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ON THE DISTANCE TO SINGULARITY VIA LOW RANK PERTURBATIONS

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Abstract. For regular matrix pencils the distance in norm to the nearest singular pencil under low rank perturbation is studied. Characterizations of this distance are derived via the Weyl function of the perturbation. Special attention is paid to the Hermitian pencil case. Estimates for the distance of a given pencil to the set of singular pencils are obtained.

1. Introduction

Consider a regular complex $n \times n$ matrix pencil $A + \lambda E$, i.e., a pencil satisfying $\det(A + \lambda E) \not\equiv 0$, and suppose the pencil is perturbed to $\widetilde{A} + \lambda \widetilde{E}$ with $\widetilde{A} = A + \Delta A$ and $\widetilde{E} = E + \Delta E$. It is a long-standing open problem in matrix analysis, see [3], to find the smallest (in some appropriate norm) perturbation $(\Delta A, \Delta E)$ such that the pencil $\widetilde{A} + \lambda \widetilde{E}$ is singular, i.e., that $\det(\widetilde{A} + \lambda \widetilde{E}) \equiv 0$. The norm of this smallest perturbation is called the *distance to singularity* of the pencil $A + \lambda E$. Determining this distance is important in many applications, in particular in the analysis of differential-algebraic equations (DAEs), see [2, 14, 15, 22] for the theory and for a large number of applications. General DAEs have the form of implicit systems of equations

$$F(t, x, \dot{x}) = 0, \tag{1}$$

where $\dot{x}$ denotes the time derivative of the state x. If such systems are linearized along a stationary solution, then one obtains a linear DAE

$$E\dot{x} + Ax = f, \tag{2}$$

with constant coefficients $A, E \in \mathbb{C}^{n \times n}$.

Consider the initial value problem of solving (2) with initial value $x(t_0) = x_0$. If the pencil $A + \lambda E$ is regular, f is sufficiently smooth, and the initial value is consistent, then the initial value problem has a unique solution. However, if $A + \lambda E$ is singular,

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then a solution only exists if the inhomogeneity f lies in the image of the DAE operator $E\dot{x}+Ax$, and even if a solution exists then it will not be unique.

In many engineering applications and in most industrial environments, modern modeling packages such as *simscape* and *modelica* [17, 20] or modeling environments such as *simulink* [18] are used to construct DAE models of the form (1) automatically. It is then necessary to analyze whether the model equations have a unique solution or not, and also to study the sensitivity of the model under small perturbations, which are inevitable due to modeling errors or uncertainty in the model parameters. This analysis is usually done by checking the regularity of the pencil $A+\lambda E$ associated with the linearization. The rule of thumb is that if the pencil $A+\lambda E$ is close to a singular pencil within the uncertainty of the data, then in a numerical simulation the DAE behaves just like a singular system, and the results of numerical simulations cannot be trusted. Unfortunately, only upper bounds are known for the distance to singularity, see [3] for a survey. One should also mention, that in many applications, in particular those arising from network analysis in electrical engineering [22] or multi-body system simulation [7], the physical application restricts the entries of the pencil $A+\lambda E$, where data uncertainties arise. This results in the effect that the perturbation matrices ΔA and ΔE typically have a rank that is small compared to the overall system size. This situation motivates the study of the distance to the nearest singular pencil under small rank perturbations, i.e., we consider perturbed pencils of the form

$$A+\tau B_1B_2^*+\lambda(E+\tau F_1F_2^*), \tag{3}$$

where τ is a complex parameter describing the perturbation level, and where the matrices

$$B_1,B_2\in\mathbb{C}^{n\times\kappa_A},\ F_1,F_2\in\mathbb{C}^{n\times\kappa_E},$$

describing the perturbations satisfy

$$\operatorname{rank}B_1=\operatorname{rank}B_2=\kappa_A\geqslant 0,\quad \operatorname{rank}F_1=\operatorname{rank}F_2=\kappa_E\geqslant 0,$$

with κ_A,κ_E being small. An important special case is the case that the matrix E is not perturbed at all, i.e., $\kappa_E=0$. This is motivated from applications in circuit simulation [22], where the matrix E is a matrix with entries $0,1,-1$ that describes the network topology and which is not influenced by possible parameter uncertainties. In this case one can study parameter uncertainties in A arising in a rank κ_A part of the matrix by a sequence of consecutive rank-one perturbations of the form

$$A+\tau uv^*+\lambda E, \tag{4}$$

where $u,v\in\mathbb{C}^n\setminus\{0\}$. In our study we will concentrate on the rank one case, i.e., rank-one perturbations of a regular pencil, for which the perturbed pencil becomes singular.

We will give particular emphasis to the class of Hermitian pencils, i.e., pencils with both A and E being Hermitian, and their Hermitian perturbations. This special type of pencils appears in a large number of applications [9, 11, 16], where the Hermitian structure arises from the physical structure of the problem. In this case it is essential that the perturbations also have the Hermitian structure, which leads to the different notion of *Hermitian distance to singularity*.

The remainder of the paper is organized as follows. In Section 2 the *Weyl function* $Q(\lambda) = v^*(A - \lambda E)^{-1}u$ associated with a rank one perturbation as in (4) is introduced. It allows to determine the spectrum of the perturbed pencil. In particular, with the help of the Weyl function one is able to see how the spectrum changes in dependence of the parameter τ. We show that $A + \tau uv^* + \lambda E$ is a singular pencil for some value of τ if and only if $Q(\lambda)$ is constant, see Theorem 4. We also observe that in this case the eigenvalues of the perturbed pencils are constant functions in the parameter τ. This peculiar behavior stands in contrast to existing results on the local behavior of the eigenvalues of matrices [12, 21, 24, 25] or regular matrix polynomials [4], but is in line with the existing results on perturbations on singular matrix pencils [6] and matrix polynomials [5].

In Section 3 we study the *rank-one distance to singularity* as the smallest norm of a rank-one perturbation that makes a given pencil singular and show in Theorem 7, that this distance can be expressed as a quadratic constrained optimization problem with quadratic constraints. This leads to a reformulation of the problem in the language of zeros of polynomials and a simple estimate, see Theorem 13.

We then specialize to Hermitian pencils in Section 4 and characterize for which canonical forms of the Hermitian pencil $A + \lambda E$ it is possible to construct a Hermitian perturbation of A of rank one, such that the perturbed pencil is singular. In Section 5 a closed formula for the rank-one distance to singularity in a special case is obtained, see Theorem 23. The formula is followed by several examples showing its limitations. In Section 6 we describe major differences between the Hermitian and non-Hermitian case, cf. Theorems 13 and 32, and illustrate these with examples.

Section 7 presents a suggestion for a numerical method of alternating projections for finding the closest rank-one singularizing perturbation. Several examples indicate the difficulties in applying the method. Finally, Section 8 presents extensions of the results from Section 2 to perturbation matrices of arbitrary rank.

2. The Weyl function and its relation to singularity of pencils

Throughout the paper, the following notation is used. For a complex matrix B of any dimension the symbol B^* denotes the conjugate transpose of B while $\|B\|_2$ denotes the operator norm and $\|B\|_F$ the Frobenius norm of B. If $X_1, \dots, X_l \in \mathbb{C}^{n\times n}$ then by span$\{X_1, \dots, X_l\}$ we denote the set of all linear combinations of matrices $X_1, \dots, X_l$. The space $\mathbb{C}^{n\times n}$ will be interpreted as a unitary space with the inner product given by

$$\langle X, Y\rangle := \operatorname{tr}(Y^*X),$$

and the corresponding norm is the Frobenius norm. The symbol '$\perp$' denotes the orthogonal complement in the space $(\mathbb{C}^{n\times n}, \langle\cdot,\cdot\rangle)$. If $\mathscr{V}$ is a subspace of $\mathbb{C}^{n\times n}$ then by $\mathfrak{P}_{\mathscr{V}}$ we denote the orthogonal projection from $\mathbb{C}^{n\times n}$ to $\mathscr{V}$.

Let $A, E \in \mathbb{C}^{n\times n}$. We say that a point $\lambda_0 \in \mathbb{C}$ is a *regular point* of the pencil $A + \lambda E$ if the matrix $A + \lambda_0 E$ is invertible. Infinity is called a regular point of $A + \lambda E$

if E is invertible. A point of the *extended complex plane* $\mathbb{C}\cup\{\infty\}$ which is not a regular point of $A+\lambda E$ will be called a *singular point*. A pencil is called *regular*, if it has regular points, otherwise it is called *singular*. The singular points are zeros of the *characteristic polynomial* $p(\lambda)=\det(A+\lambda E)$.

For a regular point λ_0 we introduce the *resolvent*

$$R(\lambda_0):=(A+\lambda_0 E)^{-1}.$$

Then for the one-parameter family of rank-one perturbations

$$A+\tau uv^*+\lambda E,\quad \tau\in\mathbb{C}, \tag{5}$$

with $u,v\in\mathbb{C}^n\setminus\{0\}$, on the set of regular points of $A+\lambda E$, we define the rational *Weyl function*

$$Q(\lambda)=v^*R(\lambda)u. \tag{6}$$

We recall the following basic proposition, an extension of this result to the case of perturbations of arbitrary rank and point $\lambda_0=\infty$ will be presented in Section 8.

PROPOSITION 1. *Let $A+\lambda E$ be a regular pencil with $A,E\in\mathbb{C}^{n\times n}$, let $u,v\in\mathbb{C}^n\setminus\{0\}$ and let $\tau_0\in\mathbb{C}$. Then*

(i) $\det(A+\tau_0uv^*+\lambda E)=\det(A+\lambda E)\cdot(1+\tau_0Q(\lambda))$*;*

(ii) a regular point $\lambda_0\in\mathbb{C}$ of $A+\lambda E$ is a singular point of $A+\tau_0uv^*+\lambda E$ if and only if $1+\tau_0Q(\lambda_0)=0$.

Moreover, the pencil $A+\lambda E+\tau uv^*$ is singular for at most one value of $\tau\in\mathbb{C}$.

Proof. The proof of (i) follows standard lines, cf. [21, 23], but we include it for the sake of completeness. We obtain

$$\begin{aligned}\det(A+\tau_0uv^*+\lambda E)&=\det(A+\lambda E)\left(I_n+\tau_0uv^*(A+\lambda E)^{-1}\right)\\&=\det(A+\lambda E)\cdot(1+\tau_0Q(\lambda)),\end{aligned}$$

using the formula $\det(I_n+BC)=\det(I_m+CB)$ for $B,C^T\in\mathbb{C}^{n\times m}$. This proves (i), and (ii) follows immediately from (i). To prove the last statement note the following. If $Q(\lambda)\equiv 0$ then the pencil $A+\lambda E+\tau uv^*$ is regular for all values of $\tau\in\mathbb{C}$. Let $Q(\lambda_0)\neq 0$ for some regular point λ_0 of $A+\lambda E$ and let the pencil $A+\lambda E+\tau uv^*$ be singular for $\tau=\tau_0$ and $\tau=\tau_1$. Then, by (ii), $1+\tau_0Q(\lambda_0)=1+\tau_1Q(\lambda_0)$. Hence, $\tau_0=\tau_1$. $\square$

Another important tool for the investigation of matrix pencils is the *Kronecker canonical form*, see e.g., [8, Ch. XII]. We only state the result for the special case of square matrix pencils.

THEOREM 2. (Kronecker canonical form) *Let $A, E \in \mathbb{C}^{n\times n}$. Then there exist invertible matrices $S, T \in \mathbb{C}^{n\times n}$ such that the pencil $S(A+\lambda E)T$ is block-diagonal with diagonal blocks of one of the following forms:*

$$\mathscr{P}_{k,\gamma}(\lambda) = \begin{bmatrix} \gamma-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \gamma-\lambda \end{bmatrix} \in \mathbb{C}^{k\times k}, \quad \gamma \in \mathbb{C}; \tag{7}$$

$$\mathscr{M}_k(\lambda) = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{k\times k}; \tag{8}$$

$$\mathscr{G}_k(\lambda) = \begin{bmatrix} 1 & & \\ -\lambda & \ddots & \\ & \ddots & 1 \\ & & -\lambda \end{bmatrix} \in \mathbb{C}^{k\times(k-1)}, \tag{9}$$

$$\mathscr{G}_k(\lambda)^\top \in \mathbb{C}^{(k-1)\times k}. \tag{10}$$

where the parameters $\gamma \in \mathbb{C}$ and $k \geqslant 1$ depend on the particular block and hence may be different in different blocks. The canonical form is unique up to permutations of blocks.

If the canonical form of $A+\lambda E$ contains a block $\mathscr{P}_{k,\gamma}(\lambda)$ ($\mathscr{M}_k(\lambda)$) then we say that γ (∞, respectively) is an *eigenvalue* of $A+\lambda E$. Note that the notion is a generalization of the notion of eigenvalue for matrices. We distinguish between eigenvalues and singular points. If $A+\lambda E$ is regular, then $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $A+\lambda E$ if and only if λ_0 is a singular point of $A+\lambda E$, but if $A+\lambda E$ is singular, then all values $\lambda_0 \in \mathbb{C}$ are singular points, but not all of them are eigenvalues of $A+\lambda E$. Furthermore, the pencil is singular if and only if it contains at least one block of the forms (9) or (10). Note that in some references, also zero blocks $\mathscr{S}(\lambda) = 0 \in \mathbb{C}^{m\times\ell}$ are introduced in the canonical form, but these are redundant because they can also be expressed by combining m blocks $\mathscr{G}_1(\lambda)$ of size 1×0 with ℓ blocks $\mathscr{G}_1(\lambda)^\top$ of size 0×1.

On the set of regular points of the pencil $A+\lambda E$, we define matrix-valued functions

$$C_j(\lambda) := R(\lambda)\,(ER(\lambda))^j = (A+\lambda E)^{-1}\left(E(A+\lambda E)^{-1}\right)^j, \quad j \in \mathbb{N}. \tag{11}$$

Then $C_0(\lambda) = R(\lambda)$ and it is straightforward to check that

$$C_j(\lambda_0) = (-1)^j \frac{1}{j!} \frac{d^j C_0(\lambda)}{d\lambda^j}\bigg|_{\lambda=\lambda_0}, \quad j \geqslant 1, \tag{12}$$

i.e. $C_j(\lambda_0)$ is the coefficient of the j-th degree term of the Taylor expansion of $R(\lambda)$ at λ_0. Furthermore, we have

$$C_{j+1}(\lambda) = R(\lambda)EC_j(\lambda), \quad i,j \geqslant 0 \tag{13}$$

and

$$C_i(\lambda)EC_j(\lambda) = C_{i+j+1}(\lambda), \quad i,j \geqslant 0. \tag{14}$$

A further important property of the matrices $C_j(\lambda)$ is given in the following lemma.

LEMMA 3. *Let $A, E \in \mathbb{C}^{n\times n}$ and let $\lambda_0 \in \mathbb{C}$ be a regular point of $A + \lambda E$. Then there exists an $n_0 \leqslant n$ such that the matrices $C_1(\lambda_0), \dots, C_{n_0}(\lambda_0)$ are linearly independent, and for all $j > n_0$, we have*

$$C_j(\lambda_0) \in \operatorname{span}\left\{C_1(\lambda_0), \dots, C_{n_0}(\lambda_0)\right\}.$$

Proof. Note that the transformation $S(A+\lambda E)T$, with S,T invertible, changes the matrices $C_j(\lambda_0)$ into $T^{-1}C_j(\lambda_0)S^{-1}$ ($j \geqslant 1$). Therefore, we may assume without loss of generality that $A + \lambda E$ is in Kronecker canonical form. If for some $j_0 > 1$,

$$C_{j_0}(\lambda_0) \in \operatorname{span}\left\{C_1(\lambda_0), \dots, C_{j_0-1}(\lambda_0)\right\}, \tag{15}$$

i.e., there exist $\alpha_1, \dots, \alpha_{j_0-1}$ such that $C_{j_0}(\lambda_0) = \sum_{k=1}^{j_0-1} \alpha_k C_k(\lambda_0)$, then by (13) we have

$$C_{j_0+1}(\lambda_0) = R(\lambda_0)EC_{j_0}(\lambda_0) = \sum_{k=1}^{j_0-1} \alpha_k C_{k+1}(\lambda_0) \in \operatorname{span}\left\{C_2(\lambda_0), \dots, C_{j_0}(\lambda_0)\right\}.$$

Thus, using (15) and induction we obtain that

$$C_j(\lambda_0) \in \operatorname{span}\left\{C_1(\lambda_0), \dots, C_{j_0-1}(\lambda_0)\right\}, \quad j \geqslant j_0. \tag{16}$$

We set n_0 to be the minimum of all j_0 satisfying (15). Observe that $C_0(\lambda)$ is block diagonal with blocks that are upper-triangular *Toeplitz matrices*, i.e., matrices with constant entries along each of the diagonals. By (12), the matrix $C_j(\lambda_0)$ ($j \geqslant 1$) has the same block structure, again with upper-triangular Toeplitz blocks. Therefore, each of the matrices $C_j(\lambda_0)$ ($j \geqslant 1$) is uniquely determined by the first rows of all the diagonal blocks, and hence the matrices $C_1(\lambda_0), \dots, C_{n+1}(\lambda_0)$ are linearly dependent. Thus, $n_0 \leqslant n$. $\square$

The following theorem presents equivalent conditions for the pencil $A + \tau uv^* + \lambda E$ to be singular for some value of τ.

