Operators
and
Matrices
Volume 9, Number 4 (2015), 831–846 doi:10.7153/oam-09-48

DISK–CYCLICITY AND CODISK–CYCLICITY OF CERTAIN SHIFT OPERATORS

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(*Communicated by P.-Y. Wu*)

Abstract. In this paper we characterize the disk-cyclicity and codisk-cyclicity of the bilateral weighted shifts on Hilbert space $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n\in\mathbb{Z}}$ of positive invertible diagonal operators on a separable complex Hilbert space $\mathscr{K}$, respectively. At last, we establish similar results for the disk-cyclic and codisk-cyclic shift operator B on $L^2(\beta)$ defined by $Bf_j = f_{j-1}$, $j\in\mathbb{Z}$, where $\{f_j\}_{j\in\mathbb{Z}}$ is a basis of $L^2(\beta)$.

1. Introduction

Let $\mathscr{H}$ be an infinite dimensional separable complex Hilbert space, and $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all linear bounded operators on $\mathscr{H}$. For $T\in\mathscr{B}(\mathscr{H})$, T is called *hypercyclic* (respectively, *supercyclic*) provided there is some $x\in\mathscr{H}$ such that the orbit $Orb(T,x)=\{T^nx : n=0,1,2\cdots\}$ (respectively, the projective orbit $\{\lambda T^nx : \lambda\in\mathbb{C}, n=0,1,2\cdots\}$) is dense in $\mathscr{H}$. Hypercyclic and supercyclic operators have received considerable attention recently, especially since they arise in familiar classes of operators, for example, weighted shifts, composition operators, adjoints of weighted composition operators. For motivation, examples and background about linear dynamics, we refer the readers to the books [3] by Bayart and Matheron, [6] by Grosse-Erdmann and Manguillot and papers [5] and [14]–[16]. Now we introduce the definition of *disk-cyclicity* from the papers [1, 8, 10]. An operator $T\in\mathscr{B}(\mathscr{H})$ is *disk-cyclic* if there exists a vector $x\in\mathscr{H}$ such that the set $\{\alpha T^nx :\ \alpha\in\mathbb{C}, 0<|\alpha|\leqslant 1,\ n\geqslant 0\}$ is norm-dense in $\mathscr{H}$. In this case x is called a *disk-cyclic* vector for T. As a consequence, every *hypercyclic* operator is *disk-cyclic*, and every *disk-cyclic* operator is *supercyclic*. Besides, T is *disk-cyclic* if and only if for all non-empty open sets U, V, there exist $n\in\mathbb{N}$, $\alpha\in\mathbb{C}$ with $0<|\alpha|\leqslant 1$ such that $T^n(\alpha U)\cap V\neq\emptyset$.

For $T\in\mathscr{B}(\mathscr{H})$, we give the notation

$$\mathbb{D}C(T)=\{x\in\mathscr{H} :\ x \text{ is a } \textit{disk-cyclic} \text{ vector for } T\}. \tag{1.1}$$

Then $\mathbb{D}C(T)=\bigcap_k\left(\bigcup_{|\alpha|\geqslant 1}\bigcup_n T^{-n}(\alpha\mathscr{U}_k)\right)$, where $\{\mathscr{U}_k\}_{k=1}^{\infty}$ is a countable base for the topology on $\mathscr{H}$. Thus *a non-empty set of disk-cyclic vectors is a dense G_δ-set in $\mathscr{H}$*.

Mathematics subject classification (2010): 47A16, 47B37.
Keywords and phrases: Disk-cyclic, codisk-cyclic, weighted shifts.
The second author is the corresponding author. The authors were supported in part by the National Natural Science Foundation of China (Grant Nos. 11371276; 11301373; 11401426).

Paper OaM-09-48

As we all know, Kitai found the Hypercyclicity Criterion in her Ph.D dissertation [11] that ensures a linear operator to be hypercyclic. A few years later it was rediscovered by Gethner and Shapiro [4]. Whether this criterion was equivalent to hypercyclicity was an open problem for many years. Bayart and Matheron [2] showed that the equivalence fails on classical Banach spaces, and even on a Hilbert space. In 2002, Jamil [8] proposed the Disk-Cyclicity Criterion in Hilbert space to show the existence of some disk-cyclic operators, which is similar to the Supercyclicity Criterion. Besides, another criterion for disk-cyclic operators—Three Open Set's Conditions has been proved by Jamil and Helal in [9]. Next we begin by recalling these two sufficient conditions for *disk-cyclic* operators.

PROPOSITION 1.1. [10, Proposition 2.4] or [9] (Three Open Set's Conditions) *Let $T \in \mathscr{B}(\mathscr{H})$, U, V be non-empty open sets in $\mathscr{H}$ and W be a neighborhood of zero in $\mathscr{H}$. If there are $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ with $0 < |\alpha| \leqslant 1$ such that $T^n(\alpha U) \cap W \neq \emptyset$ and $T^n(\alpha W) \cap V \neq \emptyset$, then T is disk-cyclic.*

PROPOSITION 1.2. [10, Proposition 2.5] (Disk-Cyclicity Criterion) *Let $T \in \mathscr{B}(\mathscr{H})$ such that*

(1) *There are dense sets $\mathscr{X}, \mathscr{Y}$ in $\mathscr{H}$ and right inverse to T (not necessary bounded) S such that $S(\mathscr{Y}) \subset \mathscr{Y}$ and $TS = I_{\mathscr{Y}}$.*

(2) *There is a sequence $(n_k) \subset \mathbb{N}$ such that*

(a) $\lim\limits_{k\to\infty} \|S^{n_k}y\| = 0$ *for all $y \in \mathscr{Y}$;*

(b) $\lim\limits_{k\to\infty} \|T^{n_k}x\|\|S^{n_k}y\| = 0$ *for all $x \in \mathscr{X}$, $y \in \mathscr{Y}$.*

Then T is disk-cyclic.

The equivalent characterizations for the *disk-cyclic* forward weighted shift were obtained in [10].

