implies $\|T\|_{\mathscr{L}(\mathscr{Y}',\mathscr{Z}')}=0$, i.e. in the case when $\|T\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}=0$ both norms are equal. Fix $T\in\mathscr{L}(\mathscr{X},\mathscr{Y})$ with $\|T\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}>0$. Notice that on account of Lemma 3 each element $y=T(x)$, where $x\in\mathscr{X}$ is arbitrary, may be treated as a continuous linear operator acting on the Banach space $\mathscr{L}(\mathscr{Y},\mathscr{Z})$ into $\mathscr{Z}$, given by

$$\mathscr{L}(\mathscr{Y},\mathscr{Z})\ni S\mapsto y(S)=S(y)=S(T(x))\in\mathscr{Z},$$

with the norm

$$\|T(x)\|_{\mathscr{L}(\mathscr{L}(\mathscr{Y},\mathscr{Z}),\mathscr{Z})}=\sup_{\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\leqslant 1}\|y(S)\|_{\mathscr{Z}}=\sup_{\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\leqslant 1}\|S(T(x))\|_{\mathscr{Z}}=\|T(x)\|_{\mathscr{Y}}.$$

Therefore, considering $x\in\mathscr{X}$, $\|x\|_{\mathscr{X}}>0$, we obtain the estimate

$$\begin{aligned}\|T\|_{\mathscr{L}(\mathscr{Y}',\mathscr{Z}')}&=\sup_{\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\leqslant 1}\left(\sup_{\|x\|_{\mathscr{X}}\leqslant 1}\|S(T)(x)\|_{\mathscr{Z}}\right)\geqslant\sup_{\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\leqslant 1}\left\|S(T)\left(\frac{x}{\|x\|_{\mathscr{X}}}\right)\right\|_{\mathscr{Z}}\\&=\frac{1}{\|x\|_{\mathscr{X}}}\cdot\sup_{\|S\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}\leqslant 1}\|T(x)(S)\|_{\mathscr{Z}}=\frac{1}{\|x\|_{\mathscr{X}}}\cdot\|T(x)\|_{\mathscr{L}(\mathscr{L}(\mathscr{Y},\mathscr{Z}),\mathscr{Z})}\\&=\frac{1}{\|x\|_{\mathscr{X}}}\cdot\|T(x)\|_{\mathscr{Y}},\end{aligned}$$

so that $\|T(x)\|_{\mathscr{Y}}\leqslant\|T\|_{\mathscr{L}(\mathscr{Y}',\mathscr{Z}')}\cdot\|x\|_{\mathscr{X}}$. The last inequality is valid as well when $x=0$, for $T(0)=0$, and since

$$\|T\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}=\inf\{M>0:\|T(x)\|_{\mathscr{Y}}\leqslant M\cdot\|x\|_{\mathscr{X}}\text{ for all }x\in\mathscr{X}\},$$

we conclude that $\|T\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}\leqslant\|T\|_{\mathscr{L}(\mathscr{Y}',\mathscr{Z}')}$. Hence and from the already verified opposite inequality, it follows finally that

$$\|T\|_{\mathscr{L}(\mathscr{L}(\mathscr{Y},\mathscr{Z}),\mathscr{L}(\mathscr{X},\mathscr{Z}))}=\|T\|_{\mathscr{L}(\mathscr{Y}',\mathscr{Z}')}=\|T\|_{\mathscr{L}(\mathscr{X},\mathscr{Y})}.\quad\square$$

On the basis of the above result we get immediately the following version of Lemma B_1.

LEMMA B_3. *Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be Banach spaces, let $T:\mathscr{X}\to\mathscr{Y}$ be a fixed continuous linear operator and let $U:\Omega\to\mathscr{L}(\mathscr{Y},\mathscr{Z})$ be an ess.sep. Bochner integrable r.e. in the Banach space $\mathscr{L}(\mathscr{Y},\mathscr{Z})$. Then for an arbitrary σ-field $\mathscr{A}\subset\mathscr{F}$,*

$$E[U(T)|\mathscr{A}]=E[T(U)|\mathscr{A}]=T(E[U|\mathscr{A}])=E[U|\mathscr{A}](T)\quad P|_{\mathscr{A}}-a.s.$$

Now we are able to formulate a version of Theorem 3 for inner transformation measurable with respect to a sub-σ-field.

THEOREM 4. *Let $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y})$, and $U:\Omega\to\mathscr{L}(\mathscr{Y},\mathscr{Z})$ be ess.sep. r.e.'s taking values Banach spaces $\mathscr{L}(\mathscr{X},\mathscr{Y})$ and $\mathscr{L}(\mathscr{Y},\mathscr{Z})$ resp., where $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ are Banach spaces. Assume that U and $U(V):\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Z})$ are Bochner integrable, and moreover, V is measurable with respect to the σ-field $\mathscr{A}\subset\mathscr{F}$. Then*