THEOREM 4. *Suppose that the pencil $A + \lambda E$ is regular, where $A, E \in \mathbb{C}^{n\times n}$, and let $u, v \in \mathbb{C}^n$. Then the following conditions are equivalent.*

(a) The pencil $A + \tau_0 uv^* + \lambda E$ is singular for some $\tau_0 \in \mathbb{C}$.

(b) The function $Q(\lambda) = v^*(A+\lambda E)^{-1}u$ is a nonzero constant function.

(c) The polynomial in two variables $p(\lambda,\tau)=\det(A+\tau uv^*+\lambda E)$ *satisfies*

$$p(\lambda,\tau)=(1+\zeta\tau)\det(A+\lambda E)$$

for some $\zeta\in\mathbb{C}\setminus\{0\}$.

(d) There exists $\tau_0\in\mathbb{C}$ *such that for all* $\tau\in\mathbb{C}\setminus\{\tau_0\}$ *the pencil is regular and the eigenvalues of* $A+\tau uv^*+\lambda E$ *and their algebraic multiplicities coincide with the eigenvalues of* $A+\lambda E$ *and their respective algebraic multiplicities.*

(e) For every regular point $\lambda_0\in\mathbb{C}$ *of* $A+\lambda E$ *we have* $v^*C_0(\lambda_0)u\neq 0$ *and*

$$v^*C_j(\lambda_0)u=0,\quad j=1,2,\ldots,n. \tag{17}$$

(f) For some regular point $\lambda_0\in\mathbb{C}$ *of* $A+\lambda E$ *we have* $v^*C_0(\lambda_0)u\neq 0$ *and the identities in* (17) *hold.*

Furthermore, if (b) *holds, then* $A+\tau_0uv^*+\lambda E$ *is singular precisely for the value* $\tau_0=-1/Q(\lambda)$, *and this* τ_0 *coincides with* τ_0 *from statements* (a) *and* (d).

Proof.

(a)⇒(b) If the pencil $A+\tau_0uv^*+\lambda E$ is singular, then every $\lambda_0\in\mathbb{C}$ is a singular point. By Proposition 1(ii) we have $Q(\lambda_0)=-1/\tau_0$ for every regular point λ_0 of $A+\lambda E$, so $Q(\lambda)$ is constant and nonzero.

(b)⇒(a) If $Q(\lambda)$ is a constant nonzero function on the set of regular points of $A+\lambda E$, then with $\tau_0=-1/Q(\lambda)$ the pencil $A+\tau_0uv^*+\lambda E$ has infinitely many finite singular points by Proposition 1(ii). Hence, it is singular.

(b)⇒(c) follows from Proposition 1(i) with $\zeta=Q(\lambda)$.

(c)⇒(d) Setting $\tau_0:=-1/\zeta$ we see that the (monic) characteristic polynomials of the pencils $A+\tau uv^*+\lambda E$ $(\tau\in\mathbb{C}\setminus\{\tau_0\})$ coincide, which is clearly equivalent to (d).

(d)⇒(b) By Proposition 1 the pencil $A+\tau uv^*+\lambda E$ is regular for all but at most one value of τ. Hence, there exists $\tau_1\in\mathbb{C}\setminus\{0,\tau_0\}$ such that the pencil $A+\tau_1uv^*+\lambda E$ is regular. Thus, if (d) holds, then $\det(A+\tau_1uv^*+\lambda E)$ and $\det(A+\lambda E)$ have the same roots as polynomials in λ, i.e.

$$\det(A+\tau_1uv^*+\lambda E)=c\det(A+\lambda E),$$

with some $c\neq 0$. Then, by Proposition 1(i), for every regular point λ of $A+\lambda E$ one has

$$1+\tau_1Q(\lambda)=c,$$

and as $\tau_1\neq 0$

$$Q(\lambda)=\frac{c-1}{\tau_1}. \tag{18}$$

Thus (b) holds.

(b)$\Rightarrow$(e) If $Q(\lambda)$ is constant, then the identities (12) imply that

$$0 = Q^{(j)}(\lambda_0) = (-1)^j j! \cdot v^* C_j(\lambda_0) u$$

for $j = 1, \ldots, n$, and for any regular point $\lambda_0 \in \mathbb{C}$ of $A + \lambda E$. Moreover, we have $0 \neq Q(\lambda_0) = v^* R(\lambda_0) u = v^* C_0(\lambda_0) u$.

(e)$\Rightarrow$(f) is trivial.

(f)$\Rightarrow$(b) If (f) holds, then by Lemma 3 and (12), all derivatives of $Q(\lambda)$ at λ_0 are zero. Since $Q(\lambda)$ is a rational function, it is constant, and it is nonzero because $Q(\lambda_0) = v^* C_0(\lambda_0) u \neq 0$. $\square$

3. The rank-one distance to singularity for general pencils

In this section we will investigate the distance to singularity for low rank perturbations in the sense of the following definition.

DEFINITION 5. Let $A + \lambda E$, $A, E \in \mathbb{C}^{n \times n}$ be a regular pencil. Consider perturbation matrices $\Delta E, \Delta A \in \mathbb{C}^{n \times n}$ with $\operatorname{rank} \Delta A \leqslant \kappa_A$, $\operatorname{rank} \Delta E \leqslant \kappa_E$, such that the perturbed pencil $\tilde{A} + \lambda \tilde{E} := A + \Delta A + \lambda(E + \Delta E)$ is singular.

Then the *rank-(κ_A, κ_E) distance to singularity* of $A + \lambda E$ is defined as

$$\delta_{\kappa_A,\kappa_E}(A,E) = \min\Big\{ \|[\Delta A, \Delta E]\|_F \;\Big|\; \Delta A, \Delta E \in \mathbb{C}^{n\times n};$$
$$\operatorname{rank} \Delta A \leqslant \kappa_A,\ \operatorname{rank} \Delta E \leqslant \kappa_E,\ \det(\tilde{A} + \lambda \tilde{E}) \equiv 0 \Big\}.$$

Here for convenience, the minimum over the empty set is defined as $+\infty$. In the particular case $\kappa_A = 1$ and $\kappa_E = 0$, we call $\delta_{1,0}$ the *rank-one distance to singularity* of $A + \lambda E$.

REMARK 6. The generalized Schur form of the pair (A, E), see, e.g., [10], states that there exist unitary matrices U, V such that both $UAV = [a_{ij}]$ and $UEV = [b_{ij}]$ are upper triangular matrices. From this we immediately deduce that $\delta_{1,1}(A,E) < +\infty$ for any regular pencil $A + \lambda E$. Indeed, $(\Delta A, \Delta E) = (-U^* a_{11} e_1 e_1^\top V^*, -U^* b_{11} e_1 e_1^\top V^*)$ is a rank-(1,1) perturbation that makes the pencil $A + \lambda E$ singular. Furthermore, note that $\delta_{\kappa_A,0}(A,E) < +\infty$ if and only if E is not invertible. Indeed, if E is singular, then there is at least one zero entry on the diagonal of UEV, say $b_{jj} = 0$. Then $\Delta A = -U^* a_{jj} e_j e_j^\top V^*$, $\Delta E = 0$ is a perturbation that makes $A + \lambda E$ singular, showing that $\delta_{\kappa_A,0}(A,E) \leqslant \delta_{1,0}(A,E) < +\infty$. If on the other hand E is nonsingular, then no perturbation of A of any rank κ_A will make the pencil singular, so $\delta_{\kappa_A,0}(A,E) = +\infty$. One should also note, that the so constructed singularizing perturbations need not to be the ones of minimal norm, see [3].

In the following we will be mainly concerned with the distance $\delta_{1,0}(A,E)$, and we begin with one of the main results of the paper that shows that the problem of determining $\delta_{1,0}(A,E)$ can be reformulated as a quadratic optimization problem with a quadratic constraint. (For convenience, we set $\max \emptyset := 0$ and $1/0 := +\infty$.)

THEOREM 7. *Let $A+\lambda E$ with $A,E\in\mathbb{C}^{n\times n}$ be a regular matrix pencil, and let λ_0 be an arbitrary regular point. Then*

$$\delta_{1,0}(A,E)^{-1}=\max\left\{|v^*R(\lambda_0)u|\,\middle|\,u,v\in\mathbb{C}^n,\ \|u\|_2\,\|v\|_2=1,\ v^*C_j(\lambda_0)u=0,\ j=1,\dots,n\right\}.$$

Proof. Recall that $\delta_{1,0}(A,E)$ equals the minimal $|\tau|$ for all $\tau\in\mathbb{C}$ for which there exist $u,v\in\mathbb{C}^n$ with $\|uv^*\|_F=1$ such that the pencil $A+\tau uv^*+\lambda E$ is singular. On the other hand, the pencil $A+\tau uv^*+\lambda E$ is singular if and only if the equations (17) are satisfied and $\tau=-1/Q(\lambda_0)=-1/v^*R(\lambda_0)u$, see Theorem 4(e). Thus

$$\delta_{1,0}(A,E)=\min\left\{|v^*R(\lambda_0)u|^{-1}\,\middle|\,u,v\in\mathbb{C}^n,\ \|uv^*\|_F=1,\ v^*C_j(\lambda_0)u=0,\ j=1,\dots,n\right\}.$$

Taking inverses of both sides and noting that $\|uv^*\|_F^2=\|u\|_2^2\,\|v\|_2^2$ finishes the proof. □

The explicit representation of the rank-one distance to singularity as a minimum of a quadratic function with quadratic constraints allows for the development of numerical methods. We will discuss such methods in the following, but first, we construct a simple estimate from below for $\delta_{1,0}$. For this, observe that the constraints $v^*C_j(\lambda_0)u=0$ of Theorem 4 can be expressed as

$$0=\operatorname{tr}\left(v^*C_j(\lambda_0)u\right)=\operatorname{tr}\left(uv^*C_j(\lambda_0)\right)=\langle C_j(\lambda_0),vu^*\rangle,$$

so the rank-one matrices vu^* and $C_j(\lambda_0)$ are orthogonal with respect to the standard inner product on $\mathbb{C}^{n\times n}$ for $j=1,\dots,n$. Thus, vu^* lies in the orthogonal complement

$$(\operatorname{span}\{C_1(\lambda_0),\dots,C_n(\lambda_0)\})^{\perp},$$

of the linear space generated by $C_1(\lambda_0),\dots,C_n(\lambda_0)$. The following Lemma shows that this space is independent of λ_0.

LEMMA 8. *The linear span with complex coefficients*

$$\operatorname{span}\{C_1(\lambda_0),\dots,C_n(\lambda_0)\}$$

does not depend on the choice of the regular point λ_0 of $A+\lambda E$.

Proof. Fix a regular point λ_0 of $A+\lambda E$. First we show that the set $\mathscr{L}$ of all regular points λ for which

$$\operatorname{span}\{C_1(\lambda),\dots,C_n(\lambda)\}\subseteq\operatorname{span}\{C_1(\lambda_0),\dots,C_n(\lambda_0)\}\tag{19}$$

is open in the set of all regular point of $A+\lambda E$. Let $\lambda_1\in\mathscr{L}$. Since $R(\lambda)$ is analytic, there exists a neighborhood $\mathscr{U}$ of λ_1 such that

$$C_0(\lambda)=R(\lambda)=\sum_{j=0}^{\infty}(\lambda-\lambda_1)^j(-1)^jC_j(\lambda_1),$$

converges absolutely for all $\lambda \in \mathscr{U}$, cf. (12). Due to Lemma 3 one has

$$R(\lambda) = \sum_{j=0}^{n} a_j(\lambda - \lambda_1)^j C_j(\lambda_1),$$

with some coefficients $a_0 = 1$, $a_1, \dots, a_n \in \mathbb{C}$, depending, possibly, on λ_1. Hence,

$$\begin{aligned} C_k(\lambda) &= R(\lambda)(ER(\lambda))^k = \left(\sum_{j=0}^{n}(\lambda-\lambda_1)^j a_j C_j(\lambda_1)\right)\left(E\sum_{j=0}^{n}(\lambda-\lambda_1)^j a_j C_j(\lambda_1)\right)^k \\ &= \sum_{l=0}^{n\cdot k}(\lambda-\lambda_1)^l \Bigg(\underbrace{\sum_{i_0+\cdots+i_k=l}(a_{i_0}\cdots a_{i_k})}_{=:b_l} C_{i_1}(\lambda_1)E\cdots EC_{i_k}(\lambda_1)\Bigg) \\ &= \sum_{l=0}^{n\cdot k}(\lambda-\lambda_1)^l\, b_l\, C_{l+k}(\lambda_1), \end{aligned}$$

where the last equality is obtained by subsequent applications of (14). By Lemma 3, we have

$$C_k(\lambda) \in \operatorname{span}\{C_k(\lambda_1), \dots, C_{nk+k}(\lambda_1)\} \subseteq \operatorname{span}\{C_1(\lambda_1), \dots, C_n(\lambda_1)\}.$$

So, $\operatorname{span}\{C_1(\lambda), \dots, C_n(\lambda)\} \subseteq \operatorname{span}\{C_1(\lambda_1), \dots, C_n(\lambda_1)\} \subseteq \operatorname{span}\{C_1(\lambda_0), \dots, C_n(\lambda_0)\}$ for all $\lambda \in \mathscr{U}$ which shows that $\mathscr{Z}$ is open.

Now observe that $\mathscr{Z}$ is also closed in the set of all regular points of $A + \lambda E$. To see this, let $\lambda_k \in \mathscr{Z}$ converge with $k \to \infty$ to a regular point z_0. By continuity, $C_j(\lambda_k)$ then converges for $k \to \infty$ to $C_j(z_0)$ for $j = 1, \dots n$. By hypothesis, we have

$$C_j(\lambda_k) \in \operatorname{span}\{C_1(\lambda_0), \dots, C_n(\lambda_0)\}$$

for $j = 1, \dots, n$, and hence, since finite-dimensional subspaces are closed, it follows that

$$C_j(z_0) \in \operatorname{span}\{C_1(\lambda_0), \dots, C_n(\lambda_0)\}$$

for $j = 1, \dots, n$, and thus $z_0 \in \mathscr{Z}$. Since the set of regular points of $A + \lambda E$ is connected, we have that $\mathscr{Z}$ is equal to the set of regular points. Finally, since λ_0 was arbitrary, a symmetry argument shows that the inclusion in (19) is an equality. $\square$

In view of Lemma 8 we introduce

$$\mathscr{D} := (\operatorname{span}\{C_1(\lambda_0), \dots, C_n(\lambda_0)\})^{\perp}, \tag{20}$$

where λ_0 is any regular point of $A + \lambda E$. Recalling that $\mathfrak{P}_{\mathscr{D}}$ denotes the orthogonal projection from $\mathbb{C}^{n\times n}$ to $\mathscr{D}$, we continue with two examples.

EXAMPLE 9. Let $A+\lambda E=\mathscr{P}_{k,\gamma}(\lambda)$ be a block of size $k>0$ associated with the finite eigenvalue γ as in (7). In this case

$$R(\lambda)=\begin{bmatrix}(\gamma-\lambda)^{-1} & -(\gamma-\lambda)^{-2} & \dots & (-1)^{k-1}(\gamma-\lambda)^{-k}\\ 0 & (\gamma-\lambda)^{-1} & \ddots & \vdots\\ \vdots & \ddots & \ddots & -(\gamma-\lambda)^{-2}\\ 0 & \dots & 0 & (\gamma-\lambda)^{-1}\end{bmatrix}.$$

Then a straightforward calculation using formula (12) shows that $C_1(\lambda),\dots,C_n(\lambda)$ are linearly independent upper triangular Toeplitz matrices with nonzero diagonals. Consequently, we have $R(\lambda)\in\operatorname{span}\{C_1(\lambda),\dots,C_n(\lambda)\}$ and $\mathfrak{P}_{\mathscr{D}}R(\lambda)=0$.

EXAMPLE 10. Let $A+\lambda E=\mathscr{M}_k(\lambda)$ be a block of size $k>0$ associated with the eigenvalue infinity as in (8). By (12) for $j\leqslant k-1$ it follows that $C_j(\lambda)$ is an upper-triangular Toeplitz matrix, with its first row equal to

$$\left[0\cdots 0\ \ a_{j+1}^{(j)}\lambda^0\ \cdots\ a_n^{(j)}\lambda^{k-j-1}\right],$$

where the zero entry is repeated j times, and $a_{j+1}^{(j)},\dots,a_n^{(j)}\in\mathbb{R}\setminus\{0\}$. Hence, we have $\mathfrak{P}_{\mathscr{D}}R(\lambda)=I_k$.

