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MULTIPLIERS OF HILBERT PRO–C^*–BIMODULES AND CROSSED PRODUCTS BY HILBERT PRO–C^*–BIMODULES

MARIA JOIŢA, RADU-B. MUNTEANU AND IOANNIS ZARAKAS

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Abstract. In this paper we introduce the notion of multiplier of a Hilbert pro-C^*-bimodule and we investigate the structure of the multiplier bimodule of a Hilbert pro-C^*-bimodule. We also investigate the relationship between the crossed product $A\times_X\mathbb{Z}$ of a pro-C^*-algebra A by a Hilbert pro-C^*-bimodule X over A, the crossed product $M(A)\times_{M(X)}\mathbb{Z}$ of the multiplier algebra $M(A)$ of A by the multiplier bimodule $M(X)$ of X and the multiplier algebra $M(A\times_X\mathbb{Z})$ of $A\times_X\mathbb{Z}$.

1. Introduction

The notion of a Hilbert C^*-module is a generalization of that of a Hilbert space in which the inner product takes its values in a C^*-algebra rather than in the field of complex numbers, but the theory of Hilbert C^*-modules is different from the theory of Hilbert spaces (for example, not every Hilbert C^*-submodule is complemented). In 1953, Kaplansky first used Hilbert C^*-modules over commutative C^*-algebras to prove that derivations of type I AW^*-algebras are inner. In 1973, the theory was extended independently by Paschke and Rieffel to non-commutative C^*-algebras and the latter author used it to construct the theory of "induced representations of C^*-algebras". Moreover, Hilbert C^*-modules gave the right context for the extension of the notion of Morita equivalence to C^*-algebras and have played a crucial role in Kasparov's KK-theory. Finally, they may be considered as a generalization of vector bundles to non-commutative $*$-algebras, therefore they play a significant role in non-commutative geometry and, in particular, in C^*-algebraic quantum group theory and groupoid C^*-algebras. The extension of such a rich in results concept, to the case of pro-C^*-algebras could not be disregarded.

In [17], Zarakas introduced the notion of a Hilbert pro-C^*-bimodule over a pro-C^*-algebra and studied its structure. In [8], Joiţa investigated the structure of the multiplier module of a Hilbert pro-C^*-module. In this paper we introduce the notion of multiplier of a Hilbert pro-C^*-bimodule and we investigate the structure of the multiplier bimodule of a Hilbert pro-C^*-bimodule.

In [11], Joiţa and Zarakas extended the construction of Abadie, Eilers and Exel [2] in the context of pro-C^*-algebras and associated to a Hilbert pro-C^*-bimodule (X,A)

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a pro-C^*-algebra $A \times_X \mathbb{Z}$, called the crossed product of A by X. It is natural to ask what is the relationship between the pro-C^*-algebras associated to a Hilbert pro-C^*-bimodule (X,A) and its multiplier bimodule $(M(X),M(A))$.

