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ON SPACES DERIVABLE FROM A SOLID SEQUENCE SPACE AND A NON–NEGATIVE LOWER TRIANGULAR MATRIX

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Abstract. The scalar field will be either the real or complex numbers. Suppose that λ is a solid sequence space over the scalar field and A is an infinite lower triangular matrix with non-negative entries and positive entries on the main diagonal such that each of its columns is in λ. For each positive integer k, the k^{th} predecessor of λ with respect to A is the solid vector space of scalar sequences x such that $A^k|x|$ is an element of λ. We denote this space by Λ_k and λ itself will be denoted by Λ_0. Under reasonable assumptions, these spaces inherit some topological properties from λ. We are interested in a projective limit of the infinite product of the Λ_k consisting of sequences of sequences $(x^{(k)})$ satisfying $Ax^{(k)} = x^{(k-1)}$ for each $k > 0$. We show that for interesting classes of situations including the cases when $\lambda = l_p$ for some $p > 1$ and A is the Cesàro matrix, the space of our interest can be non-trivial.

1. Introduction

Throughout the paper the scalars $\mathbb{F}$ will be either $\mathbb{R}$, the real numbers or $\mathbb{C}$, the complex numbers, and $\mathbb{N} = \{1,2,...\}$.

Although the only spaces to make an appearance in this paper will be spaces of sequences of scalars, in hopes of stirring interest in generalizations to a wider context, we give some basic definitions and properties concerning Riesz spaces (see [1, 2, 9]).

A real vector space E, equipped with a partial order $\leqslant$ in E^2, is a *Riesz space* or a *vector lattice* if $\sup\{x,y\}$ and $\inf\{x,y\}$ exist for all $x,y \in E$, and $x \leqslant y$ implies $\alpha x + z \leqslant \alpha y + z$ for all $z \in E$ and $0 \leqslant \alpha \in \mathbb{R}$. We define the modulus or absolute value of $x \in E$ by the formula $|x| := \sup\{x, -x\}$.

If E is a vector lattice, then the set $E^+ = \{x \in E : \ x \geqslant 0\}$ is referred to as the *positive cone* or simply the *cone* of E.

For $a \in E$, the *solid* hull of a is given by $S(a) = \{b \in E : \ |b| \leqslant |a|\}$. A subset S in a vector lattice E is said to be *solid* or an *order ideal* if it follows from $|u| \leqslant |v|$ in E and $v \in S$ that $u \in S$. In the sequel, we will use the term *solid* in preference to *order ideal*.

A norm $\|.\|$ on a vector lattice E is said to be a *lattice norm* or *solid norm* if $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for each $x,y \in E$.

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A vector lattice equipped with a solid norm is known as a *normed vector lattice*. If a normed vector lattice E is also norm complete, then it is a *Banach lattice*. It should be obvious that in a normed vector lattice E, $\|x\| = \|\,|x|\,\|$ holds for all $x \in E$.

The space of all scalar valued sequences will be denoted by $\mathbb{F}^{\mathbb{N}}$. The subspace of $\mathbb{F}^{\mathbb{N}}$ consisting of sequences with only finitely many non-zero entries will be denoted by c_{00}, whether $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. All operations on sequences will be coordinatewise. If $x = (x_n) \in \mathbb{F}^{\mathbb{N}}$, then we write $|x| = (|x_n|)$. Let $x = (x_n)$ and $y = (y_n)$ be elements of $\mathbb{R}^{\mathbb{N}}$; $x \leqslant y$ means that $x_n \leqslant y_n$ for each $n \in \mathbb{N}$. It is clear that $(\mathbb{R}^{\mathbb{N}}, \leqslant)$ is a vector lattice, and the vector lattice definition of $|x|$, $x \in \mathbb{R}^{\mathbb{N}}$ agrees with the definition given here, $|x| = (|x_n|)$.

We will denote the sequence of zeros, $(0,0,0,\ldots)$ by $\underline{0}$.

For $0 < p < \infty$, we denote

$$l_p = \{(x_n) \subset \mathbb{F} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

The usual "norm" or distance from $\underline{0}$, in l_p is defined by

$$||x||_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}},$$

for each $x \in l_p$; $\|.\|_p$ is really a norm for $p \geqslant 1$.

If λ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$, a topology on λ with which λ becomes a topological vector space is a *solid topology* if it has a basis of neighborhoods at the origin consisting of solid sets.

The spaces l_p, $1 \leqslant p < \infty$, l_∞, the space of bounded sequences and c_0, the space of null sequences, are solid subspaces of $\mathbb{F}^{\mathbb{N}}$, and their usual norms, $||.||_p$ on l_p, the sup norm on l_∞ and c_0, are solid norms. The space of convergent sequences, c, is not solid.