PROPOSITION 1.3. [10, Theorem 3.1] *Let T be a forward weighted shift with weight sequence $\{w_n\}_{n\in\mathbb{Z}}$ acting on the sequence space $\ell^2(\mathbb{Z})$. Then the following statements are equivalent:*

(1) *T is disk-cyclic.*

(2) *For all $q \in \mathbb{N}$,*

(a) $\limsup\limits_{n\to\infty} \min\left\{\prod_{h-n}^{h-1} w_k : |h| \leqslant q\right\} = \infty;$

(b) $\liminf\limits_{n\to\infty} \max\left\{\dfrac{\prod_{j}^{j+n-1} w_k}{\prod_{h-n}^{h-1} w_k} : |h|, |j| \leqslant q\right\} = 0.$

(3) *T satisfies the Disk-Cyclicity Criterion.*

For other related conclusions about weighted shifts, the readers can consult, for example, [13, 18]. Here we cite a definition which will be used to show the *conjugacy* of the backward bilateral shift and forward bilateral shift and so on.

DEFINITION 1.4. [6, Definition 1.5] Let $S : Y \to Y$ and $T : X \to X$ be dynamical systems.

(a) Then T is called *quasiconjugate* to S if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$.

(b) If ϕ can be chosen to be a homeomorphism, then S and T are called *conjugate*.

Note that *hypercyclicity* and *supercyclicity* are preserved under *quasiconjugacy*. Similarly the following proposition holds.

PROPOSITION 1.5. *Disk-cyclicity (Codisk-cyclicity) for an operator $T \in \mathscr{B}(\mathscr{H})$ is preserved under quasiconjugacy.*

The purpose of the current paper is to investigate some equivalent characterizations for the *disk-cyclic* and *codisk-cyclic* bilateral weighted shifts acting on the Hilbert space $\ell^2(\mathbb{Z},\mathscr{K})$. For the information about the space $\ell^2(\mathbb{Z},\mathscr{K})$, we refer the readers to the paper [7]. For the completeness we introduce it in detail.

As usual, $\mathbb{Z}$ is the set of all integers. Let $\mathscr{K}$ be a separable complex Hilbert space with an orthonormal basis $\{f_k\}_{k=0}^{\infty}$. Define a separable Hilbert space

$$\ell^2(\mathbb{Z},\mathscr{K}) = \{x = (\cdots, x_{-1}, [x_0], x_1, \cdots) : x_i \in \mathscr{K} \text{ and } \sum_{i\in\mathbb{Z}} \|x_i\|^2 < \infty\},$$

under the inner product

$$\langle x, y\rangle = \sum_{i\in\mathbb{Z}} \langle x_i, y_i\rangle_{\mathscr{K}},$$

where $x = (\cdots, x_{-1}, [x_0], x_1, \cdots)$ and $y = (\cdots, y_{-1}, [y_0], y_1, \cdots)$ for $x_i,\ y_i \in \mathscr{K}$.

Let $\{A_n\}$ $(n \in \mathbb{Z})$ be a uniformly bounded sequence of invertible positive diagonal operators on $\mathscr{K}$. We define two bilateral weighted shifts on $\ell^2(\mathbb{Z},\mathscr{K})$.

(i) The forward bilateral weighted shift T on $\ell^2(\mathbb{Z},\mathscr{K})$ is defined by

$$T(\cdots, x_{-1}, [x_0], x_1, \cdots) = (\cdots, A_{-2}x_{-2}, [A_{-1}x_{-1}], A_0x_0, \cdots).$$

Since $\{A_n\}_{n=-\infty}^{\infty}$ is uniformly bounded, then $\|T\| = \sup_{i\in\mathbb{Z}} \|A_i\| < \infty$. Moreover,

$$T^n(\cdots, x_{-1}, [x_0], x_1, \cdots) = (\cdots, y_{-1}, [y_0], y_1, \cdots),\ n \in \mathbb{N},$$

where

$$y_{n+j} = \prod_{s=0}^{n-1} A_{j+s}x_j \ \text{ or } \ y_j = \prod_{s=0}^{n-1} A_{j+s-n}x_{j-n}. \tag{1.2}$$

Hence

$$\|T^n\| = \sup_j \|\prod_{s=0}^{n-1} A_{j+s}\| = \sup_j \|\prod_{s=j}^{j+n-1} A_s\|. \tag{1.3}$$

If $\{A_n^{-1}\}_{n=-\infty}^{\infty}$ is also uniformly bounded, then T^{-1} is the backward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ as below

$$T^{-1}(\cdots, x_{-1}, [x_0], x_1, \cdots) = (\cdots, A_{-1}^{-1}x_0, [A_0^{-1}x_1], A_1^{-1}x_2, \cdots). \tag{1.4}$$

Then

$$T^{-n}(\cdots,x_{-1},[x_0],x_1,\cdots)=(\cdots,z_{-1},[z_0],z_1,\cdots),\ n\in\mathbb{N},$$

where

$$z_{j-n}=\prod_{s=0}^{n-1}A_{j-n+s}^{-1}x_j \text{ or } z_j=\prod_{s=0}^{n-1}A_{j+s}^{-1}x_{n+j}.$$

It implies that

$$\|T^{-n}\|=\sup_j\|\prod_{s=0}^{n-1}A_{j-n+s}^{-1}\|=\sup_j\|\prod_{s=j-n}^{j-1}A_s^{-1}\|. \tag{1.5}$$

(ii) The backward bilateral weighted shift T on $\ell^2(\mathbb{Z},\mathscr{K})$ is given by

$$T(\cdots,x_{-1},[x_0],x_1,\cdots)=(\cdots,A_{-1}x_0,[A_0x_1],A_1x_2,\cdots).$$

Then

$$T^n(\cdots,x_{-1},[x_0],x_1,\cdots)=(\cdots,y_{-1},[y_0],y_1,\cdots),$$

where

$$y_j=\prod_{s=0}^{n-1}A_{j+s}x_{n+j} \text{ or } y_{j-n}=\prod_{s=0}^{n-1}A_{j-n+s}x_j. \tag{1.6}$$