The organization of this paper is as follows. In Section 2, we recall some notations and definitions. Section 3 is devoted to investigate multipliers of a Hilbert pro-C^*-bimodule. Given a Hilbert pro-C^*-bimodule X, we show that the Hilbert pro-C^*-bimodule structure on X extends to a Hilbert pro-C^*-bimodule structure on the multiplier bimodule $M(X)$ of X. Also we define the strict topology on $M(X)$ and show that X can be identified with a Hilbert pro-C^*-sub-bimodule of $M(X)$ which is dense in $M(X)$ with respect to the strict topology. We introduce the notion of morphism of Hilbert pro-C^*-bimodules, and show that a nondegenerate morphism between Hilbert pro-C^*-bimodules is continuous with respect to the strict topology and it extends to a unique morphism between the multiplier bimodules. Finally, as in the case of Hilbert C^*-bimodules [15], we show that $(M(X),M(A))$ can be regarded as a maximal extension of (X,A). Section 4 is devoted to investigate the relationship between the crossed product $A \times_X \mathbb{Z}$ of a pro-C^*-algebra A by a Hilbert pro-C^*-bimodule X over A, the crossed product $M(A) \times_{M(X)} \mathbb{Z}$ of the multiplier algebra $M(A)$ of A by the multiplier bimodule $M(X)$ of X and the multiplier algebra $M(A \times_X \mathbb{Z})$ of $A \times_X \mathbb{Z}$. We show that the crossed product associated to a full Hilbert pro-C^*-bimodule (X,A) can be identified with a pro-C^*-subalgebra of the crossed product associated to $(M(X),M(A))$ and the crossed product associated to $(M(X),M(A))$ can be identified with a pro-C^*-subalgebra of the multiplier algebra of the crossed product associated to (X,A). Crossed products by Hilbert pro-C^*-bimodules are generalizations of crossed products of pro-C^*-algebras by inverse limit automorphism [11]. As an application, we prove that given an inverse limit automorphism α of a nonunital pro-C^*-algebra A, the crossed product of $M(A)$ by $\overline{\alpha}$, the extension of α to $M(A)$, can be identified with a pro-C^*-subalgebra of the multiplier algebra $M(A \times_\alpha \mathbb{Z})$ of $A \times_\alpha \mathbb{Z}$.

2. Preliminaries

A complete Hausdorff topological $*$-algebra A whose topology is given by a directed family of C^*-seminorms $\{p_\lambda; \lambda \in \Lambda\}$ is called a *pro-C^*-algebra*. Other terms used in the literature for pro-C^*-algebras are: locally C^*-algebras (A. Inoue, M. Fragoulopoulou, A. Mallios, etc.), *LMC**-algebras (G. Lassner, K. Schmüdgen), b^*-algebras (C. Apostol).

Let A be a pro-C^*-algebra with the topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$ and let B be a pro-C^*-algebra with the topology given by $\Gamma' = \{q_\delta; \delta \in \Delta\}$.

An approximate unit of A is a net $\{e_i\}_{i \in I}$ of positive elements in A such that $p_\lambda(e_i) \leqslant 1$ for all $i \in I$ and for all $\lambda \in \Lambda$ and the nets $\{e_i b\}_{i \in I}$ and $\{b e_i\}_{i \in I}$ converge to b for all $b \in A$.

A *pro-C^*-morphism* is a continuous $*$-morphism $\varphi : A \to B$ (that is, φ is linear, $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ for all $a, b \in A$ and for each $q_\delta \in \Gamma'$, there is $p_\lambda \in \Gamma$ such that $q_\delta(\varphi(a)) \leqslant p_\lambda(a)$ for all $a \in A$). An invertible pro-C^*-morphism $\varphi : A \to B$ is a pro-C^*-isomorphism if φ^{-1} is also pro-C^*-morphism.

$\{\left(A_\lambda, \|\cdot\|_{A_\lambda}\right); \pi_{\lambda\mu}\}_{\lambda \geqslant \mu, \lambda, \mu \in \Lambda}$ is an inverse system of C^*-algebras, then $\lim\limits_{\leftarrow \lambda} A_\lambda$ with the topology given by the family of C^*-seminorms $\{p_\lambda\}_{\lambda \in \Lambda}$, with $p_\lambda\left(\left(a_\mu\right)_{\mu \in \Lambda}\right) = \|a_\lambda\|_{A_\lambda}$ for all $\lambda \in \Lambda$, is a pro-C^*-algebra.