Let $A = [a_{ij} : i, j \geqslant 1] = [a_{ij}]$ be an infinite matrix with non-negative entries and no zero columns. The domain of A, denoted by $dom(A)$, is

$$dom(A) = \{x \in \mathbb{F}^{\mathbb{N}} : \sum_{j=1}^{\infty} a_{ij} x_j \; converges \; for \; each \; i \in \mathbb{N}\}.$$

For $x \in dom(A)$, Ax, the A-transform of x, is given by $(Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j$ for each $i \in \mathbb{N}$.

If $\lambda \subset dom(A)$,

$$A\lambda = \{Ax : x \in \lambda\}.$$

If $\lambda \subset \mathbb{F}^{\mathbb{N}}$,

$$A^{-1}(\lambda) = \{x \in dom(A) : Ax \in \lambda\}.$$

If $A = [a_{ij}]$ is a lower triangular matrix (i.e. $a_{ij} = 0$, for $i < j$) with non-negative entries and positive entries on the main diagonal (i.e. $a_{ii} > 0$, for $i \in \mathbb{N}$), then $dom(A) = \mathbb{F}^{\mathbb{N}}$. The assumption of non-zero diagonal entries implies that A has a matricial inverse

A^{-1}. This inverse A^{-1} is also lower triangular. A^{-1} will fail to have all non-negative entries, unless A is diagonal. The reader can consult the book [3] on infinite matrices.

The following definition was introduced in [5], and was inspired by [8].

DEFINITION 1. If $\lambda \subset \mathbb{F}^{\mathbb{N}}$ and A is an infinite matrix, with non-negative entries, then

$$sol - A^{-1}(\lambda) = \{x \in \mathbb{F}^{\mathbb{N}} : \ |x| \in A^{-1}(\lambda)\} = \{x \in \mathbb{F}^{\mathbb{N}} : \ |x| \in dom(A) \text{ and } A|x| \in \lambda\}.$$

The next result given in [5] justifies the name "$sol - A^{-1}(\lambda)$".

PROPOSITION 2. *Let A be an infinite matrix with non-negative entries and λ be a solid subspace of $\mathbb{F}^{\mathbb{N}}$. Then we have*

(a) $sol - A^{-1}(\lambda)$ is solid;
(b) $sol - A^{-1}(\lambda) \subset A^{-1}(\lambda)$;
(c) $sol - A^{-1}(\lambda)$ is the largest solid set of sequences contained in $A^{-1}(\lambda)$;
(d) $sol - A^{-1}(\lambda)$ is a subspace of $\mathbb{F}^{\mathbb{N}}$.

If τ is a solid topological vector space topology on λ, then it naturally induces a solid topological vector space topology on $sol - A^{-1}(\lambda)$.

Suppose λ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$ with solid topology τ, and $\mathscr{U}$ is a neighborhood base at the origin in (λ, τ) consisting of solid sets. It is shown in [5] that the sets

$$sol - A^{-1}(U) = \{x \in sol - A^{-1}(\lambda) : \ A|x| \in U\}, \quad (U \in \mathscr{U})$$

constitute a neighborhood base at the origin for a solid topological vector space topology $sol - A^{-1}(\tau)$ on $sol - A^{-1}(\lambda)$. Further, if the topology on λ is Hausdorff and A has no zero columns, then the induced topology on $sol - A^{-1}(\lambda)$ is Hausdorff. Henceforward, all our matrices will be assumed to have no zero columns.

Note that the map $x \to A|x|$ is continuous but not linear from $sol - A^{-1}(\lambda)$ into λ. But the map $x \to Ax$ is continuous and linear from $sol - A^{-1}(\lambda)$ into λ.

Clearly, if λ is equipped with a solid norm $\|.\|$, then the topology induced on $sol - A^{-1}(\lambda)$ is induced by the solid norm $x \to \| A|x| \|_{\lambda}$. The same comment holds for quasinorms, pseudonorms, and seminorms.

If λ is a solid sequence space with a solid topology, and $P_n : \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ is defined by

$$P_n(x) = (x_1, \ldots, x_n, 0, 0, \ldots)$$

for all $n \in \mathbb{N}$, then, since $P_n(x) \in S(x)$, P_n is clearly continuous on λ, for all $n \in \mathbb{N}$.

