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ASYMPTOTIC BEHAVIOR OF EIGENVALUES AND EIGENFUNCTIONS OF STURM–LIOUVILLE PROBLEMS WITH COUPLED BOUNDARY CONDITIONS AND TRANSMISSION CONDITIONS

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Abstract. The Sturm-Liouville (S-L) problems with coupled boundary conditions and transmission conditions are investigated. By defining a new Hilbert space which is related to the transmission conditions, the self-adjointness of the S-L problems in this associated Hilbert space is proved, and the asymptotic behavior of eigenvalues and eigenfunctions of the problem are described. We also give the condition for λ being the eigenvalue of the S-L problems with coupled boundary conditions.

1. Introduction

Sturm-Liouville (S-L) problems with transmission conditions, i.e. S-L operators with discontinuity conditions inside the interval appear in mathematics, mechanics, physics and in some other applications. These problems have been considered in many publications [3, 4, 5, 6, 7, 12, 16], however, these publications are only restricted in the separated boundary conditions. The self-adjointness of the S-L problems with coupled boundary conditions, as the special two-interval problems, are studied in [10]. Hence in this paper, we consider not only the self-adjointness of the S-L problems with coupled boundary conditions, but some other problems of the S-L problems with coupled boundary conditions and transmission conditions. We discuss the asymptotic behavior of eigenvalues and eigenfunctions of the S-L problem with coupled boundary conditions and transmission conditions, and give the condition for λ being the eigenvalue of the S-L problems with coupled boundary conditions.

Consider the differential equation

$$ly := -y'' + q(x)y = \lambda y,\ x \in J = [-1,0) \cup (0,1], \tag{1.1}$$

with the coupled boundary conditions

$$AY(-1) + Y(1) = 0, \quad Y(\pm 1) = \begin{pmatrix} y(\pm 1) \\ y'(\pm 1) \end{pmatrix}, \tag{1.2}$$

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and the transmission conditions

$$KY(0-)+Y(0+)=0, \quad Y(0\pm)=\begin{pmatrix} y(0\pm) \\ y'(0\pm) \end{pmatrix}, \tag{1.3}$$

where λ is complex eigenparameter, 0 is the inner discontinuity point; A, K are 2×2 matrices

$$A=e^{i\gamma}\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \quad K=\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \tag{1.4}$$

with $-\pi \leqslant \gamma \leqslant \pi$, $\alpha_1\alpha_4-\alpha_2\alpha_3>0$, $k_{11}k_{22}-k_{12}k_{21}>0$; and the matrices $(A,-I)$, $(K,-I)$ have full ranks, I is the 2×2 identity matrix; $q\in L(J,\mathbb{R})$. Note that the conditions are minimal in the sense that it is necessary and sufficient for all initial value problems of the equation (1.1) to have unique solutions on J ([9, 18]); α_j $(j=1,2,3,4)$ and k_{mj} $(m,j=1,2)$ are real numbers.

The organization of this paper is as follows: After the Introduction in Section 1, we prove the self-adjointness of the S-L problems with the coupled boundary conditions and transmission conditions in Section 2. In Section 3, we discuss the fundamental solutions of the differential equation (1.1) with the transmission conditions, and give the condition for λ being the eigenvalue of the S-L problems with coupled boundary conditions. Finally, the asymptotic formulas for eigenvalues and eigenfunctions of the S-L problems (1.1)–(1.4) are obtained in Section 4.

2. The self-adjoint operator

Let $h=\det K$, where K is the coefficient matrix in the transmission conditions (1.3), (1.4). Define a new inner product in $L^2(J)$ as follows:

$$\langle f,g\rangle = h\int_{-1}^{0} f_1\overline{g}_1 dx+\int_0^1 f_2\overline{g}_2 dx, \text{ for } f,g\in L^2(J), \tag{2.1}$$

where $f_1=f(x)\mid_{[-1,0)}, f_2=f(x)\mid_{(0,1]}$. It is easy to verify that $(L^2(J),\langle\cdot,\cdot\rangle)$ is a Hilbert space. For simplicity, we denote it by H, and the norm induced by the inner product is denoted by $\|\cdot\|_H$. Now we consider the Sturm-Liouville problems (1.1)–(1.4) in the associated Hilbert space H.

The operator L_M related to the Sturm-Liouville problems (1.1)–(1.4) is defined by

$$\mathscr{D}(L_M)=\{y\in H|y_1,y_1'\in AC_{loc}[-1,0),y_2,y_2'\in AC_{loc}(0,1],ly\in H \\ \text{and } KY(0-)+Y(0+)=0\},$$

$$L_M y=ly, y\in\mathscr{D}(L_M),$$

where $AC_{loc}[-1,0)$ and $AC_{loc}(0,1]$ denote the set of complex-valued absolutely continuous functions on whole compact subintervals of $[-1,0)$ and $(0,1]$, respectively. The operator L_0 related to the Sturm-Liouville problems (1.1)–(1.4) is defined by

$$\mathscr{D}(L_0)=\{\, y\in\mathscr{D}(L_M)|y(-1)=y'(-1)=y(1)=y'(1)=0\},$$

$$L_0 y = ly,\ y \in \mathscr{D}(L_0).$$

The S-L operator L is defined by

$$\mathscr{D}(L) = \{\, y \in \mathscr{D}(L_M) | AY(-1) + Y(1) = 0\},$$

$$Ly = ly,\ y \in \mathscr{D}(L).$$

LEMMA 2.1. *$\mathscr{D}(L)$ is dense in H.*

Proof. Let $\tilde{C}_0^\infty$ be the set of all functions defined by

$$\varphi(x) = \begin{cases} \varphi_1(x) & x \in [-1,0), \\ \varphi_2(x) & x \in (0,1], \end{cases} \tag{2.2}$$

where $\varphi_1(x) \in C_0^\infty[-1,0)$, $\varphi_2(x) \in C_0^\infty(0,1]$. Obviously $\tilde{C}_0^\infty \subset \mathscr{D}(L)$.