Let A be a pro-C^*-algebra with the topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$. For $\lambda \in \Lambda$, $\ker p_\lambda$ is a closed $*$-bilateral ideal and $A_\lambda = A/\ker p_\lambda$ is a C^*-algebra in the C^*-norm $\|\cdot\|_{p_\lambda}$ induced by p_λ (that is, $\|a + \ker p_\lambda\|_{p_\lambda} = p_\lambda(a)$, for all $a \in A$). The canonical map from A to A_λ is denoted by π_λ^A, $\pi_\lambda^A(a) = a + \ker p_\lambda$ for all $a \in A$. For $\lambda, \mu \in \Lambda$ with $\mu \leqslant \lambda$ there is a surjective C^*-morphism $\pi_{\lambda\mu}^A : A_\lambda \to A_\mu$ such that $\pi_{\lambda\mu}^A(a + \ker p_\lambda) = a + \ker p_\mu$, and then $\{A_\lambda; \pi_{\lambda\mu}^A\}_{\lambda,\mu \in \Lambda}$ is an inverse system of C^*-algebras. Moreover, the pro-C^*-algebras A and $\lim\limits_{\leftarrow \lambda} A_\lambda$ are isomorphic (Arens-Michael decomposition). For further information on pro-C^*-algebras we refer the reader to [6, 13, 14].

Here we recall some basic facts from [7] and [17] regarding Hilbert pro-C^*-modules and Hilbert pro-C^*-bimodules respectively.

Let A be a pro-C^*-algebra whose topology is given by the family of C^*-seminorms $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$.

A right Hilbert pro-C^*-module over A (or just *Hilbert A-module*), is a linear space X that is also a right A-module equipped with a right A-valued inner product $\langle\cdot,\cdot\rangle_A$, that is $\mathbb{C}$- and A-linear in the second variable and conjugate linear in the first variable, with the following properties:

1. $\langle x,x\rangle_A \geqslant 0$ and $\langle x,x\rangle_A = 0$ if and only if $x = 0$;
2. $(\langle x,y\rangle_A)^* = \langle y,x\rangle_A$

and which is complete with respect to the topology given by the family of seminorms $\{p_\lambda^A\}_{\lambda \in \Lambda}$, with $p_\lambda^A(x) = p_\lambda(\langle x,x\rangle_A)^{\frac{1}{2}}, x \in X$. A Hilbert A-module X is full if the pro-C^*- subalgebra of A generated by $\{\langle x,y\rangle_A; x,y \in X\}$ coincides with A.

A left Hilbert pro-C^*-module X over a pro-C^*-algebra A is defined in the same way, where for instance the completeness is requested with respect to the family of seminorms $\{{}^A p_\lambda\}_{\lambda \in \Lambda}$, where ${}^A p_\lambda(x) = p_\lambda({}_A\langle x,x\rangle)^{\frac{1}{2}}, x \in X$.

In the case X is a left Hilbert pro-C^*-module over $(A, \{p_\lambda\}_{\lambda \in \Lambda})$ and a right Hilbert pro-C^*-module over $(B, \{q_\lambda\}_{\lambda \in \Lambda})$, such that the following relations hold:

- ${}_A\langle x,y\rangle z = x\langle y,z\rangle_B$ for all $x,y,z \in X$,
- $q_\lambda^B(ax) \leqslant p_\lambda(a) q_\lambda^B(x)$ and ${}^A p_\lambda(xb) \leqslant q_\lambda(b)\,{}^A p_\lambda(x)$ for all $x \in X, a \in A, b \in B$ and for all $\lambda \in \Lambda$,

then we say that X is *a Hilbert $A - B$ pro-C^*-bimodule*.

A Hilbert $A - B$ pro-C^*-bimodule X is *full* if it is full as a right and as a left Hilbert pro-C^*-module. Throughout the paper we use the notation (X, A) to denote a Hilbert $A - A$ (pro-) C^*-bimodule X.