LEMMA 3. *Suppose that $\lambda \subset \mathbb{F}^{\mathbb{N}}$ is a solid topological vector space of sequences with a solid topology. Then $c_{00} \cap \lambda$ is dense in λ if and only if, for each $x \in \lambda$, $P_n(x) \to x$ as $n \to \infty$, in the topology on λ.*

Proof. Since $P_n(x) \in c_{00} \cap \lambda$ for each $x \in \lambda$, by the solidity of λ, the "if" statement is clear. Now suppose that $c_{00} \cap \lambda$ is dense in λ; suppose that $x \in \lambda$ and that U is a solid neighborhood of $\underline{0}$ in λ. Let $y \in c_{00} \cap \lambda$ be such that $x - y \in U$ and let $N \in \mathbb{N}$

be the largest index such that $y_N \neq 0$, if $y \neq \underline{0}$. If $y = \underline{0}$, let $N = 1$. In any case, for all $n \geqslant N$, $x - P_n(x) \in S(x-y) \subset U$. Since U was arbitrary, it follows that $P_n(x) \to x$ as $n \to \infty$. $\square$

Usually, we have $c_{00} \subset \lambda$, but a solid vector space of sequences derived from λ, such as $sol - A^{-1}(\lambda)$, may fail to contain c_{00}; indeed, it can happen that $sol - A^{-1}(\lambda)$ consists of the zero sequence alone. Let e_n denote the sequence with 1 in the n^{th} place and zero elsewhere, $n \in \mathbb{N}$. It is obvious that c_{00} is the linear span of $\{e_n : n \in \mathbb{N}\}$ and that Ae_n is the n^{th} column of A. The following lemma is also obvious.

LEMMA 4. *Suppose that $c_{00} \subset \lambda \subset \mathbb{F}^{\mathbb{N}}$, λ is a solid vector space of sequences, and A is an infinite matrix with non-negative entries. Then $c_{00} \subset sol - A^{-1}(\lambda)$ if and only if each column of A is in λ.*

Regarding the density of c_{00} in $sol - A^{-1}(\lambda)$, or equivalently by Lemma 3, “sectional convergence” in $sol - A^{-1}(\lambda)$, we have the following, by results in [5] and [7].

LEMMA 5. *Suppose that $c_{00} \subset \lambda \subset \mathbb{F}^{\mathbb{N}}$, λ is a solid vector space of sequences with a solid Hausdorff topological vector space topology, and A is an infinite matrix with non-negative entries and every column of A is a non-zero sequence in λ. If c_{00} is dense in λ, then c_{00} is dense in $sol - A^{-1}(\lambda)$, in the topology on that space induced by the topology on λ.*

Finally: suppose that λ is a solid vector space of sequences with a solid Hausdorff topological vector space topology τ, and A is an infinite matrix with non-negative entries, with every column a non-zero sequence in λ. If (λ, τ) is complete, is the same true for $(sol - A^{-1}(\lambda), sol - A^{-1}(\tau))$? Unfortunately, we do not know the answer to this question in total generality; the best we can say is : yes, usually. Here is the story we know, from [5].

Let $\mathscr{P}$ denote the topology of coordinatewise convergence on $\mathbb{F}^{\mathbb{N}}$; in other words, $\mathscr{P}$ is the product topology on $\mathbb{F}^{\mathbb{N}}$, considered to be the product of countably many copies of the scalar field, which bears its usual topology. Note that $\mathscr{P}$ is a solid topology. If μ is a vector subspace of $\mathbb{F}^{\mathbb{N}}$, then by $(\mu, \mathscr{P})$ we mean μ equipped with the relative topology induced by $\mathscr{P}$. If Γ is another topological vector space topology on μ, we will say that (μ, Γ) is locally coordinatewise closed, or *LCC*, for short, if there is a neighborhood base at the origin in (μ, Γ) each member of which is closed in $(\mu, \mathscr{P})$.

There do exist non-*LCC* solid spaces with solid topological vector space topologies, but they are not easy to find. All of the l_p, $0 < p \leqslant \infty$ are *LCC*, with their usual norm or quasinorm topologies, and from these we can produce many more *LCC* spaces by the following, from Theorems 2.10 and 2.13 of [5].

LEMMA 6. *Suppose that $c_{00} \subset \lambda \subset \mathbb{F}^{\mathbb{N}}$, λ is a solid vector subspace of $\mathbb{F}^{\mathbb{N}}$, τ is a solid Hausdorff topological vector space topology on $\mathbb{F}^{\mathbb{N}}$, and A is an infinite matrix with non-negative entries, with each column a non-zero sequence in λ. Then we have*

(a) If (λ, τ) is LCC, then so is $(sol - A^{-1}(\lambda), sol - A^{-1}(\tau))$;

(b) If (λ, τ) is LCC and complete, then so is $(sol - A^{-1}(\lambda), sol - A^{-1}(\tau))$.

These lemmas will be useful in what is to come, in the next section, although Lemma 6 is unnecessarily general for our purposes. Its role can be played, as well, by Proposition 10 in the next section.

2. Solid sequence spaces derived from l_p and the Cesàro matrix

Hardy [4] established the following, called *Hardy's inequality*.