Thus

$$\|T^n\|=\sup_j\|\prod_{s=0}^{n-1}A_{j-n+s}\|=\sup_j\|\prod_{s=j-n}^{j-1}A_s\|. \tag{1.7}$$

Further, if $\{A_n^{-1}\}_{n=-\infty}^{\infty}$ is also uniformly bounded, then T^{-1} is the forward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ as follows

$$T^{-n}(\cdots,x_{-1},[x_0],x_1,\cdots)=(\cdots,z_{-1},[z_0],z_1,\cdots),$$

where

$$z_j=\prod_{s=0}^{n-1}A_{j+s-n}^{-1}x_{j-n} \text{ or } z_{n+j}=\prod_{s=0}^{n-1}A_{j+s}^{-1}x_j. \tag{1.8}$$

It yeilds that

$$\|T^{-n}\|=\sup_j\|\prod_{s=0}^{n-1}A_{j+s}^{-1}\|=\sup_j\|\prod_{s=j}^{j+n-1}A_s^{-1}\|. \tag{1.9}$$

Since each A_n is an invertible diagonal operator on $\mathscr{K}$, we conclude that

$$\|A_n\|=\sup_k\|A_nf_k\|,\ \ \|A_n^{-1}\|=\sup_k\|A_n^{-1}f_k\| \text{ and } \sup_k\|A_nf_k\|=\frac{1}{\inf_k\|A_n^{-1}f_k\|}. \tag{1.10}$$

2. Disk-cyclic weighted shifts

In this section, we are concerned with the equivalent conditions for the disk-cyclic weighted shifts. For a better understanding, we cite the *supercyclicity* of the bilateral weighted shifts on $\ell^2(\mathbb{Z},\mathscr{K})$.

THEOREM 2.1. (1) [12, Theorem 3.2] *Let T be a forward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathscr{K}$. Then T is supercyclic if and only if for every $q\in\mathbb{N}$,*

$$\liminf_{n\to\infty}\max\{\|\prod_{k=j}^{j+n-1}A_k\|\|\prod_{k=h-n}^{h-1}A_k^{-1}\|:\ |j|,|h|\leqslant q\}=0. \tag{2.1}$$

(2) [12, Theorem 3.3] *Let T be a backward bilateral operator weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operator on $\mathscr{K}$. Then T is supercyclic if and only if for every $q\in\mathbb{N}$,*

$$\liminf_{n\to\infty}\max\{\|\prod_{k=j-1}^{j-n}A_k\|\|\prod_{k=h}^{h+n-1}A_k^{-1}\|:\ |j|,|h|\leqslant q\}=0.$$

The following theorem describes the disk-cyclic forward bilateral weighted shifts on $\ell^2(\mathbb{Z},\mathscr{K})$.

THEOREM 2.2. *Let T be a forward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathscr{K}$. Then the following statements are equivalent:*

(1) T is disk-cyclic.

(2) For all $q\in\mathbb{N}$,

(a) $\liminf_{n\to\infty}\max\left\{\|\prod_{k=j-n}^{j-1}A_k^{-1}\|,\ |j|\leqslant q\right\}=0;$

(b) $\liminf_{n\to\infty}\max\left\{\|\prod_{k=j}^{j+n-1}A_k\|\|\prod_{s=h-n}^{h-1}A_s^{-1}\|,\ |h|,|j|\leqslant q\right\}=0.$

(3) T satisfies the Disk-Cyclicity Criterion.

Proof. $(3)\Rightarrow(1)$. This implication is obvious.

$(1)\Rightarrow(2)$. Suppose T is disk-cyclic. Let $\varepsilon>0$, $q\in\mathbb{N}$ be given. Choose $\delta>0$ such that $\delta/(1-\delta)<\varepsilon$. For an arbitrary fixed non-negative integer i, consider the vector $z=(\cdots,z_{-1},[z_0],z_1,\cdots)\in\ell^2(\mathbb{Z},\mathscr{K})$ defined by $z_j=f_i$ if $|j|\leqslant q$ and $z_j=0$ if $|j|>q$.

Since T is disk-cyclic, by the density of the disk-cyclic vectors, there exist a vector $x=(\cdots,x_{-1},[x_0],x_1,\cdots)$ and a complex number α with $0<|\alpha|\leqslant 1$ such that the set $\{\alpha T^n x:\ \alpha\in\mathbb{C},0<|\alpha|\leqslant 1,\ n\geqslant 0\}$ is norm-dense in $\ell^2(\mathbb{Z},\mathscr{K})$ and

$$\|x-z\|<\delta. \tag{2.2}$$

Meanwhile, there exists a subsequence of $\{\alpha T^n x\}_n$ converging to z. Then we can choose a positive integer $n > 2q$ such that

$$\|\alpha T^n x - z\| < \delta, \tag{2.3}$$

where $T^n x = (\cdots, y_{-1}, [y_0], y_1, \cdots)$. It implies that

$$\begin{cases} \|x_j\| < \delta, & \text{for all } |j| > q; \\ \|x_j - f_i\| < \delta, & \text{for all } |j| \leqslant q. \end{cases}$$

$$\begin{cases} \|\alpha y_j\| < \delta, & \text{if } |j| > q; \\ \|\alpha y_j - f_i\| < \delta, & \text{if } |j| \leqslant q. \end{cases}$$

Since $x \in \ell^2(\mathbb{Z}, \mathscr{K})$, there exist scalars $\alpha_k^{(j)}$ such that $x_j = \sum_{k=0}^{\infty} \alpha_k^{(j)} f_k$. Then

$$\begin{cases} |\alpha_k^{(j)}| < \delta, & |j| > q \text{ and for all } k; \\ |\alpha_k^{(j)}| < \delta, & |j| \leqslant q \text{ and for } k \neq i; \\ |\alpha_k^{(j)}| > 1 - \delta, & |j| \leqslant q \text{ and for } k = i. \end{cases} \tag{2.4}$$

From the hypothesis $n > 2q$, we verify that $j + n > q$ for all $|j| \leqslant q$ and hence it follows that

$$\|\alpha y_{j+n}\| < \delta, \text{ for all } |j| \leqslant q.$$

By (1.2),

$$y_{j+n} = \prod_{s=0}^{n-1} A_{j+s} x_j = \sum_{k=0}^{\infty} \alpha_k^{(j)} \prod_{s=0}^{n-1} A_{j+s} f_k.$$