Let Λ be an upward directed set and $\{A_\lambda; B_\lambda; X_\lambda; \pi_{\lambda\mu}; \chi_{\lambda\mu}; \sigma_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geqslant \mu\}$ an inverse system of Hilbert C^*-bimodules, that is:

- $\{A_\lambda;\pi_{\lambda\mu};\lambda,\mu\in\Lambda,\lambda\geqslant\mu\}$ and $\{B_\lambda;\chi_{\lambda\mu};\lambda,\mu\in\Lambda,\lambda\geqslant\mu\}$ are inverse systems of C^*-algebras;
- $\{X_\lambda;\sigma_{\lambda\mu};\lambda,\mu\in\Lambda,\lambda\geqslant\mu\}$ is an inverse system of Banach spaces;
- for each $\lambda\in\Lambda$, X_λ is a Hilbert $A_\lambda-B_\lambda$ C^*-bimodule;
- $\left\langle\sigma_{\lambda\mu}(x),\sigma_{\lambda\mu}(y)\right\rangle_{B_\mu}=\chi_{\lambda\mu}\left(\langle x,y\rangle_{B_\lambda}\right)$ and ${}_{A_\mu}\left\langle\sigma_{\lambda\mu}(x),\sigma_{\lambda\mu}(y)\right\rangle=\pi_{\lambda\mu}\left({}_{A_\lambda}\langle x,y\rangle\right)$ for all $x,y\in X_\lambda$ and for all $\lambda,\mu\in\Lambda$ with $\lambda\geqslant\mu$.
- $\sigma_{\lambda\mu}(x)\chi_{\lambda\mu}(b)=\sigma_{\lambda\mu}(xb)$, $\pi_{\lambda\mu}(a)\sigma_{\lambda\mu}(x)=\sigma_{\lambda\mu}(ax)$ for all $x\in X_\lambda$, $a\in A_\lambda$, $b\in B_\lambda$ and for all $\lambda,\mu\in\Lambda$ with $\lambda\geqslant\mu$.

Let $A=\lim\limits_{\leftarrow\lambda}A_\lambda$, $B=\lim\limits_{\leftarrow\lambda}B_\lambda$ and $X=\lim\limits_{\leftarrow\lambda}X_\lambda$. Then X has a structure of a Hilbert $A-B$ pro-C^*-bimodule with

$$(x_\lambda)_{\lambda\in\Lambda}(b_\lambda)_{\lambda\in\Lambda}=(x_\lambda b_\lambda)_{\lambda\in\Lambda} \text{ and } \left\langle(x_\lambda)_{\lambda\in\Lambda},(y_\lambda)_{\lambda\in\Lambda}\right\rangle_B=\left(\langle x_\lambda,y_\lambda\rangle_{B_\lambda}\right)_{\lambda\in\Lambda}$$
and
$$(a_\lambda)_{\lambda\in\Lambda}(x_\lambda)_{\lambda\in\Lambda}=(a_\lambda x_\lambda)_{\lambda\in\Lambda} \text{ and } {}_A\left\langle(x_\lambda)_{\lambda\in\Lambda},(y_\lambda)_{\lambda\in\Lambda}\right\rangle=\left({}_{A_\lambda}\langle x_\lambda,y_\lambda\rangle\right)_{\lambda\in\Lambda}.$$

Let X be a Hilbert $A-B$ pro-C^*-bimodule. Then, for each $\lambda\in\Lambda$, ${}^Ap_\lambda(x)=q_\lambda^B(x)$ for all $x\in X$, and the normed space $X_\lambda=X/N_\lambda^B$, where $N_\lambda^B=\{x\in X;q_\lambda^B(x)=0\}$, is complete in the norm $||x+N_\lambda^B||_{X_\lambda}=q_\lambda^B(x),x\in X$. Moreover, X_λ has a canonical structure of a Hilbert $A_\lambda-$ B_λ C^*-bimodule with $\left\langle x+N_\lambda^B,y+N_\lambda^B\right\rangle_{B_\lambda}=\langle x,y\rangle_B+\ker q_\lambda$ and ${}_{A_\lambda}\left\langle x+N_\lambda^B,y+N_\lambda^B\right\rangle={}_A\langle x,y\rangle+\ker p_\lambda$ for all $x,y\in X$. The canonical surjection from X to X_λ is denoted by σ_λ^X. For $\lambda,\mu\in\Lambda$ with $\lambda\geqslant\mu$, there is a canonical surjective linear map $\sigma_{\lambda\mu}^X:X_\lambda\to X_\mu$ such that $\sigma_{\lambda\mu}^X\left(x+N_\lambda^B\right)=x+N_\mu^B$ for all $x\in X$. Then $\{A_\lambda;B_\lambda;X_\lambda;\pi_{\lambda\mu}^A;\pi_{\lambda\mu}^B;\sigma_{\lambda\mu}^X;\lambda,\mu\in\Lambda,\lambda\geqslant\mu\}$ is an inverse system of Hilbert C^*-bimodules in the above sense.