THEOREM H. *For any non-zero scalar sequence $x=(x_n)\in l_p$ and $1<p<\infty$, we have*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p<\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}|x_n|^p.$$

Furthermore, the constant $(\frac{p}{p-1})^p$ appearing in this inequality is the best (smallest) possible.

Let $A=[a_{ij}]$ be the Cesàro matrix, defined by

$$a_{ij}=\begin{cases}\frac{1}{i} : i\geqslant j\\ 0 : i<j\end{cases};$$

that is to say,

$$A=\begin{bmatrix}1 & 0 & 0 & . & . & . & . & . \\ \frac12 & \frac12 & 0 & . & . & . & . & . \\ \frac13 & \frac13 & \frac13 & 0 & . & . & . & . \\ \frac14 & \frac14 & \frac14 & \frac14 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}.$$

It is straightforward to verify that its inverse $A^{-1}=[x_{ij}]$ is defined by $x_{ij}=0$ $(j\neq i,$ $j\neq i-1)$, $x_{ii}=i$ and $x_{i,i-1}=-(i-1)$. We also have $dom(A)=\mathbb{F}^{\mathbb{N}}$ by its lower triangularity; $Ax=(\frac{1}{i}\sum_{j=1}^{i}x_j)_i$ for all $x\in\mathbb{F}^{\mathbb{N}}$. One may view A as a linear operator from the space $\mathbb{F}^{\mathbb{N}}$ into itself.

From now on, A denotes the Cesàro matrix if not otherwise stated.

By using Hardy's inequality, we have that

$$||Ax||_p\leqslant||\,A|x|\,||_p\leqslant\frac{p}{p-1}||x||_p$$

for $1<p<\infty$, and $\frac{p}{p-1}$ cannot be replaced by any smaller constant. So the Hardy operator $H:l_p\to l_p$ defined by

$$H((x_n)):=A(x_n)=\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)_{n\in\mathbb{N}}$$

is linear and continuous with operator norm $||H|| = \frac{p}{p-1}$.

Thus the Cesàro matrix multiplies l_p into l_p. Will the same hold for each matrix of Cesàro type,

$$B = \begin{bmatrix} b_1 & 0 & 0 & . & . & . \\ b_2 & b_2 & 0 & . & . & . \\ b_3 & b_3 & b_3 & 0 & . & . \\ b_4 & b_4 & b_4 & b_4 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} = \begin{bmatrix} b_1 & 0 & 0 & . & . & . \\ 0 & b_2 & 0 & . & . & . \\ 0 & 0 & b_3 & 0 & . & . \\ 0 & 0 & 0 & b_4 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & . & . & . \\ 1 & 1 & 1 & 0 & . & . \\ 1 & 1 & 1 & 1 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}.$$

in which $(b_k) \in l_p$ is a positive sequence? The answer is no, not necessarily, as the next example shows.

EXAMPLE 7. Fix $r \in (0,1)$ and let $b_k = k^{-r}$, $k = 1,2,\ldots$ Then $(b_k)_{k\geqslant 1} \in l_p$ for all $1 < p < \infty$ such that $p > \frac{1}{r}$. Note that

$$\sum_{k=1}^{n} \frac{1}{k^r} > \int_1^n \frac{1}{x^r} dx = \frac{n^{1-r}-1}{1-r}.$$

Therefore, with the inequality understood to hold coordinate wise,

$$B(b_k) = \begin{bmatrix} b_1 & 0 & 0 & . & . & . \\ 0 & b_2 & 0 & . & . & . \\ 0 & 0 & b_3 & 0 & . & . \\ 0 & 0 & 0 & b_4 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & . & . & . \\ 1 & 1 & 1 & 0 & . & . \\ 1 & 1 & 1 & 1 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ . \\ . \\ . \end{bmatrix}$$

$$\geqslant \frac{1}{r-1} \begin{bmatrix} 1 & 0 & 0 & . & . & . \\ 0 & 2^{-r} & 0 & . & . & . \\ 0 & 0 & 3^{-r} & 0 & . & . \\ 0 & 0 & 0 & 4^{-r} & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 0 \\ 2^{1-r}-1 \\ 3^{1-r}-1 \\ 4^{1-r}-1 \\ . \\ . \\ . \end{bmatrix}$$

$$= \frac{1}{1-r} \begin{bmatrix} 0 \\ 2^{1-2r}-2^{-r} \\ 3^{1-2r}-3^{-r} \\ 4^{1-2r}-4^{-r} \\ . \\ . \\ . \end{bmatrix} \notin l_\infty,$$

if $r \in (0, \frac{1}{2}]$, and in some other cases.

PROPOSITION 8. *For* $1 < p < \infty$, $A(l_p)$ *is dense in* l_p.