Next, for proving $\mathscr{D}(L)$ is dense in H, we show that $\tilde{C}_0^\infty$ is dense in H. Let f be a function in $L^2(J)$ with $f(x) = f_1(x)$, $x \in [-1,0)$, and $f(x) = f_2(x)$, $x \in (0,1]$. Since $C_0^\infty[-1,0)$ is dense in $L^2[-1,0)$ as Theorem 2.19 in [1], for $f_1 \in L^2[-1,0)$, there exists a function $g_1 \in C_0^\infty[-1,0)$ such that

$$h\int_{-1}^{0} |f_1(x) - g_1(x)|^2 dx < \frac{\varepsilon}{2}.$$

Similarly, for $f_2 \in L^2(0,1]$, there exists a function $g_2 \in C_0^\infty(0,1]$ such that

$$\int_{0}^{1} |f_2(x) - g_2(x)|^2 dx < \frac{\varepsilon}{2}.$$

Then for any $f \in H$ and $\varepsilon > 0$, there exists $g \in \tilde{C}_0^\infty$ with

$$g(x) = \begin{cases} g_1(x) & x \in [-1,0), \\ g_2(x) & x \in (0,1], \end{cases}$$

such that

$$\|f-g\|_H^2 = h\int_{-1}^{0} |f_1(x) - g_1(x)|^2 dx + \int_{0}^{1} |f_2(x) - g_2(x)|^2 dx < \varepsilon,$$

where $h = \det K$. Thus, $\tilde{C}_0^\infty$ is dense in H. Therefore $\mathscr{D}(L)$ is dense in H. $\square$

The next theorem shows that the operator L defined by the S-L problems (1.1)–(1.4) is self-adjoint.

THEOREM 2.2. *If the matrices A and K satisfy*

$$AEA^* = hE, \quad KEK^* = hE,$$

with $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then the operator L is self-adjoint.

Proof. Firstly, we prove the operator L is symmetric. For any $f,g \in \mathscr{D}(L)$, by twice partial integrating

$$\begin{aligned}\langle lf,g\rangle =&h\int_{-1}^{0}(lf)\overline{g}dx+\int_{0}^{1}(lf)\overline{g}dx \\ =&\langle f,lg\rangle +hW(f,\overline{g},0-)-hW(f,\overline{g},-1)-W(f,\overline{g},0+)+W(f,\overline{g},1),\end{aligned} \tag{2.3}$$

where $W(f,g,x)=f(x)g'(x)-f'(x)g(x)$. Since the matrices A,K satisfy $AEA^{*}=hE$, $KEK^{*}=hE$, it follows that

$$\alpha_1\alpha_4-\alpha_2\alpha_3=h,\quad k_{11}k_{22}-k_{12}k_{21}=h.$$

From the transmission conditions (1.3), we get

$$\begin{aligned}W(f,\overline{g},0+)&=f(0+)\overline{g'}(0+)-f'(0+)\overline{g}(0+) \\ &=(k_{11}f(0-)+k_{12}f'(0-))(k_{21}\overline{g}(0-)+k_{22}\overline{g'}(0-)) \\ &\quad -(k_{21}f(0-)+k_{22}f'(0-))(k_{11}\overline{g}(0-)+k_{12}\overline{g'}(0-)) \\ &=(k_{11}k_{22}-k_{12}k_{21})(f(0-)\overline{g'}(0-)-f'(0-)\overline{g}(0-))=hW(f,\overline{g},0-).\end{aligned} \tag{2.4}$$

And from the boundary conditions (1.2), we get

$$\begin{aligned}W(f,\overline{g},1)=&(\alpha_1e^{i\gamma}f(-1)+\alpha_2e^{i\gamma}f'(-1))(\alpha_3e^{-i\gamma}\overline{g}(-1)+\alpha_4e^{-i\gamma}\overline{g'}(-1)) \\ &+(\alpha_3e^{i\gamma}f(-1)+\alpha_4e^{i\gamma}f'(-1))(\alpha_1e^{-i\gamma}\overline{g}(-1)+\alpha_2e^{-i\gamma}\overline{g'}(-1)) \\ =&(\alpha_1\alpha_4-\alpha_2\alpha_3)(f(-1)\overline{g'}(-1)-f'(-1)\overline{g}(-1)) \\ =&hW(f,\overline{g},-1).\end{aligned} \tag{2.5}$$

By using (2.4), (2.5), the equation (2.3) becomes

$$\langle lf,g\rangle =\langle f,lg\rangle . \tag{2.6}$$

This means that the operator L is symmetric. Next we prove the operator L is self-adjoint.

We show that if $\langle lf,g\rangle =\langle f,w\rangle$ for $f\in\mathscr{D}(L)$, then $g\in\mathscr{D}(L)$ and $lg=w$. Since $\langle lf,g\rangle =\langle f,w\rangle$ for $f\in\tilde{C}_0^{\infty}\subset\mathscr{D}(L)$, by the classical S-L theory there exist $g_1,g_1'\in AC_{loc}[-1,0)$ and $g_2,g_2'\in AC_{loc}(0,1]$ such that $lg\in H$ and $w=lg$, where $g(x)=g_1(x)$ for $x\in[-1,0)$, and $g(x)=g_2(x)$ for $x\in(0,1]$. In the following we will prove that g satisfy the boundary conditions (1.2) and the transmission conditions (1.3).