That is,

$$|\alpha||\alpha_k^{(j)}|\|\prod_{s=0}^{n-1} A_{j+s} f_k\| < \delta, \text{ for all } k \text{ and } |j| \leqslant q.$$

If $k = i$, by the third inequality in (2.4), it yields that

$$|\alpha|\|\prod_{s=0}^{n-1} A_{j+s} f_i\| < \frac{\delta}{1-\delta}, \text{ for all } |j| \leqslant q.$$

Thus

$$\|\prod_{s=j}^{j+n-1} A_s\| = \sup_i \|\prod_{s=0}^{n-1} A_{j+s} f_i\| \leqslant \frac{\delta}{|\alpha|(1-\delta)}, \quad \text{for all } |j| \leqslant q. \tag{2.5}$$

Again from (1.2), it leads that

$$y_j = \prod_{s=0}^{n-1} A_{j+s-n} x_{j-n} = \sum_{k=0}^{\infty} \alpha_k^{(j-n)} \prod_{s=0}^{n-1} A_{j+s-n} f_k.$$

Then by (2.3), we derive that

$$\begin{cases} (i)\ \ \|\alpha\alpha_k^{(j-n)}\prod_{s=0}^{n-1}A_{j+s-n}f_k - f_k\| < \delta, \text{ for } k=i; \\ (ii)\ \ \|\alpha\alpha_k^{(j-n)}\prod_{s=0}^{n-1}A_{j+s-n}f_k\| < \delta, \qquad \text{for } k \neq i. \end{cases} \tag{2.6}$$

Since $n > 2q$, it holds that $|j-n| > q$ for all $|j| \leqslant q$. Employing the second inequality in (2.4), we conclude that $|\alpha_k^{(j-n)}| < \delta$ for all k. If $k = i$,

$$|\alpha_i^{(j-n)}| < \delta, \text{ for all } |j| \leqslant q. \tag{2.7}$$

Rewrite (2.6) (i) as

$$|\alpha||\alpha_i^{(j-n)}|\|\prod_{s=0}^{n-1}A_{j+s-n}f_i\| > 1-\delta, \text{ for all } |j| \leqslant q. \tag{2.8}$$

Combining (2.7) and (2.8), it implies that

$$|\alpha|\|\prod_{k=j-n}^{j-1}A_k f_i\| > \frac{1-\delta}{\delta}, \text{ for all } |j| \leqslant q.$$

Since the above inequality holds for an arbitrary i, then

$$\inf_i \|\prod_{k=j-n}^{j-1}A_k f_i\| \geqslant \frac{1-\delta}{|\alpha|\delta}, \text{ for all } |j| \leqslant q,$$

and further by the third equation in (1.10), it follows that

$$\sup_i \|\prod_{k=j-n}^{j-1}A_k^{-1} f_i\| = \frac{1}{\inf_i \|\prod_{k=j-n}^{j-1}A_k f_i\|} \leqslant \frac{|\alpha|\delta}{1-\delta}, \text{ for all } |j| \leqslant q.$$

Therefore

$$\|\prod_{k=j-n}^{j-1}A_k^{-1}\| \leqslant \frac{|\alpha|\delta}{1-\delta} \leqslant \frac{\delta}{1-\delta} < \varepsilon, \text{ for all } |j| \leqslant q. \tag{2.9}$$

Consequently, multiplying (2.5) and (2.9), we formulate

$$\|\prod_{k=j}^{j+n-1}A_k\|\|\prod_{s=h-n}^{h-1}A_s^{-1}\| < \frac{\delta}{|\alpha|(1-\delta)}\frac{|\alpha|\delta}{1-\delta} < \varepsilon^2, \quad \text{for all } \ |h|,|j| \leqslant q.$$

Considering the above two inequalities and ε is arbitrary, we obtain (2).

$(2) \Rightarrow (3)$. Denote

$$\alpha(n,q) = \max\left\{\|\prod_{k=j-n}^{j-1}A_k^{-1}\|, \ |j| \leqslant q\right\},$$

$$\beta(n,q) = \max\left\{\|\prod_{k=j}^{j+n-1}A_k\|\|\prod_{s=h-n}^{h-1}A_s^{-1}\|, \ |h|,|j| \leqslant q\right\}.$$

By the condition (2) and the definition of $\liminf_{n\to\infty}$, there exist integers $1 \leqslant n_1 < n_2 < n_3 < \cdots$ satisfying, for each $q \in \mathbb{N}$, that

$$\alpha(n_q, q) < \frac{1}{q} \quad \text{and} \quad \beta(n_q, q) < \frac{1}{q}. \tag{2.10}$$

For the above $q \in \mathbb{N}$, define

$$G_q = \{(x_j) \in \ell^2(\mathbb{Z}, \mathscr{K}) : x_j = 0 \text{ for } |j| > q\}.$$

Then choose $X_0 = Y_0 = \bigcup_{q\in\mathbb{N}} G_q$, which are dense subsets of $\ell^2(\mathbb{Z}, \mathscr{K})$. Define the mapping $S : Y_0 \to \ell^2(\mathbb{Z}, \mathscr{K})$ by

$$S(\cdots, x_{-1}, [x_0], x_1, \cdots) = (\cdots, A_{-1}^{-1}x_0, [A_0^{-1}x_1], A_1^{-1}x_2, \cdots),$$

and let $S_{n_k} = S^{n_k}$. It holds trivially that $S(Y_0) \subset Y_0$ and

$$TS = Id_{Y_0}.$$

For $g, h \in G_q$, by (1.3),(1.5) and (2.10), it yields that

$$\|T^{n_q} g\| \leqslant \max\left\{ \| \prod_{k=j}^{j+n_q-1} A_k \|, \ |j| \leqslant q \right\} \|g\|,$$

$$\begin{aligned} \|S_{n_q} h\| &\leqslant \max\left\{ \| \prod_{k=l-n_q}^{l-1} A_k^{-1} \|, \ |l| \leqslant q \right\} \|h\| \\ &= \alpha(n_q, q)\|h\| < \frac{1}{q}\|h\|. \end{aligned}$$

$$\begin{aligned} \|T^{n_q} g\|\|S_{n_q} h\| &\leqslant \max\left\{ \| \prod_{k=j}^{j+n_q-1} A_k \| \| \prod_{k=l-n_q}^{l-1} A_k^{-1} \|, \ |j| \leqslant q, \ |l| \leqslant q \right\} \|g\|\|h\| \\ &= \beta(n_q, q)\|g\|\|h\| < \frac{1}{q}\|g\|\|h\|. \end{aligned}$$