Proof. For each $n \in \mathbb{N}$, A multiplies $(\underbrace{1,1,...,1}_{n},-n,0,...)$ into $(\underbrace{1,1,...,1}_{n},0,...)$. Therefore, $A(l_p)$ contains a sequence of vectors which span c_{00}, which is dense in l_p. □

Let C be the set of all sequences x in $\mathbb{R}^{\mathbb{N}}$ with non-negative terms such that $Ax \in l_p$. Clearly, C is a cone in $\mathbb{R}^{\mathbb{N}}$. Let

$$ces_p = sol - A^{-1}(l_p) = \{x \in \mathbb{R}^{\mathbb{N}} : |x| \in C\} = \{x \in \mathbb{R}^{\mathbb{N}} : A|x| \in l_p\}.$$

Then ces_p is a solid subspace in the vector lattice $\mathbb{R}^{\mathbb{N}}$. This space was studied by Shiue [10] and Leibowitz [8]. They showed that ces_p is trivial if $0 < p \leqslant 1$, and contains l_p as a proper subspace if $1 < p \leqslant \infty$. One may induce a norm on ces_p via $||x||_{ces_p} = ||\, A|x|\, ||_p$; clearly this is a solid norm. Leibowitz [8] proved that ces_p with this norm is complete, that is, ces_p is a Banach lattice.

ces_p is not a subspace of l_∞ as the next example shows.

EXAMPLE 9. Let $x = (x_n)$ be such that

$$x_n = \begin{cases} k : n = 2^k \\ 0 : \text{otherwise} \end{cases}$$

Then $x \in ces_p$ but $x \notin l_\infty$. This shows that $ces_p \not\subset l_\infty$.

We now define $\Lambda_0, \Lambda_1, \Lambda_2, ...$ by $\Lambda_0 = l_p$, $\Lambda_1 = ces_p$ ($1 < p < \infty$, p fixed) and for $k > 1$,

$$\Lambda_k = sol - A^{-1}(\Lambda_{k-1}) = \{x \in \mathbb{F}^{\mathbb{N}} : A|x| \in \Lambda_{k-1}\}.$$

By induction on k, Λ_k is a solid subspace of $\mathbb{F}^{\mathbb{N}}$ for each $k \geqslant 0$.

Since $\Lambda_0 = l_p \subset sol - A^{-1}(l_p) = ces_p = \Lambda_1$, we have

$$\Lambda_k = \{x \in \mathbb{F}^{\mathbb{N}} : A|x| \in \Lambda_{k-1}\} \subset \{x \in \mathbb{F}^{\mathbb{N}} : A|x| \in \Lambda_k\} = \Lambda_{k+1},$$

by the induction hypothesis that $\Lambda_{k-1} \subset \Lambda_k$.

Therefore $(\Lambda_k)_{k \geqslant 0}$ is an increasing sequence with respect to set inclusion, so $\Lambda_0 = l_p \subset \Lambda_k$ for all $k \geqslant 0$.

The proof of the following proposition is very similar to the proof of Proposition 1.1 in [6] so that we omit its proof.

PROPOSITION 10. *Suppose that* $(\lambda, ||.||_\lambda)$ *is a solid Banach sublattice of* $\mathbb{F}^{\mathbb{N}}$. *Let* A *be an infinite lower triangular matrix with non-negative entries and positive entries on the main diagonal. Then we have that*

$$sol - A^{-1}(\lambda) = \{x \in \mathbb{F}^{\mathbb{N}} : A|x| \in \lambda\}$$

is a solid Banach sublattice of $\mathbb{F}^{\mathbb{N}}$ *if equipped with the solid norm* $\|.\|$ *defined by* $\|x\| = \|\, A|x|\, \|_\lambda$ *where* $x \in sol - A^{-1}(\lambda)$.

We have defined $\|x\|_{ces_p} = \|\, A|x|\, \|_p$. Similarly, we define $\|x\|_{\Lambda_k} = \|\, A|x|\, \|_{\Lambda_{k-1}}$. From the proposition above, it follows that all the Λ_k's are Banach lattices.

Let $\{e_k\}$ be the sequence of basic unit vectors in l_p so that $(e_k)_N = \delta_{k,N}$ for all N where δ is the Kronecker delta. The next two propositions and their proofs are similar to results and proofs in [8].

PROPOSITION 11. *(a) If $x \in \Lambda_k$, then $Ax \in \Lambda_{k-1}$, if $k > 0$.*
(b) $x \in \Lambda_k$ if and only if $A^k|x| \in \Lambda_0 = l_p$ for each $k \geqslant 0$.
(c) Let $k > l \geqslant 0$ where k and l are integers. Then $A^{k-l}(\Lambda_k) \subset \Lambda_l$.
(d) c_{00} is dense in Λ_k for each $k \geqslant 0$. Equivalently, for each $k \geqslant 0$ if $x = (x_j)_{j\in\mathbb{N}} \in \Lambda_k$, then $\|x - \sum_{j=1}^{n} x_j e_j\|_{\Lambda_k} \to 0$ as $n \to \infty$.
(e) Λ_k is a separable Banach lattice for each $k \geqslant 0$.

