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MAPS ON OPERATORS STRONGLY PRESERVING SHARP ORDER

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Abstract. We discuss the structure of additive transformations on $B(H)$ that are strongly monotone with respect to the $\overset{\sharp}{\leqslant}$-order and characterize them under the assumption of bijectivity. We also characterize bijective transformations on the sets of linear combinations of idempotents in $B(H)$ that are strongly monotone with respect to the $\overset{\sharp}{\leqslant}$-order.

1. Introduction

Let $\mathbb{F}$ be a field of real or complex numbers and H denote a Hilbert space on the field $\mathbb{F}$. Let $M_n(\mathbb{F})$ denote the space of square matrices of order n with coefficients from the field $\mathbb{F}$. We denote by $B(H)$ the space of linear bounded operators on the space H.

DEFINITION 1.1. A matrix $A \in M_n(\mathbb{F})$ has the *index* l ($\mathrm{Ind}A = l$) if $\mathrm{rk}A^l = \mathrm{rk}A^{l+1}$ and l is the smallest positive number with this property.

In particular, we note that any diagonalizable matrix A has index 1, i.e. $\mathrm{rk}A = \mathrm{rk}A^2$.

DEFINITION 1.2. [22] Let $A \in M_n(\mathbb{F})$. The system of matrix equations

$$AXA = A, \quad XAX = X, \quad AX = XA$$

has a solution X if and only if $\mathrm{Ind}A = 1$. This solution is unique. It is called the *group inverse* of A and is denoted by $A^\sharp$.

Group inverse is one of the matrix generalized inverses, which have many useful properties and applications. A more detailed description of this topic can be found for example in [2, 23]. An interesting application of generalized inverses is the fact that they can be utilized to introduce order relations on matrices. In particular, the group inverse leads to the following order relation called the sharp order and denoted by the symbol $\overset{\sharp}{\leqslant}$.

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DEFINITION 1.3. [22] Let $A,B \in M_n(\mathbb{F})$. Then $A \overset{\sharp}{\leqslant} B$ if and only if $A = B$ or $\mathrm{Ind}A = 1$ and $AA^\sharp = BA^\sharp = A^\sharp B$. Moreover, if $A \overset{\sharp}{\leqslant} B$ and $A \neq B$, then $A \overset{\sharp}{<} B$.

We remark that there are many orders that can be introduced on the matrix algebra. In particular, well-known minus order defined below is related to the introduced order, see [23].

DEFINITION 1.4. [17] Let $A,B \in M_n(\mathbb{F})$. Then $A \overset{-}{\leqslant} B$ if and only if $\mathrm{rk}(B-A) = \mathrm{rk}B - \mathrm{rk}A$.

Let us note that $\overset{\sharp}{\leqslant}$-order is stronger than $\overset{-}{\leqslant}$-order.

LEMMA 1.5. [18], [23, Chapter 4] *Let* $A,B \in M_n(\mathbb{F})$, $A \overset{\sharp}{\leqslant} B$. *Then* $A \overset{-}{\leqslant} B$.

The detailed and self-contained information on the matrix partial orders can be found in [23].

Recently Šemrl [29] extended the minus partial order from $M_n(\mathbb{F})$ to $B(H)$. More precisely, he defined this order on $B(H)$ in the following way: for $A,B \in B(H)$ we have $A \overset{-}{\leqslant} B$ if and only if there exist idempotent operators $P,Q \in B(H)$ such that $\mathrm{Im}\,P = \overline{\mathrm{Im}A}$, $\mathrm{Ker}\,Q = \mathrm{Ker}A$, $PA = PB$ and $AQ = BQ$. It is proved in [29] that this definition reduces to the standard definition (see Definition 1.4) in the finite dimensional case.

Several other matrix partial orders have been generalized to linear bounded operators on the Hilbert or Banach spaces, see [8, 9]. The analog of the sharp order in infinite dimensional case was introduced recently in [12] by the first author.

If we have a partially ordered set, it is natural to ask about its automorphisms, i.e. those maps on this set which keep the order relation invariant.

Let $\leqslant$ be a certain partial order relation on the set S.

DEFINITION 1.6. A map $T : S \to S$ is called *monotone* with respect to $\leqslant$-order, if for arbitrary two elements $A,B \in S$ it holds that $A \leqslant B$ implies $T(A) \leqslant T(B)$.

DEFINITION 1.7. A map $T : S \to S$ is called *strongly monotone* with respect to $\leqslant$-order, if for arbitrary two elements $A,B \in S$ it holds that the conditions $A \leqslant B$ and $T(A) \leqslant T(B)$ are equivalent.

Monotone transformations were investigated during recent decades, see for example [1, 3, 10, 11, 12, 13, 14, 19, 20, 26, 28] and references therein for finite dimensional case. In particular, monotone transformations on matrices defined via the group inverse were also previously characterized in [3, 10, 11, 13, 14].

In parallel, the study of monotone maps was continued in the infinite dimensional case, in particular, monotone transformations of operators on Hilbert spaces are investigated intensively. Ovchinnikov in [25] obtained the characterization of bijective maps on the sets of skew projectors on Hilbert spaces, which are monotone with respect to minus order, see Theorem 3.6 below. In [29] Šemrl extended this result to the whole space $B(H)$. Recently in the work [7] Dolinar, Marovt and the second author characterized bijective additive continuous maps on the set of compact operators on the Hilbert space which are strongly monotone with respect to the Drazin star partial order. In [8] they obtain the characterization of bijective additive maps on $B(H)$ which are strongly monotone with respect to either left star or right star orders. Characterizations of the partial orderings for bounded operators were also studied in [6] by Deng and Wang. In [21] Marovt, Rakić, and Djordjević investigated star, left-star, and right-star partial orders in Rickart $*$-rings. Bohata and Hamhalter classified nonlinear maps on von Neumann algebras preserving the star order in [4] and investigated star order on JBW algebras in [5]. There are several interesting results by Rakić and Djordjević concerning the space pre-order and minus partial order for operators on Banach spaces, see [27]; by Hamhalter concerning isomorphisms of ordered structures of abelian C^*-subalgebras of C^*-algebras, see [15]; and by Hamhalter and Turilova concerning automorphisms of order structures of abelian parts of operator algebras and their applications in quantum theory, see [16].

At the same time, $\overset{\sharp}{\leqslant}$-order and corresponding monotone transformations, which were investigated a lot in finite dimensional case, see the monographs [2, 23], were not studied in infinite dimension. The first author in the paper [11] defined an analog of the $\overset{\sharp}{\leqslant}$-order in the infinite dimensional case. The aim of the present paper is to study the corresponding monotone transformations. Our paper contains the characterization of bijective additive transformations on $B(H)$ that are strongly monotone with respect to the $\overset{\sharp}{\leqslant}$-order. We also characterize bijective not necessarily additive transformations on finite linear combinations of idempotents from $B(H)$ that are strongly monotone with respect to the $\overset{\sharp}{\leqslant}$-order. Observe that in the first case we obtain as a corollary that the maps are automatically semi-linear.

Our paper is organized as follows. In Section 2 we introduce $\overset{\sharp}{\leqslant}$-order for linear bounded operators on Hilbert spaces and recall several its properties. Section 3 is devoted to the characterization of monotone additive bijective maps. Section 4 deals with the bijective monotone transformations on the finite linear combinations of idempotent operators. In Section 5 we collect several examples showing that our assumptions are indispensable and further problems.

2. Definition of the sharp order

DEFINITION 2.1. The operator $P \in B(H)$ is called an *idempotent* if $P^2 = P$. By $\mathfrak{I} = \{P \in B(H) \mid P^2 = P\}$ we denote the set of all idempotents. By $I \in \mathfrak{I}$ we denote the identity operator.

For bounded linear operators over Hilbert space the following analog of the minus order was introduced by Šemrl [29]:

DEFINITION 2.2. [29] Let $A, B \in B(H)$. It is said that $A \overset{-}{\leqslant} B$ if there exist idempotents $P, Q \in B(H)$ such that $\overline{\operatorname{Im} A} = \operatorname{Im} P$, $\operatorname{Ker} A = \operatorname{Ker} Q$, $PA = PB$, $AQ = BQ$.

The analog of $\overset{\sharp}{\leqslant}$-order for linear bounded operators was defined in [12]. We provide definitions and several results from that paper, without proofs.

DEFINITION 2.3. [12, Definition 6] Let $A,B \in B(H)$. We say that $A \overset{\sharp}{\leqslant} B$ if $A=B$ or there exists an idempotent $P \in B(H)$ such that $\overline{\operatorname{Im} A} = \operatorname{Im} P$, $\operatorname{Ker} A = \operatorname{Ker} P$, $PA = PB$, $AP = BP$.

The following lemma is straightforward.