$$E[U(V)|\mathscr{A}]=E[U|\mathscr{A}](V)\quad P|_{\mathscr{A}}-a.s.$$

Proof. By analogy to the above arguments, consider r.e. $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y})$ as a random linear transformation taking values in the closed linear (Banach) subspace $\mathscr{L}(\mathscr{X},\mathscr{Y})$ of the Banach space $\mathscr{L}(\mathscr{L}(\mathscr{Y},\mathscr{Z}),\mathscr{L}(\mathscr{X},\mathscr{Z}))$, defined for each $\omega\in\Omega$ and $S\in\mathscr{L}(\mathscr{Y},\mathscr{Z})$ by $V(\omega)(S)=S(V(\omega))$. Then it is clear that r.e.'s $U:\Omega\to\mathscr{Y}'=\mathscr{L}(\mathscr{Y},\mathscr{Z})$ and $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y})\subseteq\mathscr{L}(\mathscr{Y}',\mathscr{Z}')$, where $\mathscr{Z}'=\mathscr{L}(\mathscr{X},\mathscr{Z})$, satisfy all the requirements of Theorem 3, provided their roles are interchanged. Hence it follows that

$$E[U(V)|\mathscr{A}]=E[V(U)|\mathscr{A}]=V(E[U|\mathscr{A}])=E[U|\mathscr{A}](V)\quad P|_{\mathscr{A}}-a.s.\quad\Box$$

The methods used for the proof of the above theorems suggest that for composition of a greater number of random linear operations it should be possible to separate at the same time outer and inner transformations measurable with respect to a sub-σ-field. In this way we obtain the following result.

COROLLARY 1. *Let $\xi:\Omega\to\mathscr{X}$, $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y})$, and $U:\Omega\to\mathscr{L}(\mathscr{Y},\mathscr{Z})$ be ess.sep. r.e.'s taking values in Banach spaces $\mathscr{X}$, $\mathscr{L}(\mathscr{X},\mathscr{Y})$ and $\mathscr{L}(\mathscr{Y},\mathscr{Z})$ resp., such that V, the composition $U(V(\xi)):\Omega\to\mathscr{Z}$, and either $V(\xi):\Omega\to\mathscr{Y}$, or $U(V):\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Z})$ are Bochner integrable. If r.e.'s ξ and U are measurable with respect to the σ-field $\mathscr{A}\subset\mathscr{F}$, then*

$$E[U(V(\xi))|\mathscr{A}]=U(E[V|\mathscr{A}](\xi))\quad P|_{\mathscr{A}}-a.s.$$

Proof. In the case when $E\|V(\xi)\|_{\mathscr{Y}}<\infty$, on account of Theorems 1 and 2 we have

$$E[U(V(\xi))|\mathscr{A}]=U\left(E[V(\xi)|\mathscr{A}]\right)=U(E[V|\mathscr{A}](\xi))\quad P|_{\mathscr{A}}-a.s.$$

On the other hand, if $E\|U(V)\|_{\mathscr{L}(\mathscr{X},\mathscr{Z})}<\infty$, then applying Theorems 2 and 3 we conclude that

$$E[U(V(\xi))|\mathscr{A}]=E[U(V)|\mathscr{A}](\xi)=U(E[V|\mathscr{A}](\xi))\quad P|_{\mathscr{A}}-a.s.\quad\Box$$

Obviously, we may treat a greater number of random linear operators in a similar manner.

COROLLARY 2. *Let $V:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Y}), U:\Omega\to\mathscr{L}(\mathscr{Y},\mathscr{Z}), W:\Omega\to\mathscr{L}(\mathscr{Z},\mathscr{W})$ be ess.sep. r.e.'s with values in spaces of continuous linear operators $\mathscr{L}(\mathscr{X},\mathscr{Y})$, $\mathscr{L}(\mathscr{Y},\mathscr{Z})$ and $\mathscr{L}(\mathscr{Z},\mathscr{W})$ resp., where $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$, $\mathscr{W}$ are Banach spaces. Assume that r.e.'s U and the composition $WUV:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{W})$ are Bochner integrable, and moreover, either $UV:\Omega\to\mathscr{L}(\mathscr{X},\mathscr{Z})$, or $WU:\Omega\to\mathscr{L}(\mathscr{Y},\mathscr{W})$ are also Bochner integrable. If r.e.'s V and W are measurable with respect to the σ-field $\mathscr{A}\subset\mathscr{F}$, then*

$$E[WUV|\mathscr{A}]=WE[U|\mathscr{A}]V\quad P|_{\mathscr{A}}-a.s.$$

Proof. For the proof of the above equation we use analogous argumentation as in the proof of Corollary 1, but instead of Theorems 1 and 2 we apply Theorems 3 and 4. Namely, if $E\|UV\|_{\mathscr{L}(\mathscr{X},\mathscr{Z})} < \infty$, then on account of Theorems 3 and 4, we have

$$E[WUV|\mathscr{A}] = WE[UV|\mathscr{A}] = WE[U|\mathscr{A}]V \quad P|_{\mathscr{A}} - a.s.,$$

and if $E\|WU\|_{\mathscr{L}(\mathscr{Y},\mathscr{W})} < \infty$, then using first Theorem 4, and next Theorem 3, we obtain

$$E[WUV|\mathscr{A}] = E[WU|\mathscr{A}]V = WE[U|\mathscr{A}]V \quad P|_{\mathscr{A}} - a.s. \quad \square$$

REMARK. Part II of this paper is devoted to limit theorems for martingales formed by means of composition of random linear operators.

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