Proof. (a) Let $x \in \Lambda_k$. Then $A|x| \in \Lambda_{k-1}$. Since $|Ax| \leqslant A|x|$ coordinate–wise and Λ_{k-1} is a solid subspace of $\mathbb{F}^{\mathbb{N}}$, these imply that $Ax \in \Lambda_{k-1}$.

(b) Let $x \in \Lambda_k$. Therefore $A|x| \in \Lambda_{k-1} \Rightarrow A^2|x| \in \Lambda_{k-2} \Rightarrow A^3|x| \in \Lambda_{k-3}$, etc. It follows that $A^k|x| \in \Lambda_0 = l_p$ for each $k \geqslant 0$. The reverse implication follows by induction on k.

(c) If $k > 0$ and $x \in \Lambda_k$, then $Ax \in S(A|x|) \subset \Lambda_{k-1}$; thus $A(\Lambda_k) \subset \Lambda_{k-1}$. Therefore, $A^2(\Lambda_k) = A(A(\Lambda_k)) \subset A(\Lambda_{k-1}) \subset \Lambda_{k-2}$, if $k > 1$, and so on.

(d) The conclusion follows from Lemma 3 and Lemma 5.

(e) The conclusion follows from Lemma 6 or from Proposition 10, and from part (d). □

For $1 < p < \infty$, there is a natural mapping from Λ_k into Λ_{k-1} given by averaging. Specifically, we have the following result, which is very similar to Proposition 5 of [8].

PROPOSITION 12. *Let $1 < p < \infty$ and $k \in \mathbb{N}$. Define σ on Λ_k by $\sigma(x) = Ax$. Then σ is a one-to-one bounded linear operator from Λ_k into Λ_{k-1} with operator norm 1. Furthermore, the range of σ is a proper dense linear subspace of Λ_{k-1}.*

Proof. Since $|Ax| \leqslant A|x|$ and Λ_{k-1} is solid, we have

$$\|\sigma(x)\|_{\Lambda_{k-1}} = \|Ax\|_{\Lambda_{k-1}} \leqslant \|\, A|x|\, \|_{\Lambda_{k-1}} = \|x\|_{\Lambda_k}.$$

Clearly, σ is linear, so σ is a bounded linear operator from Λ_k into Λ_{k-1} with $\|\sigma\| \leqslant 1$. But if all the coordinates of x are non-negative, then we have,

$$\|\sigma(x)\|_{\Lambda_{k-1}} = \|Ax\|_{\Lambda_{k-1}} = \|\, A|x|\, \|_{\Lambda_{k-1}} = \|x\|_{\Lambda_k};$$

thus $\|\sigma\| = 1$.

Each e_k belongs to the range of σ. Indeed, one can compute directly that $e_k = \sigma(ke_k - ke_{k+1})$ for each $k \in \mathbb{N}$. Since c_{00}, the linear span of the e_k is dense in Λ_{k-1}, the range of σ is a dense linear subspace of Λ_{k-1}. On the other hand, it is not all of Λ_{k-1}. For example, let y be the sequence $(\frac{(-1)^{N+1}}{N})_N$. Then $y \in l_p \subset \Lambda_{k-1}$ for every $p > 1$

and $k \geqslant 0$. We claim that $y \in \Lambda_{k-1} \setminus A(\Lambda_k)$. Suppose the contrary that $y = \sigma(x) = Ax$ for some $x \in \Lambda_k$. Then $x = A^{-1}y = (1,-2,2,-2,\ldots)$ and $|x| = (1,2,2,2,\ldots)$. It is easy to see that $(A^k|x|)_n \to 1$ as $n \to \infty$ for each k. This shows that $A^k|x| \notin l_p$ so that $x \notin \Lambda_k$, which is a contradiction.

Since A is lower triangular and invertible, σ is one-to-one on all of $\mathbb{F}^{\mathbb{N}}$, and thus on Λ_k for each $k > 0$. $\square$

3. A special projective limit of $\prod_{k=0}^{\infty} \Lambda_k$

PROPOSITION 13. *Let $B = [b_{ij}]$ be a lower triangular matrix such that each column of it is in c_{00}. Then $B^k x$ is in c_{00} for any $x = (x_j)_{j\in\mathbb{N}} \in c_{00}$ and any $k \in \mathbb{N}$.*

Proof. If $x \in c_{00}$, then Bx is a finite linear combination of columns of B, and is therefore also in c_{00}, since each column of B is in c_{00}. Then $Bx \in c_{00}$ implies $B(Bx) = B^2x \in c_{00}$, which implies $B^3x \in c_{00}$, etc. $\square$

We now define

$$X = \{(x^{(k)}) \in \prod_{k=0}^{\infty} \Lambda_k : \; Ax^{(k)} = x^{(k-1)}, \; k > 0\}.$$

which is called the projective limit of $\prod_{k=0}^{\infty} \Lambda_k$ with respect to the maps $A : \Lambda_k \to \Lambda_{k-1}$.