Combining the above two inequalities, we prove that

$$\lim_{q\to\infty} \|T^{n_q} g\|\|S_{n_q} h\| = 0 \quad \text{and} \quad \lim_{q\to\infty} \|S_{n_q} h\| = 0.$$

That is, the operator T satisfies Proposition 1.2 with respect to the sequence $\{n_q\}_q$, then we get (3). This proves the desired results. □

We further have the following corollary.

COROLLARY 2.3. *Let T be a disk-cyclic forward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathscr{K}$. Then there is a sequence $\{n_r\}\subset\mathbb{N}$ such that*

$$\lim_{r\to\infty}\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|=0 \text{ and } \lim_{r\to\infty}\|\prod_{k=1}^{n_r}A_k\|\|\prod_{s=1}^{n_r}A_{-s}^{-1}\|=0. \tag{2.11}$$

Proof. Take $q=0$ in Theorem 2.2 (2), then $j=h=0$. Hence

$$\liminf_{n\to\infty}\|\prod_{k=-n}^{-1}A_k^{-1}\|=0 \text{ and } \liminf_{n\to\infty}\|\prod_{k=0}^{n-1}A_k\|\|\prod_{k=-n}^{-1}A_k^{-1}\|=0.$$

In particular, there is a sequence $\{n_r\}\subset\mathbb{N}$ such that

$$\lim_{r\to\infty}\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|=0 \text{ and } \lim_{r\to\infty}\|\prod_{k=0}^{n_r-1}A_k\|\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|=0.$$

Let $\varepsilon>0$, then there is $N>0$ such that for all $r>N$,

$$\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|<\varepsilon \text{ and } \|\prod_{k=0}^{n_r-1}A_k\|\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|<\varepsilon\frac{1}{\|A_0^{-1}\|\|A_{n_r}\|}.$$

That is,

$$\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|<\varepsilon \text{ and } \|\prod_{k=1}^{n_r}A_k\|\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|<\varepsilon.$$

Consequently, (2.11) is true. □

THEOREM 2.4. *Let T be a forward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathscr{K}$ and $\{A_n^{-1}\}$ is also a uniformly bounded sequence. Then the following statements are equivalent:*

(1) T is disk-cyclic.

(2) There is a sequence $\{n_r\}\subset\mathbb{N}$, $n_r\to\infty$, such that

(a) $\lim_{r\to\infty}\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|=0$;

(b) $\lim_{r\to\infty}\|\prod_{k=1}^{n_r}A_k\|\|\prod_{s=1}^{n_r}A_{-s}^{-1}\|=0$.

Proof. $(1)\Rightarrow(2)$. This implication follows from Corollary 2.3.

$(2)\Rightarrow(1)$. Suppose that (2) holds, we will require Theorem 2.2 to show (1). Denote $M:=\max\{\sup_{n\in\mathbb{Z}}\|A_n\|,\ \sup_{n\in\mathbb{Z}}\|A_n^{-1}\|\}<\infty$. Let $\varepsilon>0$ and $q\in\mathbb{N}$. For a $\delta>0$ (we will prescribe δ later), there is an arbitrarily large n_r such that

$$\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|<\delta \text{ and } \|\prod_{k=1}^{n_r}A_k\|\|\prod_{s=1}^{n_r}A_{-s}^{-1}\|<\delta. \tag{2.12}$$

Let $n=n_r+q+2$ and $h\in\mathbb{Z}$ with $|h|\leqslant q$. Then $n_r+1\leqslant n-h$. Employing (2.12), we conclude that

$$\|\prod_{h-n}^{h-1}A_k^{-1}\|=\|\prod_{1-h}^{n-h}A_{-k}^{-1}\|\leqslant M_h\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|\|\prod_{n_r+1}^{n-h}A_{-k}^{-1}\|, \tag{2.13}$$

here $\{M_h\}$ is a finite collection of positive constants depending only on q. Since $n-h-(n_r+1)\leqslant 2q+1$, then

$$\|\prod_{n_r+1}^{n-h}A_{-k}^{-1}\|\leqslant M^{2q+1}.$$

It follows from (2.13) that

$$\|\prod_{h-n}^{h-1}A_k^{-1}\|\leqslant M_hM^{2q+1}\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|. \tag{2.14}$$

Let $M_0=\max\{M_h:|h|\leqslant q\}$, by (2.12), we deduce that

$$\|\prod_{h-n}^{h-1}A_k^{-1}\|\leqslant M_0M^{2q+1}\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|\leqslant M_0M^{2q+1}\delta. \tag{2.15}$$

On the other hand, let $j\in\mathbb{Z}$ with $|j|\leqslant q$. Then $j+n-1\geqslant n_r+1>n_r\geqslant 1$. Thus

$$\|\prod_{k=j}^{j+n-1}A_k\|\leqslant \widetilde{M}_j\|\prod_{k=1}^{n_r}A_k\|\|\prod_{n_r+1}^{j+n-1}A_k\|,$$

where $\{\widetilde{M}_j\}$ is a finite collection of positive constants depending only on q. Since $j\leqslant q$, then $j+n-1-(n_r+1)\leqslant 2q$ and

$$\|\prod_{n_r+1}^{j+n-1}A_k\|\leqslant M^{2q}.$$

Let $\widetilde{M}_0=\max\{\widetilde{M}_j:|j|\leqslant q\}$, consequently,

$$\|\prod_{k=j}^{j+n-1}A_k\|\leqslant \widetilde{M}_jM^{2q}\|\prod_{k=1}^{n_r}A_k\|\leqslant \widetilde{M}_0M^{2q}\|\prod_{k=1}^{n_r}A_k\|. \tag{2.16}$$