Let X and Y be Hilbert pro-C^*-modules over B. A morphism $T:X\to Y$ of right modules is *adjointable* if there is another morphism of modules $T^*:Y\to X$ such that $\langle Tx,y\rangle_B=\langle x,T^*y\rangle_B$ for all $x\in X,y\in Y$. The vector space $L_B(X,Y)$ of all adjointable module morphisms from X to Y has a structure of locally convex space under the topology given by the family of seminorms $\{q_{\lambda,L_B(X,Y)}\}_{\lambda\in\Lambda}$, where $q_{\lambda,L_B(X,Y)}(T)=\sup\{q_\lambda^B(Tx);x\in X,q_\lambda^B(x)\leqslant 1\}$. Moreover, $\{L_{B_\lambda}(X_\lambda,Y_\lambda);\chi_{\lambda\mu}^{L_B(X,Y)};\lambda,\mu\in\Lambda,\lambda\geqslant\mu\}$ where $\chi_{\lambda\mu}^{L_B(X,Y)}:L_{B_\lambda}(X_\lambda,Y_\lambda)\to L_{B_\mu}(X_\mu,Y_\mu)$ is given by $\chi_{\lambda\mu}^{L_B(X,Y)}(T)\left(\sigma_\mu^X(x)\right)=\sigma_{\lambda\mu}^Y\left(T(\sigma_\lambda^X(x))\right)$, is an inverse system of Banach spaces and $L_B(X,Y)=\lim\limits_{\leftarrow\lambda}L_{B_\lambda}(X_\lambda,Y_\lambda)$ up to an isomorphism of locally convex spaces. The canonical projections $\chi_\lambda^{L_B(X,Y)}:L_B(X,Y)\to L_{B_\lambda}(X_\lambda,Y_\lambda)$, $\lambda\in\Lambda$ are given by $\chi_\lambda^{L_B(X,Y)}(T)\left(\sigma_\lambda^X(x)\right)=\sigma_\lambda^Y(T(x))$ for all $x\in X$. For $x\in X$ and $y\in Y$, the map $\theta_{y,x}:X\to Y$ given by $\theta_{y,x}(z)=y\langle x,z\rangle_B$ is an adjointable module morphism and the closed subspace of $L_B(X,Y)$ generated by $\{\theta_{y,x};x\in X$ and $y\in Y\}$ is denoted by $K_B(X,Y)$, whose elements are usually called *compact operators*. For $Y=X$, $L_B(X)=L_B(X,X)$ is a pro-C^*-algebra with $(L_B(X))_\lambda=$

$L_{B_\lambda}(X_\lambda)$ for each $\lambda \in \Lambda$, and $K_B(X) = K_B(X,X)$ is a closed two-sided $*$-ideal of $L_B(X)$ with $(K_B(X))_\lambda = K_{B_\lambda}(X_\lambda)$ for each $\lambda \in \Lambda$.