Examples 9 and 10 suggest that $\mathfrak{P}_{\mathscr{D}}R(\lambda_0)$ may also be independent of the choice of the regular point λ_0 and this is indeed the case as the next result shows.

PROPOSITION 11. *If $A+\lambda E$ is a regular pencil, then the matrix $\mathfrak{P}_{\mathscr{D}}R(\lambda_0)$ does not depend on the choice of the regular point λ_0. Furthermore, $\mathfrak{P}_{\mathscr{D}}R(\lambda_0)=0$ if and only if infinity is a regular point of $A+\lambda E$.*

Proof. As in the proof of Lemma 8 fix a regular point λ_0 and define the set $\mathscr{W}$ of all those regular points λ for which $\mathfrak{P}_{\mathscr{D}}R(\lambda_0)=\mathfrak{P}_{\mathscr{D}}R(\lambda)$. We show that $\mathscr{W}$ is open in the set of regular points. Take $\lambda_1\in\mathscr{W}$ and let $\mathscr{U}$ be a neighborhood of λ_1 such that for all $\lambda\in\mathscr{U}$ we have

$$R(\lambda)=\sum_{j=0}^{n}(\lambda-\lambda_1)^j\,a_j\,C_j(\lambda_1),$$

with some coefficients $a_0=1$, $a_1,\dots,a_n\in\mathbb{C}$, see the proof of Lemma 8. By Lemma 8, we have

$$\mathfrak{P}_{\mathscr{D}}R(\lambda)=\sum_{j=0}^{n}a_j(\lambda-\lambda_1)^j\mathfrak{P}_{\operatorname{span}\{C_1(\lambda_1),\dots,C_n(\lambda_1)\}^\perp}C_j(\lambda_1)=\mathfrak{P}_{\mathscr{D}}R(\lambda_1)$$

for all $\lambda\in\mathscr{U}$ and thus $\mathscr{W}$ is open. The set $\mathscr{W}$ is clearly closed in the set of the regular points, hence, it is equal to the set of regular points which is a connected set.

For the 'furthermore' part note, that without loss of generality one may assume that $A+\lambda E$ is in Kronecker canonical form, and using (12), one may consider each block separately. If $A+\lambda E = \mathscr{P}_{k,\gamma}(\lambda)$, then by Example 9 one has $\mathfrak{P}_{\mathscr{D}}R(\lambda_0)=0$ and if $A+\lambda E = \mathscr{M}_k(\lambda)$, then $\mathfrak{P}_{\mathscr{D}}R(\lambda_0)=I_k$. □

In view of Proposition 11 we define the quantity

$$\rho(A,E) := \|\mathfrak{P}_{\mathscr{D}}R(\lambda_0)\|_F^{-1}.$$

for a regular point λ_0 of $A+\lambda E$. By Proposition 11, $\rho(A,E)$ is well defined if and only if infinity is an eigenvalue of $A+\lambda E$. If this is not the case, then we set $\rho(A,E) := +\infty$.

The following observation is a key step in showing that $\rho(A,E)$ can be used as a lower bound for $\delta_{1,0}(A,E)$.

PROPOSITION 12. *Let $A+\lambda E$ be a regular pencil, let λ_0 be a regular point and let $G \in \mathbb{C}^{n\times n}$ be of rank one. Then*

$$A+\tau G+\lambda E$$

is singular for some $\tau_0 \in \mathbb{C}$ if and only if $\mathfrak{P}_{\mathscr{D}}G^* = G^*$ and $\operatorname{tr}(GR(\lambda_0)) \neq 0$.

If these conditions hold, then the only value of τ for which $A+\tau G+\lambda E$ is singular is

$$\tau_0 = -\frac{1}{\operatorname{tr}(GR(\lambda_0))}.$$

Proof. Let $G = uv^*$, with some $u,v \in \mathbb{C}^n$. Note that $\mathfrak{P}_{\mathscr{D}}(G^*) = G^*$ is equivalent to

$$0 = \langle C_j(\lambda_0), G^* \rangle = \operatorname{tr}(uv^*C_j(\lambda_0)) = v^*R(\lambda_0)(ER(\lambda_0))^j u, \quad j=1,\dots,n,$$

for any regular point λ_0 of $A+\lambda E$. Together with the fact that $v^*R(\lambda_0)u = \operatorname{tr}(uv^*R(\lambda_0)) = \operatorname{tr}(GR(\lambda_0)) \neq 0$, the assertion follows by Theorem 4, equivalence (a)⇔(e). The identity

$$\tau_0 = -\frac{1}{v^*R(\lambda_0)u} = -\frac{1}{\operatorname{tr}\big(GR(\lambda_0)\big)}$$

then follows from the second part of Theorem 4. □

As a consequence we obtain the following lower bound for the rank-one distance to singularity.

THEOREM 13. *Let $A+\lambda E$ be a regular pencil and let E be singular, i.e., infinity is an eigenvalue. Let the linear subspace $\mathscr{D}$ of $\mathbb{C}^{n\times n}$ be defined by* (20) *and* (11)*, and let $D_0,\dots,D_k$ be an orthonormal basis of $\mathscr{D}$ with*

$$D_0 = \frac{\mathfrak{P}_{\mathscr{D}}R(\lambda_0)}{\|\mathfrak{P}_{\mathscr{D}}R(\lambda_0)\|_F}.$$

Then the set

$$\Xi := \left\{ [\alpha_0, \dots, \alpha_k]^T \in \mathbb{C}^{k+1} \,\middle|\, \alpha_0 \neq 0,\ \sum_{j=0}^{k} |\alpha_j|^2 = 1,\ \operatorname{rank}\left(\sum_{j=0}^{k} \alpha_j D_j \right) = 1 \right\}$$

is not empty and

$$\delta_{1,0}(A,E) = \rho(A,E) \min_{(\alpha_0,\dots,\alpha_k)\in\Xi} |\alpha_0|^{-1}.$$

In particular, we have

$$\delta_{1,0}(A,E) \geqslant \rho(A,E).$$

Proof. By assumption, there exists a matrix G of rank one and Frobenius norm one, such that $A + \tau G + \lambda E$ is singular for some $\tau \in \mathbb{C}$, see Remark 6. By Proposition 12 we have $G^* = \sum_{j=0}^{k} \alpha_j D_j$ for some $\alpha_0, \dots, \alpha_k \in \mathbb{C}$, with $\sum_{j=0}^{k} |\alpha_j|^2 = 1$, as well as

$$0 \neq \operatorname{tr}\big(GR(\lambda_0)\big) = \langle R(\lambda_0), G^* \rangle = \alpha_0 \langle R(\lambda_0), D_0 \rangle = \alpha_0 \left\| \mathfrak{P}_{\mathscr{D}} R(\lambda_0) \right\|_F,$$

which implies that $\alpha_0 \neq 0$. This shows that $\Xi \neq \emptyset$.

Conversely, if $G^* = \sum_{j=0}^{k} \alpha_j D_j$ for some $\alpha_0, \dots, \alpha_k \in \Xi$ then by Proposition 12 the pencil $A + \tau G + \lambda E$ is singular for

$$\tau = -\frac{1}{\operatorname{tr}\big(GR(\lambda_0)\big)} = -\frac{1}{\alpha_0 \left\| \mathfrak{P}_{\mathscr{D}} R(\lambda_0) \right\|_F} = -\alpha_0^{-1} \rho(A,E).$$

Taking $\delta_{1,0}(A,E)$ as the minimum of $|\tau|$ over all pairs (τ, G) with $G^* \in \Xi$ and such that $A + \tau G + \lambda E$ is singular, finishes the proof. □

REMARK 14. The quantity $\min_{(\alpha_0,\dots,\alpha_k)\in\Xi} |\alpha_0|^{-1}$ may be expressed in terms of zeros of multivariate polynomials. Indeed, considering the 2×2 minors of the matrices D_i, we immediately see that the rank condition in the definition of Ξ is equivalent to

$$p_{i_1,i_2,j_1,j_2}(\alpha_0, \dots, \alpha_k) := \sum_{i,j=0}^{k} \big((\alpha_i D_i)_{i_1,j_1} (\alpha_j D_j)_{i_2,j_2} - \alpha_i \alpha_j (D_i)_{i_2,j_1} (D_j)_{i_1,j_2} \big) = 0$$

for every $i_1, i_2, j_1, j_2 = 1, \dots, n$, $i_1 \neq i_2$, $j_1 \neq j_2$, with $(D_i)_{p,q}$ denoting the (p,q) entry of the matrix D_i. In other words, to compute $\delta_{1,0}(A,E)$ one needs to find a common zero of the polynomials p_{i_1,i_2,j_1,j_2} on the unit sphere, with the largest absolute value of the α_0 coordinate.

The inequality $\rho(A,E) \leqslant \delta_{1,0}(A,E)$ may be strict as the following example demonstrates.

EXAMPLE 15. Let $A + \lambda E = \mathscr{M}_k(\lambda)$ with $\mathscr{M}_k(\lambda)$ as in (8) and $k > 0$. Since by Example 10 we have $\mathfrak{P}_{\mathscr{D}} R(\lambda) = I_k$, it follows that $\rho(A,E) = k^{-1/2}$. On the other hand, $\delta_{1,0}(A,E)$ is clearly greater than or equal to the minimal singular value of A, which is one in this case.

Further examples can be found in Section 6, where the analogue of Theorem 13 for Hermitian pencils is discussed.

4. Singularizing perturbations for Hermitian pencils

We call a pencil $A+\lambda E$ *Hermitian* if both matrices A and E are Hermitian, which we denote by $A,E\in\mathbb{C}_H^{n\times n}$.

DEFINITION 16. Let $A+\lambda E$, $A,E\in\mathbb{C}_H^{n\times n}$ be a regular Hermitian pencil. Then the *Hermitian rank-* (κ_A,κ_E) *distance to singularity* of $A+\lambda E$ is defined as

$$\delta^H_{\kappa_A,\kappa_E}(A,E) = \min\Big\{ \|[\Delta A,\Delta E]\|_F \;\Big|\; \Delta A,\Delta E\in\mathbb{C}_H^{n\times n},\ \operatorname{rank}\Delta A\leqslant\kappa_A,\ \operatorname{rank}\Delta E\leqslant\kappa_E,\ \det\big(A+\Delta A+\lambda(E+\Delta E)\big)\equiv 0\Big\}.$$

In the particular case $\kappa_A=1$ and $\kappa_E=0$, we call $\delta^H_{1,0}(A,E)$ the *Hermitian rank-one distance to singularity* of $A+\lambda E$.

Clearly, for any κ_A,κ_E, we have

$$\delta_{\kappa_A,\kappa_E}(A,E)\leqslant\delta^H_{\kappa_A,\kappa_E}(A,E).$$

and if $\kappa_A\geqslant\kappa_1$, $\kappa_E\geqslant\kappa_2$ then $\delta^H_{\kappa_A,\kappa_E}\leqslant\delta^H_{\kappa_1,\kappa_2}$. These inequalities may be strict as demonstrated in Example 18 below which presents a pencil for which we have $\delta_{\kappa_A,\kappa_E}(A,E)<\delta^H_{\kappa_A,\kappa_E}(A,E)$ and $\delta^H_{2,0}(A,E)<\delta^H_{1,0}(A,E)$.

Next, we show that for any Hermitian pencil $A+\lambda E$ we have $\delta^H_{1,1}(A,E)<+\infty$ and we will characterize all Hermitian pencils for which $\delta^H_{1,0}(A,E)<+\infty$. We will also present a family of Hermitian pencils for which

$$\delta_{1,0}(A,E)<\delta^H_{1,0}(A,E)=+\infty.$$

The analysis is based on the canonical form for Hermitian pencils under congruence, see [26].

THEOREM 17. (Hermitian canonical form) *Let $A,E\in\mathbb{C}_H^{n\times n}$. Then there exists an invertible matrix $S\in\mathbb{C}^{n\times n}$ such that the pencil $S^*(A+\lambda E)S$ is block-diagonal with diagonal blocks of one of the following forms:*

i) blocks corresponding to a real eigenvalue $\gamma\in\mathbb{R}$:

$$\mathscr{J}^s_{k,\gamma}(\lambda):=s\begin{bmatrix} & & & \gamma-\lambda\\ & & \iddots & 1\\ & \iddots & \iddots & \\ \gamma-\lambda & 1 & & \end{bmatrix}\in\mathbb{C}^{k\times k},\ s\in\{-1,1\};\tag{21}$$

ii) blocks corresponding to a pair of conjugate complex eigenvalues $\gamma,\overline{\gamma}$, where $\gamma\in\mathbb{C}^+:=\{z:\operatorname{Im}z>0\}$:

$$\mathscr{J}_{2k,\gamma}(\lambda):=\begin{bmatrix} 0 & \mathscr{J}^1_{k,\gamma}(\lambda)\\ \mathscr{J}^1_{k,\overline{\gamma}}(\lambda) & 0\end{bmatrix}\in\mathbb{C}^{2k\times 2k},\tag{22}$$

where $\mathscr{J}^1_{k,\gamma}(\lambda)$ and $\mathscr{J}^1_{k,\overline{\gamma}}(\lambda)$ are defined as in (21);

iii) blocks corresponding to the eigenvalue infinity:

$$\mathcal{N}_k^s(\lambda) := s \begin{bmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & -\lambda \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & -\lambda & & \end{bmatrix} \in \mathbb{C}^{k\times k},\ s\in\{-1,1\}; \tag{23}$$

iv) singular blocks:

$$\mathcal{L}_{2k-1}(\lambda) := \begin{bmatrix} 0 & \mathcal{G}_k(\lambda) \\ \mathcal{G}_k^\top(\lambda) & 0 \end{bmatrix} \in \mathbb{C}^{(2k-1)\times(2k-1)}, \tag{24}$$

where $\mathcal{G}_k(\lambda)$ is given by (9).

The parameters $\gamma\in\mathbb{C}$, $s\in\{-1,1\}$, and $k\geqslant 1$ depend on the particular block and hence may be different in different blocks. Moreover, the canonical form is unique up to permutation of diagonal blocks.

A Hermitian pencil is singular if and only if it contains blocks of the form (24) and the notion of the eigenvalue agrees with the one introduced in Section 2. In contrast to the unstructured canonical form, besides the eigenvalues and the sizes of the blocks, the numbers s in the blocks (21) and (23) associated with real eigenvalues and the eigenvalue infinity are additional invariants. They are called the *signs* of the blocks and their collection is called the *sign characteristic* of the Hermitian pencil $A+\lambda E$, see, e.g., [9].

If $A+\lambda E$ is a regular Hermitian pencil, then the canonical form $S^*(A+\lambda E)S$ reduces to a block-diagonal pencil of the form

$$\left(\bigoplus_{\gamma\in\sigma_{\mathbb{R}}}\bigoplus_{j=1}^{N_\gamma}\mathcal{J}^{s(j,\gamma)}_{k_j(\gamma),\gamma}\right)\oplus\left(\bigoplus_{\gamma\in\sigma_{\mathbb{C}^+}}\bigoplus_{j=1}^{N_\gamma}\mathcal{J}_{2k_j(\gamma),\gamma}\right)\oplus\left(\bigoplus_{j=1}^{N_\infty}\mathcal{N}^{s(j,\infty)}_{k_j(\infty)}\right), \tag{25}$$

where $\sigma_{\mathbb{R}}$, $\sigma_{\mathbb{C}^+}$ denote the sets of real eigenvalues and eigenvalues with positive imaginary part of $A+\lambda E$, respectively, and where $N_\gamma\geqslant 0$, $k_j(\gamma)>0$, $s(k_j,\gamma)\in\{-1,1\}$, $j=1,\dots,N_\gamma$, $\gamma\in\sigma_{\mathbb{R}}\cup\sigma_{\mathbb{C}^+}\cup\{\infty\}$.

If $\kappa_A=1$ and $\kappa_E=0$, then the general form of a Hermitian rank-one perturbation of a Hermitian pencil is given by

$$A+\tau uu^*+\lambda E,\quad \tau\in\mathbb{R}.$$

In this case the Weyl function takes the form $Q(\lambda)=u^*R(\lambda)u$, cf. (5) and (6).