Combining (2.12), (2.14) and (2.16), for $j,h\in\mathbb{Z}$ with $|j|,\ |h|\leqslant q$, we show that

$$\begin{aligned}\|\prod_{k=j}^{j+n-1}A_k\|\|\prod_{s=h-n}^{h-1}A_s^{-1}\|&=\|\prod_{k=j}^{j+n-1}A_k\|\|\prod_{s=1-h}^{n-h}A_{-s}^{-1}\|\\&\leqslant M_0\widetilde{M_0}M^{2q+1}M^{2q}\|\prod_{k=1}^{n_r}A_{-k}^{-1}\|\|\prod_{k=1}^{n_r}A_k\|<\alpha\delta,\end{aligned} \tag{2.17}$$

where $\alpha = M_0\widetilde{M}_0M^{4q+1}$ is a constant depending only on q. Let

$$\delta = \min\left\{\frac{\varepsilon}{\alpha}, \frac{\varepsilon}{M_0M^{2q+1}}\right\}.$$

Observing from (2.15) and (2.17), we arrive at

(a) $\liminf_{n\to\infty} \max\left\{\|\prod_{k=j-n}^{j-1} A_k^{-1}\|, |j| \leqslant q\right\} = 0;$

(b) $\liminf_{n\to\infty} \max\left\{\|\prod_{k=j}^{j+n-1} A_k\|\|\prod_{s=h-n}^{h-1} A_s^{-1}\|, |h|,|j| \leqslant q\right\} = 0.$

An application of Theorem 2.2 tells us that T is disk-cyclic. The proof of the theorem is complete. $\square$

Through the mapping $\phi : \ell^2(\mathbb{Z},\mathscr{K}) \to \ell^2(\mathbb{Z},\mathscr{K}), (x_n) \to (x_{-n})$, it turns out that the backward bilateral shift T with weight sequence $\{A_n\}$ is *conjugate* to a forward bilateral shift S with weight sequence $\{A_{-n-1}\}$. We get down to the details as follows:

$$\begin{aligned} & S\circ\phi(\cdots,x_{-1},[x_0],x_1,\cdots) \\ = & S(\cdots,x_1,[x_0],x_{-1},\cdots) \\ = & (\cdots,A_1x_2,[A_0x_1],A_{-1}x_0,\cdots). \\ & \phi\circ T(\cdots,x_{-1},[x_0],x_1,\cdots) \\ = & \phi(\cdots,A_{-1}x_0,[A_0x_1],A_1x_2,\cdots) \\ = & (\cdots,A_1x_2,[A_0x_1],A_{-1}x_0,\cdots). \end{aligned}$$

That is, $S\circ\phi = \phi\circ T$. Hence the characterizations for the disk-cyclic (codisk-cyclic) forward bilateral shift S with weight sequence $\{A_{-n-1}\}$ can be applied to the disk-cyclicity (codisk-cyclicity) of the backward bilateral shift T with weight sequence $\{A_n\}$. By Proposition 1.5, Theorems 2.5 and 2.6 can be deduced from Theorems 2.2 and 2.4, just replacing the weight sequence $\{A_n\}$ by $\{A_{-n-1}\}$, respectively. So we omit the details here.

THEOREM 2.5. *Let T be a backward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operator on $\mathscr{K}$. Then the following statements are equivalent:*

(1) T is disk-cyclic.

(2) For all $q\in\mathbb{N}$,

(a) $\liminf_{n\to\infty} \max\{\|\prod_{k=j}^{j+n-1} A_k^{-1}\| : |j| \leqslant q\} = 0;$

(b) $\liminf_{n\to\infty} \max\{\|\prod_{k=j-n}^{j-1} A_k\|\|\prod_{k=h}^{h+n-1} A_k^{-1}\|, |j|,|h| \leqslant q\} = 0.$

(3) T satisfies the Disk-Cyclicity Criterion.

THEOREM 2.6. *Let T be a backward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operator on $\mathscr{K}$ and $\{A_n^{-1}\}$ is also a uniformly bounded sequence. Then the following statements are equivalent:*

(1) T is disk-cyclic.

(2) There is a sequence $\{n_r\} \subset \mathbb{N}$, $n_r \to \infty$, *such that*

(a) $\lim\limits_{r\to\infty} \|\prod_{k=1}^{n_r} A_k^{-1}\| = 0$;

(b) $\lim\limits_{r\to\infty} \|\prod_{k=1}^{n_r} A_{-k}\| \|\prod_{k=1}^{n_r} A_k^{-1}\| = 0$.

3. Codisk-cyclic weighted shifts

In this section, we turn our attention to study the *codisk-cyclic* bilateral weighted shifts on $\ell^2(\mathbb{Z}, \mathscr{K})$. An operator $T \in \mathscr{B}(\mathscr{H})$ is *codisk-cyclic* if there exists $x \in \mathscr{H}$ such that

$$\{\alpha T^n x : \alpha \in \mathbb{C}, |\alpha| \geqslant 1, n \geqslant 0\} \text{ is norm-dense in } \mathscr{H},$$

and x is said to be a *codisk-cyclic* vector for T. Clearly every *codisk-cyclic* operator is *supercyclic* as well. Also, T is *codisk-cyclic* if and only if for all non-empty open sets U, V, there exist $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant 1$ such that $T^n(\alpha U) \cap V \neq \emptyset$. Due to the fact $T^n(\alpha U) \cap V \neq \emptyset$ if and only if $U \cap T^{-n}\left(\frac{1}{\alpha}V\right) \neq \emptyset$, therefore an inverse operator T is *codisk-cyclic* if and only if T^{-1} is *disk-cyclic*. Analogously, the *Codisk-Cyclicity Criterion* is the main ingredient to discover *codisk-cyclic* operators, which provides a sufficient condition for bounded linear operators to have *codisk-cyclic* vectors.