LEMMA 2.4. [12, Lemma 1] *Let* $A,P \in B(H)$ *and* P *be an idempotent* $\overline{\operatorname{Im} A} = \operatorname{Im} P$, $\operatorname{Ker} A = \operatorname{Ker} P$. *Then* $AP = PA = A$.

For the $\overset{\sharp}{\leqslant}$-order in the finite-dimensional case we have:

PROPOSITION 2.5. [12, Statement 1] *If* H *is finite-dimensional, then Definitions* 1.3 *and* 2.3 *are equivalent.*

The following lemma is analogous to the corresponding matrix result (see [2]).

THEOREM 2.6. [12, Lemma 2] *Let* H *be a Hilbert space,* $A,B \in B(H)$, $A \overset{\sharp}{\leqslant} B$. *Then* $A \overline{\leqslant} B$.

We denote by $\mathfrak{G}$ the set of $A \in B(H)$ such that there exists an idempotent P satisfying the condition $\overline{\operatorname{Im} A} = \operatorname{Im} P$, $\operatorname{Ker} A = \operatorname{Ker} P$. An idempotent is completely determined by its image and kernel. Hence for any $A \in \mathfrak{G}$ the idempotent P is uniquely determined. Let us denote this idempotent P by $\pi(A)$.

Further we denote

$$\overset{\circ}{\mathfrak{G}} = \{A \in \mathfrak{G} \mid \operatorname{Ker} A \neq 0\},$$

$$\overset{\circ}{\mathfrak{I}} = \{P \in \mathfrak{I} \mid \operatorname{Ker} P \neq 0\} = \mathfrak{I} \setminus \{I\}.$$

LEMMA 2.7. [12, Lemma 3] *Let* $A \in B(H)$. *Then* $A \in \mathfrak{G}$ *iff* $\overline{\operatorname{Im} A} \oplus \operatorname{Ker} A = H$.

Now we define the orthogonality relation for operators.

DEFINITION 2.8. Operators $A,B \in B(H)$ are orthogonal ($A \perp B$) if $AB = BA = 0$.

Similarly to the matrix case the $\overset{\sharp}{\leqslant}$-order has the following characterization in terms of orthogonality and direct decompositions:

THEOREM 2.9. [12, Theorem 1] *Let* $A,B \in B(H)$. *Then the following conditions are equivalent:*

1) $A \overset{\sharp}{\leqslant} B$;

2) $A = B$ or there exists a direct decomposition of the space H into the sum of its closed subspaces $H = X_1 \oplus X_2$ such that linear operators $A, B\colon X_1 \oplus X_2 \to X_1 \oplus X_2$ allow the representations:

$$A = A_1 \oplus O, \qquad B = A_1 \oplus B_1,$$

where $A_1 : X_1 \to X_1$ and $B_1 : X_2 \to X_2$ are bounded linear operators, the operator A_1 is injective and $\overline{\mathrm{Im} A} = X_1$, $O\colon X_2 \to X_2$ is a zero operator;

3) $A = B$ or $A \in \mathfrak{G}$, $A \perp (B - A)$.

LEMMA 2.10. [12, Lemma 4] *Let $A, B, C \in B(H)$, $A \overset{\sharp}{\leqslant} B$, $B \perp C$. Then $A \perp C$.*

THEOREM 2.11. [12, Theorem 2] *The relation $\overset{\sharp}{\leqslant}$ is a partial order.*

Let us prove several supplementary statements for the set of operators that are orthogonal to a given operator. For $A \in B(H)$ we denote

$$\mathscr{O}(A) = \{C \in B(H) \mid C \perp A\},$$

$$\mathscr{O}_{\mathfrak{G}}(A) = \mathscr{O}(A) \cap \mathfrak{G}, \quad \mathscr{O}_{\overset{\circ}{\mathfrak{G}}}(A) = \mathscr{O}(A) \cap \overset{\circ}{\mathfrak{G}},$$

$$\mathscr{O}_{\mathfrak{J}}(A) = \mathscr{O}(A) \cap \mathfrak{J}, \quad \mathscr{O}_{\overset{\circ}{\mathfrak{J}}}(A) = \mathscr{O}(A) \cap \overset{\circ}{\mathfrak{J}}.$$

LEMMA 2.12. *Let $A, B \in \mathfrak{G}$, $\dim H > 1$. Then the following conditions are equivalent:*

1) $\pi(A) = \pi(B)$;

2) $\overline{\mathrm{Im} A} = \overline{\mathrm{Im} B}$ and $\mathrm{Ker} A = \mathrm{Ker} B$;

3) $\mathscr{O}(A) = \mathscr{O}(B)$;

4) $\mathscr{O}_{\mathfrak{G}}(A) = \mathscr{O}_{\mathfrak{G}}(B)$;

5) $\mathscr{O}_{\overset{\circ}{\mathfrak{G}}}(A) = \mathscr{O}_{\overset{\circ}{\mathfrak{G}}}(B)$;

6) $\mathscr{O}_{\overset{\circ}{\mathfrak{J}}}(A) = \mathscr{O}_{\overset{\circ}{\mathfrak{J}}}(B)$;

7) $\mathscr{O}_{\mathfrak{J}}(A) = \mathscr{O}_{\mathfrak{J}}(B)$.

Proof.

1 → 2. $\mathrm{Im}\,\pi(A) = \overline{\mathrm{Im} A}$ and $\mathrm{Ker}\,\pi(A) = \mathrm{Ker} A$ by the definition of idempotents. Also $\mathrm{Im}\,\pi(B) = \overline{\mathrm{Im} B}$ and $\mathrm{Ker}\,\pi(B) = \mathrm{Ker} B$. Then $\overline{\mathrm{Im} A} = \overline{\mathrm{Im} B}$, $\mathrm{Ker} A = \mathrm{Ker} B$.

2 → 3. Since $\mathrm{Ker} C = \overline{\mathrm{Ker} C}$ for any $C \in B(H)$ the following sequence of equalities holds:

$$\begin{aligned}\mathscr{O}(A) &= \{C \in B(H) \mid C \perp A\} \\ &= \{C \in B(H) \mid \mathrm{Im} C \subseteq \mathrm{Ker} A, \mathrm{Im} A \subseteq \mathrm{Ker} C\} \\ &= \{C \in B(H) \mid \mathrm{Im} C \subseteq \mathrm{Ker} A, \overline{\mathrm{Im} A} \subseteq \mathrm{Ker} C\} \\ &= \{C \in B(H) \mid \mathrm{Im} C \subseteq \mathrm{Ker} B, \overline{\mathrm{Im} B} \subseteq \mathrm{Ker} C\} \\ &= \{C \in B(H) \mid C \perp B\} = \mathscr{O}(B).\end{aligned}$$

3 → 4.
$$\mathscr{O}_{\mathfrak{G}}(A) = \mathscr{O}(A) \cap \mathfrak{G} = \mathscr{O}(B) \cap \mathfrak{G} = \mathscr{O}_{\mathfrak{G}}(B).$$

4 → 5.
$$\mathscr{O}_{\overset{\circ}{\mathfrak{G}}}(A) = \mathscr{O}_{\mathfrak{G}}(A) \cap \overset{\circ}{\mathfrak{G}} = \mathscr{O}_{\mathfrak{G}}(B) \cap \overset{\circ}{\mathfrak{G}} = \mathscr{O}_{\overset{\circ}{\mathfrak{G}}}(B).$$

5 → 6.
$$\mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(A) = \mathscr{O}_{\overset{\circ}{\mathfrak{G}}}(A) \cap \mathfrak{I} = \mathscr{O}_{\overset{\circ}{\mathfrak{G}}}(B) \cap \mathfrak{I} = \mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(B).$$

6 → 7. Let $\mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(A) = \mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(B)$. Observe that for any operator C we have: $\mathscr{O}_{\mathfrak{I}}(C) \subseteq \mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(C) \cup \{I\}$.

Assume in the contrary that $\mathscr{O}_{\mathfrak{I}}(A) \neq \mathscr{O}_{\mathfrak{I}}(B)$. Without loss of generality we can assume $\mathscr{O}_{\mathfrak{I}}(A) = \mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(A) \cup \{I\}$, $\mathscr{O}_{\mathfrak{I}}(B) = \mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(B)$. Then $A \perp I$, hence $A = 0$ and $\mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(B) = \mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(A) = \overset{\circ}{\mathfrak{I}}$.

Since $\dim H > 1$ then $\overset{\circ}{\mathfrak{I}} \neq \{0\}$ and there exists $P \in \overset{\circ}{\mathfrak{I}} \setminus \{0\}$. We have P, $(I - P) \in \overset{\circ}{\mathfrak{I}} = \mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(B)$ i.e., $B \perp P$, $B \perp (I-P)$ thus $B \perp I$ and $\mathscr{O}_{\mathfrak{I}}(B) = \mathscr{O}_{\overset{\circ}{\mathfrak{I}}}(B) \cup \{I\}$. The obtained contradiction shows that $\mathscr{O}_{\mathfrak{I}}(A) = \mathscr{O}_{\mathfrak{I}}(B)$.