EXAMPLE 18. Let $a>0$ and

$$A+\lambda E = a\mathcal{N}_2^+(\lambda)\oplus\mathcal{N}_1^+(\lambda) = \begin{bmatrix} 0 & a & 0 \\ a & -a\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We show that for $a < 1/\sqrt{2}$ we have $\delta^H_{2,0}(A,E) < \delta^H_{1,0}(A,E) < +\infty$, and $\delta_{1,0}(A,E) < \delta^H_{1,0}(A,E)$. It is clear that

$$\delta^H_{2,0}(A,E) \leqslant a \left\| \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\|_F = a\sqrt{2} < 1.$$

On the other hand let $u = [u_1, u_2, u_3]^\top \in \mathbb{C}^3$ be such that $A + \lambda E + \tau uu^*$ is singular for some $\tau \in \mathbb{R}$. Then, according to Theorem 4, the Weyl function

$$Q(\lambda) = u^*(A+\lambda E)^{-1}u = a^{-1}\lambda \bar{u}_1 u_1 + a^{-1}(\bar{u}_2 u_1 + \bar{u}_1 u_2) + \bar{u}_3 u_3$$

is nonzero and constant. This implies that $u_1 = 0$ and consequently

$$\det(A + \tau uu^* + \lambda E) = \det \begin{bmatrix} 0 & a & 0 \\ a & \tau \bar{u}_2 u_2 - a\lambda & \tau \bar{u}_2 u_3 \\ 0 & \tau \bar{u}_3 u_2 & \tau \bar{u}_3 u_3 + 1 \end{bmatrix} = -a^2(1 + \tau |u_3|^2).$$

Hence, $u = (0, u_2, u_3)$ and $\tau = -1/|u_3|^2$. Conversely, for any such u, τ the pencil $A + \lambda E + \tau uu^*$ is singular. From this we obtain that $\delta^H_{1,0}(A,E) = 1$. Finally, note that

$$A - a \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda E$$

is singular and hence $\delta_{1,0}(A,E) \leqslant a < 1/\sqrt{2} < 1 = \delta^H_{1,0}(A,E)$.

In the following we present necessary and sufficient conditions in terms of the Hermitian canonical form for the Hermitian rank-one distance to singularity to be finite. Let $A + \lambda E$ be a regular pencil in Hermitian canonical form (25) and let $u = [u_1^\top, u_2^\top, u_3^\top]^\top$ be an associated conformable partition of the entries of a vector $u \in \mathbb{C}^n$. Introducing the functions

$$Q_f(\lambda) := u_1^* \left(\bigoplus_{\gamma \in \sigma_{\mathbb{R}}} \bigoplus_{j=1}^{N_\gamma} \left(\mathscr{J}^{s(j,\gamma)}_{k_j(\gamma),\gamma}(\lambda) \right)^{-1} \right) u_1 + u_2^* \left(\bigoplus_{\gamma \in \sigma_{\mathbb{C}+}} \bigoplus_{j=1}^{N_\gamma} \left(\mathscr{J}_{2k_j(\gamma),\gamma}(\lambda) \right)^{-1} \right) u_2, \tag{26}$$

and

$$Q_\infty(\lambda) := u_3^* \left(\bigoplus_{j=1}^{N_\infty} \left(\mathscr{N}^{s(j,\infty)}_{k_j(\infty)}(\lambda) \right)^{-1} \right) u_3, \tag{27}$$

we obtain the following result.

LEMMA 19. *Let $A + \lambda E$ be a regular Hermitian pencil in canonical form (25) and let the correspondingly partitioned vector $u = [u_1^\top, u_2^\top, u_3^\top]^\top \in \mathbb{C}^n$ be such that $A + \tau uu^* + \lambda E$ is singular for some value $\tau = \tau_0 \in \mathbb{R}$. Then $u_3 \neq 0$, $Q_f(\lambda) \equiv 0$, and $Q_\infty(\lambda) \equiv -\frac{1}{\tau_0}$, where Q_f and Q_∞ are as in (26) and (27), respectively.*

If, additionally, $A+\lambda E$ has only real eigenvalues (including infinity), all finite eigenvalues are semi-simple and for each fixed real eigenvalue all corresponding blocks are of the same sign, then $u_1 = 0$, and u_2 is void.

Proof. Introducing the partitioning

$$\begin{aligned} u_1 &= [u_{\gamma,j}]_{\gamma\in\sigma_{\mathbb{R}},\ j=1,\dots,N_\gamma}, \\ u_2 &= [u_{\gamma,j}]_{\gamma\in\sigma_{\mathbb{C}+},\ j=1,\dots,N_\gamma}, \\ u_3 &= [u_{\infty,j}]_{j=1,\dots,N_\infty}, \end{aligned}$$

of the vectors u_1, u_2, u_3, we have that

$$Q_f(\lambda) = \sum_{\gamma\in\sigma_{\mathbb{R}}}\sum_{j=1}^{N_\gamma} u_{\gamma,j}^*\left(\mathscr{J}_{k_j(\gamma),\gamma}^{s(j,\gamma)}(\lambda)\right)^{-1} u_{\gamma,j} + \sum_{\gamma\in\sigma_{\mathbb{C}+}}\sum_{j=1}^{N_\gamma} u_{\gamma,j}^*\left(\mathscr{J}_{2k_j(\gamma),\gamma}(\lambda)\right)^{-1} u_{\gamma,j}.$$

Furthermore, for $\gamma\in\sigma_{\mathbb{R}}$, $j=1,\dots,N_\gamma$, we obtain

$$\begin{aligned} &u_{\gamma,j}^*\left(\mathscr{J}_{k,\gamma}^{s(j,\gamma)}(\lambda)\right)^{-1} u_{\gamma,j} \\ &= s(j,\gamma)\, u_{\gamma,j}^* \begin{bmatrix} (-1)^{k-1}(\gamma-\lambda)^{-k} & \cdots & -(\gamma-\lambda)^{-2} & (\gamma-\lambda)^{-1} \\ \vdots & \iddots & \iddots & \\ -(\gamma-\lambda)^{-2} & \iddots & & \\ (\gamma-\lambda)^{-1} & & & \end{bmatrix} u_{\gamma,j}, \end{aligned}$$

where we abbreviated $k = k_j(\gamma)$ for simplicity. Analogously, partitioning conformably $u_{\gamma,j}^\top = [u_{\gamma,j,1}^\top, u_{\gamma,j,2}^\top]$, for $\gamma\in\sigma_{\mathbb{C}^+}$, $j=1,\dots,N_\gamma$, we get that

$$\begin{aligned} u_{\gamma,j}^*\left(\mathscr{J}_{2k,\gamma}(\lambda)\right)^{-1} u_{\gamma,j} &= \begin{bmatrix} u_{\gamma,j,1} \\ u_{\gamma,j,2} \end{bmatrix}^* \begin{bmatrix} 0 & \left(\mathscr{J}_{k,\bar\gamma}^1(\lambda)\right)^{-1} \\ \left(\mathscr{J}_{k,\gamma}^1(\lambda)\right)^{-1} & 0 \end{bmatrix} \begin{bmatrix} u_{\gamma,j,1} \\ u_{\gamma,j,2} \end{bmatrix} \\ &= u_{\gamma,j,1}^*\left(\mathscr{J}_{k,\bar\gamma}^1(\lambda)\right)^{-1} u_{\gamma,j,2} + u_{\gamma,j,2}^*\left(\mathscr{J}_{k,\gamma}^1(\lambda)\right)^{-1} u_{\gamma,j,1}, \end{aligned}$$

where again we used $k = k_j(\gamma)$ for simplicity. Note that for each $\gamma\in\sigma_{\mathbb{R}}$ the function $Q_f(\lambda)$ has either a pole in γ or

$$\sum_{j=1}^{N_\gamma} u_{\gamma,j}^*\left(\mathscr{J}_{k_j(\gamma),\gamma}^{s(j,\gamma)}(\lambda)\right)^{-1} u_{\gamma,j} \equiv 0. \tag{28}$$

Analogously, for each $\gamma\in\sigma_{\mathbb{C}^+}$, the function $Q_f(\lambda)$ has either a pole in γ or

$$\sum_{j=1}^{N_\gamma} u_{\gamma,j}^*\left(\mathscr{J}_{2k_j(\gamma),\gamma}(\lambda)\right)^{-1} u_{\gamma,j} \equiv 0. \tag{29}$$

Hence, $Q_f(\lambda)$ is either a zero function or a rational function, which is not a polynomial. On the other hand, for $j = 1,\ldots,N_\infty$ we have that

$$u_{\infty,j}^* \left(\mathscr{N}_{k_j(\infty)}^{s(j,\infty)}(\lambda)\right)^{-1} u_{\infty,j} = s(j,\infty)\, u_{\infty,j}^* \begin{bmatrix} \lambda^{k-1} & \cdots & \lambda & 1 \\ \vdots & & \iddots & \iddots \\ \lambda & \iddots & & \\ 1 & & & \end{bmatrix} u_{\infty,j}, \tag{30}$$

where we used the abbreviation $k = k_j(\infty)$. Hence, $Q_\infty(\lambda)$ is a polynomial. By Theorem 4 the function $Q(\lambda)$ is constantly equal to $-1/\tau_0$. Therefore, $Q_f(z) \equiv 0$ and $Q_\infty(z) \equiv -1/\tau_0$. Hence, $u_3 \neq 0$.

To prove the second assertion, note that if the assumption (37) holds, then u_2 is void and the left hand side of (28) is zero if and only if $u_{\gamma,j} = 0$, $j = 1,\ldots,N_\gamma$. $\square$

The following theorem presents the main result of this section, classifying several low-rank distances for Hermitian pencils.

THEOREM 20. *Let $A+\lambda E$, with $A,E \in \mathbb{C}_H^{n\times n}$, be a regular Hermitian pencil.*

(i) There exist $u \in \mathbb{C}^n$ and $\tau_0 \in \mathbb{R}\setminus\{0\}$ such that the pencil

$$A + \tau_0 uu^* + \lambda E$$

is singular if and only if the Hermitian canonical form of $A+\lambda E$ contains either an odd sized block associated with the eigenvalue infinity, or two even sized blocks of possibly distinct dimensions with opposite signs, i.e., it contains at least one of the following two blocks:

(i.1) $\mathscr{N}_{2k+1}^s(\lambda)$, $k \geqslant 0$, $s \in \{1,-1\}$,

(i.2) $\mathscr{N}_{2l}^1(\lambda) \oplus \mathscr{N}_{2l'}^{-1}(\lambda)$, $l,l' \geqslant 1$.

(ii) There exist matrices $B \in \mathbb{C}^{n\times 2}$, $H = H^* \in \mathbb{C}^{2\times 2}$ and $\tau_0 \in \mathbb{R}\setminus\{0\}$ such that

$$A + \tau_0 BHB^* + \lambda E$$

is singular if and only if infinity is an eigenvalue of the pencil, i.e., if and only if the Hermitian canonical form of $A+\lambda E$ contains at least one block of the form

(ii.1) $\mathscr{N}_k^s(\lambda)$, $k \geqslant 1$, $s \in \{1,-1\}$.

(iii) There exist $v \in \mathbb{C}^n$ and $\tau_0 \in \mathbb{R}\setminus\{0\}$ such that the pencil

$$A + \lambda(E + \tau_0 vv^*)$$

is singular if and only if the Hermitian canonical form of $A+\lambda E$ contains either an odd sized block associated with the eigenvalue zero, or two even sized blocks of possibly distinct dimensions with opposite signs, i.e., it contains at least one of the following two blocks:

(iii.1) $\mathscr{J}^{s}_{2k+1,0}(\lambda)$, $k \geqslant 0$, $s \in \{1,-1\}$,

(iii.2) $\mathscr{J}^{1}_{2l,0}(\lambda) \oplus \mathscr{J}^{-1}_{2l',0}(\lambda)$, $l,l' \geqslant 1$.

(iv) There exist matrices $C \in \mathbb{C}^{n\times 2}$, $K = K^* \in \mathbb{C}^{2\times 2}$ *and* $\tau_0 \in \mathbb{R}\setminus\{0\}$ *such that*

$$A+\lambda(E+\tau_0 CKC^*)$$

is singular if and only if zero is an eigenvalue of the pencil, i.e., if and only if the Hermitian canonical form of $A+\lambda E$ *contains at least one block of the form*

(iv.1) $\mathscr{J}^{s}_{k,0}(\lambda)$, $k \geqslant 1$, $s \in \{1,-1\}$.

(v) There exist vectors $u,v \in \mathbb{C}^n$, $h \in \{-1,1\}$ *and* $\tau_0 \in \mathbb{R}\setminus\{0\}$ *such that*

$$A+h\tau_0 uu^* + \lambda(E+\tau_0 vv^*)$$

is singular, regardless of the particular Hermitian canonical form of $A+\lambda E$.

Proof. In the complete proof we assume without loss of generality that $A+\lambda E$ is in the Hermitian canonical form (25) and that $u = [u_1^\top, u_2^\top, u_3^\top]^\top$ is partitioned correspondingly.

(i) Assume that $A+\tau_0 uu^* + \lambda E$ is singular for some $\tau_0 \in \mathbb{R}\setminus\{0\}$. Then by Lemma 19 and $\widetilde{u} := [0,0,u_3^\top]^\top$, we have

$$\widetilde{u}^*(A+\lambda E)^{-1}\widetilde{u} = Q_\infty(\lambda) \equiv \text{const} \neq 0.$$

Hence, by Theorem 4, the pencil $A+\tau_0\widetilde{u}\widetilde{u}^* + \lambda E$ is singular as well. Therefore, we may assume that $A+\lambda E$ only has the eigenvalue infinity.

Suppose now that there are no odd sized blocks corresponding to infinity and that all even sized blocks have the same sign, i.e.,

$$A+\lambda E = \mathscr{N}^{s}_{2l_1}(\lambda) \oplus \cdots \oplus \mathscr{N}^{s}_{2l_{N_\infty}}(\lambda), \tag{31}$$

where $s \in \{\pm 1\}$. We also assume that $l_1 \geqslant \cdots \geqslant l_{N_\infty}$, and let $u = [u_1^\top, \ldots, u_{N_\infty}^\top]^\top$ be the corresponding partition of the entries of the vector u. Denoting by $u_{j,i}$ the i-th (scalar) entry of u_j we obtain from (30) that

$$Q(\lambda) = \sum_{j=1}^{N_\infty} u_j^* \left(\mathscr{N}^{s}_{2l_j}(\lambda)\right)^{-1} u_j = s\sum_{j=1}^{N_\infty}\sum_{i=0}^{2l_j-1} d_i^{(j)}\lambda^i, \tag{32}$$

where

$$d_i^{(j)} = \sum_{i_1+i_2=2l_j-i+1} \bar{u}_{j,i_1}\, u_{j,i_2}, \quad j=1,\ldots,N_j, \quad i=0,1,\ldots,2l_j-1. \tag{33}$$

Hence, the leading coefficient of the polynomial $Q(\lambda)$ is

$$s\sum_{j\in\{j:l_j=l_1\}} |u_{j,1}|^2\lambda^{2l_1-1}.$$

On the other hand, $Q(\lambda)$ is constant, therefore, $u_{j,1} = 0$ for j such that $l_j = l_1$. Subsequently, according to (32) and (33), the coefficient of λ^{2l_1-3} in $Q(\lambda)$ equals

$$s\left[\sum_{j\in\{j:l_j=l_1\}} \left(\bar{u}_{j,1}u_{j,3} + |u_{j,2}|^2 + u_{j,1}\bar{u}_{j,3}\right) + \sum_{j\in\{j:l_j=l_1-1\}} |u_{j,1}|^2\right].$$

Since $Q(\lambda)$ is constant and $u_{j,1} = 0$ if $l_j = l_1$, we get $u_{j,2} = 0$ if $l_j = l_1$ and also $u_{j,1} = 0$ if $l_j = l_1 - 1$. Proceeding in this way by induction for $j = 1,\dots,N_\infty$, we obtain $u_{j,r} = 0$ for $r = 1,\dots,l_j$. Hence, $\mathscr{N}^s_{2l_1}(\lambda)$ and u_1 are of the form

$$\mathscr{N}^s_{2l_1}(\lambda) = \begin{bmatrix} 0 & \mathscr{N}^s_{l_1}(\lambda) \\ \mathscr{N}^s_{l_1}(\lambda) & -\lambda E^{l_1}_{11} \end{bmatrix} \quad \text{and} \quad u_1 = \begin{bmatrix} 0 \\ \widetilde{u}_1 \end{bmatrix},$$

respectively, with some $\widetilde{u}_1 \in \mathbb{C}^{l_1}$. Here $E^{l_1}_{11}$ denotes the $l_1 \times l_1$ matrix with one in the $(1,1)$-position and zeros elsewhere, and then we obtain that

$$A + \tau uu^* + \lambda E = \begin{bmatrix} 0 & \mathscr{N}^s_{l_1}(\lambda) & 0 \\ \mathscr{N}^s_{l_1}(\lambda) & -\lambda E^{l_1}_{11} + \tau\widetilde{u}_1\widetilde{u}_1^* & \tau\widetilde{u}_1\widetilde{u}^* \\ 0 & \tau\widetilde{u}\widetilde{u}_1^* & \widetilde{A} + \lambda\widetilde{E} + \tau\widetilde{u}\widetilde{u}^* \end{bmatrix},$$

where $\widetilde{u} = [u_2^\top,\dots,u_{N_\infty}^\top]^\top$ and

$$\widetilde{A} + \lambda\widetilde{E} = \mathscr{N}^s_{2l_2}(\lambda) \oplus \cdots \oplus \mathscr{N}^s_{2l_{N_\infty}}(\lambda).$$

Thus, by Laplace expansion, induction, and using that $|\det \mathscr{N}^s_{l_j}(\lambda)| = 1$, we obtain

$$|\det(A + \tau uu^* + \lambda E)| = |\det(\widetilde{A} + \lambda\widetilde{E} + \tau\widetilde{u}\widetilde{u}^*)| = \cdots = 1,$$

and hence the pencil $A + \tau uu^* + \lambda E = A + \lambda E$ is regular for all τ, which contradicts the main assumption that $A + \tau_0 uu^* + \lambda E$ is singular for some $\tau_0 \in \mathbb{R}\setminus\{0\}$. As a consequence, (31) is false, which finishes the proof of one direction of (i).

