Operators and Matrices
Volume 9, Number 4 (2015), 847–851
doi:10.7153/oam-09-49

ON EXTENSIONS OF J–SKEW–SYMMETRIC AND J–ISOMETRIC OPERATORS

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(*Communicated by V. V. Peller*)

Abstract. In this paper it is proved that each densely defined J-skew-symmetric operator (or each J-isometric operator with $\overline{D(A)} = \overline{R(A)} = H$) in a separable Hilbert space H has a J-skew-self-adjoint (respectively J-unitary) extension in a separable Hilbert space $\widetilde{H} \supseteq H$. We follow the ideas of Galindo in [A. Galindo, On the existence of J-self-adjoint extensions of J-symmetric operators with adjoint, Comm. Pure Appl. Math., Vol. XV, 423–425 (1962)] with necessary modifications.

1. Introduction

Last years an increasing number of papers was devoted to the investigations of operators related to a conjugation in a Hilbert space, see, e.g. [2], [3], [6], [5] and references therein. A conjugation J in a separable Hilbert space H is an *antilinear* operator on H such that $J^2x = x$, $x \in H$, and $(Jx,Jy)_H = (y,x)_H$, $x,y \in H$. The conjugation J generates the following bilinear form:

$$[x,y]_J := (x,Jy)_H, \qquad x,y \in H.$$

For J there always exists an orthonormal basis $\{f_k\}$ in H such that $Jf_k = f_k$ for all k, see, e.g., [2, Lemma 1]. We shall say that such a basis is *corresponding to* J. A linear operator A in H is said to be J-symmetric (J-skew-symmetric) if

$$[Ax,y]_J = [x,Ay]_J, \qquad x,y \in D(A), \tag{1}$$

or, respectively,

$$[Ax,y]_J = -[x,Ay]_J, \qquad x,y \in D(A). \tag{2}$$

A linear operator A in H is said to be J-isometric if

$$[Ax,Ay]_J = [x,y]_J, \qquad x,y \in D(A). \tag{3}$$

If $\overline{D(A)} = H$, then conditions (1), (2) and (3) are equivalent to the following conditions:

$$JAJ \subseteq A^*, \tag{4}$$

Mathematics subject classification (2010): 47A20.
Keywords and phrases: Extension, conjugation, J-skew-symmetric operator, J-isometric operator.

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$$JAJ \subseteq -A^*, \tag{5}$$

and

$$JA^{-1}J \subseteq A^*, \tag{6}$$

respectively. A linear operator A in H is called J-self-adjoint (J-skew-self-adjoint, or J-unitary) if

$$JAJ = A^*, \tag{7}$$

$$JAJ = -A^*, \tag{8}$$

or

$$JA^{-1}J = A^*, \tag{9}$$

respectively.

We shall prove that each densely defined J-skew-symmetric operator (each J-isometric operator with $\overline{D(A)} = \overline{R(A)} = H$) in a separable Hilbert space H has a J-skew-self-adjoint (respectively J-unitary) extension in a separable Hilbert space $\widetilde{H} \supseteq H$. We shall follow the ideas of Galindo in [1] with necessary modifications. In particular, Lemma in [1] can not be applied in our case, since its assumptions can never be satisfied with T: $T^2 = I$, if $H \neq \{0\}$. In fact, in this case T would be a conjugation in H. Choosing an element $f \in H$ of an orthonormal basis in H corresponding to T we would get $(f, Tf) = (f, f) = 1 \neq 0$. Moreover, an exit out of the original space can appear in our case.

We notice that under stronger assumptions on a J-skew-symmetric operator the existence of a J-skew-self-adjoint extension was proved by Kalinina in [4].