7 → 1. Observe that for any operator $A \in \mathfrak{G}$ we have the equality $\pi(A) = \pi(\pi(A))$. By the implication $1 \to 7$, which is proved already, we have $\mathscr{O}_{\mathfrak{I}}(A) = \mathscr{O}_{\mathfrak{I}}(\pi(A))$. Similarly, $\mathscr{O}_{\mathfrak{I}}(B) = \mathscr{O}_{\mathfrak{I}}(\pi(B))$. Denote $P = \pi(A)$, $Q = \pi(B)$. Then

$$\mathscr{O}_{\mathfrak{I}}(P) = \mathscr{O}_{\mathfrak{I}}(A) = \mathscr{O}_{\mathfrak{I}}(B) = \mathscr{O}_{\mathfrak{I}}(Q).$$

Since $P \perp (I-P)$, $Q \perp (I-Q)$ then $P \perp (I-Q)$, $Q \perp (I-P)$. Thus $P = PQ = Q$ i.e., $\pi(A) = P = Q = \pi(B)$. $\square$

Recall that on the set of idempotent operators the following order relation can be introduced:

DEFINITION 2.13. [25] Let $P, Q \in \mathfrak{I}$. Define the *minus order* relation by $P \leqslant Q$ if $P = PQ = QP$.

On the set of idempotents this order coincides with the minus partial order introduced in Definition 2.2, see [29] for the details. The following lemma relates the $\overset{\sharp}{\leqslant}$-order with the standard order on idempotents.

LEMMA 2.14. *Let $A, B \in B(H)$, $B \in \mathfrak{G}$. Then the following conditions are equivalent:*

1) $A \overset{\sharp}{\leqslant} B$;

2) there exists an idempotent P such that $A = BP = PB$;

3) there exists an idempotent Q such that $Q \leqslant \pi(B)$, $A = BQ = QB$.

Proof.

1 → 2. If $A=B$ then substitute $P=I$. In the other case there exists an idempotent P such that $AP=BP$, $PA=PB$, $P=\pi(A)$. Then $A=AP=PA=BP=PB$.

2 → 3. Denote $X_1=\overline{\operatorname{Im} B}$, $X_2=\operatorname{Ker} B$. Let $x\in\overline{\operatorname{Im} B}$. Then there exists a sequence $\{y_n\}_{n=1}^{\infty}\subseteq H$ such that $x_n=B(y_n)\to x$ as $n\to\infty$. Then $B(Py_n)=P(By_n)=Px_n\to Px$ i.e. $Px\in\overline{\operatorname{Im} B}$. If $z\in\operatorname{Ker} B$ then $B(Pz)=PBz=0$ and thus $Pz\in\operatorname{Ker} B$. In other words, $P(X_1)\subseteq X_1$, $P(X_2)\subseteq X_2$ and $P\colon X_1\oplus X_2\to X_1\oplus X_2$ has a matrix representation $P=\begin{pmatrix}P_1&0\\0&P_2\end{pmatrix}$. Observe that $\pi(B)=\begin{pmatrix}I&0\\0&0\end{pmatrix}$. Denote $Q=\pi(B)P$. Then $Q=\begin{pmatrix}P_1&0\\0&0\end{pmatrix}$, $Q^2=Q$, $\pi(B)Q=Q\pi(B)=Q$ i.e. $Q\leqslant\pi(B)$. We have

$$A=BP=B\pi(B)P=BQ,\quad A=PB=P\pi(B)B=QB.$$

3 → 1. Observe that $B-A=B(I-Q)\perp BQ=A$. Let us show that $A\in\mathfrak{G}$. Since $Q\leqslant\pi(B)$, we obtain using the block matrix representation of the operators that

$$Q=\begin{pmatrix}I&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad B=\begin{pmatrix}B_1&B_2&0\\B_3&B_4&0\\0&0&0\end{pmatrix},\quad A=\begin{pmatrix}B_1&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

where I, B_1,B_2,B_3,B_4 are the operators on the spaces of appropriate dimension. Then $BQ=QB$ implies that $B_2=0$ and $B_3=0$, so the result follows. □

From this lemma we have:

COROLLARY 2.15. *If $P\in\mathfrak{I}$ and $A\overset{\sharp}{\leqslant}P$ then $A\in\mathfrak{I}$ and $A=PA=AP$, i.e., $\overset{\sharp}{\leqslant}$-order coincides with the standard order on the idempotents and in the case when the bigger operator is an idempotent.*

Proof. There exists an operator $Q\in\mathfrak{I}$ such that

$$Q\leqslant\pi(P)=P,\quad A=PQ=QP.$$

Moreover by the arguments from the proof of Lemma 2.14 we have $Q=\pi(A)$. In other words $A^2=A$, $A\in\mathfrak{I}$ and $A=\pi(A)=Q$. Thus $A\leqslant P$. □

3. Additive monotone maps on operators on the Hilbert space

DEFINITION 3.1. Let $M\subseteq B(H)$. The map $T: M\to M$ is called *0-additive* if for any two operators $A,B\in M$ such that $A\perp B$, $A\in\mathfrak{G}$ we have:
a) $T(A)\perp T(B)$; b) $T(A+B)=T(A)+T(B)$.

Below we investigate monotone maps on $B(H)$.

LEMMA 3.2. *Let $T: B(H)\to B(H)$ be an additive bijective map which is strongly monotone with respect to $\overset{\sharp}{<}$-order. Then $T(\overset{\circ}{\mathfrak{G}})=\overset{\circ}{\mathfrak{G}}$, and the restrictions $T|_{\overset{\circ}{\mathfrak{G}}}$, $T^{-1}|_{\overset{\circ}{\mathfrak{G}}}$ are 0-additive maps.*

Proof. Let $A \in \mathring{\mathfrak{G}}$. Then $\overline{\operatorname{Im} A} \oplus \operatorname{Ker} A = H$, $\operatorname{Ker} A \neq 0$. Substitute $P = \pi(A)$, $Q = I - P$. We have $P \perp Q$, therefore $A \perp Q$. Moreover since $\operatorname{Ker} A \neq 0$ then $P \neq I$ and $Q \neq 0$.

It follows that $A \overset{\sharp}{<} (A+Q)$ and $T(A) \overset{\sharp}{<} (T(A)+T(Q))$. Therefore $T(A) \in \mathfrak{G}$, $T(A) \perp T(Q)$. Since $Q \neq 0$ then $T(Q) \neq 0$ and $\operatorname{Ker} T(A) \neq 0$, $T(A) \in \mathring{\mathfrak{G}}$. Thus $T(\mathring{\mathfrak{G}}) \subseteq \mathring{\mathfrak{G}}$.

Let
$$A, B \in \mathring{\mathfrak{G}}, \quad A \perp B, \quad A \neq 0, \quad B \neq 0.$$
Then
$$A \overset{\sharp}{<} A+B, \quad T(A) \overset{\sharp}{<} T(A)+T(B), \quad T(A) \perp T(B).$$
If $A = 0$ or $B = 0$ then $T(A) \perp T(B)$. Moreover, $T(A+B) = T(A)+T(B)$ by the additivity of the map T. Therefore the restriction $T|_{\mathring{\mathfrak{G}}}$ of the map T on the set $\mathring{\mathfrak{G}}$ is 0-additive map.

Since the map T is bijective and strongly monotone with respect to $\overset{\sharp}{<}$-order then the same arguments are applicable to its inverse T^{-1}. Thus $T^{-1}(\mathring{\mathfrak{G}}) \subseteq \mathring{\mathfrak{G}}$ and $T^{-1}|_{\mathring{\mathfrak{G}}}$ is a 0-additive map. It follows that $T(\mathring{\mathfrak{G}}) = \mathring{\mathfrak{G}}$ and the lemma is proved. $\square$

LEMMA 3.3. *Let the map $T\colon \mathring{\mathfrak{G}} \to \mathring{\mathfrak{G}}$ be bijective, T and T^{-1} be 0-additive. Then the condition $\pi(A) = \pi(B)$ holds true for some operators $A, B \in \mathring{\mathfrak{G}}$ iff $\pi(T(A)) = \pi(T(B))$.*

Proof. Let $\pi(A) = \pi(B)$. Our goal is to show that $\pi(T(A)) = \pi(T(B))$. By Lemma 2.12 we have $\mathscr{O}_{\mathring{\mathfrak{G}}}(A) = \mathscr{O}_{\mathring{\mathfrak{G}}}(B)$. Therefore $T(\mathscr{O}_{\mathring{\mathfrak{G}}}(A)) = T(\mathscr{O}_{\mathring{\mathfrak{G}}}(B))$ by the bijectivity of the map T. Moreover, since the maps T and T^{-1} are 0-additive then T strongly preserves the orthogonality of operators. In other words,
$$T(\mathscr{O}_{\mathring{\mathfrak{G}}}(A)) = T(\mathscr{O}(A) \cap \mathring{\mathfrak{G}}) = \mathscr{O}(T(A)) \cap \mathring{\mathfrak{G}} = \mathscr{O}_{\mathring{\mathfrak{G}}}(T(A)).$$
Similarly, $T(\mathscr{O}_{\mathring{\mathfrak{G}}}(B)) = \mathscr{O}_{\mathring{\mathfrak{G}}}(T(B))$. Therefore $\mathscr{O}_{\mathring{\mathfrak{G}}}(T(A)) = \mathscr{O}_{\mathring{\mathfrak{G}}}(T(B))$. By Lemma 2.12 we have $\pi(T(A)) = \pi(T(B))$.