Alternatively,

$$X = \{(x^{(k)}) \in \prod_{k=0}^{\infty} \Lambda_k : \; x^{(k)} = (A^{-1})^k x^{(0)}, \; k > 0\}.$$

Let $x^{(0)} \in c_{00}$. Since A^{-1} is a lower triangular matrix with columns in c_{00}, it follows from Proposition 13 that $(A^{-1})^k x^{(0)}$ is in c_{00}, and therefore $(A^{-1})^k x^{(0)} \in \Lambda_k$ for each $k \geqslant 0$. Therefore,

$$(x^{(k)})_{k\geqslant 0} = ((A^{-1})^k x^{(0)})_{k\geqslant 0} = (x^{(0)}, A^{-1}(x^{(0)}), \ldots, (A^{-1})^k (x^{(0)}), \ldots) \in X.$$

Hence X is not trivial. Since each Λ_k is complete with respect to $||.||_{\Lambda_k}$ and the map $x \to Ax$ from Λ_k into Λ_{k-1} is continuous, X is complete in the product topology on $\prod_{k=0}^{\infty} \Lambda_k$. Therefore X is a Fréchet space, when equipped with that product topology.

PROPOSITION 14. *If $(x^{(k)})_{k\geqslant 0} \in X$, then $A^k|x^{(k)}| \geqslant |x^{(0)}|$ coordinate–wise for each k.*

Proof. Let $(x^{(k)})_{k\geqslant 0} \in X$. Then $Ax^{(k)} = x^{(k-1)}$, $k > 0$ and $A|x^{(k)}| \in \Lambda_{k-1}$. We also have $A|x^{(k)}| \geqslant |x^{(k-1)}| = |Ax^{(k)}| \Rightarrow A^2|x^{(k)}| \geqslant A|Ax^{(k)}| = A|x^{(k-1)}| \geqslant |Ax^{(k-1)}| = |x^{(k-2)}|$, and so on. It follows that $A^k|x^{(k)}| \geqslant |x^{(0)}|$ coordinatewise, for each $k \geqslant 0$. $\square$

Let $P_0, P_1, \ldots,$ be the projections on X, where, for example, $P_0((x^{(k)})) = x^{(0)}$, $P_1((x^{(k)})) = x^{(1)}, \ldots,$ for $(x^{(k)}) \in X$. Note that we can write

$$X = \{(x, A^{-1}x, A^{-2}x, \ldots, A^{-k}x, \ldots) : \, x \in P_0(X)\}.$$

It is clear that $c_{00} \subset P_0(X) \subset l_p$.

PROPOSITION 15. *$P_0(X)$ is not solid and also not closed in $(l_p, \|.\|_p)$.*

Proof. Let $x = (1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots)$ and $y = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$. Then $|x| \leqslant |y|$, coordinatewise, $x \in l_p$, $y \in P_0(X)$, because $A^{-1}y = e_1$, so $A^{-k}y \in c_{00} \subset \Lambda_k$, $k \in \mathbb{N}$. But $x \notin P_0(X)$, because $A^{-1}x \notin \Lambda_1$. This shows that $P_0(X)$ is not solid in l_p. Let us show that $P_0(X)$ is not closed in $(l_p, \|.\|_p)$. Let

$$x_j^{(n)} = \begin{cases} \frac{(-1)^{j-1}}{j} & : 1 \leqslant j \leqslant n \\ 0 & : j > n \end{cases}.$$

Then $x^{(n)} = (x_j^{(n)}) \in c_{00} \subset P_0(X)$ for each n. We also have that

$$\|x - x^{(n)}\|_p^p = \sum_{j=n+1}^{\infty} \frac{1}{j^p} \to 0 \text{ as } n \to \infty.$$

Hence x is the limit of $x^{(n)}$ in l_p. But $x \notin P_0(X)$. $\square$

PROPOSITION 16. *The onto map $T : P_0(X) \to P_1(X)$ defined by*

$$Tx = A^{-1}x, \; x \in P_0(X)$$

is not continuous with respect to the norms $\|.\|_{\Lambda_0}$ and $\|.\|_{\Lambda_1}$.