Since $\langle lf,g\rangle =\langle f,w\rangle$, we have

$$\langle lf,g\rangle =h\int_{-1}^{0}fl\overline{g}dx+\int_{0}^{1}fl\overline{g}dx.$$

And by the integration by parts, we have

$$\begin{aligned}\langle lf,g\rangle =&h\int_{-1}^{0}fl\overline{g}dx+\int_{0}^{1}fl\overline{g}dx+hW(f,\overline{g},0-)-hW(f,\overline{g},-1) \\ &-W(f,\overline{g},0+)+W(f,\overline{g},1).\end{aligned}$$

Hence

$$hW(f,\overline{g},0-) - hW(f,\overline{g},-1) - W(f,\overline{g},0+) + W(f,\overline{g},1) = 0. \tag{2.7}$$

By the Naimark-Patching Lemma, there exists a function $f \in \mathscr{D}(L)$ such that $f(0-) = f'(0-) = f(0+) = f'(0+) = 0$, and $f(-1) = \alpha_2 e^{-i\gamma}$, $f'(-1) = -\alpha_1 e^{-i\gamma}$, $f(1) = 0$, $f'(1) = -h$. Substituting above into (2.7), we have

$$\alpha_1 e^{i\gamma} g(-1) + \alpha_2 e^{i\gamma} g'(-1) - g(1) = 0.$$

If $f(-1) = \alpha_4 e^{-i\gamma}$, $f'(-1) = -\alpha_3 e^{-i\gamma}$, $f(1) = h$, $f'(1) = 0$, then from (2.7) we obtain

$$\alpha_3 e^{i\gamma} g(-1) + \alpha_4 e^{i\gamma} g'(-1) - g'(1) = 0.$$

Hence the function g satisfies the boundary conditions (1.2).

Similarly, choosing a function $f \in \mathscr{D}(L)$ with $f(-1) = f'(-1) = f(1) = f'(1) = 0$, and $f(0-) = k_{12}$, $f'(0-) = -k_{11}$, $f(0+) = 0$, $f'(0+) = -h$, then (2.7) becomes

$$k_{11} g(0-) + k_{12} g'(0-) - g(0+) = 0.$$

By choosing $f(0-) = k_{22}$, $f'(0-) = -k_{21}$, $f(0+) = h$, $f'(0+) = 0$, we obtain

$$k_{21} g(0-) + k_{22} g'(0-) - g(0+) = 0.$$

Hence g satisfies the transmission conditions (1.3). The proof is completed. $\square$

3. The basic solutions and their asymptotic approximations

Below, we consider the S-L problems (1.1)–(1.4) with the conditions

$$AEA^* = hE, \qquad KEK^* = hE.$$

That is, the S-L operator L generated by the S-L problems (1.1)–(1.4) is self-adjoint. Define the fundamental solution

$$\phi(x,\lambda) = \begin{cases} \phi_1(x,\lambda),\ x \in [-1,0), \\ \phi_2(x,\lambda),\ x \in (0,1], \end{cases}$$

of the differential equation (1.1), which satisfies

$$y(-1) = 1, \qquad y'(-1) = 0,$$

and the transmission conditions (1.3). By virtue of Theorem 1.5 in [14], there is a unique solution $\phi_1(x,\lambda)$ for each $\lambda \in \mathbb{C}$, which is an entire function of λ for each fixed $x \in [-1,0)$.

Define the other fundamental solution

$$\chi(x,\lambda) = \begin{cases} \chi_1(x,\lambda),\ x \in [-1,0), \\ \chi_2(x,\lambda),\ x \in (0,1], \end{cases}$$

of the differential equation (1.1), which satisfies

$$y(-1)=0, \qquad y'(-1)=1,$$

and the transmission conditions (1.3). Similarly, there is a unique solution $\chi_1(x,\lambda)$ which is an entire function of λ for each fixed $x\in[-1,0)$.

It is well known, from the ordinary linear differential equation theory, the Wronskian $W(\phi_j(x,\lambda),\ \chi_j(x,\lambda))$ is independent of the variable x. Let

$$\omega_j(\lambda):=W(\phi_j(x,\lambda),\chi_j(x,\lambda))=\begin{vmatrix}\phi_j(x,\lambda) & \chi_j(x,\lambda)\\ \phi_j'(x,\lambda) & \chi_j'(x,\lambda)\end{vmatrix},$$

then we have

$$\begin{aligned}
&\omega_1(\lambda)=\omega_1(\lambda)|_{x=-1}=1,\\
&\omega_2(\lambda)=\omega_2(\lambda)|_{x=0+}=\begin{vmatrix}\phi_2(0+,\lambda) & \chi_2(0+,\lambda)\\ \phi_2'(0+,\lambda) & \chi_2'(0+,\lambda)\end{vmatrix}\\
&\quad=\begin{vmatrix}k_{11}\phi_1(0-,\lambda)+k_{12}\phi_1'(0-,\lambda) & k_{11}\chi_1(0-,\lambda)+k_{12}\chi_1'(0-,\lambda)\\ k_{21}\phi_1(0-,\lambda)+k_{22}\phi_1'(0-,\lambda) & k_{21}\chi_1(0-,\lambda)+k_{22}\chi_1'(0-,\lambda)\end{vmatrix}=h\omega_1(\lambda)=h.
\end{aligned}$$

LEMMA 3.1. *Let*

$$y(x,\lambda)=\begin{cases} y_1(x,\lambda),\ x\in[-1,0),\\ y_2(x,\lambda),\ x\in(0,1],\end{cases}$$

be a solution of the equation (1.1), then the solution can be expressed in the following form

$$y(x,\lambda)=\begin{cases} c_1\phi_1(x,\lambda)+c_2\chi_1(x,\lambda),\ x\in[-1,0),\\ d_1\phi_2(x,\lambda)+d_2\chi_2(x,\lambda),\ x\in(0,1].\end{cases} \tag{3.1}$$

If $y(x,\lambda)$ satisfies the transmission conditions (1.3), then $c_1=d_1, c_2=d_2$.