Conversely, let $\pi(T(A)) = \pi(T(B))$. It is straightforward to check that the map T^{-1} satisfies the conditions of the lemma. Therefore
$$\pi(T^{-1}(T(A))) = \pi(T^{-1}(T(B))),$$
i.e. $\pi(A) = \pi(B)$. $\square$

LEMMA 3.4. *Let the map $T\colon \mathring{\mathfrak{G}} \to \mathring{\mathfrak{G}}$ be bijective, T and T^{-1} be 0-additive. Let the map $\varphi\colon \mathfrak{I} \to \mathfrak{I}$ be defined by the following rule: $\varphi(P) = \pi(T(P))$ for all $P \in \mathring{\mathfrak{I}}$, $\varphi(I) = I$. Then φ is bijective and the maps φ and φ^{-1} are 0-additive.*

Proof. Let us show at first that the map φ is injective. Indeed, assume that $P,Q \in \mathfrak{I}$, $\varphi(P) = \varphi(Q)$. If $\varphi(P) = I$ then $P = I = Q$ since $\varphi(\mathring{\mathfrak{I}}) \subseteq \mathring{\mathfrak{I}}$. If $\varphi(P) \in \mathring{\mathfrak{I}}$ then

$$\pi(T(P)) = \varphi(P) = \varphi(Q) = \pi(T(Q)).$$

By Lemma 3.3 we have $\pi(P) = \pi(Q)$. Moreover since $P,Q \in \mathfrak{I}$ then $P = \pi(P) = \pi(Q) = Q$ and injectivity of the map φ is proved.

Let us prove the surjectivity of φ. Since

$$\varphi(\mathfrak{I}) = \varphi(\mathring{\mathfrak{I}} \cup \{I\}) = \pi(T(\mathring{\mathfrak{I}})) \cup \{I\},$$

it is sufficient to check the equality $\pi(T(\mathring{\mathfrak{I}})) = \mathring{\mathfrak{I}}$. Observe that $\mathring{\mathfrak{I}} = \pi(\mathring{\mathfrak{G}})$. Moreover $\pi(A) = \pi(\pi(A))$ for any operator $A \in \mathfrak{G}$, therefore by Lemma 3.3 we have that

$$\pi(T(A)) = \pi(T(\pi(A))).$$

It follows that

$$\pi(T(\mathring{\mathfrak{I}})) = \pi(T(\pi(\mathring{\mathfrak{G}}))) = \pi(T(\mathring{\mathfrak{G}})) = \pi(\mathring{\mathfrak{G}}) = \mathring{\mathfrak{I}}.$$

This implies that $\varphi(\mathfrak{I}) = \mathfrak{I}$ hence the map φ is surjective. Thus φ is bijective.

Let us show 0-additivity of φ. Assume that $P,Q \in \mathfrak{I}$, $P \perp Q$. If $P = I$ then $Q = 0$,

$$\varphi(P) = I, \quad \varphi(Q) = 0, \quad \varphi(P+Q) = \varphi(P) + \varphi(Q), \quad \varphi(P) \perp \varphi(Q).$$

Similar formulas can be obtained if $Q = I$.

Let $P \neq I$, $Q \neq I$, $P \perp Q$. Then $P,Q \in \mathring{\mathfrak{I}}$,

$$T(P+Q) = T(P) + T(Q), \quad T(P) \perp T(Q).$$

Therefore $\pi(T(P)) \perp \pi(T(Q))$,

$$\varphi(P+Q) = \pi(T(P) + T(Q)) = \pi(T(P)) + \pi(T(Q)) = \varphi(P) + \varphi(Q),$$

i.e. φ is 0-additive map.

Since φ is bijective, there exists φ^{-1}. Assume $P \in \mathring{\mathfrak{I}}$. Then $\varphi(\varphi^{-1}(P)) = P$. On the other side

$$\varphi(\pi(T^{-1}(P))) = \pi(T(\pi(T^{-1}(P)))) = \pi(T(T^{-1}(P))) = \pi(P) = P,$$

i.e. $\varphi^{-1}(P) = \pi(T^{-1}(P))$ for any $P \in \mathring{\mathfrak{I}}$, $\varphi^{-1}(I) = I$.

Applying the statement of the lemma to T^{-1}, we obtain that φ^{-1} is 0-additive. □

THEOREM 3.5. *Let* $\dim H \geqslant 2$ *and* $T\colon B(H) \to B(H)$ *be an additive map with the following properties: for any* $A \in B(H)$ *and* $Q \in \mathfrak{I}$ *satisfying* $A \perp Q$ *we have* $T(A) \perp Q$. *Then there exists an* $\alpha \in \mathbb{F}$ *such that* $T(A) = \alpha A$ *for any* $A \in B(H)$.

Proof. 1. *Assume* $P_1 \in \mathfrak{I}$, $\dim(\operatorname{Im} P_1) = 1$, $\lambda \in \mathbb{F} \setminus \{0\}$. *Let us show that* $T(\lambda P_1) = \mu P_1$ *for some* $\mu \in \mathbb{F}$.

Since $\lambda P_1 \perp (I - P_1)$, by the conditions of our theorem one gets that $T(\lambda P_1) \perp (I - P_1)$. Therefore

$$\operatorname{Im}(T(\lambda P_1)) \subseteq \operatorname{Ker}(I - P_1) = \operatorname{Im} P_1,$$

$$\operatorname{Ker} P_1 = \operatorname{Im}(I - P_1) \subseteq \operatorname{Ker}(T(\lambda P_1)).$$

There are two possible variants: either

$$\operatorname{Im}(T(\lambda P_1)) = \{0\},$$

or

$$\operatorname{Im}(T(\lambda P_1)) = \operatorname{Im} P_1, \quad \operatorname{Ker}(T(\lambda P_1)) = \operatorname{Ker} P_1.$$

In the first case $T(\lambda P_1) = 0 = 0 \times P_1$. In the second case there exists $\mu \in \mathbb{F} \setminus \{0\}$ such that $T(\lambda P_1) = \mu P_1$.

It follows that for any idempotent P_1 satisfying the condition $\dim(\operatorname{Im} P_1) = 1$ and $\lambda \in \mathbb{F}$ there exists $\mu \in \mathbb{F}$ such that $T(\lambda P_1) = \mu P_1$. We denote by $\sigma_1(\lambda, P_1)$ the above value of a parameter μ.

2. *Let us prove that the value* $\sigma_1(\lambda, P_1)$ *for* $P_1 \in \mathfrak{I}$ *satisfying* $\dim(\operatorname{Im} P_1) = 1$ *and* $\lambda \in \mathbb{F}$ *depends only on the vector generating* $\operatorname{Im} P_1$.

Let $x \in H$, $P_x \in \mathfrak{I}$ be some idempotent satisfying the condition $\operatorname{Im} P_x = \langle x \rangle$. Then $\lambda(I - P_x) \perp P_x$ and $T(\lambda(I - Px)) \perp P_x$. Therefore

$$T(\lambda(I - P_x))x = T(\lambda(I - P_x))P_x x = 0.$$

Consider the expression $T(\lambda I)x$:

$$T(\lambda I)x = T(\lambda P_x + \lambda(I - P_x))x =$$

$$= T(\lambda P_x)x + T(\lambda(I - P_x))x = \sigma_1(\lambda, P_x)P_x x = \sigma_1(\lambda, P_x)x.$$

From the formula above we have that $\sigma_1(\lambda, P_x)$ does not depend on the particular idempotent P_x but only on x. Let us define the function $\sigma_2(\lambda, x)$ in such a way that for all $\lambda \in \mathbb{F}$ and $x \in H$ we have the equality $T(\lambda I)x = \sigma_2(\lambda, x)x$.

3. *Let us show that the values of the function* $\sigma_2(\lambda, x)$ *does not depend on* $x \in H$.