To prove the converse implication, it is enough to consider the two cases that $A+\lambda E$ equals the block(s) in (i.1) or (i.2), respectively. If $A+\lambda E = \mathscr{N}^s_{2k+1}(\lambda)$, $k \geqslant 0$, $s \in \{1,-1\}$, then setting $u = e_{(k+1)/2}$ we get that the pencil $A - suu^* + \lambda E$ singular. In the other case, we have $A + \lambda E = \mathscr{N}^{-1}_{2l}(\lambda) \oplus \mathscr{N}^1_{2l'}(\lambda)$, $l, l' \geqslant 0$, and we first consider the situation when $l = l'$. Let $u = [u_1^\top, u_2^\top]^\top = [u_{1,1},\dots,u_{1,2l},u_{2,1},\dots,u_{2,2l}]$ be such that

$$u_{1,i} = u_{2,i} \quad \text{for } i = 1,\dots,l-1, \quad \text{and} \quad \mathrm{Re}(u_{1,1}u_{1,l}) \neq \mathrm{Re}(u_{2,1}u_{2,l}).$$

Then, due to (32), (33) and the fact that the signs of the two blocks of $A+\lambda E$ are opposite, we have

$$Q(\lambda) = -2\,\mathrm{Re}(u_{1,1}u_{1,l}) + 2\,\mathrm{Re}(u_{2,1}u_{2,l}),$$

which is a nonzero constant. Application of Theorem 4 finishes the proof in the situation $l = l'$.

If $l \neq l'$, then we assume for simplicity that $l' > l$, the case $l < l'$ can be treated similarly. Let $\tilde{l} = l' - l$ and let $u \in \mathbb{C}^{2l'}$ be a vector such that

$$u_1 = \cdots = u_{\tilde{l}} = u_{\tilde{l}+2l+1} = \cdots = u_{2l'} = 0.$$

Then

$$u^*(\mathscr{N}_{2l'}^1(\lambda))^{-1}u = v^*(\mathscr{N}_{2l}^1(\lambda))^{-1}v$$

with $v = [u_{\tilde{l}+1}, \ldots, u_{\tilde{l}+2l}]^\top$. This reduces the situation to the case $l' = l$ and thus finishes the proof of (i).

(ii) If infinity is not an eigenvalue of $A + \lambda E$ then, due to the regularity of $A + \lambda E$, the matrix E is invertible. Hence, $A + \tilde{B} + \lambda E$ is regular for all $\tilde{B} \in \mathbb{C}^{n\times n}$. Assume now, that infinity is an eigenvalue of $A + \lambda E$. If the canonical Hermitian form of $A + \lambda E$ contains one of the structures listed in (i.1) or (i.2), then we set $B = [u, 0]$ and $H = I_2$, where u is constructed as in the proof of (i). By (i) the pencil $A + \tau_0 BHB^* + \lambda E$ is singular for some τ_0. The remaining case to consider is that $A + \lambda E$ has only even blocks of the same sign corresponding to infinity. It is enough to consider the case that $A + \lambda E = \mathscr{N}_{2l}^s(\lambda)$, $l \geqslant 1$, $s \in \{-1, 1\}$. Setting

$$B = [e_1, e_{2l}], \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we then have that the pencil $A - BHB^* + \lambda E$ is singular.

(iii) and (iv) follow from (i) and (ii) by interchanging the roles of A and E.

(v) Without loss of generality we may assume that $A + \lambda E$ is a single block of one of the forms (21), (22), or (23). The cases $A + \lambda E = \mathscr{N}_1^s(\lambda)$ and $A + \lambda E = \mathscr{J}_{1,0}^s(\lambda)$, $s \in \{-1, 1\}$, are trivial. Consider the case

$$A + \lambda E = \mathscr{N}_k^s(\lambda), \quad k \geqslant 2,\ s \in \{-1, 1\}, \tag{34}$$

let $w = e_1 + e_{k-1} + e_k$, with the special case $w = e_1 + e_2$ if $k = 2$, and let $\tau_1 = -s/2$. According to Proposition 1(i), the fact that $\det(A + \lambda E) = \pm 1$, and formulas (32) and (33), the characteristic polynomial of $A + \tau_1 ww^* + \lambda E$ equals

$$\pm\big(1 - \frac{1}{2}(2\lambda + 2)\big) = \pm\lambda,$$

with the special case $\pm\big(1 - \frac{1}{2}(\lambda + 2)\big) = \pm\frac{1}{2}\lambda$ if $k = 2$, and it clearly has a simple zero at $\lambda = 0$. Therefore, the pencil $A + \tau_1 ww^* + \lambda E$ has in its Hermitian canonical form a block $\mathscr{J}_{1,0}^s(\lambda)$. By (iii) there exists $v \in \mathbb{C}^k$ and $\tau_0 \in \mathbb{R} \setminus \{0\}$ such that the pencil $A + \tau_1 ww^* + \lambda(E + \tau_0 vv^*)$ is singular. Setting $h = \operatorname{sgn}(\tau_0\tau_1)$ and $u = |\tau_0|^{-\frac{1}{2}}|\tau_1|^{\frac{1}{2}}w$ finishes the proof in this case.

Consider next the case

$$A + \lambda E = \mathscr{J}_{k,\gamma}^s(\lambda), \quad \gamma \in \mathbb{R} \setminus \{0\},\ s \in \{1, -1\}. \tag{35}$$

Taking $u = e_1$, $\tau_1 = s(-1)^k\gamma^k$, then according to Proposition 1(i) and the fact that $\det(A + \lambda E) = \pm(\gamma - \lambda)^k$, the characteristic polynomial of $A + \tau_1 uu^* + \lambda E$ equals

$$\pm(\gamma - \lambda)^k \left(1 - \frac{\gamma^k}{(\gamma - \lambda)^k}\right) = \pm\big((\gamma - \lambda)^k - \gamma^k\big),$$

and clearly has a simple zero at $\lambda = 0$. As in case (34), application of (iii) finishes the proof.

Next, consider the case

$$A + \lambda E = \mathscr{J}_{2k,\gamma}(\lambda), \quad \gamma \in \mathbb{C}^+,\ k > 0. \tag{36}$$

Taking $u = -(\frac{1}{2}\gamma^k)e_1 + e_{k+1}$, $\tau_1 = (-1)^{k+1}$, then according to Proposition 1(i) and the fact that $\det(A+\lambda E) = (-1)^k(\gamma-\lambda)^k(\bar{\gamma}-\lambda)^k$, the characteristic polynomial of $A + \tau_1 uu^* + \lambda E$ equals

$$(-1)^k(\gamma-\lambda)^k(\bar{\gamma}-\lambda)^k \left(1 - \frac{\gamma^k}{2(\gamma-\lambda)^k} - \frac{\bar{\gamma}^k}{2(\bar{\gamma}-\lambda)^k}\right),$$

and clearly has a simple zero at $\lambda = 0$. As in cases (34), (35), application of statement (iii) finishes the proof.

The remaining case to consider is

$$A + \lambda E = \mathscr{J}^s_{k,0}(\lambda), \quad s \in \{1,-1\}, k \geqslant 2.$$

In this situation the statement follows from (34) by interchanging the roles of A and E. $\square$

REMARK 21. Observe that if $A + \lambda E = \mathscr{N}^s_{2k+1}(\lambda)$, then the sign of τ_0, as constructed in the proof of (i), is opposite to s. In the case $A + \lambda E = \mathscr{N}^+_{2l}(\lambda) \oplus \mathscr{N}^-_{2l'}(\lambda)$, the sign τ_0 can be arbitrary, depending on the choice of u. This observation remains valid under congruence transformations $W^*(A + \tau uu^* + \lambda E)W$, since both the sign characteristic and the sign of τ_0 stay invariant.

REMARK 22. Note that Theorem 20 presents two different methods of making the pencil $\mathscr{N}^s_{2k+1}(\lambda)$ singular. In (i) the matrix A is perturbed by a rank-one matrix so that the perturbed pencil equals $\mathscr{L}_{2k+1}(\lambda)$. On the other hand in (v) first A is perturbed so that zero is an eigenvalue, and then the matrix E is perturbed to get the block $\mathscr{L}_1(\lambda)$ in the Hermitian canonical form.

5. An explicit formula for the rank-one distance to singularity for a special Hermitian pencil

In this section, we present an explicit formula for $\delta^H_{1,0}(A,E)$ in a specially simple case. For this, we will make use of several assumptions on the Hermitian canonical form (25) of a regular Hermitian pencil which we list for later reference:

i) there are no non-real eigenvalues, each fixed $\gamma \in \sigma_{\mathbb{R}}$ is semi-simple and all corresponding blocks have the same sign, i.e.,

$$\sigma_{\mathbb{C}^+} = \emptyset \quad \text{and} \quad s(j,\gamma) =: s(\gamma),\ k_j(\gamma) = 1, \quad j = 1,\dots,N_\gamma, \quad \gamma \in \sigma_{\mathbb{R}}; \tag{37}$$

ii) infinity is a semi-simple eigenvalue, i.e.,

$$k_j(\infty) = 1, \quad j = 1, \ldots, N_\infty; \tag{38}$$

iii) all blocks corresponding to infinity have the same sign, i.e.,

$$s(j,\infty) =: s(\infty), \ j = 1, \ldots, N_\infty. \tag{39}$$

THEOREM 23. *Let $A+\lambda E$ be a regular Hermitian pencil that has only real eigenvalues (including infinity) that are all semi-simple and such that for each fixed finite eigenvalue all corresponding blocks in the Hermitian canonical form have the same sign, i.e., the Hermitian canonical form* (25) *of $A+\lambda E$ satisfies* (37) *and* (38)*. Then for every invertible matrix S, such that $S(A+\lambda E)S^*$ is in Hermitian canonical form one has*

$$\delta_{1,0}^H(A,E) \geqslant \left\| S|_{\ker(ES^*S)} \right\|_2^{-2}. \tag{40}$$

Furthermore, if all blocks corresponding to the infinity eigenvalue are of the same sign s_∞, i.e., the Hermitian canonical form (25) *of $A+\lambda E$ satisfies* (37)–(39)*, then*

$$\delta_{1,0}^H(A,E) = \left\| S|_{\ker(ES^*S)} \right\|_2^{-2}.$$

and for every $u \in \ker(ES^*S)$, $u \neq 0$ the pencil $A - s_\infty \|Su\|_2^{-2} uu^* + \lambda E$ is singular.

Proof. Fix an invertible matrix S, such that $S(A+\lambda E)S^*$ is in Hermitian canonical form and let $u \in \mathbb{C}^n$. Observe that the pencil $A + \tau_0 uu^* + \lambda E$ is singular if and only if $u \in \ker(ES^*S)$ and

$$-1/\tau_0 = u^*(A+\lambda E)^{-1}u. \tag{41}$$

To see this, set $u_J = Su$, $A_J = SAS^*$, and $E_J = SES^*$. If $A + \tau uu^* + \lambda E$ is singular for some $\tau = \tau_0 \in \mathbb{R}$, then the pencil $A_J + \tau_0 u_J u_J^* + \lambda E_J$ is singular as well. By Lemma 19 applied to A_J, E_J and u_J and the fact that all blocks corresponding to the eigenvalue infinity are of size one, we have $u_J \in \ker E_J$, or equivalently $u \in \ker(ES^*S)$. Then (41) follows from Theorem 4. The converse implication is immediate.

Using this observation, we have

$$\begin{aligned}
&\delta_{1,0}^H(A,E)^{-1} \\
&\quad = \max\left\{ |\tau|^{-1} \,\middle|\, (u,\tau) \in \mathbb{C}^n \times \mathbb{R},\ A + \tau uu^* + \lambda E \text{ is singular},\ \|u\|_2 = 1 \right\} \\
&\quad = \max\left\{ |u^*(A+\lambda E)^{-1}u| \,\middle|\, u \in \ker(ES^*S),\ \|u\|_2 = 1 \right\} \\
&\quad = \max\left\{ |u_J^*(A_J+\lambda E_J)^{-1}u_J| \,\middle|\, u_J \in \ker E_J,\ \left\|S^{-1}u_J\right\|_2 = 1 \right\}.
\end{aligned}$$

Since the pencil $A_J + \lambda E_J$ is in Hermitian canonical form and infinity is a semi-simple eigenvalue, the part of A_J corresponding to $\ker E_J$ is diagonal with entries ± 1 on the diagonal, depending on the signs of the blocks corresponding to infinity. Thus, for $u_J \in \ker E_J$ we have that $|u_J^*(A_J+\lambda E_J)^{-1}u_J| \leqslant \|u_J\|_2^2$. Hence,

$$\begin{aligned}
\delta_{1,0}^H(A,E)^{-1} &\leqslant \max\left\{ \|u_J\|_2^2 \,\middle|\, u_J \in \ker E_J,\ \left\|S^{-1}u_J\right\|_2 = 1 \right\} \\
&= \max\left\{ \|Su\|_2^2 \,\middle|\, u \in \ker ES^*S,\ \|u\|_2 = 1 \right\} = \left\| S|_{\ker(ES^*S)} \right\|_2^2.
\end{aligned}$$

If all blocks corresponding to the eigenvalue infinity have the same sign s_∞, then we have $|u_J^*(A_J+\lambda E_J)^{-1}u_J| = \|u_J\|_2^2$ for $u_J \in \ker E_J$, and thus $\delta_{1,0}^H(A,E)^{-1} = \left\|S|_{\ker(ES^*S)}\right\|_2^2$. If in this case $u \in \ker(ES^*S)\setminus\{0\}$, then the pencil $A+\tau uu^*+\lambda E$ is singular for

$$\begin{aligned}\tau &= \frac{-1}{u^*(A+\lambda E)^{-1}u} = \frac{-1}{u_J^*(A_J+\lambda E_J)^{-1}u_J} = \frac{-1}{s_\infty \|u_J\|_2^2} \\ &= \frac{-s_\infty}{\|Su\|_2^2} = \frac{-s_\infty}{\|Su\|_2^2}. \qquad \square\end{aligned}$$

The assumption of equal signs of blocks corresponding to finite real eigenvalues is essential for the first part of Theorem 23. This is demonstrated in the following example.

EXAMPLE 24. Let

$$S(A+\lambda E)S^* = \begin{bmatrix} 1 & & \\ & \lambda & \\ & & -\lambda \end{bmatrix} = A_J + \lambda E_J,$$

where S is any invertible matrix satisfying

$$\left\|S^{-1}[1,1,1]^\top\right\|_2 < \left\|S^{-1}[1,0,0]^\top\right\|_2. \tag{42}$$

Observe that

$$\begin{aligned}\left\|S|_{\ker(ES^*S)}\right\|_2^2 &= \max\left\{\|u_J\|_2^2 \,\middle|\, u_J \in \ker E_J,\ \left\|S^{-1}u_J\right\|_2 = 1\right\} \\ &= \left\|S^{-1}[1,0,0]^\top\right\|_2^{-2}.\end{aligned} \tag{43}$$

On the other hand, the pencil $A_J - u_0u_0^* + \lambda E_J$ is singular also for $u_0 = [1,1,1]^\top$, i.e., the pencil $A - uu^* + \lambda E$ is singular with $u_1 = S^{-1}u_0$, and hence

$$\delta_{1,0}^H(A,E) \leqslant \|u_1u_1^*\|_F = \left\|S^{-1}[1,1,1]^\top\right\|_2^2. \tag{44}$$

Equations (42), (43) and (44) are in contradiction with (40).

Similarly, one can construct examples showing the necessity of the assumption that the pencil has no non-real eigenvalues.