Proof. Since $y(x,\lambda)$ satisfies the transmission conditions (1.3), i.e.

$$\begin{aligned}
&k_{11}(c_1\phi_1(0-,\lambda)+c_2\chi_1(0-,\lambda))+k_{12}(c_1\phi_1'(0-,\lambda)+c_2\chi_1'(0-,\lambda))\\
&\quad-(d_1\phi_2(0+,\lambda)+d_2\chi_2(0+,\lambda))=0,\\
&k_{21}(c_1\phi_1(0-,\lambda)+c_2\chi_1(0-,\lambda))+k_{22}(c_1\phi_1'(0-,\lambda)+c_2\chi_1'(0-,\lambda))\\
&\quad-(d_1\phi_2'(0+,\lambda)+d_2\chi_2'(0+,\lambda))=0.
\end{aligned}$$

Since ϕ,χ satisfy the transmission conditions, the last equation system becomes

$$\begin{cases}(c_1-d_1)\phi_2(0+,\lambda)+(c_2-d_2)\chi_2(0+,\lambda)=0,\\ (c_1-d_1)\phi_2'(0+,\lambda)+(c_2-d_2)\chi_2'(0+,\lambda)=0.\end{cases}$$

Since the determinant of the coefficient matrix of the equation system is

$$\begin{vmatrix}\phi_2(0+,\lambda) & \chi_2(0+,\lambda)\\ \phi_2'(0+,\lambda) & \chi_2'(0+,\lambda)\end{vmatrix}=\omega_2(\lambda)\neq 0,$$

we get $c_1 = d_1, c_2 = d_2$. $\square$

Let

$$\Phi_j(x,\lambda) = \begin{pmatrix} \phi_j(x,\lambda) & \chi_j(x,\lambda) \\ \phi_j'(x,\lambda) & \chi_j'(x,\lambda) \end{pmatrix}, \quad j = 1,2,$$

and let

$$\Phi(x,\lambda) = \begin{cases} \Phi_1(x,\lambda),\ x \in [-1,0), \\ \Phi_2(x,\lambda),\ x \in (0,1]. \end{cases} \tag{3.2}$$

Then we have the following theorem.

THEOREM 3.2. *Let $\lambda_0 \in \mathbb{C}$. Then λ_0 is the eigenvalue of the S-L problems (1.1)–(1.4) if and only if*

$$\Delta(\lambda_0) := \det(A - \Phi(1,\lambda_0)) = 0,$$

where $A = e^{i\gamma}\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$.

Proof. Let λ_0 be an eigenvalue of the S-L problems (1.1)–(1.4) and $y(x,\lambda_0)$ the corresponding eigenfunction. From Lemma 3.1, there exist c_1, c_2 such that

$$y(x,\lambda_0) = \begin{cases} c_1\phi_1(x,\lambda_0) + c_2\chi_1(x,\lambda_0),\ x \in [-1,0), \\ c_1\phi_2(x,\lambda_0) + c_2\chi_2(x,\lambda_0),\ x \in (0,1], \end{cases} \tag{3.3}$$

where at least one of the constants c_1, c_2 is not zero. Substituting (3.3) into the boundary conditions (1.2) we obtain

$$e^{i\gamma}\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}\begin{pmatrix} c_1\phi_1(-1,\lambda_0) + c_2\chi_1(-1,\lambda_0) \\ c_1\phi_1'(-1,\lambda_0) + c_2\chi_1'(-1,\lambda_0) \end{pmatrix} - \begin{pmatrix} c_1\phi_2(1,\lambda_0) + c_2\chi_2(1,\lambda_0) \\ c_1\phi_2'(1,\lambda_0) + c_2\chi_2'(1,\lambda_0) \end{pmatrix} = 0,$$

that is,

$$\left[e^{i\gamma}\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} - \begin{pmatrix} \phi_2(1,\lambda_0) & \chi_2(1,\lambda_0) \\ \phi_2'(1,\lambda_0) & \chi_2'(1,\lambda_0) \end{pmatrix}\right]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

Since at least one of the constants c_1, c_2 is not zero, we obtain

$$\Delta(\lambda_0) = \det(A - \Phi(1,\lambda_0)) = 0, \tag{3.4}$$

where $A = e^{i\gamma}\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ with $-\pi < \gamma \leqslant \pi$ and $\alpha_1\alpha_4 - \alpha_2\alpha_3 > 0$.

Conversely, if $\det(A - \Phi(1,\lambda_0)) = 0$, then the equation

$$(A - \Phi(1,\lambda_0))\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0,$$

has a nonzero solution (c_1', c_2'). Let

$$y(x,\lambda_0) = \begin{cases} c_1'\phi_1(x,\lambda_0) + c_2'\chi_1(x,\lambda_0),\ x \in [-1,0), \\ c_1'\phi_2(x,\lambda_0) + c_2'\chi_2(x,\lambda_0),\ x \in (0,1]. \end{cases} \tag{3.5}$$

Then $y(x,\lambda_0)$ is a nonzero solution of the equation (1.1) and satisfies both boundary and transmission conditions (1.2), (1.3). Hence λ_0 is an eigenvalue of the S-L problems (1.1)–(1.4), and $y(x,\lambda_0)$ is the corresponding eigenfunction. □

In order to give the asymptotic formulas for eigenvalues and eigenfuntions, we consider the asymptotic approximations of the solutions of ϕ and χ.