NOTATIONS. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. Set $\overline{0,d} = \{0, 1, ..., d\}$, if $d \in \mathbb{N}$; $\overline{0,\infty} = \mathbb{Z}_+$. If H is a Hilbert space then $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ mean the scalar product and the norm in H, respectively. Indices may be omitted in obvious cases. For a linear operator A in H, we denote by $D(A)$ its domain, by $R(A)$ its range, and A^* means the adjoint operator if it exists. If A is invertible then A^{-1} means its inverse. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of M in the norm of H. By $\mathrm{Lin} M$ we denote the set of all linear combinations of elements of M, and $\mathrm{span} M := \overline{\mathrm{Lin} M}$. By E_H we denote the identity operator in H, i.e. $E_H x = x$, $x \in H$. In obvious cases we may omit the index H. All appearing Hilbert spaces are assumed to be separable.

2. Extensions of J-skew-symmetric and J-isometric operators

We shall make use of the following lemma.

LEMMA 1. *Let H be a separable Hilbert space with a positive even or infinite dimension, and J be a conjugation on H. Then there exists a subspace M in H such that*

$$M \oplus JM = H.$$

Proof. Let $\{f_n\}_{n=0}^{2d+1}$ be an orthonormal basis in H corresponding to J, i.e. such that $Jf_n = f_n$, $0 \leqslant n \leqslant 2d+1$; $d \in Z_+ \cup \{+\infty\}$ ($2d+2 = \dim H$). Set

$$f^+_{2k,2k+1} = \frac{1}{\sqrt{2}}(f_{2k} + if_{2k+1}), \quad f^-_{2k,2k+1} = \frac{1}{\sqrt{2}}(f_{2k} - if_{2k+1}), \qquad k \in \overline{0,d}.$$

It is easy to see that $\{f^+_{2k,2k+1}, f^-_{2k,2k+1}\}_{k=0}^d$ is an orthonormal basis in H. Set $M := \mathrm{span}\{f^+_{2k,2k+1}\}_{k=0}^d$. It remains to notice that $JM = \mathrm{span}\{f^-_{2k,2k+1}\}_{k=0}^d$. □

THEOREM 1. *Let H be a separable Hilbert space and J be a conjugation on H. Let A be a J-skew-symmetric (J-isometric) operator in H. Suppose that $\overline{D(A)} = H$ (respectively $\overline{D(A)} = \overline{R(A)} = H$). Then there exists a J-skew-self-adjoint (respectively J-unitary) extension of A in a separable Hilbert space $\widetilde{H} \supseteq H$ (with an extension of J to a conjugation on $\widetilde{H}$).*

Proof. Let A be such an operator as that in the statement of the theorem. The operator A admits the closure which is J-skew-symmetric (respectively J-isometric) (see, e.g. [6, p. 18]). Thus, without loss of generality we shall assume that A is closed. In what follows, in the case of a J-skew-symmetric (J-isometric) A, we shall say about case (a) (respectively case (b)). Set $H_2 = H \oplus H$, and consider the following transformations on H_2:

$$J_2\{x,y\} = \{Jx,Jy\},\ V\{x,y\} = \{y,-x\},\ U\{x,y\} = \{y,x\}, \quad \forall \{x,y\} \in H_2,$$

and $R := UJ_2 = J_2U$, $K := VR$. Observe that R and K are conjugations on H_2. The graph of an arbitrary linear operator C in the Hilbert space H will be denoted by G_C ($\subseteq H_2$). Observe that

$$J_2G_C = G_{JCJ}, \quad RG_C = UG_{JCJ}. \tag{10}$$

If $\overline{D(C)} = H$, then

$$G_{C^*} = H_2 \ominus VG_C. \tag{11}$$

In the case (a) we may write:

$$(\{x,Ax\},\{JAJy,y\}) = (x,JAJy) + (Ax,y) = 0, \quad \forall x \in D(A), y \in D(JAJ).$$

Then

$$G_A \perp RG_A. \tag{12}$$

In the case (b), we have

$$(\{x,Ax\},\{JA^{-1}Jy,-y\}) = 0, \quad \forall x \in D(A), y \in D(JA^{-1}J),$$

and therefore

$$G_A \perp KG_A. \tag{13}$$

Set $D = \begin{cases} H_2 \ominus [G_A \oplus RG_A] & \text{in the case (a)} \\ H_2 \ominus [G_A \oplus KG_A] & \text{in the case (b)} \end{cases}$. If $D = \{0\}$ then it means that A is J-skew-self-adjoint (respectively J-unitary), see considerations for the operator B below. In the opposite case, we have $RD = D$ (respectively $KD = D$).