A pro-C^*-algebra A has a natural structure of Hilbert pro-C^*-module, and the multiplier algebra $M(A)$ has a structure of pro-C^*-algebra which is isomorphic to $L_A(A)$ [14]. Moreover, pro-C^*-algebras A and $K_A(A)$ are isomorphic and A is a closed bilateral ideal of $M(A)$ which is dense in $M(A)$ with respect to the strict topology. The strict topology on $M(A)$ is given by the family of seminorms $\{p_{(\lambda,a)}\}_{(\lambda,a)\in\Lambda\times A}$, where $p_{(\lambda,a)}(b) = p_\lambda(ab) + p_\lambda(ba)$ for all $b \in M(A)$.

A pro-C^*-morphism $\varphi : A \to M(B)$ is nondegenerate if $[\varphi(A)B] = B$, where $[\varphi(A)B]$ denotes the closed subspace of B generated by $\{\varphi(a)b; a \in A, b \in B\}$. A nondegenerate pro-$C^*$-morphism $\varphi : A \to M(B)$ extends to a unique pro-C^*-morphism $\overline{\varphi} : M(A) \to M(B)$ which is strictly continuous on bounded sets.

Throughout this paper, A and B will denote two pro-C^*-algebras whose topologies are given by the families of C^*-seminorms $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$, respectively $\Gamma' = \{q_\delta; \delta \in \Delta\}$.

3. Multipliers of Hilbert pro-C^*-bimodules

Let X and Y be two Hilbert pro-C^*-modules over A.

PROPOSITION 3.1. *The vector space $L_A(X,Y)$ of all adjointable module maps from X to Y has a natural structure of Hilbert $L_A(Y) - L_A(X)$ pro-C^*-bimodule with the bimodule structure given by*

$$S \cdot T = S \circ T \text{ and } T \cdot R = T \circ R$$

for all $T \in L_A(X,Y), S \in L_A(Y)$ and $R \in L_A(X)$ and the inner products given by

$${}_{L_A(Y)}\langle T_1, T_2\rangle = T_1 \circ T_2^* \text{ and } \langle T_1, T_2\rangle_{L_A(X)} = T_1^* \circ T_2$$

for all $T_1, T_2 \in L_A(X,Y)$.

Proof. It is a simple calculation to verify that $L_A(X,Y)$ has a structure of pre-right Hilbert $L_A(X)$-pro-C^*-module with

$$T \cdot R = T \circ R \text{ and } \langle T_1, T_2\rangle_{L_A(X)} = T_1^* \circ T_2$$

and $L_A(X,Y)$ has a structure of pre-left Hilbert $L_A(Y)$-pro-C^*-module with

$$S \cdot T = S \circ T \text{ and } {}_{L_A(Y)}\langle T_1, T_2\rangle = T_1 \circ T_2^*.$$

Moreover,

$$\begin{aligned} p_\lambda^{L_A(X)}(T)^2 &= p_{\lambda,L_A(X)}\left(\langle T,T\rangle_{L_A(X)}\right) = p_{\lambda,L_A(X)}(T^* \circ T) \\ &= \left\|\chi_\lambda^{L_A(X,Y)}(T)^* \chi_\lambda^{L_A(X,Y)}(T)\right\|_{L_{A_\lambda}(X_\lambda)} \\ &\quad \text{(see, for example, the proof of Proposition 1.10 [5])} \\ &= \left\|\chi_\lambda^{L_A(X,Y)}(T)\right\|^2_{L_{A_\lambda}(X_\lambda,Y_\lambda)} = p_{\lambda.L_A(X,Y)}(T)^2 \end{aligned}$$

and

$$
\begin{aligned}
{}^{L_A(Y)}p_\lambda(T)^2 &= p_{\lambda,L_A(Y)}\left({}_{L_A(Y)}\langle T,T\rangle\right) = p_{\lambda,L_A(Y)}(T\circ T^*)\\
&= \left\|\chi_\lambda^{L_A(X,Y)}(T)\,\chi_\lambda^{L_A(X,Y)}(T)^*\right\|_{L_{A_\lambda}(Y_\lambda)}\\
&= \left\|\chi_\lambda^{L_A(X,Y)}(T)^*\right\|^2_{L_{A_\lambda}(Y_\lambda,X_\lambda)}\\
&\quad \text{(see, for example, the proof of Proposition 1.10 [5])}\\
&= \left\|\chi_\lambda^{L_A(X,Y)}(T)\right\|^2_{L_{A_\lambda}(X_\lambda,Y_\lambda)} = p_{\lambda.L_A(X,Y)}(T)^2
\end{aligned}
$$