EXAMPLE 25. Consider the pencil

$$A_J + \lambda E_J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda-\alpha-\imath\beta \\ 0 & \lambda-\alpha+\imath\beta & 0 \end{bmatrix}, \quad \alpha,\beta \in \mathbb{R}.$$

Then not only the perturbation $A_J - u_Ju_J^* + \lambda E_J$ with $u_J = [1,0,0]^\top \in \ker E_J$ is singular, but also $A_J - u_0u_0^* + \lambda E_J$ with $u_0 = [1,0,1]^\top$. Choosing an appropriate transformation matrix S leads, as in Example 24, to a contradiction with (40).

The assumption of equal signs of blocks corresponding to the eigenvalue infinity is essential for the second claim of Theorem 23, as the following example demonstrates.

EXAMPLE 26. Let

$$A+\lambda E=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Then for every $a\in\mathbb{R}\setminus\{0\}$ and the matrix

$$S=\frac{\sqrt{2}}{2}\begin{bmatrix}1/a&a\\-1/a&a\end{bmatrix},$$

we have $S(A+\lambda E)S^*=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$. Due to the fact that $\ker E=\mathbb{C}^2$, we obtain

$$\left\|S|_{\ker(ES^*S)}\right\|_2=\|S\|_2\geqslant\frac{\sqrt{2}}{2}|a|,$$

which may be chosen arbitrarily large by varying a, while $\delta^H_{1,0}(A,E)=1$.

6. Computing the rank-one distance to singularity for Hermitian pencils

In this section we present the analogues of the results from Section 3 for the case of Hermitian pencils. We begin with the Hermitian version of Theorem 7. The proof follows the same lines as in Section 3 and is not repeated here.

THEOREM 27. *Let $\lambda_0\in\mathbb{C}$ be an arbitrary regular point of the Hermitian pencil $A+\lambda E$. Then*

$$\delta^H_{1,0}(A,E)^{-1}=\max\left\{|u^*R(\lambda_0)u|\;\middle|\;u\in\mathbb{C}^n,\ u^*u=1,\ u^*C_j(\lambda_0)u=0,\ j=1,\dots,n\right\}.$$

To measure distances, consider the real orthogonal space of Hermitian matrices $\mathbb{C}^{n\times n}_H$ with the inner product

$$\langle X,Y\rangle_H:=\operatorname{tr}(Y^*X).$$

Note that this is indeed an inner product on the real space of Hermitian matrices, because $\operatorname{tr}(Y^*X)=\operatorname{tr}(YX)=\operatorname{tr}(XY)=\operatorname{tr}(X^*Y)$ and hence $\langle X,Y\rangle_H$ is real.

The corresponding norm is the Frobenius norm and the inner product equals $\langle X,Y\rangle$ restricted to the set of Hermitian matrices, but we use the subscript H to avoid confusion. The adjective 'H-orthogonal' and the symbol '$\perp_H$' will refer to orthogonality and orthogonal complement in the space $\left(\mathbb{C}^{n\times n}_H,\langle\cdot,\cdot\rangle_H\right)$. The matrices $C_j(\lambda_0)$ are defined as before by (11). Clearly, if λ_0 is real, then the matrices $C_j(\lambda_0)$ are all Hermitian.

Then Lemma 8 takes the form below. Note that we have to give a different proof, because the set of real regular points need not be a connected set and this fact was used in the proof of Lemma 8.

LEMMA 28. *For any regular point $\lambda_0 \in \mathbb{R}$ of the regular Hermitian pencil $A+\lambda E$ we have*

$$\operatorname{span}_{\mathbb{R}}\{C_1(\lambda_0),\dots,C_n(\lambda_0)\} = \operatorname{span}_{\mathbb{R}}\left\{C_j(\lambda_0)\,|\,j\in\mathbb{N},\ j\geqslant 1\right\} \tag{45}$$

and the span does not depend on the particular choice of λ_0.

Proof. Let the regular point $\lambda_0 \in \mathbb{R}$ be fixed. Assume that there exists a regular point $\lambda_1 \in \mathbb{R}$ and $k \in \mathbb{N}\setminus\{0\}$ such that

$$C_k(\lambda_1) \notin \operatorname{span}_{\mathbb{R}}\{C_1(\lambda_0),\dots,C_n(\lambda_0)\}.$$

By Lemma 8, we have nevertheless $C_k(\lambda_1) \in \operatorname{span}_{\mathbb{C}}\{C_1(\lambda_0),\dots,C_n(\lambda_0)\}$. Choosing a basis $C_1(\lambda_0),\dots,C_{n_0}(\lambda_0)$ of the complex linear span, see Lemma 8, there exist coefficients $\alpha_1,\dots,\alpha_{n_0} \in \mathbb{C}$ such that

$$\sum_{i=1}^{n_0} \alpha_i C_i(\lambda_0) = C_k(\lambda_1) = \left(C_k(\lambda_1)\right)^* = \sum_{i=1}^{n_0} \bar{\alpha}_i C_i(\lambda_0),$$

where we have used that C_k and C_{j_i}, $i=1,\dots,n_0$ are Hermitian. This implies that

$$0 = \sum_{i=1}^{n_0} (\alpha_i - \bar{\alpha}_i) C_i(\lambda_0),$$

and hence all $\alpha_1,\dots,\alpha_{n_0}$ are real, contradicting the assumption. This proves (45) and

$$\{C_1(\lambda_1),\dots,C_n(\lambda_1)\} \subseteq \{C_1(\lambda_0),\dots,C_n(\lambda_0)\}.$$

By symmetry of the argument, the latter inclusion is an identity. $\square$

As in the unstructured case we define

$$\mathscr{D}_H := \left(\operatorname{span}_{\mathbb{R}}\{C_1(\lambda_0),\dots,C_n(\lambda_0)\}\right)^{\perp_H} \subseteq \mathbb{C}_H^{n\times n}, \tag{46}$$

where λ_0 is any real regular point of the Hermitian pencil $A+\lambda E$. If $\mathscr{V}$ is a real subspace of $\mathbb{C}_H^{n\times n}$, then by $\mathfrak{P}_{\mathscr{V}}^H$ we denote the orthogonal projection from $\mathbb{C}_H^{n\times n}$ to $\mathscr{V}$.

LEMMA 29. *Let $A+\lambda E$ be a regular Hermitian pencil and let G be a Hermitian matrix. Then*

$$\mathfrak{P}_{\mathscr{D}_H}^H G = \mathfrak{P}_{\mathscr{D}} G. \tag{47}$$

Proof. Recall that if A and E are Hermitian, then so are the matrices $C_j(\lambda_0)$, $j=1,\dots,n$. Let $\tilde{C}_1(\lambda_0),\dots,\tilde{C}_{\tilde{n}}(\lambda_0)$ be the result of Gram-Schmidt orthonormalization of the matrices $C_1(\lambda_0),\dots,C_n(\lambda_0)$ with respect to the complex inner product $\langle\cdot,\cdot\rangle$. However, note that $\langle X,Y\rangle \in \mathbb{R}$ for Hermitian X,Y and therefore, $\tilde{C}_1(\lambda_0),\dots,\tilde{C}_{\tilde{n}}(\lambda_0)$

are Hermitian matrices and thus identical to those obtained by Gram-Schmidt orthonormalization applied to $C_1(\lambda_0),\dots,C_n(\lambda_0)$ with respect to the real inner product $\langle\cdot,\cdot\rangle_H$. Hence,

$$\mathfrak{P}_{\mathscr{D}}G = G - \sum_{j=1}^{\tilde{n}} \langle G, \tilde{C}_j(\lambda_0)\rangle \tilde{C}_j(\lambda_0) = G - \sum_{j=1}^{\tilde{n}} \langle G, \tilde{C}_j(\lambda_0)\rangle_H \tilde{C}_j(\lambda_0) = \mathfrak{P}^H_{\mathscr{D}_H}G. \quad \square$$

Lemma 29 immediately allows to deduce Hermitian versions of Propositions 11 and 12.

PROPOSITION 30. *If $A+\lambda E$ is a regular Hermitian pencil then the matrix $\mathfrak{P}^H_{\mathscr{D}}R(\lambda_0)$ does not depend on the particular choice of the regular point $\lambda_0 \in \mathbb{R}$. Furthermore, $\mathfrak{P}^H_{\mathscr{D}}R(\lambda_0) = 0$ if and only if infinity is not an eigenvalue of $A+\lambda E$.*

PROPOSITION 31. *Let $\lambda_0 \in \mathbb{R}$ be a regular point of the Hermitian $n\times n$ pencil $A+\lambda E$ and let $G \in \mathbb{C}^{n\times n}$ be a Hermitian matrix of rank one. Then*

$$A + \tau G + \lambda E$$

is singular for some $\tau \in \mathbb{R}$ if and only if $\mathfrak{P}^H_{\mathscr{D}}G = G$ and $\operatorname{tr}(GR(\lambda_0)) \neq 0$. If the latter is the case, then

$$\tau = -\frac{1}{\operatorname{tr}(GR(\lambda_0))}.$$

Observe that by (47) we have

$$\rho(A,E) = \|\mathfrak{P}_{\mathscr{D}}R(\lambda_0)\|_F^{-1} = \left\|\mathfrak{P}^H_{\mathscr{D}_H}R(\lambda_0)\right\|_F^{-1},$$

where λ_0 is any real regular point of the Hermitian pencil $A+\lambda E$. By Proposition 30, we furthermore have that $\rho(A,E) < +\infty$ if and only if infinity is an eigenvalue of $A+\lambda E$.

We now present an analogue of Theorem 13. Note that $\rho(A,E) \leqslant \delta^H_{1,0}(A,E)$ follows directly from $\rho(A,E) \leqslant \delta_{1,0}(A,E)$ due to Theorem 13 and the fact that it holds $\delta_{1,0}(A,E) \leqslant \delta^H_{1,0}(A,E)$.

THEOREM 32. *Let $A+\lambda E$ be a regular Hermitian pencil for which infinity is an eigenvalue, and let $\lambda_0 \in \mathbb{R}$ be any regular point. Furthermore, introduce the linear subspace $\mathscr{D}_H$ of $\mathbb{C}^{n\times n}$ defined by (46) and (11), and let $D^H_0,\dots,D^H_l$ be an orthonormal basis of $\mathscr{D}_H$ with*

$$D^H_0 = \frac{\mathfrak{P}^H_{\mathscr{D}_H}R(\lambda_0)}{\left\|\mathfrak{P}^H_{\mathscr{D}_H}R(\lambda_0)\right\|_F}.$$

Introduce also

$$\Xi_H = \left\{ [\alpha_0,\dots,\alpha_l]^\top \in \mathbb{R}^{l+1} \;\middle|\; \alpha_0 \neq 0,\ \sum_{j=0}^{l} |\alpha_j|^2 = 1,\ \operatorname{rank}\left(\sum_{j=0}^{l} \alpha_j D^H_j\right) = 1 \right\}.$$

Then $\Xi_H \neq \emptyset$ *if and only if* $\delta^H_{1,0}(A,E) < +\infty$. *In this case*

$$\delta^H_{1,0}(A,E) = \rho(A,E) \min_{(\alpha_0,\dots,\alpha_l)\in\Xi_H} |\alpha_0|^{-1}.$$

Proof. Besides the fact that we have to additionally assume that $\delta^H_{1,0}(A,E)$ is finite, the proof follows the same lines as the proof of Theorem 13, with the use of Proposition 31 instead of Proposition 12, G being additionally Hermitian, $D^H_0,\dots,D^H_l$ replacing $D_0,\dots,D_k$ and $\tau,\alpha_0,\dots,\alpha_l \in \mathbb{R}$ replacing $\tau,\alpha_0,\dots,\alpha_k \in \mathbb{C}$. $\square$

REMARK 33. Note that Theorem 20(i) gives a criterion for $\delta^H_{1,0}(A,E) < +\infty$ in terms of the canonical form of $A+\lambda E$. Therefore, Theorem 32 may be also viewed as a method for revealing the structure at infinity of a given Hermitian pencil. Similar to Remark 14, the rank condition in the definition of Ξ_H is equivalent to

$$p^H_{i_1,i_2,j_1,j_2}(\alpha_0,\dots,\alpha_k) := \sum_{i,j=0}^{l} \left((\alpha_i D^H_i)_{i_1,j_1}(\alpha_j D^H_j)_{i_2,j_2} - (\alpha_i D^H_i)_{i_2,j_1}(\alpha_j D^H_j)_{i_1,j_2}\right) = 0$$

for every $i_1,i_2,j_1,j_2 = 1,\dots,n$, $i_1 \neq i_2$, $j_1 \neq j_2$, with $(D^H_i)_{pq}$ denoting the (p,q) entry of the matrix D^H_i.

EXAMPLE 34. Let

$$A+\lambda E = \mathscr{N}_2^1(\lambda) = \begin{bmatrix} 0 & 1 \\ 1 & \lambda \end{bmatrix}.$$

Then $R(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C_2 = 0$. Hence,

$$D^H_0 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D^H_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and, thus

$$p_{1,2,1,2}(\alpha_0,\alpha_1) = \det\begin{bmatrix} 0 & \alpha_0/\sqrt{2} \\ \alpha_0/\sqrt{2} & \alpha_1 \end{bmatrix} = -\frac{\alpha_0^2}{2},$$

which clearly has no zeros on the real unit sphere. This confirms that by Theorem 20(i) the Hermitian rank one distance to singularity is infinite.

EXAMPLE 35. Let $A+\lambda E$ be defined as in Example 18, where a is sufficiently small. By Theorem 32, we have $\delta^H_{1,0}(A,E) = 1 > a = \delta_{1,0}(A,E) \geqslant \rho(A,E)$ showing that $\rho(A,E) \ll \delta^H_{1,0}(A,E)$ is possible.

EXAMPLE 36. In Examples 15 and 35 the reason for the estimate $\rho(A,E)$ being significantly smaller than $\delta_{1,0}(A,E)$ or $\delta^H_{1,0}(A,E)$, respectively, was the presence of Jordan chains. However, even if we start with a pencil satisfying $\rho(A,E) = \delta^H_{1,0}(A,E)$, then a simple congruence transformation $(A+\lambda E) \mapsto T(A+\lambda E)T^*$ can also cause $\rho(TAT^*, TET^*) < \delta^H_{1,0}(TAT^*, TET^*)$. With

$$A+\lambda E = \begin{bmatrix} 1 & \\ & \lambda \end{bmatrix}, \quad T = \begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix}, \; (\det T = -c \neq 0),$$

we obviously have $\rho(A,E) = \delta_{1,0}(A,E) = \delta^H_{1,0}(A,E) = 1$. Consider the transformed pencil

$$T(A+\lambda E)T^* = \begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{c} \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

let $u = [u_1,\, u_2]^\top \in \mathbb{C}^2$ be arbitrary and let $\tau \in \mathbb{R}$ be such that

$$\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{c} \end{bmatrix} + \tau u u^* + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} d+\lambda & 0 \\ 0 & 0 \end{bmatrix} \tag{48}$$

with some $d \in \mathbb{C}$. For this we need $\tau < 0$, hence we may just take $\tau = -1$. Then $u_2 = e^{\imath\theta} c$ for some $\theta \in [0, 2\pi)$ and consequently $u_1 = e^{\imath\theta} a$. Since the vector u was chosen arbitrarily and any singular pencil $\tilde{A} + \lambda TET^*$ has to be of the form (48), we have shown that

$$\delta^H_{1,0}(TAT^*, TET^*) = |e^{\imath\theta}| \left\| \begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{c} \end{bmatrix} \right\|_F = |a|^2 + |c|^2. \tag{49}$$

On the other hand observe that

$$R(\lambda) = T^{-*} \begin{bmatrix} 1 & \\ & \lambda^{-1} \end{bmatrix} T^{-1}, \quad C_j(\lambda) = T^{-*} \begin{bmatrix} 0 & \\ & \lambda^{-(j+1)} \end{bmatrix} T^{-1},$$

so that we can write the resolvent as

$$\begin{aligned} R(\lambda) &= T^{-*} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} T^{-1} + \lambda T^{-*} \begin{bmatrix} 0 & \\ & \lambda^{-2} \end{bmatrix} T^{-1} \\ &= T^{-*} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} T^{-1} + \lambda C_1(\lambda). \end{aligned}$$

Hence, $\operatorname{span}_{\mathbb{R}}\{C_1(\lambda), \ldots, C_n(\lambda)\} = \operatorname{span}_{\mathbb{R}}\{C_1(\lambda)\}$ and thus,

$$\begin{aligned} \mathfrak{P}_{\mathscr{D}} R(\lambda) &= \mathfrak{P}_{\operatorname{span}_{\mathbb{R}}\{C_1(\lambda)\}^{\perp_H}} R(\lambda) = \mathfrak{P}_{\operatorname{span}_{\mathbb{R}}\{C_1(\lambda)\}^{\perp_H}} \left(T^{-*} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} T^{-1} \right) \\ &= \frac{1}{|c|^2} \mathfrak{P}_{\operatorname{span}\{C_1(\lambda)\}^{\perp_H}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{50}$$

Furthermore, $\operatorname{span}_{\mathbb{R}}\{C_1(\lambda)\} = \operatorname{span}_{\mathbb{R}}\{\tilde{C}_1(\lambda)\}$, where

$$\tilde{C}_1(\lambda) = \begin{bmatrix} -\bar{c} \\ \bar{a} \end{bmatrix} \begin{bmatrix} -c & a \end{bmatrix}, \quad \|\tilde{C}_1(\lambda)\|_F^2 = (|c|^2 + |a|^2)^2.$$

With this we obtain

$$\begin{aligned}\mathfrak{P}_{\mathrm{span}_{\mathbb{R}}\{C_1(\lambda)\}}\begin{bmatrix}0&0\\0&1\end{bmatrix} &= \|\tilde{C}_1(\lambda)\|_F^{-2}\,\mathrm{tr}\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\tilde{C}_1(\lambda)\right)\tilde{C}_1(\lambda)\\ &= \frac{|a|^2}{(|c|^2+|a|^2)^2}\tilde{C}_1(\lambda).\end{aligned}$$

Using (50) we have

$$\begin{aligned}\mathfrak{P}_{\mathscr{D}}R(\lambda) &= \frac{1}{|c|^2}\left(\begin{bmatrix}0&0\\0&1\end{bmatrix} - \frac{|a|^2}{(|c|^2+|a|^2)^2}\tilde{C}_1(\lambda)\right)\\ &= \frac{|a|^2}{|c|^2(|c|^2+|a|^2)^2}\begin{bmatrix}|c|^2 & -\bar{c}a\\ -\bar{a}c & \frac{|c|^4+2|c|^2|a|^2}{|a|^2}\end{bmatrix}.\end{aligned}$$

and an easy calculation gives

$$\rho(TAT^*,TET^*)^{-1} = \frac{|c|(|c|^2+|a|^2)}{\sqrt{2|a|^2+|c|^2}}.$$

Comparing this with (49), we see that for $a\to\infty$ and c constant, the estimate $\rho(A,E)$ is significantly smaller then $\delta^H_{1,0}(A,E)$. Also note that with $a\to\infty$ the condition number of T grows to infinity.