PROPOSITION 3.1. [8] (Codisk-Cyclicity Criterion) *Let* $T \in \mathscr{B}(\mathscr{H})$ *such that*

(1) *There are dense sets* $\mathscr{X}, \mathscr{Y}$ *in* $\mathscr{H}$ *and a right inverse to* T *(not necessarily bounded)* S *such that* $S(\mathscr{Y}) \subset \mathscr{Y}$ *and* $TS = I_{\mathscr{Y}}$.

(2) *There is a sequence* $\{n_k\} \subset \mathbb{N}$ *such that*

(*a*) $\lim\limits_{k\to\infty} \|T^{n_k}x\| = 0$ *for all* $x \in \mathscr{X}$;

(*b*) $\lim\limits_{k\to\infty} \|T^{n_k}x\|\|S^{n_k}y\| = 0$ *for all* $x \in \mathscr{X}$, $y \in \mathscr{Y}$.

Then T *is codisk-cyclic.*

Depending on Proposition 3.1, we can arrive at the following theorems corresponding to ones in Section 2.

THEOREM 3.2. *Let* T *be a forward bilateral weighted shift on* $\ell^2(\mathbb{Z}, \mathscr{K})$ *with weight sequence* $\{A_n\}_{n=-\infty}^{\infty}$, *where* $\{A_n\}$ *is a uniformly bounded sequence of positive invertible diagonal operators on* $\mathscr{K}$. *Then the following statements are equivalent:*

(1) T *is codisk-cyclic.*

(2) For all $q \in \mathbb{N}$,

(a) $\liminf\limits_{n\to\infty} \max\left\{\|\prod_{k=j}^{j+n-1} A_k\|,\ |j| \leqslant q\right\} = 0$;

(b) $\liminf\limits_{n\to\infty} \max\left\{\|\prod_{k=j}^{j+n-1} A_k\|\|\prod_{s=h-n}^{h-1} A_s^{-1}\|,\ |h|, |j| \leqslant q\right\} = 0$.

(3) T *satisfies the Codisk-Cyclicity Criterion.*

THEOREM 3.3. *Let* T *be a forward bilateral weighted shift on* $\ell^2(\mathbb{Z}, \mathscr{K})$ *with weight sequence* $\{A_n\}_{n=-\infty}^{\infty}$, *where* $\{A_n\}$ *is a uniformly bounded sequence of positive invertible diagonal operators on* $\mathscr{K}$ *and* $\{A_n^{-1}\}$ *is also a uniformly bounded sequence. Then the following statements are equivalent:*

(1) T is codisk-cyclic.

(2) There is a sequence $\{n_r\} \subset \mathbb{N}$, $n_r \to \infty$, *such that*

(a) $\lim\limits_{r\to\infty} \|\prod_{k=1}^{n_r} A_k\| = 0;$

(b) $\lim\limits_{r\to\infty} \|\prod_{k=1}^{n_r} A_k\|\|\prod_{s=1}^{n_r} A_{-s}^{-1}\| = 0.$

THEOREM 3.4. *Let T be a backward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operator on $\mathscr{K}$. Then the following statements are equivalent:*

(1) T is codisk-cyclic.

(2) For all $q \in \mathbb{N}$,

$(a) \liminf\limits_{n\to\infty} \max\left\{\|\prod_{k=j-n}^{j-1} A_k\|,\ |j| \leqslant q\right\} = 0;$

$(b) \liminf\limits_{n\to\infty} \max\{\|\prod_{k=j-n}^{j-1} A_k\|\|\prod_{k=h}^{h+n-1} A_k^{-1}\|,\ |j|,|h| \leqslant q\} = 0.$

(3) T satisfies the Codisk-Cyclicity Criterion.

THEOREM 3.5. *Let T be a backward bilateral weighted shift on $\ell^2(\mathbb{Z},\mathscr{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operator on $\mathscr{K}$ and $\{A_n^{-1}\}$ is also a uniformly bounded sequence.Then the following statements are equivalent:*

(1) T is codisk-cyclic.

(2) There is a sequence $\{n_r\} \subset \mathbb{N}$, $n_r \to \infty$, *such that*

(a) $\lim\limits_{r\to\infty} \|\prod_{k=1}^{n_r} A_{-k}\| = 0;$

(b) $\lim\limits_{r\to\infty} \|\prod_{k=1}^{n_r} A_{-k}\|\|\prod_{k=1}^{n_r} A_k^{-1}\| = 0.$

4. Disk-cyclicity and codisk-cyclicity of B on $L^2(\beta)$

Given a sequence of positive numbers $\{\beta(n)\}_{n=-\infty}^{\infty}$ with $\beta(0)=1$, the space of formal Laurent series consists of the sequences $f=\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ such that

$$\|f\|^2 = \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\beta(n)^2 < \infty.$$

We will use the notation $f(z)=\sum_{n=-\infty}^{\infty}\hat{f}(n)z^n$ whether or not the series converges for any z. These are called formal Laurent series. As we all know, they are called formal power series and are denoted by $H^2(\beta)$, when n ranges on $\mathbb{N}\cup\{0\}$. The space $H^2(\beta)$ is a Hilbert space under the inner product

$$\langle f,g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2,$$

where $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ and $g(z)=\sum_{n=0}^{\infty}\hat{g}(n)z^n$.