LEMMA 3.3. *Let $\lambda = s^2$ with $s = \sigma + it$, then the following asymptotic formulas for ϕ_1, χ_1 hold as $|\lambda| \to \infty$:*

$$\frac{d^m}{dx^m}\phi_1(x,\lambda) = \frac{d^m}{dx^m}\cos s(x+1) + O(|s|^{m-1}e^{|t|(x+1)}), \tag{3.6}$$

$$\frac{d^m}{dx^m}\chi_1(x,\lambda) = \frac{1}{s}\frac{d^m}{dx^m}\sin s(x+1) + O(|s|^{m-2}e^{|t|(x+1)}), \quad m = 0,1. \tag{3.7}$$

Proof. The asymptotic formulas for $\phi_1(x,\lambda)$, $\chi_1(x,\lambda)$ follow immediately from the similar formulas of Lemma 1.7 in [14], hence is omitted here. □

Next, we give the asymptotic formulas for $\phi_2(x,\lambda)$ and $\chi_2(x,\lambda)$ which read as the following lemmas.

LEMMA 3.4. *Let $\lambda = s^2$ with $s = \sigma + it$, then $\phi_2(x,\lambda)$ has the following asymptotic formulas:*

1. *If $k_{12} \neq 0$, then*

$$\frac{d^m}{dx^m}\phi_2(x,\lambda) = -k_{12}s\sin s\frac{d^m}{dx^m}\cos sx + O(|s|^m e^{|t|x}), \tag{3.8}$$

2. *if $k_{12} = 0$, then*

$$\frac{d^m}{dx^m}\phi_2(x,\lambda) = k_{11}\cos s\frac{d^m}{dx^m}\cos sx - k_{22}\sin s\frac{d^m}{dx^m}\sin sx + O(|s|^{m-1}e^{|t|x}), \tag{3.9}$$

as $|\lambda| \to \infty$, $m = 0,1$.

Proof. Let $\lambda = s^2$ with $s = \sigma + it$. Then

$$\begin{aligned}\phi_2(x,\lambda) = {} & (k_{11}\phi_1(0-,\lambda) + k_{12}\phi_1'(0-,\lambda))\cos sx + \frac{1}{s}(k_{21}\phi_1(0-,\lambda) \\ & + k_{22}\phi_1'(0-,\lambda))\sin sx + \frac{1}{s}\int_0^x \sin[s(x-z)]q_2(z)\phi_2(z,\lambda)dz.\end{aligned} \tag{3.10}$$

For $m = 0$, substituting (3.6) into (3.10), we have

$$\begin{aligned}\phi_2(x,\lambda) = {} & -k_{12}s\sin s\cos sx + \Big(k_{11}\cos s\cos sx - k_{22}\sin s\sin sx\Big) \\ & + \frac{1}{s}k_{21}\cos s\sin sx + \frac{1}{s}\int_0^x \sin[s(x-z)]q_2(z)\phi_2(z,\lambda)dz + O\left(\frac{1}{|s|^2}e^{|t|(x+1)}\right).\end{aligned} \tag{3.11}$$

Multiplying (3.11) by $|s|^{-1}e^{-|t|x}$ and by denoting $F_2(x,\lambda) = |s|^{-1}e^{-|t|x}\phi_2(x,\lambda)$, (3.11) becomes

$$F_2(x,\lambda) = -k_{12}e^{-|t|x}\sin s\cos sx + \frac{1}{|s|}e^{-|t|x}\Big(k_{11}\ \cos s\cos sx - k_{22}\sin s\sin sx\Big) \qquad (3.12)$$
$$+\frac{1}{|s|^2}k_{21}e^{-|t|x}\cos s\sin sx + \frac{1}{|s|^2}\int_0^x \sin[s(x-z)]q_2(z)F_2(z,\lambda)e^{-|t|(x-z)}dz$$
$$+O\left(\frac{1}{|s|^3}e^{|t|}\right).$$

Let $\mu(\lambda) = \max\limits_{0<x\leqslant 1}|F_2(x,\lambda)|$, then

$$\mu(\lambda) \leqslant |k_{12}| + \frac{1}{|s|}(|k_{11}|+|k_{22}|) + \frac{M_0}{|s|^2}, \qquad (3.13)$$

for some $M_0 > 0$. Consequently, $\mu(\lambda) = O(1)$ as $|\lambda| \to \infty$ for $k_{12} \neq 0$. Therefore

$$\phi_2(x,\lambda) = O(|s|e^{|t|x}), \qquad (3.14)$$

as $|\lambda| \to \infty$. Substituting the asymptotic equality (3.14) into (3.11) gives (3.8) for $m = 0$.

Differentiating (3.11) and by the similar calculation, we obtain (3.8) for $m = 1$.