The theoretical results of this section are not very suitable for numerical computation, since in the neighborhood of a singular pencil the standard eigenvalue methods may behave very erratically. This is demonstrated in the following example.

EXAMPLE 37. To illustrate potential numerical errors in eigenvalue computation in the neighborhood of singular pencils, we used `matlab` [19] to evaluate the formula for the projection

$$\mathfrak{P}_{\mathrm{span}\{C_1(\lambda),\dots,C_n(\lambda)\}^\perp}X = X - \sum_{j=1}^{\tilde{n}}\left\langle X,\tilde{C}_j(\lambda)\right\rangle,$$

where $\tilde{C}_1(\lambda),\dots,\tilde{C}_{\tilde{n}}(\lambda)$ is determined via Gram-Schmidt orthonormalization applied to the matrices $C_1(\lambda_0),\dots,C_n(\lambda_0)$ and $A+\lambda E = S^*\big(\mathscr{N}_3^1(\lambda)\oplus\mathscr{J}_{1,5}(\lambda)\big)S$, where S is some invertible random matrix.

We present the results for the theoretically constant function

$$f(\lambda) := \left\|\mathfrak{P}_{\mathrm{span}\{C_1(\lambda),\dots,C_n(\lambda)\}^\perp}R(\lambda)\right\|_F^{-1} \tag{51}$$

in Figure 1.

Note that $f(\lambda)$ is not only deviating from the constant function at the singular point $\lambda=5$. For this system we have $\rho(A,E)=0.45$.

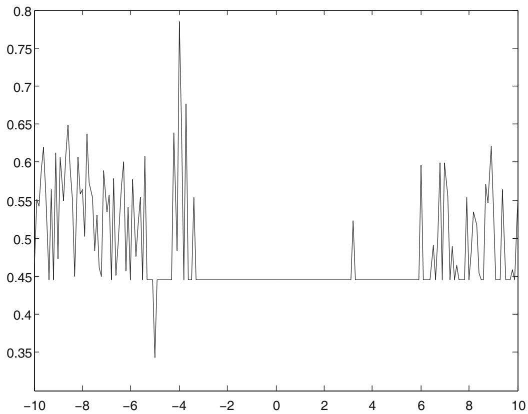

Figure 1: *The numerically obtained plot of the constant function* $f(\lambda)$ *as in* (51).

7. The method of alternating projections

In this section we consider both the situation that $A+\lambda E$ is an arbitrary unstructured pencil or a Hermitian pencil. By Proposition 12 and 31, we have that a singularizing rank-one perturbation of the regular $n\times n$ pencil $A+\lambda E$ is given by a matrix $G\in\mathbb{C}^{n\times n}$ satisfying $\mathscr{P}_{\mathscr{D}}G^*=G^*$ and $\operatorname{rank}(G)=1$, where in the Hermitian case we assume in addition that G is Hermitian.

To derive a numerical method to compute such a singularizing perturbation matrix G, we start with an arbitrary matrix $R_0\in\mathbb{C}^{n\times n}$. It can then be anticipated that the projected matrix $R_1:=\left(\mathfrak{P}_{\mathscr{D}}R_0^*\right)^*$ will be closer to a singularizing perturbation matrix. However, R_1 need not be of rank one, not even in the case that the initial matrix R_0 was of rank one. So, the idea for the construction of a numerical method is to project R_1 to the nearest rank-one matrix. This will most likely move the projected matrix out of $\mathscr{D}$ again, but alternating this process using the two mentioned projections, we can hope to converge to a rank-one matrix G satisfying $\mathfrak{P}_{\mathscr{D}}G^*=G^*$. We will call this procedure the *method of alternating projections*.

The orthogonal projection of a matrix $X\in\mathbb{C}^{n\times n}$ to the set of matrices of rank one can be easily performed using the singular value decomposition $X=U\Sigma V^*$ of X, where $\Sigma=\operatorname{diag}(\sigma_1,\ldots,\sigma_n)$, with $\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_n$ and where U and V are unitary. Setting

$$\mathfrak{Q}(X)=U\operatorname{diag}(\sigma_1,0,,\ldots,,0)V^* \tag{52}$$

uniquely defines the matrix $\mathfrak{Q}(X)$ if $\sigma_1>\sigma_2$, but depends on the actual singular value decomposition if $\sigma_1=\sigma_2$. Choosing in each case a particular SVD then fixes a mapping $\mathfrak{Q}:\mathbb{C}^{n\times n}\to\mathbb{C}^{n\times n}$, so that the matrix $\mathfrak{Q}(X)$ satisfies (52) for every $X\in\mathbb{C}^{n\times n}$. Similarly, we define by $\mathfrak{Q}_H$ a mapping satisfying (52) and such that $\mathfrak{Q}_HX$ is Hermitian for a Hermitian matrix X.

Given an arbitrary initial nonzero matrix $R_0\in\mathbb{C}^{n\times n}$, we then define the sequence of matrices

$$R_{2k+1} := \left(\mathfrak{P}_{\mathscr{D}} R_{2k}^*\right)^*, \quad R_{2k+2} = \mathfrak{Q}(R_{2k+1}), \quad k = 0, 1, \ldots.$$

In the Hermitian case we set

$$R_{2k+1}^H = \mathfrak{P}_{\mathscr{D}} R_{2k}^H, \quad R_{2k+2}^H = \mathfrak{Q}_H(R_{2k+1}^H), \quad k = 0, 1, \ldots,$$

if the initial matrix R_0^H is a nonzero Hermitian matrix. The choice of R_0, as well as the choice of $\mathfrak{Q}$ as one of the mappings satisfying (52) will affect the limiting behavior of the sequence, see Example 41.

In view of Theorems 13 and 32 the natural candidate for the initial value is $R_0 = R(\lambda_0)$, where λ_0 is a regular point of $A + \lambda E$. By Lemma 8 and Proposition 11, the matrices R_j, $j > 1$, are independent on the choice of λ_0.

PROPOSITION 38. *Let $A + \lambda E$ be a regular pencil with $A, E \in \mathbb{C}^{n\times n}$. If the alternating projection sequence $(R_k)_{k\in\mathbb{N}}$ converges to some $G \neq 0$ with $k \to \infty$, then G is of rank one, the pencil $A - (\mathrm{tr}(GR(\lambda_0)))^{-1}G + \lambda E$ is singular, and we have $\delta_{1,0}(A,E) \leqslant |\,\mathrm{tr}(GR(\lambda_0))|^{-1}\,\|G\|_F$.*

If, additionally, A and E are Hermitian and the Hermitian alternating projection sequence $(R_k^H)_{k\in\mathbb{N}}$ converges for $k \to \infty$ to some $G \neq 0$, then G is an Hermitian matrix of rank one, the pencil $A - (\mathrm{tr}(GR(\lambda_0)))^{-1}G + \lambda E$ is singular, and $\delta_{1,0}^H(A,E) \leqslant |\,\mathrm{tr}(GR(\lambda_0))|^{-1}\,\|G\|_F$.

Proof. Note that the matrix G satisfies $\mathfrak{P}_{\mathscr{D}} G^* = G^*$, $\mathfrak{Q}G = G$. Application of Proposition 12 finishes the proof. The Hermitian case follows analogously. $\square$

If R_0 is sufficiently close to the set of singularizing perturbations of the form τuv^*, then convergence of the sequence $(R_k)_{k\in\mathbb{N}}$ follows from general results in [1]. The following two examples illustrate the convergence behavior of the method.

EXAMPLE 39. Let $A + \lambda E = (A' + \lambda E') \oplus \mathscr{E}(\lambda)$ be a regular pencil, where

$$A' + \lambda E' = \begin{bmatrix} \varepsilon_1 & \lambda & & & \\ & \varepsilon_2 & \lambda & & \\ & & \ddots & \ddots & \\ & & & \ddots & \lambda \\ & & & & \varepsilon_k \end{bmatrix}$$

with $0 < |\varepsilon_{l_0}| < |\varepsilon_l|$, $l \neq l_0$ for some $l_0 \in \{1, \ldots, k\}$, and where $\mathscr{E}(\lambda)$ is a regular pencil with only finite and nonzero eigenvalues. Then

$$R_1 = (\mathfrak{P}_{\mathscr{D}} R(0)^*)^* = \begin{bmatrix} \varepsilon_1^{-1} & & & \\ & \varepsilon_2^{-1} & & \\ & & \ddots & \\ & & & \varepsilon_k^{-1} \end{bmatrix} \oplus 0$$

and $R_2 = \mathfrak{Q}\left(\mathfrak{P}_{\mathscr{D}}R(0)^*\right)^* = \varepsilon_{l_0}^{-1}e_{l_0}e_{l_0}^*$, where e_{l_0} denotes the l_0-th vector of the canonical basis in $\mathbb{C}^n$. Observe that $R_2 \in \mathscr{D}$, i.e., the sequence (R_k) becomes constant for $k \geqslant 2$. On the other hand, one has

$$\delta_{1,0}(A,E) = \delta_{1,0}(A',E') \geqslant \delta(A',E') \geqslant \sigma_{\min}(A) = |\varepsilon_{l_0}|,$$

where the inequality $\delta(A',E') \geqslant \sigma_{\min}(A)$ results from [3, Section 5.2]. Hence, $\delta_{1,0}(A,E) = |\varepsilon_{l_0}|$ and $\varepsilon_{l_0}^2 R_2$ realizes this distance.

EXAMPLE 40. We apply the alternating projection method to Example 36 with $a = 30$, $c = 1$, which has a unique rank-one singularizing perturbation of $A + \lambda E$. After 10^6 iterations performed with `Matlab` [19], for $\lambda_0 = 0.5$ the computed singularizing perturbation is

$$\begin{bmatrix} -863.3295 & -29.3888 \\ -29.3888 & -1.0004 \end{bmatrix},$$

while the only rank-one perturbation G that makes the pencil $A + G + \lambda E$ singular equals

$$G = \begin{bmatrix} -900 & -30 \\ -30 & -1 \end{bmatrix}.$$

A plot of the convergence behavior is presented in Figure 2.

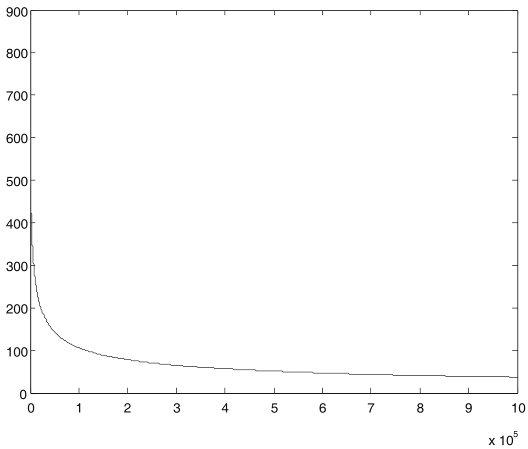

Figure 2: *Graph of* $\|R_{2n} - G\|_F$

EXAMPLE 41. For

$$A + \lambda E = \begin{bmatrix} 0 & 1 \\ 1 & \lambda \end{bmatrix}$$

the sequence (R_j^H) with the initial value $R_0^H = R(0.5)$ computed via the `Matlab` program

```
[U,Sigma]=eig(V1);
[m,id]=max(abs(diag(Sigma)));
Sigma2=zeros(n,n); Sigma2(id,id)=Sigma(id,id);
V=U*Sigma2*U';
```

diverges.

On the other hand, to compute the mapping $\mathfrak{Q}(X)$, the `Matlab` program

```
[U1,Sigma,U2]=svd(X);
Sigma2=zeros(n,n); Sigma2(1,1)=Sigma(1,1);
QX=U1*Sigma2*U2';
```

with the same initial value $R_0 = R(0.5)$ yields that the sequence R_j is constant for $j \geqslant 2$ and with

$$G = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Both results confirm the theory as there is no Hermitian rank-one perturbation that singularizes the given Hermitian pencil.

For the Hermitian pencil

$$A + \lambda E = \begin{bmatrix} 0 & 1 \\ 1 & \lambda \end{bmatrix} \oplus \begin{bmatrix} 10 \end{bmatrix}$$

one obtains the sequence

$$\lim_{j\to\infty} R_j^H = \begin{bmatrix} 0 & -0.5 \\ -0.5 & +\infty \end{bmatrix} \oplus \begin{bmatrix} 0 \end{bmatrix},$$

while

$$\lim_{j\to\infty} R_j = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \end{bmatrix}.$$

Thus, while the unstructured method converges to a singularizing perturbation matrix of minimal norm, the Hermitian method fails to find a Hermitian singularizing perturbation.

On the other hand, setting $R_0 = R_0^H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 10 \end{bmatrix}$ one gets

$$R_j = R_j^H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} -10 \end{bmatrix}, \quad j \geqslant 1,$$

so now the Hermitian method finds the unique Hermitian singularizing perturbation matrix of rank one while the unstructured method does not find the singularizing perturbation matrix of minimal norm.

Examples 40 and 41 show the difficulties with the alternating projection method. It is still an **open problem** to derive a monotonically convergent sequence to the smallest singularizing rank-one perturbation.

8. Other low-rank distances

In this section, we present some further results on singularizing perturbations of arbitrary rank. The central role in Sections 3 and 5 is played by Theorem 7, which further refers to Theorem 4, condition (e). Here we show an analogue of the aforementioned condition (e) in Theorem 4 in the Hermitian case with $\kappa_A = \kappa_E = 1$. The condition is, however, significantly more complicated and thus harder to apply.

For the pencil $A+\lambda E$, we consider the perturbations

$$A+\tau B_1B_2^* + \lambda(E+\tau F_1F_2^*), \quad \tau\in\mathbb{C},$$

with the two choices

$$B_1, B_2 \in \mathbb{C}^{n\times\kappa_A},\ F_1, F_2 \in \mathbb{C}^{n\times\kappa_E}, \tag{53}$$

$$\operatorname{rank} B_1 = \operatorname{rank} B_2 = \kappa_A \geqslant 0, \quad \operatorname{rank} F_1 = \operatorname{rank} F_2 = \kappa_E \geqslant 0. \tag{54}$$

Similarly as in Section 2, we define the matrix valued Weyl functions

$$Q(\lambda) = \begin{bmatrix} B_2^* \\ \lambda F_2^* \end{bmatrix} R(\lambda) \begin{bmatrix} B_1 & F_1 \end{bmatrix}, \tag{55}$$

and

$$Q(\infty) = \begin{bmatrix} 0 & 0 \\ F_2^*E^{-1}B_1 & F_2^*E^{-1}F_1 \end{bmatrix}, \tag{56}$$

if infinity is a regular point of $A+\lambda E$. Observe that $Q(\lambda)$ is analytic on the set of regular points of $A+\lambda E$.