Assume that for some $\lambda \in \mathbb{F}$ and $x, y \in H$ we have inequality $\sigma_2(\lambda, x) \neq \sigma_2(\lambda, y)$. Then

$$\sigma_2(\lambda, x + y)(x + y) = T(\lambda I)(x + y) = \sigma_2(\lambda, x)x + \sigma_2(\lambda, y)y.$$

Since $\sigma_2(\lambda, x) \neq \sigma_2(\lambda, y)$ we have a non-trivial linear combination of vectors x and y equal to zero. Therefore x and y are linear dependent. Without loss of generality $y = \gamma x$ for some $\gamma \in \mathbb{F}$. On the other side

$$T(\lambda I)y = \gamma T(\lambda I)x = \gamma \sigma_2(\lambda, x)x = \sigma_2(\lambda, x)y,$$

and $\sigma_2(\lambda, x) = \sigma_2(\lambda, y)$. The obtained contradiction shows that $\sigma_2(\lambda, x) = \sigma_2(\lambda, y)$ for any $\lambda \in \mathbb{F}$ and $x, y \in H$. By $\sigma(\lambda)$ we denote such a function that $\sigma(\lambda) = \sigma_2(\lambda, x)$ for any $\lambda \in \mathbb{F}$ and $x \in H$.

4. *Let us show that* $\sigma(\lambda) = \alpha\lambda$ *for all* $\lambda \in \mathbb{F}$ *and a certain element* $\alpha \in \mathbb{F}$.

Let $P \in \mathfrak{I}$ be a certain idempotent with $\dim(\operatorname{Im} P) = 1$. We choose an operator $N \neq 0$ such that $N^2 = 0$, $PN = N$, $NP = 0$ (it does exist since $\dim H \geqslant 2$). Then $P' = P + \nu N$ is also an idempotent with $\dim(\operatorname{Im} P') = 1$ for all $\nu \in \mathbb{F}$. For all $\lambda \in \mathbb{F} \setminus \{0\}$ we consider $T(\lambda P + N)$:

$$T(\lambda P + N) = T\left(\lambda\left(P + \frac{1}{\lambda}N\right)\right) = \sigma(\lambda)\left(P + \frac{1}{\lambda}N\right) = \sigma(\lambda)P + \frac{\sigma(\lambda)}{\lambda}N,$$

$$T(\lambda P + N) = T(\lambda P) + T(N) = \sigma(\lambda)P + T(N),$$

thus $T(N) = \dfrac{\sigma(\lambda)}{\lambda}N$ for all $\lambda \in \mathbb{F} \setminus \{0\}$. Therefore the expression $\dfrac{\sigma(\lambda)}{\lambda}$ does not depend on λ, and there exists $\alpha \in \mathbb{F}$ such that $\dfrac{\sigma(\lambda)}{\lambda} = \alpha$ for all $\lambda \in \mathbb{F} \setminus \{0\}$.

5. *Assume* $A \in B(H)$, $\dim(\operatorname{Im} A) < \infty$. *Let us show that* $T(A) = \alpha A$.

Consider an operator A as a linear combination

$$A = \sum_{i=1}^{k} \lambda_i P_i,$$

where P_i are idempotents and $\dim(\operatorname{Im} P_i) = 1$. From the previous items we obtain $T(\lambda_i P_i) = \alpha\lambda_i P_i$. By the additivity of T we have:

$$T(A) = \sum_{i=1}^{k} T(\lambda_i P_i) = \alpha \sum_{i=1}^{k} \lambda_i P_i = \alpha A.$$

6. *Let* $A \in B(H)$ *be an arbitrary bounded linear operator. Let us show that* $T(A) = \alpha A$.

Assume $x \in H$, $P_x \in \mathfrak{I}$ is a certain idempotent satisfying the condition $\operatorname{Im} P_x = \langle x \rangle$. Denote $B = (I - P_x)A(I - P_x)$. Since $B \perp P_x$ then $T(B) \perp P_x$ therefore $T(B)x = T(B)P_x x = 0$. Moreover

$$A = AP_x + P_x A(I - P_x) + B.$$

Since $\dim(\operatorname{Im} P_x) = 1 < \infty$ then

$$\dim(\operatorname{Im} AP_x) < \infty, \quad \dim(\operatorname{Im} P_x A(I - P_x)) < \infty,$$

and by item 5 we obtain:

$$T(AP_x) = \alpha AP_x, \quad T(P_x A(I - P_x)) = \alpha P_x A(I - P_x).$$

Consider the expression $T(A)x$:

$$\begin{aligned} T(A)x &= T(AP_x)x + T(P_x A(I - P_x))x + T(B)x \\ &= \alpha AP_x x + \alpha P_x A(I - P_x)x + 0 = \alpha Ax. \end{aligned}$$

Therefore $T(A) = \alpha A$ for all $A \in B(H)$ and the theorem is proved. $\square$

Let us recall a theorem of Ovchinnikov [25] about monotone maps on idempotents.

THEOREM 3.6. [25] *Let* $\dim H \geqslant 3$ *and* $\varphi\colon \mathfrak{I} \to \mathfrak{I}$ *be a bijective map which is strongly monotone with respect to the standard order on idempotents. Then there exists a linear or semilinear invertible bounded operator* $S\colon H \to H$ *such that* $\varphi(P) = SPS^{-1}$ *for all* $P \in \mathfrak{I}$ *or* $\varphi(P) = SP^*S^{-1}$ *for all* $P \in \mathfrak{I}$.

Our characterization result can be formulated as follows:

THEOREM 3.7. *Let* $\dim H \geqslant 3$ *and* $T\colon B(H) \to B(H)$ *be an additive bijective map which is strongly monotone with respect to* $\overset{\sharp}{<}$*-order. Then there exists* $\alpha \in \mathbb{F} \setminus \{0\}$ *and a linear or semilinear invertible bounded operator* $S\colon H \to H$ *such that* $T(A) = \alpha SAS^{-1}$ *for all* $A \in B(H)$ *or* $T(A) = \alpha SA^*S^{-1}$ *for all* $A \in B(H)$.

Proof. 1. Applying Lemma 3.2 to the transformation T we obtain that $T(\overset{\circ}{\mathfrak{G}}) = \overset{\circ}{\mathfrak{G}}$, the map $T\colon \overset{\circ}{\mathfrak{G}} \to \overset{\circ}{\mathfrak{G}}$ is bijective, and the restrictions $T|_{\overset{\circ}{\mathfrak{G}}}$, $T^{-1}|_{\overset{\circ}{\mathfrak{G}}}$ are 0-additive maps.

2. Let us define a map $\varphi\colon \mathfrak{I} \to \mathfrak{I}$ as follows: $\varphi(P) = \pi(T(P))$ for all $P \in \overset{\circ}{\mathfrak{I}}$, $\varphi(I) = I$. Then by Lemma 3.4 we have that φ is bijective, maps φ and φ^{-1} are 0-additive.

3. From Item 2 we have that the map φ is bijective and strongly monotone with respect to $\overset{\sharp}{<}$-order. By the Ovchinnikov's theorem (Theorem 3.6) there exists a linear or semilinear invertible bounded operator $S\colon H \to H$ such that $\varphi(P) = SPS^{-1}$ for all $P \in \mathfrak{I}$ or $\varphi(P) = SP^*S^{-1}$ for all $P \in \mathfrak{I}$.

Applying if necessary the map $P \mapsto S^{-1}PS$ and the conjugation to the map T we obtain an additive bijective map T_1 which is strongly monotone with respect to $\overset{\sharp}{<}$-order and satisfies the condition $\pi(T_1(P)) = P$ for any $P \in \mathfrak{I}$.

4. For any operator $A \in B(H)$ and idempotent $Q \in \mathfrak{I}$, $A \perp Q$, we have $Q \overset{\sharp}{\leqslant} Q + A$. Therefore

$$T_1(Q) \overset{\sharp}{\leqslant} T_1(Q) + T_1(A), \quad T_1(A) \perp T_1(Q).$$

It follows that $T_1(A) \perp \pi(T_1(Q)) = Q$ and we can apply the theorem 3.5 to the map T_1. Consequently there exists $\alpha \in \mathbb{F}$ such that $T_1(A) = \alpha A$ for any operator $A \in B(H)$.

5. By the bijectivity of the map T_1 we have that $\alpha \neq 0$. The map T_1 is a composition of the map T with the map $P \mapsto S^{-1}PS$ and the conjugation transformation. Therefore, the map T has the required form. □

In the above theorem the conditions of bijectivity and additivity are indispensable as the examples in Section 5 show.