Let $L^2(\beta)$ denote a Hilbert space endowed with the inner product

$$\langle f,g\rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2,$$

where $f(z)=\sum_{n=-\infty}^{\infty}\hat{f}(n)z^n$ and $g(z)=\sum_{n=-\infty}^{\infty}\hat{g}(n)z^n$. Moreover, the norm on the space $L^2(\beta)$ is $\|.\|_\beta$. Let $\hat{f}_k(n)=\delta_k(n)$, then $f_k(z)=z^k$ and $\{f_k\}_{k\in\mathbb{Z}}$ is a basis for $L^2(\beta)$ such that $\|f_k\|=\beta(k)$. The shift operator B on $L^2(\beta)$ is defined by

$$Bf_j=f_{j-1},\ \ j\in\mathbb{Z}.$$

A straightforward calculation shows that B is bounded if and only if the sequence $\{\beta(k)/\beta(k+1)\}_k$ is bounded. The *hereditarily hypercyclicity* of B on $L^p(\beta)$ $(p\geqslant 1)$ with respect to the *entire sequence* was discussed in [17, Theorem 2.7]. Recently, a companion result for the supercyclic shift operator B on $L^p(\beta)$ $(p\geqslant 1)$ was investigated as follows:

THEOREM 4.1. [12, Theorem 4.1] *Suppose the shift operator B is bounded on $L^p(\beta)$. Then B is supercyclic on $L^p(\beta)$ if and only if*

$$\liminf_{n\to\infty}\max\{\beta(k-n)\beta(j+n):\ \ |j|,|k|\leqslant q\}=0, \tag{4.1}$$

for all $q\in\mathbb{N}$.

Now we proceed to show the disk-cyclicity and codisk-cyclicity of the shift operator B on $L^2(\beta)$. On the one hand, the results below can be proved by Proposition 1.2 and Proposition 3.1 (this method is omitted here). On the other hand, we note that the shift operator B on $L^2(\beta)$ is *conjugate* to the backward bilateral weighted shift T with sequence $\left\{\frac{\beta_n}{\beta_{n+1}}\right\}$ on $\ell^2(\mathbb{Z},\mathbb{C})$ via the mapping $\phi:\ell^2(\mathbb{Z},\mathbb{C})\to L^2(\beta),(x(n))\to\left(\frac{x_n}{\beta_n}\right)$ as below:

$$\begin{aligned}
&B\circ\phi(\cdots,x_{-1},[x_0],x_1,\cdots)\\
&=B\left(\cdots,\frac{x_{-1}}{\beta_{-1}},\left[\frac{x_0}{\beta_0}\right],\frac{x_1}{\beta_1},\cdots\right)\\
&=\left(\cdots,\frac{x_0}{\beta_0},\left[\frac{x_1}{\beta_1}\right],\frac{x_2}{\beta_2},\cdots\right).\\
&\phi\circ T(\cdots,x_{-1},[x_0],x_1,\cdots)\\
&=\phi\left(\cdots,\frac{\beta_{-1}}{\beta_0}x_0,\left[\frac{\beta_0}{\beta_1}x_1\right],\frac{\beta_1}{\beta_2}x_2,\cdots\right)\\
&=\left(\cdots,\frac{x_0}{\beta_0},\left[\frac{x_1}{\beta_1}\right],\frac{x_2}{\beta_2},\cdots\right).
\end{aligned}$$

Thus $B\circ\phi=\phi\circ T$. Employing Proposition 1.5 and Theorem 2.5, we introduce a brief proof for the disk-cyclicity of the shift operator B on $L^2(\beta)$.

THEOREM 4.2. *Suppose the shift operator B is bounded on $L^2(\beta)$. Then the following statements are equivalent:*

(1) *B is disk-cyclic.*

(2) *For all $q\in\mathbb{N}$,*

(a) $\liminf_{n\to\infty} \max\{\beta(j+n): \ |j| \leqslant q\} = 0;$
(b) $\liminf_{n\to\infty} \max\{\beta(k-n)\beta(j+n): \ |j|,|k| \leqslant q\} = 0.$
(3) T satisfies the Disk-Cyclicity Criterion.

Proof. Replacing the weight A_k by $\frac{\beta_k}{\beta_{k+1}}$ in (2) of Theorem 2.5, it guarantees that 'For all $q \in \mathbb{N}$,
(a) $\liminf_{n\to\infty} \max\{\frac{\beta(j+n)}{\beta(j)} : |j| \leqslant q\} = 0;$
(b) $\liminf_{n\to\infty} \max\{\frac{\beta(j-n)}{\beta(j)} \frac{\beta(h+n)}{\beta(h)}, \ |j|,|h| \leqslant q\} = 0.$'.
Obviously, the above results are equivalent to
'For all $q \in \mathbb{N}$,
(a) $\liminf_{n\to\infty} \max\{\beta(j+n): \ |j| \leqslant q\} = 0;$
(b) $\liminf_{n\to\infty} \max\{\beta(k-n)\beta(j+n): \ |j|,|k| \leqslant q\} = 0.$'.
Therefore we derive the results from Theorem 2.5. This completes the proof. □

Next corollary is an application of Theorem 4.2.

COROLLARY 4.3. *Suppose the shift operator B is bounded on $L^2(\beta)$. If B is disk-cyclic on $L^2(\beta)$, then there is a sequence $\{n_r\} \subset \mathbb{N}$ such that*
(a) $\lim_{r\to\infty} \beta(n_r) = 0;$
(b) $\lim_{r\to\infty} \beta(-n_r)\beta(n_r) = 0.$

The equivalent descriptions for the *codisk-cyclicity* of B on $L^2(\beta)$ are also given below, which can be deduced from Theorem 3.4.

THEOREM 4.4. *Suppose the shift operator B is bounded on $L^2(\beta)$. Then the following statements are equivalent:*
(1) *B is codisk-cyclic.*
(2) *For all $q \in \mathbb{N}$,*
(a) $\liminf_{n\to\infty} \max\{\beta(k-n): \ |k| \leqslant q\} = 0;$
(b) $\liminf_{n\to\infty} \max\{\beta(k-n)\beta(j+n): \ |j|,|k| \leqslant q\} = 0.$
(3) T satisfies the Codisk-Cyclicity Criterion.

COROLLARY 4.5. *Suppose the shift operator B is bounded on $L^2(\beta)$. If B is codisk-cyclic on $L^2(\beta)$, then there is a sequence $\{n_r\} \subset \mathbb{N}$ such that*
(a) $\lim_{r\to\infty} \beta(-n_r) = 0;$
(b) $\lim_{r\to\infty} \beta(-n_r)\beta(n_r) = 0.$

Acknowledgements. The authors would like to thank the editor and referees for the useful comments and suggestions which improved the presentation of this paper.

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(Received July 5, 2014)

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