PROPOSITION 42. *Let the $n\times n$ pencil $A+\lambda E$ be regular, let B_1, B_2, F_1, F_2 be as in (53) and (54), and let $\tau_0\in\mathbb{C}\setminus\{0\}$. Then*

(i) the characteristic polynomial $p(\lambda)$ of $A+\tau_0B_1B_2^*+\lambda(E+\tau_0F_1F_2^*)$ is given by

$$p(\lambda) = \det(A+\lambda E)\cdot\det\left(I_{\kappa_A+\kappa_E} + \tau_0 Q(\lambda)\right);$$

(ii) a regular point $\lambda_0\in\mathbb{C}\cup\{\infty\}$ of $A+\lambda E$ is a singular point of the perturbed pencil $A+\tau_0B_1B_2^*+\lambda(E+\tau_0F_1F_2^*)$ if and only if $-1/\tau_0$ is an eigenvalue of the matrix $Q(\lambda_0)$.

Proof. (i) If $\lambda_0\in\mathbb{C}$ is a regular point of $A+\lambda E$, then one has

$$\begin{aligned} p(\lambda) &= \det\left(A+\tau_0B_1B_2^*+\lambda_0(E+\tau_0F_2F_2^*)\right) \\ &= \det(A+\lambda_0E)\cdot\det\left(I_n + \tau_0(A+\lambda_0E)^{-1}\begin{bmatrix} B_1 & F_1\end{bmatrix}\begin{bmatrix} B_2^* \\ \lambda F_2^*\end{bmatrix}\right) \\ &= \det(A+\lambda_0E)\cdot\det\left(I_{\kappa_A+\kappa_E}+\tau_0Q(\lambda_0)\right). \end{aligned}$$

(ii) For $\lambda_0 \in \mathbb{C}$ the proof is a direct consequence of (i). The case $\lambda_0 = \infty$ follows from

$$\begin{aligned}\det(E + \tau_0 F_1 F_2^*) &= \det(E) \cdot \det(I_{\kappa_E} + \tau_0 F_2^* E^{-1} F_1) \\ &= \det(E) \det\left(I_{\kappa_A+\kappa_E} + \tau_0 Q(\infty)\right). \quad \square\end{aligned}$$

From Proposition 42, we obtain immediately that a regular point $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ of $A + \lambda E$ is a singular point of the pencil $A + \tau B_1 B_2^* + \lambda(E + \tau F_1 F_2^*)$ for at most $\kappa_A + \kappa_E$ values of the parameter τ.

The statements (a')–(c') in the following theorem directly generalize respective statements from Theorem 4.

THEOREM 43. *Let the $n \times n$ pencil $A + \lambda E$ be regular and let B_1, B_2, F_1, F_2 be as in* (53) *and* (54). *Then the following statements are equivalent.*

(a') The pencil $A + \tau_0 B_1 B_2^* + \lambda(E + \tau_0 F_1 F_2^*)$ is singular for some $\tau_0 \in \mathbb{C}$.

(b') On the set of regular points of $A + \lambda E$, all matrices $Q(\lambda)$ have common, nonzero eigenvalues $\zeta_1, \ldots, \zeta_k$ (independent of λ), where k is some integer satisfying $1 \leqslant k \leqslant \kappa_A + \kappa_E$.

(c') The polynomial in two variables

$$p(\tau, \lambda) = \det(A + \tau B_1 B_2^* + \lambda(E + \tau F_1 F_2^*))$$

is divisible by the polynomial $q(\lambda, \tau) = (1 + \zeta_1 \tau) \cdots (1 + \zeta_k \tau)$ with some $\zeta_1, \ldots, \zeta_k \neq 0$, for some $k \in \mathbb{N}$ with $1 \leqslant k \leqslant \kappa_A + \kappa_E$.

Furthermore, the numbers $\zeta_1, \ldots, \zeta_k$ in (b') *and* (c') *coincide and the perturbed pencil $A + \tau_0 B_1 B_2^* + \lambda(E + \tau_0 F_1 F_2^*)$ is singular precisely if and only if $\tau_0 = -1/\zeta_j$ for some $j \in \{1, \ldots, k\}$.*

Proof. (a')$\Leftrightarrow$(b') If the pencil $A + \tau_0 B_1 B_2^* + \lambda(E + \tau_0 F_1 F_2^*)$ is singular, then each $\lambda \in \mathbb{C} \cup \{\infty\}$ is a singular point. By Proposition 42(i) we get that $-1/\tau_0$ is an eigenvalue of the matrix $Q(\lambda_0)$ for all regular points $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ of $A + \lambda E$, so (b') is satisfied for some $k \geqslant 1$. The reversed argument proves the converse implication.

(b')$\Rightarrow$(c') Let $\zeta_1, \ldots, \zeta_k$ be the common nonzero eigenvalues of the matrices $Q(\lambda_0)$, where λ_0 is a regular point of $A + \lambda E$. Then

$$\det(I_{\kappa_A+\kappa_E} + \tau Q(\lambda)) = (1 + \zeta_1 \tau) \cdots (1 + \zeta_k \tau) Q_1(\tau, \lambda)$$

for some function $Q_1(\tau, \lambda)$ polynomial in τ and rational in λ. Hence, by Proposition 42(i),

$$p(\tau, \lambda) = (1 + \zeta_1 \tau) \cdots (1 + \zeta_k \tau) \det(A + \lambda E) Q_1(\tau, \lambda).$$

Since $p(\tau, \lambda)$ is a polynomial in λ, the function $\det(A + \lambda E) Q_1(\tau, \lambda)$ is polynomial in λ and τ.

(c')⇒(a') Assume that (c') holds. Then

$$p(-1/\zeta_j,\lambda)=0,\quad \lambda\in\mathbb{C},\quad j=1,\dots,k$$

and hence the pencil $A+\tau_0B_1B_2^*+\lambda(E+\tau_0F_1F_2^*)$ is singular with $\tau_0=-1/\zeta_j$, $j=1,\dots,k$. $\square$

EXAMPLE 44. Let $\kappa_A=2$, $\kappa_E=0$, $B_1=B_2=I_2$, and let

$$A=\begin{bmatrix}2&0\\0&1\end{bmatrix},\quad E=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

Then the function $Q(\lambda)\equiv A^{-1}$ has two constant eigenvalues. Furthermore, this simple example shows that the number k appearing in (b') and (c') need not be equal to the number of singular blocks in the perturbed pencil.

EXAMPLE 45. In this example we show that point (d) of Theorem 4, i.e., the fact that the eigenvalues are constant in τ when the perturbed pencil is regular, cannot easily be generalized to the case of perturbations of arbitrary rank. Consider the perturbed pencil

$$\begin{bmatrix}1+\tau&0\\0&\lambda+\tau\end{bmatrix}$$

which is singular for $\tau=-1$, but for $\tau\neq-1$ the eigenvalues are ∞ and $-\tau$.

The perturbed pencil

$$\begin{bmatrix}1+\tau\lambda&0\\0&\lambda+\lambda\tau\end{bmatrix}$$

is singular for $\tau=-1$, but for $\tau\neq-1$ the eigenvalues are zero and $-1/\tau$.

The perturbed pencil

$$\begin{bmatrix}\lambda(\tau+1)&0&1\\0&(\tau+1)&\lambda\\1&\lambda&(\tau+1)\end{bmatrix}$$

is singular for $\tau=-1$, but for $\tau\neq-1$ the eigenvalues are roots of the polynomial $\lambda^3-(1+\tau)\lambda+1$, i.e., they are nonconstant functions of τ. A detailed discussion on fractional power series expansions of eigenvalues of a perturbed singular pencil is given in [6].

REMARK 46. Observe that Theorem 43 (b') is a generalization of Theorem 4 (b). However, the condition of Theorem 4 that is most useful for applications is statement (e). Unfortunately, there seems to be no simple condition generalizing Theorem 4(e). In this remark we present a condition equivalent to Theorem 43 (b') for a Hermitian $n\times n$ pencil $A+\lambda E$ and perturbations of the forms

$$A+\tau_0uu^*+\lambda(E+\tau_0vv^*),\quad A-\tau_0uu^*+\lambda(E+\tau_0vv^*).\tag{57}$$

Our choice is motivated by Theorem 20(v), which says that every Hermitian pencil can be made singular by one of the perturbations given in (57). Observe that

$$Q(\lambda)=\begin{bmatrix} u^*R(\lambda)u & u^*R(\lambda)v \\ \lambda v^*R(\lambda)u & \lambda v^*R(\lambda)v \end{bmatrix}$$

and the eigenvalues of $Q(\lambda)$ equal

$$\zeta_{1,2}(\lambda)=-\frac{1}{2}\left(t(\lambda)\pm\sqrt{\Delta(\lambda)}\right),$$

where

$$t(\lambda)=t(\lambda;u,v)=\operatorname{tr}Q(\lambda),\quad d(\lambda)=d(\lambda,u,v)=\det Q(\lambda).$$

and

$$\Delta(\lambda)=\Delta(\lambda;u,v)=t^2(\lambda;u,v)-4d(\lambda;u,v).$$

Fixing a regular point λ_0 with $\Delta(\lambda_0)\neq 0$ and expressing the derivatives of the above functions in terms of A, E, u and v, we obtain

$$\begin{aligned}
t_k(u,v) &:= \frac{\partial^k t}{\partial\lambda^k}(\lambda_0)=\operatorname{tr}Q_{u,v}^{(k)}(\lambda_0)\\
&= u^*R^{(k)}(\lambda_0)u+\lambda_0 v^*R^{(k)}(\lambda_0)v+kv^*R^{(k-1)}(\lambda_0)v;\\
d_k(u,v) &:= \frac{\partial^k d}{\partial\lambda^k}(\lambda_0)\\
&= \sum_{i=1}^{k}\binom{k}{i}\left[u^*R^{(i)}(\lambda_0)u\left(\lambda_0 v^*R^{(k-i)}(\lambda_0)v+(k-i)v^*R^{(k-i-1)}v\right)\right.\\
&\qquad\left.-\,u^*R^{(i)}(\lambda_0)v\left(\lambda_0 v^*R^{(k-i)}(\lambda_0)u+(k-i)v^*R^{(k-i-1)}u\right)\right].
\end{aligned}$$

For $l=1,2\ldots$ we set

$$\Delta_l(u,v):=\frac{\partial^l(t^2-4d)}{\partial\lambda^l}(\lambda_0)=2\sum_{i=0}^{\lfloor l/2\rfloor}\frac{t_{l-i}(u,v)}{(l-i)!}\frac{t_i(u,v)}{i!}-4d_l(u,v).$$

Differentiating the formula for the eigenvalue $\zeta_1(\lambda)$ with the help of the Faá di Bruno formula (see [13, Theorem 1.3.2]), and finding the zeros of the derivatives of the eigenvalues, we get the following necessary and sufficient condition on the pair (u,v) for the eigenvalue $\zeta_1(\lambda)$ of $Q(\lambda)$ to be constant.

$$\begin{aligned}
0=t_k(u,v)+\sum\frac{k!}{i_1!\cdots i_k!}\left(\frac{1}{2}\right)_{i_1+\cdots+i_k}\Delta^{\frac{1-2(i_1+\cdots i_k)}{2}}(u,v)\\
\cdot\left(\frac{\Delta_1(u,v)}{1!}\right)^{i_1}\left(\frac{\Delta_1(u,v)}{2!}\right)^{i_2}\cdots\left(\frac{\Delta_k(u,v)}{k!}\right)^{i_k}
\end{aligned}$$

for $k=1,2\ldots$, where we used the *Pochhammer symbol* $(x)_j:=x(x-1)\cdots(x-j+1)$, and where the sum is taken over all sequences $i_1,i_2,i_3,\ldots,i_{k-j+1}$ of non-negative integers such that

$$i_1+2i_2+3i_3+\cdots ki_k=k.$$

The necessary and sufficient condition on the pair (u,v) for the eigenvalue $\zeta_2(\lambda)$ of $Q(\lambda)$ being constant is analogous.

9. Conclusions

We have studied low rank perturbations of unstructured and Hermitian pencils matrix with the goal to find smallest norm perturbations that make the pencil singular. Motivated by the fact that most 'smallest distance perturbations' can be realized by small rank perturbations, we have identified several cases which allow characterizations to these smallest distance and contributed with partial results to the open problem of finding the distance to singularity for matrix pencils.

REFERENCES

[1] H. ATTOUCH, J. BOLTE, P. REDONT, AND A. SOUBEYRAN, *Proximal alternating minimization and projection methods for nonconvex problems: an approach based on the Kurdyka-łojasiewicz inequality* Math. Oper. Res., 35: 438–457, 2010.

[2] K. E. BRENAN, S. L. CAMPBELL, AND L. R. PETZOLD, *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations*, SIAM Publications, Philadelphia, PA, 2nd edition, 1996.

[3] R. BYERS, C. HE, AND V. MEHRMANN, *Where is the nearest non-regular pencil*, Linear Algebra Appl., 285: 81–105, 1998.

[4] F. DE TERÁN AND F. DOPICO, *Low rank perturbation of regular matrix polynomials*, Linear Algebra Appl., 430: 579–586, 2009.

[5] F. DE TERÁN AND F. DOPICO, *First order spectral perturbation theory of square singular matrix polynomials*, Linear Algebra Appl., 432: 892–910, 2010.

[6] F. DE TERÁN, F. DOPICO, AND J. MORO, *First order spectral perturbation theory of square singular matrix pencils*, Linear Algebra Appl., 429: 548–576, 2008.

[7] E. EICH-SOELLNER AND C. FÜHRER, *Numerical Methods in Multibody Systems*, B. G. Teubner Stuttgart, 1998.

[8] F. R. GANTMACHER, *Theory of Matrices*, Chelsea, New York, 1959.

[9] I. GOHBERG, P. LANCASTER, AND L. RODMAN, *Indefinite Linear Algebra and Applications*, Birkhäuser, Basel, 2005.

[10] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, Johns Hopkins University Press, Baltimore, 3rd edition, 1996.

[11] N. J. HIGHAM, D. S. MACKEY, N. MACKEY, AND F. TISSEUR, *Symmetric linearizations for matrix polynomials*, SIAM J. Matrix Anal. Appl., **29** (1): 143–159, 2006.

[12] T. KATO, *Perturbation Theory for Linear Operators*, Springer Verlag, New York, N.Y., 1966.

[13] S. KRANTZ AND H. PARKS, *A Primer of Real Analytic Functions*, Birkhäuser, Boston, MA, 2nd edition, 2002.

[14] P. KUNKEL AND V. MEHRMANN, *Differential-Algebraic Equations. Analysis and Numerical Solution*, EMS Publishing House, Zürich, Switzerland, 2006.

[15] R. LAMOUR, R. MÄRZ, AND C. TISCHENDORF, *Differential-Algebraic Equations: A Projector Based Analysis*, Differential-Algebraic Equations Forum, Springer Verlag, 2013.

[16] D. MACKEY, N. MACKEY, C. MEHL, AND V. MEHRMANN, *Structured polynomial eigenvalue problems: Good vibrations from good linearizations*, SIAM J. Matrix Anal. Appl., **28** (4): 1029–1051, 2006.

[17] *The MathWorks*, Inc., Natick, MA, Simscape, 2013., `http://www.mathworks.de/products/simscape/`.

[18] *The MathWorks*, Inc., Natick, MA, Simulink, 2013., `http://www.mathworks.de/products/simulink/`.

[19] *MATLAB*, Version r2013b, The MathWorks, inc., 24 Prime Park Way, Natick, MA 01760-1500, USA, 2013.

[20] *Modelica Association*, Modelica standard library 3.2.1., `https://www.modelica.org/`.
[21] A. C. M. RAN AND M. WOJTYLAK, *Eigenvalues of rank one perturbations of unstructured matrices*, Linear Algebra Appl., 437: 589–600, 2012.
[22] R. RIAZA, *Differential-algebraic systems. Analytical aspects and circuit applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ., 2008.
[23] S. SAVCHENKO, *On the Perron roots of principal submatrices of co-order one of irreducible nonnegative matrices*, Linear Algebra Appl., 361: 257–277, 2003.
[24] S. SAVCHENKO, *Typical changes in spectral properties under perturbations by a rank-one operator*, Mat. Zametki, 74: 590–602, 2003., (Russian). Translation in Mathematical Notes, 74: 557–568, 2003.
[25] S. SAVCHENKO, *On the change in the spectral properties of a matrix under a perturbation of a sufficiently low rank*, Funktsional. Anal. i Prilozhen, 38: 85–88, 2004., (Russian). Translation in Funct. Anal. Appl. 38: 69–71, 2004.
[26] R. THOMPSON, *The characteristic polynomial of a principal subpencil of a Hermitian matrix pencil*, Linear Algebra Appl., 14: 135–177, 1976.

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