for all $T\in L_A(X,Y)$ and for all $\lambda\in\Lambda$. Therefore, $L_A(X,Y)$ is a left Hilbert $L_A(Y)$-module and a right Hilbert $L_A(X)$-module.

Also it is easy to check that ${}_{L_A(Y)}\langle T_1,T_2\rangle\cdot T_3 = T_1\cdot\langle T_2,T_3\rangle_{L_A(X)}$ for all $T_1,T_2,T_3\in L_A(X,Y)$, and since $p_\lambda^{L_A(X)}(T) = {}^{L_A(Y)}p_\lambda(T) = p_{\lambda.L_A(X,Y)}(T)$ for all $T\in L_A(X,Y)$ and for all $\lambda\in\Lambda$, $L_A(X,Y)$ has a structure of Hilbert $L_A(Y)-L_A(X)$ pro-C^*-bimodule. $\square$

REMARK 3.2. Suppose that (X,A) is a full Hilbert pro-C^*-bimodule. Then there is a pro-C^*-isomorphism $\Phi_A : A\to K_A(X)$ given by $\Phi_A(a)(x) = a\cdot x$ which extends to a pro-C^*-isomorphism $\overline{\Phi_A} : M(A)\to L_A(X)$. Moreover, $p_{\lambda,L_A(X)}(\Phi_A(a)) = p_\lambda(a)$ for all $a\in A$ and $\lambda\in\Lambda$. Identifying $M(A)$ with $L_A(A)$ and using Proposition 3.1 and [15, Proposition 2.5], we obtain a natural structure of Hilbert $M(A)-M(A)$ pro–C^*-bimodule on $L_A(A,X)$ with

$$m\cdot T = \overline{\Phi_A}(m)\circ T \text{ and } {}_{M(A)}\langle T_1,T_2\rangle = \overline{\Phi_A^{-1}}(T_1\circ T_2^*)$$

and

$$T\cdot m = T\circ m \text{ and } \langle T_1,T_2\rangle_{M(A)} = T_1^*\circ T_2$$

for all $T,T_1,T_2\in L_A(A,X)$ and $m\in M(A)$.

DEFINITION 3.3. Let (X,A) be a full Hilbert pro-C^*-bimodule. The Hilbert $M(A)-M(A)$ pro-C^*-bimodule $L_A(A,X)$ is called the multiplier bimodule of X and it is denoted by $M(X)$.

The following definition is a generalization of [5, Definition 1.25].

DEFINITION 3.4. The strict topology on $M(X)$ is given by the family of seminorms $\{p_{(\lambda,a)}\}_{(\lambda,a)\in\Lambda\times A}$, where $p_{(\lambda,a)}(T) = p_\lambda^{M(A)}(T\cdot a) + p_\lambda^{M(A)}(a\cdot T)$ for all $T\in M(X)$ and $a\in A$.

REMARK 3.5. Let $\{T_n\}_n$ be a sequence in $M(X)$.