4. Monotone maps on orthogonal idempotents and their linear combinations

Denote by $\mathfrak{L}$ the set of all finite linear combinations of orthogonal idempotents. In other words,

$$\mathfrak{L} = \{A \in B(H) \mid A = \sum_{i=1}^{n} \lambda_i P_i,\ P_i \in \mathfrak{I},\ \lambda_i \in \mathbb{F},\ P_i \perp P_j \text{ if } i \neq j,\ 1 \leqslant i, j \leqslant n\}.$$

LEMMA 4.1. *Let* $A,B \in \mathfrak{L}$, $A \overset{\sharp}{\leqslant} B$. *Assume that* $\lambda_1,\ldots,\lambda_n \in \mathbb{F}\setminus\{0\}$ *are such that* $A = \sum_{i=1}^{n} \lambda_i P_i$, $B = \sum_{i=1}^{n} \lambda_i P_i'$, *for all* $i \neq j$ *we have* $P_i \perp P_j$, $P_i' \perp P_j'$, $\lambda_i \neq \lambda_j$. *Then* $P_j \overset{\sharp}{\leqslant} P_j'$ *for all* $j \in \{1,2,\ldots,n\}$.

Proof. Fix a certain $j \in \{1,2,\ldots,n\}$. Then

$$\lambda_j P_j \overset{\sharp}{\leqslant} A \overset{\sharp}{\leqslant} B.$$

Since $B \in \mathfrak{L} \subseteq \mathfrak{G}$ then we can apply Lemma 2.14 to the operators $\lambda_j P_j$ and B. Therefore there exists an idempotent $P \in \mathfrak{I}$ such that

$$\lambda_j P_j = BP = PB.$$

Assume that $f \in \mathbb{F}[t]$ is a certain polynomial satisfying the condition $f(0) = 0$. Then

$$f(\lambda_j P_j) = f(BP) = f(B)P = Pf(B),$$

and $f(\lambda_j P_j) \overset{\sharp}{\leqslant} f(B)$ by Lemma 2.14.

Consider a polynomial f such that $f(0) = 0$, $f(\lambda_i) = 0$ for $i \neq j$, $f(\lambda_j) \neq 0$. We have

$$f(\lambda_j)P_j = f(\lambda_j P_j) \overset{\sharp}{\leqslant} f(B) = f\left(\sum_{i=1}^{n} \lambda_i P_i'\right) = \sum_{i=1}^{n} f(\lambda_i)P_i' = f(\lambda_j)P_j',$$

therefore $P_j \overset{\sharp}{\leqslant} P_j'$ and the lemma is proved. □

The following corollary is straightforward.

COROLLARY 4.2. *Let* $A_1, A_2, B \in \mathfrak{L}$ *satisfy* $A_1 \overset{\sharp}{\leqslant} B$, $A_2 \overset{\sharp}{\leqslant} B$. *Assume that* $A_1 = \sum_{i=1}^{n} \lambda_i P_i$, $P_i \in \mathfrak{I}$, $\lambda_i \in \mathbb{F}$, $P_i \perp P_j$, $A_2 = \sum_{i=1}^{n} \mu_i Q_i$, $Q_i \in \mathfrak{I}$, $\mu_i \in \mathbb{F}$, $Q_i \perp Q_j$, *and* $\{\lambda_1,\ldots,\lambda_n\} \cap \{\mu_1,\ldots,\mu_n\} \subseteq \{0\}$. *Then* $A_1 \perp A_2$.

THEOREM 4.3. *Let* $\dim H \geqslant 3$ *and a bijective map* $T\colon \mathfrak{L} \to \mathfrak{L}$ *be strongly monotone with respect to the* $\overset{\sharp}{<}$*-order. Then there exists a bijection* $\sigma\colon \mathbb{F} \to \mathbb{F}$ *such that* $\sigma(0) = 0$ *and a linear or semilinear bounded invertible operator* $S\colon H \to H$ *such that*

$$T\left(\sum_{i=1}^{n} \lambda_i P_i\right) = S\left(\sum_{i=1}^{n} \sigma(\lambda_i) P_i\right) S^{-1}$$

for all P_i, $P_i \perp P_j$ *if* $i \neq j$, *or*

$$T\left(\sum_{i=1}^{n} \lambda_i P_i\right) = S\left(\sum_{i=1}^{n} \sigma(\lambda_i) P_i^*\right) S^{-1}$$

for all P_i, $P_i \perp P_j$ *if* $i \neq j$.

Proof. 1. For any operator $A \in \mathfrak{L}$ it is natural to consider sequences of operators that are below or above A with respect to the order $\overset{\sharp}{<}$. The sequence $0 \neq A_1, A_2, \dots, A_l$ is called a *left chain* for A with respect to $\overset{\sharp}{<}$-order if

$$A_1 \overset{\sharp}{<} A_2 \overset{\sharp}{<} \cdots \overset{\sharp}{<} A_l \overset{\sharp}{<} A.$$

Similarly, $A_1, A_2, \dots, A_r$ is called a *right chain* for A if

$$A \overset{\sharp}{<} A_1 \overset{\sharp}{<} A_2 \overset{\sharp}{<} \cdots \overset{\sharp}{<} A_r.$$

The numbers l, r here are called the *lengths* of the chains. If values of l are bounded, we denote by L_A the maximal value of l. Otherwise $L_A = \infty$. In the same way, we denote by R_A the maximal value of r (or ∞).

Since the map T is strongly monotone with respect to $\overset{\sharp}{<}$-order then for an arbitrary operator $A \in \mathfrak{L}$ we have $L_A = L_{T(A)}$, $R_A = R_{T(A)}$. So, L_A and R_A are invariants of T. Corresponding invariants for matrices are introduced and investigated in [14, Section 3].

In particular, $T(0) = 0$ since the maximal length of the left chain of A is equal to 0 iff $A = 0$. Moreover for any $\lambda \in \mathbb{F} \setminus \{0\}$ and any idempotent $P \in \mathfrak{I}$ satisfying the condition $\dim(\operatorname{Im} P) = 1$ we have $L_{\lambda P} = 1$ and therefore $L_{T(\lambda P)} = 1$ and there exist $\mu \in \mathbb{F} \setminus \{0\}$ and $P' \in \mathfrak{I}$ such that $T(\lambda P) = \mu P'$.

2. Observe that for $A \in \mathfrak{L}$ the equality $L_A = 2$ is true iff there are $\lambda_1, \lambda_2 \in \mathbb{F} \setminus \{0\}$ and $P_1, P_2 \in \mathfrak{I}$ satisfying

$$\dim(\operatorname{Im} P_1) = \dim(\operatorname{Im} P_2) = 1, \quad P_1 \perp P_2,$$

such that $A = \lambda_1 P_1 + \lambda_2 P_2$. Therefore for $T(A)$ we have that $T(A) = \lambda_1' P_1' + \lambda_2' P_2'$ for some $\lambda_1', \lambda_2' \in \mathbb{F} \setminus \{0\}$ and $P_1', P_2' \in \mathfrak{I}$ such that

$$\dim(\operatorname{Im} P_1') = \dim(\operatorname{Im} P_2') = 1, \quad P_1' \perp P_2'.$$

Note that if $\lambda_1 \neq \lambda_2$ then the number of operators $C \in \mathfrak{L}$ such that $C \overset{\sharp}{<} A$ is finite. If $\lambda_1 = \lambda_2$ then the number of operators $C \in \mathfrak{L}$ such that $C \overset{\sharp}{<} A$ is infinite. This implies that $\lambda_1' = \lambda_2'$ iff $\lambda_1 = \lambda_2$.

3. Let $\lambda \in \mathbb{F} \setminus \{0\}$. Consider the operator λI. Since $R_{\lambda I} = 0$ then $R_{T(\lambda I)} = 0$. Moreover any operator $C \in \mathfrak{L}$ satisfying the conditions $C \overset{\sharp}{<} \lambda I$ and $L_C = 2$ has equal eigenvalues. By Item 2 we have that for any operator $C' \in \mathfrak{L}$ satisfying conditions $C' \overset{\sharp}{<} T(\lambda I)$ and $L_{C'} = 2$ its non-zero eigenvalues are equal.

We stress that $T(\lambda I) \in \mathfrak{L}$, and there exist scalars $\mu_1, \mu_2, \dots, \mu_n \in \mathbb{F} \setminus \{0\}$ and non-zero idempotent operators $Q_1, Q_2, \dots, Q_n \in B(H)$ such that $T(\lambda I) = \sum\limits_{i=1}^{n} \mu_i Q_i$, $Q_i \perp Q_j$ if $i \neq j$. Assume that $\mu_i \neq \mu_j$ for a certain $i \neq j$. There are idempotent operators

$\hat{Q}_i, \hat{Q}_j$ such that $\hat{Q}_i \overset{\sharp}{\leqslant} Q_i$, $\hat{Q}_j \overset{\sharp}{\leqslant} Q_j$, $\dim(\operatorname{Im}\hat{Q}_i) = \dim(\operatorname{Im}\hat{Q}_j) = 1$. We denote $C' = \mu_i\hat{Q}_i + \mu_j\hat{Q}_j$. Indeed, $L_{C'} = 2$, its non-zero eigenvalues are not equal, and we obtain a contradiction. Consequently $\mu_i = \mu$ for any $i \in \{1,2,\ldots,n\}$, $T(\lambda I) = \mu Q$. In addition, $R_{T(\lambda I)} = R_{\mu Q} = 0$, and $Q = I$.