If $k_{12} = 0$, then $\mu(\lambda) = O(\frac{1}{|s|})$ as $|\lambda| \to \infty$ from (3.13). Then $\phi_2(x,\lambda) = O(e^{|t|x})$ as $|\lambda| \to \infty$. Substituting the asymptotic equality into (3.11) gives (3.9) for $m = 0$. Similarly, we can obtain the formula for the case $m = 1$. □

LEMMA 3.5. *Let $\lambda = s^2$ with $s = \sigma + it$, then $\chi_2(x,\lambda)$ has the following asymptotic formulas:*

1. *If $k_{12} \neq 0$, then*

$$\frac{d^m}{dx^m}\chi_2(x,\lambda) = k_{12}\cos s\frac{d^m}{dx^m}\cos sx + O(|s|^{m-1}e^{|t|x}), \qquad (3.15)$$

2. *if $k_{12} = 0$, then*

$$\frac{d^m}{dx^m}\chi_2(x,\lambda) = \frac{1}{s}\Big(k_{11}\sin s\frac{d^m}{dx^m}\cos sx + k_{22}\cos s\frac{d^m}{dx^m}\sin sx\Big) + O(|s|^{m-2}e^{|t|x}), \qquad (3.16)$$

as $|\lambda| \to \infty$, $m = 0,1$.

Proof. The proof is similar to the one of Lemma 3.4, hence it is omitted. □

4. The asymptotic formulas for eigenvalues and eigenfunctions

The eigenvalues of the separated boundary value problems with transmission conditions at an interior point of the interval are studied in [17]. Some other authors also considered the eigenvalues and eigenfunctions of the S-L problems with transmission conditions, however, the boundary conditions are not coupled ([15]).

In this section, we will obtain the asymptotic formulas for eigenvalues and eigenfunctions of the S-L problems (1.1)–(1.4) by using the asymptotic expressions of the fundamental solutions.

The asymptotic formulas of $\phi_2(x,\lambda)$ and $\chi_2(x,\lambda)$ are obtained in Lemmas 3.4 and 3.5. Substituting the asymptotic formulas of the solutions into the representation of $\Delta(\lambda)$ we can establish the following theorem.

THEOREM 4.1. *Let $\lambda = s^2$ with $s = \sigma + it$, then $\Delta(\lambda)$ has the following asymptotic formulas:*

1. *If $k_{12} \neq 0,\ \alpha_2 \neq 0$, then*

$$\Delta(\lambda) = \alpha_2 k_{12} s^2 e^{i\gamma} \sin^2 s + O(|s|e^{|t|}).$$

2. *If $k_{12} \neq 0,\ \alpha_2 = 0$, then*

$$\Delta(\lambda) = \frac{1}{2} k_{12} s e^{i\gamma}(\alpha_1 + \alpha_4)\sin 2s + O(e^{|t|}).$$

3. *If $k_{12} = 0,\ \alpha_2 \neq 0$, then*

$$\Delta(\lambda) = -\frac{1}{2}\alpha_2 s e^{i\gamma}(k_{11} + k_{22})\sin 2s + O(e^{(|t|}).$$

4. *If $k_{12} = 0,\ \alpha_2 = 0$, then*

$$\begin{aligned}\Delta(\lambda) =& \alpha_1 e^{i\gamma}[k_{22}\cos s\cos s - k_{11}\sin s\sin s] - \alpha_4 e^{i\gamma}[k_{11}\cos s\cos s - k_{22}\sin s\sin s] \\ &+ O(|s|^{-1}e^{|t|}).\end{aligned}$$

Proof. Substituting the asymptotic formulas of $\phi_2(x,\lambda)$ and $\chi_2(x,\lambda)$ which are given as in Lemmas 3.4, 3.5 into $\Delta(\lambda) = \det(A - \Phi(1,\lambda_0))$, we have

$$\Delta(\lambda) = \alpha_2 k_{12} s^2 e^{i\gamma}\sin^2 s + \frac{1}{2}k_{12} s e^{i\gamma}(\alpha_1 + \alpha_4)\sin 2s + O(e^{|t|}),$$

for $k_{12} \neq 0$ and

$$\begin{aligned}\Delta(\lambda) =& -\frac{1}{2}\alpha_2 s e^{i\gamma}(k_{11} + k_{22})\sin 2s - \alpha_1 e^{i\gamma}[k_{22}\cos s\cos s - k_{11}\sin s\sin s] \\ &- \alpha_4 e^{i\gamma}[k_{11}\cos s\cos s - k_{22}\sin s\sin s] + O(|s|^{-1}e^{|t|}),\end{aligned}$$

for $k_{12} = 0$. By some calculations similar to the proof of Lemma 3.4, we obtain four distinct kinds of asymptotic formulas of $\Delta(\lambda)$, which are classified according to k_{12} and α_2. $\square$

Now we can obtain the asymptotic approximation formulas for eigenvalues of the operator L defined by S-L problems (1.1)–(1.4). Since the eigenvalues of the operator are zeros of the function $\Delta(\lambda)$, we can find the asymptotic formulas for the eigenvalues using the asymptotic formulas of $\Delta(\lambda)$ and the well known Rouche's theorem. Furthermore, the asymptotic formulas of the corresponding eigenfunctions can be obtained by using the asymptotic formulas of eigenvalues.