Proof. It is easy to see that T is a linear map. Consider the sequence $x^{(n)} = (x_j^{(n)}) \in P_0(X)$ and $x = (x_j)$ in the proof of Proposition 15; $(x^{(n)})$ is a Cauchy sequence in $(P_0(X), \|.\|_{\Lambda_0})$ since $x^{(n)} \to x \in l_p$. But $(A^{-1}(x^{(n)}))_n$ is not a Cauchy sequence in $(P_1(X), \|.\|_{\Lambda_1})$. Let $\varepsilon > 0$. For each $n, m \in \mathbb{N}$ such that $n \geqslant m$, $A^{-1}(x^{(n)}) = (1, -2, 2, \ldots, \mp 2, \mp 1, 0, \ldots)$ and

$$|A^{-1}(x^{(n)}) - A^{-1}(x^{(m)})| = (0, \ldots, 0, 1, 2, \ldots, 2, 1, 0, \ldots).$$

so that

$$\|A|A^{-1}(x^{(n)} - A^{-1}(x^{(m)})|\|_p \geqslant \|(0, \ldots, 0, 1-\varepsilon, 2-\varepsilon, \ldots, 2-\varepsilon, 1-\varepsilon, 0, \ldots)\|_p \geqslant 1,$$

for some $\varepsilon \in (0,1)$. $\square$

We wonder if $(X,\mathscr{P})$, where $\mathscr{P}$ is the product topology on $\prod_{k=0}^{\infty}\Lambda_k$ (and so is not the same $\mathscr{P}$ as in section 1), is normable? We think not, although we cannot prove it. But we can prove that the obvious norm on X, $\|.\|_X$ defined by

$$\|(x^{(k)})\|_X = \|P_0((x^{(k)}))\|_{\Lambda_0} = \|x^{(0)}\|_p,$$

for all $(x^{(k)})\in X$, does not give us the topology on X induced by the product topology on X as a subset $\prod_{k=0}^{\infty}\Lambda_k$. By the proof of Proposition 15, we can find a sequence in X which is Cauchy with respect to $\|.\|_X$ which does not converge, in the topology defined by $\|.\|_X$, to any element of X, so $(X,\|.\|_X)$ is not complete, whereas $(X,\mathscr{P})$, by previous remarks, is complete.

4. A generalization of the space X

We can generalize the space X for an infinite lower triangular matrix of Cesàro type

$$A=\begin{bmatrix} a_1 & 0 & 0 & . & . & . & . \\ a_2 & a_2 & 0 & . & . & . & . \\ a_3 & a_3 & a_3 & 0 & . & . & . \\ a_4 & a_4 & a_4 & a_4 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix},\ (a_k>0\ \ \forall k\in\mathbb{N})$$

and a solid sequence subspace λ of $\mathbb{F}^{\mathbb{N}}$ such that $A\lambda\subset\lambda$. We need to require the property $A\lambda\subset\lambda$ since a generalized Cesàro matrix may not multiply λ into λ as shown in Example 7. We can write A as a product,

$$A=\begin{bmatrix} a_1 & 0 & 0 & . & . & . \\ 0 & a_2 & 0 & . & . & . \\ 0 & 0 & a_3 & 0 & . & . \\ 0 & 0 & 0 & a_4 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & . & . & . \\ 1 & 1 & 1 & 0 & . & . \\ 1 & 1 & 1 & 1 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}.$$

So the inverse of the matrix A is the following infinite lower triangular matrix with columns in c_{00},

$$A^{-1}=\begin{bmatrix} a_1^{-1} & 0 & 0 & . & . & . & . \\ -a_1^{-1} & a_2^{-1} & 0 & . & . & . & . \\ 0 & -a_2^{-1} & a_3^{-1} & 0 & . & . & . \\ 0 & 0 & -a_3^{-1} & a_4^{-1} & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}\ (a_k>0\ \ \forall k\in\mathbb{N}).$$

Let us denote the space X derived from A and λ as the space X in the previous section was derived from the Cesàro matrix and l_p by $X(A,\lambda)$. By using Proposition 13, it follows that

$$X_{c_{00}} = \{(x, A^{-1}x, A^{-2}x, \dots) : x \in c_{00}\} \subset X(A,\lambda).$$

Therefore, $X(A,\lambda)$ is non-trivial.

Finally, we put some problems for further study on the spaces $X = X(A,\lambda)$.

PROBLEM 1. Let $(x^{(j)}) \in X \setminus \{0\}$. Can there be an element $(y^{(j)}) \in X$ such that $|x_k^{(j)}| \leqslant y_k^{(j)}$ for each $j \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$?

PROBLEM 2. A Hausdorff locally convex topological vector space is called *normable* if and only if it has a bounded neighborhood of zero. Can it happen that X is normable in the product topology on $\prod_{k=0}^{\infty} \Lambda_k$?

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