Hence for any $\lambda \in \mathbb{F}\setminus\{0\}$ there exists $\mu \in \mathbb{F}\setminus\{0\}$ such that $T(\lambda I) = \mu I$.

4. Define the map $\sigma\colon \mathbb{F} \to \mathbb{F}$ by the following condition: $\sigma(0) = 0$ and for $\lambda \neq 0$ the value $\sigma(\lambda)$ is equal to μ such that $T(\lambda I) = \mu I$. Note that items 1 – 3 hold for the map T^{-1} as well. Then σ is point-wise invertible and hence it is surjective. Moreover, σ is injective since $T(\lambda_1 I) = \sigma(\lambda_1)I = \sigma(\lambda_2)I = T(\lambda_2 I)$. Thus bijectivity of the map σ is proved.

5. Denote $T_1(A) = \dfrac{1}{\sigma(1)}T(A)$, $\sigma_1(\lambda) = \dfrac{\sigma(\lambda)}{\sigma(1)}$. Then $\sigma_1(1) = 1$, $T_1(I) = I$. For any $P \in \mathfrak{I}$ we have $P \overset{\sharp}{<} I$. Therefore $T_1(P) \overset{\sharp}{<} I$ and $T_1(P) \in \mathfrak{I}$, and then we can apply the Ovchinnikov's theorem (Theorem 3.6) to $(T_1)|_{\mathfrak{I}}$.

Therefore there exists a linear or semilinear invertible bounded operator $S\colon H \to H$ such that $T_1(P) = SPS^{-1}$ for all $P \in \mathfrak{I}$ or $T_1(P) = SP^*S^{-1}$ for all $P \in \mathfrak{I}$. Applying to T_1 the map $P \mapsto S^{-1}PS$ and conjugation if necessary (in the latter case we use $\sigma_2 = \overline{\sigma_1}$ instead of σ_1) we obtain a bijective map T_2 which is strongly monotone with respect to $\overset{\sharp}{<}$-order and satisfies the property $T_2(P) = P$ for any $P \in \mathfrak{I}$.

6. Let $\lambda \in \mathbb{F}\setminus\{0,1\}$, $P \in \mathfrak{I}$ is arbitrary. Then $T_2(\lambda P) = \sigma_1(\lambda)Q$ for a certain $Q \in \mathfrak{L}$. Let us show that Q is idempotent. Indeed, $T_2(\lambda P)$ has the form $\sigma_1(\lambda)Q$ because of $\lambda P \overset{\sharp}{\leqslant} \lambda I$, so $T_2(\lambda P) \overset{\sharp}{\leqslant} \sigma_1(\lambda)I$. Therefore, $Q = \frac{1}{\sigma_1(\lambda)}T_2(\lambda P) \overset{\sharp}{\leqslant} I$, i.e., $Q \in \mathfrak{I}$.

Moreover $\lambda P \perp (I-P)$ and

$$\sigma_1(\lambda)Q = T_2(\lambda P) \overset{\sharp}{\leqslant} T_2(\lambda P + (I-P)),$$

$$I - P = T_2(I-P) \overset{\sharp}{\leqslant} T_2(\lambda P + (I-P)).$$

Since the map σ_1 is bijective, by Items 3 and 5 correspondingly we get that $\sigma_1(\lambda) \neq 0$, $\sigma_1(\lambda) \neq 1$. Hence, we can apply Corollary 4.2 and obtain that $\sigma_1(\lambda)Q \perp (I-P)$.

Thus $T_2(\lambda P) \perp (I-P)$ and $T_2(\lambda P) \overset{\sharp}{\leqslant} \sigma_1(\lambda)P$. Moreover if $\lambda \in \{0,1\}$ then $T_2(\lambda P) \overset{\sharp}{\leqslant} \sigma_1(\lambda)P$. Therefore $T_2(\lambda P) \overset{\sharp}{\leqslant} \sigma_1(\lambda)P$ for all $\lambda \in \mathbb{F}$, $P \in \mathfrak{I}$.

7. The statement of the theorem is also applicable to the map T_2^{-1}. Thus, applying the same arguments as in Items 1–6 to the map T_2^{-1} we obtain $T_2^{-1}(\lambda P) \overset{\sharp}{\leqslant} \sigma_1^{-1}(\lambda)P$ for all $\lambda \in \mathbb{F}$, $P \in \mathfrak{I}$. Let $\lambda = \sigma_1(\mu)$ then $T_2^{-1}(\sigma_1(\mu)P) \overset{\sharp}{\leqslant} \mu P$.

Since the map T_2 is monotone we have that $\sigma_1(\mu)P \overset{\sharp}{\leqslant} T_2(\mu P)$ for all $\mu \in \mathbb{F}$, $P \in \mathfrak{I}$. Therefore $T_2(\lambda P) = \sigma_1(\lambda)P$ for all $\lambda \in \mathbb{F}$, $P \in \mathfrak{I}$.

8. Let $\lambda_i \in \mathbb{F}$, $P_i \in \mathfrak{I}$, $P_i \perp P_j$ for $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$. Then

$$\sigma_1(\lambda_j)P_j = T_2(\lambda_j P_j) \overset{\sharp}{\leqslant} T_2\left(\sum_{i=1}^n \lambda_i P_i\right)$$

for all $j \in \{1, 2, \ldots, n\}$, therefore,

$$\sum_{i=1}^n \sigma_1(\lambda_i)P_i \overset{\sharp}{\leqslant} T_2\left(\sum_{i=1}^n \lambda_i P_i\right).$$

Moreover,

$$T_2\left(\sum_{i=1}^n \lambda_i P_i\right) \perp \left(I - \sum_{i=1}^n P_i\right) \tag{1}$$

Indeed, $T_2\left(\sum\limits_{i=1}^n \lambda_i P_i\right) = \sum\limits_{i=1}^k \mu_i P_i'$ for some $\mu_i \in \mathbb{F}$, $P_i' \in \mathfrak{I}$. Due to the infinity of the field $\mathbb{F}$ there exists $\mu \in \mathbb{F}$ such that $\mu \neq \mu_i$ for all $i = 1, \ldots, n$. By Item 7 it holds that $\mu\left(I - \sum\limits_{i=1}^n P_i\right) = T_2\left(\sigma_1^{-1}(\mu)\left(I - \sum\limits_{i=1}^n P_i\right)\right)$. Then similarly to Item 6 we have

$$\sum_{i=1}^k \mu_i P_i' = T_2\left(\sum_{i=1}^n \lambda_i P_i\right) \overset{\sharp}{\leqslant} T_2\left(\sum_{i=1}^n \lambda_i P_i + \sigma_1^{-1}(\mu)\left(I - \sum_{i=1}^n P_i\right)\right)$$

and

$$\mu\left(I - \sum_{i=1}^n P_i\right) = T_2\left(\sigma_1^{-1}(\mu)\left(I - \sum_{i=1}^n P_i\right)\right) \overset{\sharp}{\leqslant} T_2\left(\sum_{i=1}^n \lambda_i P_i + \sigma_1^{-1}(\mu)\left(I - \sum_{i=1}^n P_i\right)\right),$$

Hence, by Corollary 4.2, Formula (1) holds. Thus

$$T_2\left(\sum_{i=1}^n \lambda_i P_i\right) = \sum_{i=1}^n \sigma_1(\lambda_i)P_i.$$

Since the map T_2 can be obtained as a composition of T, the map $P \mapsto S^{-1}PS$, conjugation and the scalar multiplication then the map T has the required form. $\square$

5. Examples

In this section we are going to show that the conditions of the presented theorems are indispensable.

Assume that the space H is separable and the sequence $\{x_n\}_{n=1}^\infty \subseteq H$ is a basis in H. By R and L we denote the operators representing left and right shift in this basis correspondingly. It is straightforward to show that $LR = I$.

The following example shows that there are injective linear maps on $B(H)$ strongly monotone with respect to $\overset{\sharp}{<}$-order, which are not surjective and not of the form presented in Theorem 3.7.

EXAMPLE 5.1. Define the map $T\colon B(H) \to B(H)$ as follows: $T(A) = RAL$ for all $A \in B(H)$. Then T is a linear injective map strongly monotone with respect to $\overset{\sharp}{<}$-order.

Proof. 1. The linearity of the map T is straightforward.

2. Let $\hat{T}(A) = LAR$ for $A \in B(H)$. Since $LR = I$ then

$$\hat{T}(T(A)) = L(RAL)R = A$$

for all $A \in B(H)$. Moreover

$$T(A) \cdot T(B) = (RAL) \cdot (RBL) = R(AB)L = T(AB)$$

for any $A, B \in B(H)$.