THEOREM 4.2. *Let $\lambda = s^2$ with $s = \sigma + it$, then the following asymptotic formulas hold for eigenvalues and eigenfunctions of the operator L defined by the S-L problems (1.1)–(1.4):*

1. *If $k_{12} \neq 0$, $\alpha_2 \neq 0$, then*

$$\sqrt{\lambda_n} = s_n = (n-1)\pi + O\left(\frac{1}{n}\right),$$

and

$$u(x,\lambda_n) = \begin{cases} \alpha_4 k_{12} s_n \cos s_n(x+1) + he^{i\gamma} \sin s_n(x+1) + O(\frac{1}{n}), & x \in [-1,0), \\ (-1)^{n-1} k_{12} s_n h e^{i\gamma} \cos s_n x + O(\frac{1}{n}), & x \in (0,1]. \end{cases}$$

2. *If $k_{12} \neq 0$, $\alpha_2 = 0$, then for $\alpha_1 = \alpha_4$,*

$$\sqrt{\lambda_n} = s_n = \frac{n-1}{2}\pi + O\left(\frac{1}{n}\right).$$

And if n is even,

$$u(x,\lambda_n) = \begin{cases} he^{i\gamma} \cos s_n(x+1) + \alpha_1 k_{12} s_n \sin s_n(x+1) + O(\frac{1}{n}), & x \in [-1,0), \\ O(1), & x \in (0,1]; \end{cases}$$

if n is odd,

$$u(x,\lambda_n) = \begin{cases} -k_{12} \cos s_n(x+1) + \alpha_1 e^{i\gamma} \sin s_n(x+1) + O(\frac{1}{n}), & x \in [-1,0), \\ (-1)^{\frac{n-1}{2}} \alpha_1 k_{12} s_n e^{i\gamma} \cos s_n x + O(\frac{1}{n}), & x \in (0,1]. \end{cases}$$

3. *If $k_{12} = 0$, $\alpha_2 \neq 0$, then for $k_{11} = k_{22}$*

$$\sqrt{\lambda_n} = s_n = \frac{n-1}{2}\pi + O\left(\frac{1}{n}\right).$$

And if n is even,

$$u(x,\lambda_n) = \begin{cases} \alpha_2 k_{11} s_n \cos s_n(x+1) + (he^{i\gamma} + \alpha_4 k_{11}) \sin s_n(x+1) + O(\frac{1}{n}), \ x \in [-1,0), \\ -\alpha_2 k_{11}^2 s_n \sin s_n x, \ x \in (0,1]; \end{cases}$$

if n is odd,

$$u(x,\lambda_n)=\begin{cases}\alpha_2 k_{11}s_n\cos s_n(x+1)+(he^{i\gamma}-\alpha_4 k_{11})\sin s_n(x+1)+O(\frac{1}{n}),\ x\in[-1,0),\\ (-1)^{\frac{n-1}{2}}\alpha_2 k_{11}^2 s_n\cos s_n x+O(\frac{1}{n}),\ x\in(0,1].\end{cases}$$

4. *If $k_{12}=0,\ \alpha_2=0,$ then for $\alpha_1=\alpha_4$*

$$\sqrt{\lambda_n}=s_n=\frac{2n-1}{4}\pi+O\left(\frac{1}{n}\right),$$

and

$$u(x,\lambda_n)=\begin{cases}(-1)^n k_{11}s_n\cos s_n(x+1)+\alpha_1 e^{i\gamma}\sin s_n(x+1)+O(\frac{1}{n}),\quad x\in[-1,0),\\ (-1)^n k_{11}^2\cos s_n(x+1)+\alpha_1 k_{11}e^{i\gamma}\sin s_n(x+1)+O(\frac{1}{n}),\quad x\in(0,1].\end{cases}$$

Proof. Let $\lambda=s^2$ with $s=\sigma+it$. We will use a similar method mentioned in [2, 8, 11]. From Theorem 4.1, for $k_{12}\neq 0,\ \alpha_2\neq 0,$

$$\Delta(\lambda)=\alpha_2 k_{12}s^2 e^{i\gamma}\sin^2 s+O(|s|e^{|t|}).$$

Denote $\Delta^*(\lambda)=\alpha_2 k_{12}s^2 e^{i\gamma}\sin^2 s$. Let

$$\Gamma_n'=\left\{\lambda=s^2=(\sigma+it)^2\mid |\sigma|=\left(n+\frac{1}{2}\right)\pi,\quad 0<t<\left(n+\frac{1}{2}\right)\pi\right\},$$

$$\Gamma_n''=\left\{\lambda=s^2=(\sigma+it)^2\mid |\sigma|\leqslant\left(n+\frac{1}{2}\right)\pi,\quad t=\left(n+\frac{1}{2}\right)\pi\right\}.$$

By applying the Rouche's Theorem in [13] and some calculations, we can get that $\Delta(\lambda)$ and $\Delta^*(\lambda)$ have the same zeros interior of $\Gamma_n=\Gamma_n'\cup\Gamma_n''$. Yet $\Delta^*(\lambda)$ has the zeros

$$0^2,\pi^2,(2\pi)^2,\cdots,(n\pi)^2,$$

interior of Γ_n. Therefore $\Delta(\lambda)$ has a sequence of zeros:

$$\sqrt{\lambda_n}=s_n=(n-1)\pi+O\left(\frac{1}{n}\right).$$

Let

$$u(x,\lambda_n)=\begin{cases}c_{n,1}\phi_1(x,\lambda_n)+c_{n,2}\chi_1(x,\lambda_n),\ x\in[-1,0),\\ c_{n,1}\phi_2(x,\lambda_n)+c_{n,2}\chi_2(x,\lambda_n),\ x\in(0,1],\end{cases}\tag{4.1}$$

where at least one of the constants $c_{n,1}$, $c_{n,2}$ is not zero. Then $u(x,\lambda_n)$ is the eigenfunction of the S-L problems (1.1)–(1.4) corresponding to the eigenvalue λ_n. ϕ_1 and χ_1 satisfy the conditions

$$y(-1)=1,\qquad y'(-1)=0\quad\text{and}\quad y(-1)=0,\qquad y'(-1)=1,$$

respectively. Substituting (4.1) into the boundary condition (1.2), we have

$$\begin{cases} c_{n,1}(\alpha_1 e^{i\gamma} - \phi_2(1,\lambda_n)) + c_{n,2}(\alpha_2 e^{i\gamma} - \chi_2(1,\lambda_n)) = 0, \\ c_{n,1}(\alpha_3 e^{i\gamma} - \phi_2'(1,\lambda_n)) + c_{n,2}(\alpha_4 e^{i\gamma} - \chi_2'(1,\lambda_n)) = 0. \end{cases}$$