At first, suppose that D has a positive even or infinite dimension. By Lemma 1 we obtain that there exists a subspace $X \subseteq D$ such that $X \oplus RX = D$ (respectively $X \oplus KX = D$). Since each element of X is orthogonal to $RG_A = VG_{-JAJ}$ ($KG_A = VG_{JA^{-1}J}$), by (11) it follows that

$$X \subseteq G_{-JA^*J} \quad (\text{respectively } X \subseteq G_{J(A^{-1})^*J}). \tag{14}$$

Set $G' = G_A \oplus X$. Suppose that $\{0,y\} \in G'$. Then there exist $\{x,Ax\} \in G_A$ such that $\{0,y\} - \{x,Ax\} = \{-x, y-Ax\} \in X$. By (14) we get $y - Ax = JA^*Jx$ (respectively $y - Ax = -J(A^{-1})^*Jx$), and therefore $y = 0$. Thus, G' is a graph G_B of a densely defined linear operator B. Moreover, we have

$$G_B \oplus RG_B = H_2 \quad (\text{respectively } G_B \oplus KG_B = H_2).$$

In the case (a) we get

$$UG_B \oplus URG_B = H_2;$$

$$G_{(-B)^*} = H_2 \ominus VG_{-B} = H_2 \ominus UG_B = URG_B = J_2G_B = G_{JBJ}.$$

In the case (b) we get

$$VG_B \oplus VKG_B = H_2;$$

$$G_{B^*} = H_2 \ominus VG_B = VKG_B = -RG_B = G_{JB^{-1}J}.$$

Suppose now that D has a positive odd dimension. In this case we consider a linear operator $\mathscr{A} = A \oplus A$, with $D(\mathscr{A}) = D(A) \oplus D(A)$, in a Hilbert space $\mathscr{H} = H \oplus H$ with a conjugation $\mathscr{J} = J \oplus J$. Observe that $\mathscr{A}$ is a closed $\mathscr{J}$-skew-symmetric ($\mathscr{J}$-isometric) operator with $\overline{D(\mathscr{A})} = \mathscr{H}$ (respectively $\overline{D(\mathscr{A})} = \overline{R(\mathscr{A})} = \mathscr{H}$). Its graph $G_{\mathscr{A}}$ in a Hilbert space $\mathscr{H}_2 = \mathscr{H} \oplus \mathscr{H}$ may be identified with $G_A \oplus G_A$ in $H_2 \oplus H_2$:

$$G_{\mathscr{A}} = \{\{(f,Af),(g,Ag)\},\ f,g \in D(A)\}.$$

Let $\mathscr{R}$, $\mathscr{K}$ be constructed for $\mathscr{A}$ as R and K for A. In the case (a) we see that

$$\mathscr{H}_2 \ominus [G_{\mathscr{A}} \oplus \mathscr{R}G_{\mathscr{A}}] = (H_2 \ominus [G_A \oplus RG_A]) \oplus (H_2 \ominus [G_A \oplus RG_A]),$$

has a positive even dimension. In the case (b), $\mathscr{H}_2 \ominus [G_{\mathscr{A}} \oplus \mathscr{K}G_{\mathscr{A}}]$ has a positive even dimension. Thus, we may apply the above construction with $\mathscr{A}$ instead of A. □

REMARK 1. In the proof of the last theorem one may choose various subspaces X to construct required extensions. However, we do not know whether all possible extensions can be constructed on this way. We think that it is an interesting question for further investigations.

Acknowledgements. The author is grateful to the referee for useful comments and suggestions.

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(Received July 29, 2014)

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