3. Let us prove the injectivity of the map T. Indeed, assume that $T(A) = T(B)$ for some $A, B \in B(H)$. Then

$$A = \hat{T}(T(A)) = \hat{T}(T(B)) = B,$$

and the map T is injective.

4. Let us show that the map T strongly preserves the orthogonality relation between the operators. Let $A \perp B$ then $AB = BA = 0$ and

$$T(A) \cdot T(B) = RALRBL = RABL = 0 = RBAL = T(B) \cdot T(A),$$

i.e. $T(A) \perp T(B)$. If $T(A) \perp T(B)$ then $T(AB) = T(BA) = 0$ and $AB = BA = 0$ by the injectivity of T, i.e. $A \perp B$.

5. Let us prove that $T(\mathfrak{G}) = \mathfrak{G}$.

Note that we have $LH = H$, $\operatorname{Ker} L \oplus \operatorname{Im} R = H$. Moreover, for any operator $A \in B(H)$ we have

$$\overline{\operatorname{Im} T(A)} = \overline{(RAL)H} = \overline{RAH} = R(\overline{\operatorname{Im} A}),$$

$$\operatorname{Ker} T(A) = \operatorname{Ker} L \oplus R(\operatorname{Ker} A).$$

Let $A \in \mathfrak{G}$ then $\overline{\operatorname{Im} A} \oplus \operatorname{Ker} A = H$, it follows that

$$\begin{aligned}\overline{\operatorname{Im} T(A)} \oplus \operatorname{Ker} T(A) &= \operatorname{Ker} L \oplus R(\overline{\operatorname{Im} A} \oplus \operatorname{Ker} A) \\ &= \operatorname{Ker} L \oplus \operatorname{Im} R = H,\end{aligned}$$

and $T(A) \in \mathfrak{G}$.

Moreover if $A \notin \mathfrak{G}$ then either $\overline{\operatorname{Im} A} \oplus \operatorname{Ker} A \neq H$ or the sum $\overline{\operatorname{Im} A} + \operatorname{Ker} A$ is not direct. In both cases it turns out that $T(A) \notin \mathfrak{G}$. Indeed, if the sum $\overline{\operatorname{Im} A} + \operatorname{Ker} P$ is not direct, then the set $\overline{\operatorname{Im} A} \cap \operatorname{Ker} P \neq \{0\}$. Hence, $R(\overline{\operatorname{Im} A} \cap \operatorname{Ker} P) = R(\overline{\operatorname{Im} A}) \cap R(\operatorname{Ker} P) \neq \{0\}$, and the sum $\overline{\operatorname{Im} T(A)} + \operatorname{Ker} T(P)$ is not direct. Similarly, if $\overline{\operatorname{Im} A} \oplus \operatorname{Ker} P \neq H$, then $\overline{\operatorname{Im} A} \oplus \operatorname{Ker} P$ is a proper subset in H. Hence its R-image is a proper subset in RH and thus $\overline{\operatorname{Im} T(A)} + \operatorname{Ker} T(P)$ is a proper subset in H.

Hence, $T^{-1}(\mathfrak{G}) \subseteq \mathfrak{G}$, where by $T^{-1}(V)$ we denote the full inverse image of the set V. Hence, $T(\mathfrak{G}) = \mathfrak{G}$.

6. Let us show that T is strongly monotone with respect to $\overset{\sharp}{<}$-order.

Indeed, let $A,B \in B(H)$, $A \overset{\sharp}{<} B$. Then

$$A \in \mathfrak{G}, \quad A \perp (B-A).$$

Therefore

$$T(A) \in \mathfrak{G}, T(A) \perp (T(B)-T(A))$$

due to the linearity. Moreover, $T(B-A) \neq 0$ and $T(A) \overset{\sharp}{<} T(B)$.

Similarly we obtain that if $T(A) \overset{\sharp}{<} T(B)$ then $A \overset{\sharp}{<} B$. $\square$

The following example shows that there are bijective maps on $B(H)$ which are strongly monotone with respect to $\overset{\sharp}{<}$-order, but not additive and not of the form presented in Theorem 3.7.

EXAMPLE 5.2. Define the map $T: B(H) \to B(H)$ as follows: $T(A) = A$ for all $A \in B(H) \setminus \{I+L, I+R\}$, $T(I+L) = I+R$, $T(I+R) = I+L$. Then T is a bijective map strongly monotone with respect to the $\overset{\sharp}{<}$-order.

Proof. 1. Assume that $A \overset{\sharp}{<} I+R$. Let us show that $A = 0$.

Since $A \overset{\sharp}{<} I+R$ then $A \in \mathfrak{G}$, $A+B = I+R$ for a certain operator $B \in B(H)$, $A \perp B$. Denote $P = \pi(A)$. Then

$$A = AP = (I+R-B)P = P+RP$$

and

$$A = PA = P(I+R-B) = P+PR.$$

It follows that $RP = PR$. Let $v \in \operatorname{Im} P$. we have

$$Pv = v, \quad P(Rv) = RPv = Rv,$$

i.e. $Rv \in \operatorname{Im} P$.

1. a) Assume that $\operatorname{Im} P \neq \{0\}$, $\operatorname{Im} P \neq H$ and denote by s the minimal natural number such that in $\operatorname{Im} P$ there are vectors with nonzero values on the s-th position in the basis $\{x_n\}_{n=1}^{\infty} \subseteq H$. Moreover by $v_s \in \operatorname{Im} P$ we denote the vector on which minimum is attained.

Let us prove that $s = 1$.

Indeed, assume that $s > 1$ then $RLv_s = v_s$. From $LR = I$ and $RP = PR$ we have that $P = LPR$. Consider $P(Lv_s)$:

$$P(Lv_s) = (LPR)(Lv_s) = LPv_s = Lv_s,$$

it follows that $Lv_s \in \operatorname{Im} P$ which contradicts the definition of s.

Therefore $s=1$, $v_1 \in \operatorname{Im} P$. Moreover for any natural number n we have $R^n v_1 \in \operatorname{Im} P$. Since $\operatorname{Im} P$ is closed linear subspace we have $x_1 \in \operatorname{Im} P$. Therefore $x_n = R^{n-1}x_1 \in \operatorname{Im} P$ for any natural number n and $\operatorname{Im} P = H$. However $\operatorname{Im} P \neq H$ by the assumption. So, we arrive to the contradiction.

1. b) Since the case 1. a) is non-realizable then either $\operatorname{Im} P = \{0\}$ or $\operatorname{Im} P = H$. If $\operatorname{Im} P = H$ then $P = I$, $A = I + R$ which is not true. Thus $\operatorname{Im} P = \{0\}$, $P = 0$, $A = 0$, the result follows.

2. Let $A \overset{\sharp}{<} I+L$. Then

$$A^* \overset{\sharp}{<} I + L^* = I + R,$$

and $A^* = 0$ by item 1. Therefore $A = 0$.

3. Let us prove that the map T is monotone with respect to $\overset{\sharp}{<}$-order. Indeed, assume $A, B \in B(H)$, $A \overset{\sharp}{<} B$.

If $A \in \{I+L, I+R\}$ then $\operatorname{Ker} A = \{0\}$ and the inequality $A \overset{\sharp}{<} B$ is impossible. If $B \in \{I+L, I+R\}$ then $A = 0$ by above. Therefore $T(A) = 0 \overset{\sharp}{<} T(B)$ in this case.

Moreover, if $A \notin \{I+L, I+R\}$, $B \notin \{I+L, I+R\}$ then

$$T(A) = A \overset{\sharp}{<} B = T(B).$$

4. Since $T^{-1} = T$ then T is strongly monotone with respect to $\overset{\sharp}{<}$-order. □

So, both assumptions: additivity and bijectivity, are necessary in order to obtain the result.

Let us compare the statement of Theorem 4.3 with its matrix analog ([13, Theorem 1.12]). In the last Theorem we have considered only injective monotone maps on the set $\mathscr{D}_n(\mathbb{F})$ of diagonalizable matrices and in Theorem 4.3 we have considered the bijective strongly monotone maps on the set $\mathfrak{L}$.

Observe that if $H = \mathbb{F}^n$ then $\mathfrak{L}$ are linear operators with matrices, which are in some fixed basis are the elements of the set of diagonalizable matrices $\mathscr{D}_n(\mathbb{F})$. In the finite dimensional case the characterization results coincide up to the fact that in Theorem 4.3 there are more restrictions on the map, and any nonzero endomorphism f of the field $\mathbb{F}$ should be replaced by the identity map or by the complex conjugation.

QUESTION 5.3. In [12] the $\overset{\sharp}{\leqslant}$-order is defined for the algebra of bounded linear operators over an arbitrary Banach space. So, the problem of characterization for monotone maps with respect to $\sharp$-order for operators on general Banach spaces is open.

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