Since the determinant of the coefficients $\Delta(\lambda) = 0,$ the equation system at least have a non-zero solution $(c_{n,1}, c_{n,2})$. And from the values of ϕ_2, χ_2, we have

$$\begin{cases} c_{n,1}\alpha_1 e^{i\gamma} + c_{n,2}(\alpha_2 e^{i\gamma} - k_{12}) = 0, \\ c_{n,1}\alpha_3 e^{i\gamma} + c_{n,2}\alpha_4 e^{i\gamma} = 0. \end{cases}$$

Multiplying the first and second equation by α_4 and $-\alpha_2$ respectively, and then adding these new two equations together, we obtain

$$c_{n,1} h e^{i\gamma} + c_{n,2}\alpha_4 k_{12} = 0$$

Let $c_{n,2} = he^{i\gamma} s_n$, then $c_{n,1} = \alpha_4 k_{12} s_n$. Hence (4.1) becomes

$$u(x,\lambda_n) = \begin{cases} \alpha_4 k_{12} s_n \cos s_n(x+1) + he^{i\gamma} \sin s_n(x+1) + O(\frac{1}{n}), & x \in [-1,0), \\ (-1)^{n-1} k_{12} s_n h e^{i\gamma} \cos s_n x + O(\frac{1}{n}), & x \in (0,1]. \end{cases}$$

The other cases can be proved in the same way. □

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REFERENCES

[1] R. A. ADAMS AND J. J. F. FOURNIER, *Sobolev Spaces*, Volume 140, Second Edition (Pure and Applied Mathematics), 2003.

[2] N. ALTINISIK, M. KADAKAL, O. SH. MUKHTAROV, *Eigenvalues and eigenfunctions of discontinuous Sturm-Liouville problems with eigenparameter-dependent boundary conditions*, Acta Mathematica Hungarica, **102** (2004), 159–193.

[3] N. ALTINISIK, O. SH. MUKHTAROV, M. KADAKAL, *Asymptotic formulas for eigenfunctions of the Sturm-Liouville problems with eigenvalue parameter in the boundary conditions*, Kuwait J. Sci. **39** (1A) (2012), 1–17.

[4] R. KH. AMIROV, *Eigenvalues and normalized eigenfunctions of discontinuous Sturm-Liouville problem with transmission conditions*, J. Math. Anal. Appl., **317** (1) (2006), 163–176.

[5] K. AYDEMIR, O. SH. MUKHTAROV, *Green's function method for self-adjoint realization of boundary-value problems with interior singularities*, Abstract and Applied Analysis, (2013), Article ID 503267, 7 pages.

[6] K. AYDEMIR, *Boundary value problems with eigenvalue-dependent boundary and transmission conditions*, Boundary value problems, **1** (2014), 131.

[7] E. BAIRAMOV, E. UGURLU, *The determinants of dissipative Sturm-Liouville operators with transmission conditions*, Mathematical and Computer Modelling, **53** (2011), 805–813.

[8] M. DEMIRCI, Z. AKDOĞAN, O. S. MUKHTAROV, *Asymptotic behavior of eigenvalues and eigenfunctions of one discontinuous boundary-value problem*, International Journal of Computational Cognition, **2** (3) 2004, 101–113.

[9] W. N. EVERITT, D. RACE, *On necessary and sufficient conditions for the existence of Caratheodory solutions of ordinary differential equations*, Quaest. Math., **3** (1976), 507–512.
[10] W. N. EVERITT, A. ZETTL, *Sturm-Liouville differential operators in direct sum spaces*, Rocky Mountain J. Math., **16** (3) (1986), 497–516.
[11] O. SH. MUKHTAROV, M. KADAKAL AND F. S. MUHTAROV, *Eigenvalues and normalized eigenfunctions of discontinuous Sturm-Liouville problem with transmission conditions*, Reports on Mathematical Physics, **54** (1) (2004), 41–56.
[12] C. SHIEH, V. A. YURKO, *Inverse nodal and inverse spectral problems for discontinuous boundary value problems*, J. Math. Anal. Appl., **347** (2008), 266–272.
[13] E. C. TITCHMARSH, *The Theory of Functions* (2nd ed.), Oxford University Press, 1939.
[14] E. C. TITCHMARSH, *Eigenfunction Expansions Associated with Second-order Differential Equations*, Clarendon Press, Oxford (1946).
[15] A. WANG, J. SUN, X. HAO, AND S. YAO, *Completeness of eigenfunctions of Sturm-Liouville problems with transmission conditions*, Methods Appl. Anal., **16** (3) (2009), 299–312.
[16] C. YANG, *Inverse nodal problems of discontinuous Sturm-Liouville operator*, J. Differential Equations, **254** (2013), 1992–2014.
[17] V. A. YURKO, *Boundary value problems with discontinuity conditions in an interior point of the interval*, Diff. Equa., **36** (8) (2000), 1266–1269.
[18] A. ZETTL, *Sturm-Liouville Theory*, AMS, Mathematical Surveys and Monographs vol. **121** (2005).

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