1. If $\{T_n\}_n$ is strictly convergent, then it is bounded. Indeed, if $\{T_n\}_n$ converges strictly to $T \in M(X)$, then for each $\lambda \in \Lambda$, since

$$\begin{aligned}\left\|\chi_\lambda^{M(X)}(T_n)\pi_\lambda^A(a) - \chi_\lambda^{M(X)}(T)\pi_\lambda^A(a)\right\|_{X_\lambda} &= p_\lambda^A(T_n(a) - T(a)) \\ &= p_\lambda^{M(A)}(T_n \cdot a - T \cdot a),\end{aligned}$$

the sequence $\{\chi_\lambda^{M(X)}(T_n)\pi_\lambda^A(a)\}_n$ converges to $\chi_\lambda^{M(X)}(T)\pi_\lambda^A(a)$ for all $a \in A$ and by the Banach-Steinhaus theorem there is $M_\lambda > 0$ such that

$$p_\lambda^{M(A)}(T_n) = p_{\lambda, L_A(A,X)}(T_n) = \left\|\chi_\lambda^{M(X)}(T_n)\right\|_{L_{A_\lambda}(A_\lambda, X_\lambda)} \leqslant M_\lambda$$

for all $n \in \mathbb{N}$.

2. If $\{T_n\}_n$ converges strictly to 0, then the sequences $\{\langle T_n, T_n\rangle_{M(A)}\}_n$ and $\{_{M(A)}\langle T_n, T_n\rangle\}_n$ are strictly convergent to 0 in $M(A)$.

Suppose that X is a Hilbert pro-C^*-module over A. In [8, Definition 3.2], the strict topology on $L_A(A,X)$ is given by the family of seminorms $\{p_{(\lambda,a,x)}\}_{(\lambda,a,x)\in\Lambda\times A\times X}$, where $p_{(\lambda,a,x)}(T) = p_\lambda^A(T(a)) + p_\lambda(T^*(x))$. We will show that this definition coincides with the above definition of the strict topology on $M(X)$ on bounded subsets when X is a full Hilbert $A-A$ pro-C^*-bimodule. To show this, we will use the following result.

LEMMA 3.6. *Let X be a Hilbert pro-C^*-module over A. For each x in X there is a unique element y in X such that $x = y\langle y, y\rangle_A$.*

Proof. Let $x \in X$. For each $\lambda \in \Lambda$, there is a unique element $y_\lambda \in X_\lambda$ such that $\sigma_\lambda^X(x) = y_\lambda \langle y_\lambda, y_\lambda\rangle_{A_\lambda}$ (see, for example, [16, Proposition 2.31]). Let $\lambda, \mu \in \Lambda$ with $\lambda \geqslant \mu$. From

$$\sigma_\mu^X(x) = \sigma_{\lambda\mu}^X(\sigma_\lambda^X(x)) = \sigma_{\lambda\mu}^X(y_\lambda)\left\langle \sigma_{\lambda\mu}^X(y_\lambda), \sigma_{\lambda\mu}^X(y_\lambda)\right\rangle_{A_\mu}$$

and [16, Proposition 2.31], we deduce that $\sigma_{\lambda\mu}^X(y_\lambda) = y_\mu$. Therefore, there exists $y \in X$ such that $\sigma_\lambda^X(y) = y_\lambda$ for all $\lambda \in \Lambda$ and $x = y\langle y, y\rangle_A$. Moreover, y is unique with this property. □

PROPOSITION 3.7. *Let (X, A) be a full Hilbert pro-C^*-bimodule and $\{T_i\}_{i\in I}$ a net in $M(X)$.*

1. *If $\{T_i\}_{i\in I}$ converges strictly to 0, then $\{p_{(\lambda,a,x)}(T_i)\}_{i\in I}$ converges to 0 for all $a \in A$, for all $x \in X$ and for all $\lambda \in \Lambda$.*

2. *If $\{T_i\}_{i\in I}$ is bounded and $\{p_{(\lambda,a,x)}(T_i)\}_{i\in I}$ converges to 0 for all $a \in A$, for all $x \in X$ and for all $\lambda \in \Lambda$, then $\{T_i\}_{i\in I}$ converges strictly to 0.*

Proof. (1) If the net $\{T_i\}_{i\in I}$ converges strictly to 0, then $\{p_\lambda^A(T_i(a))\}_{i\in I}$ converges to 0 for all $a\in A$ and $\lambda\in\Lambda$. Let $x\in X$ and $\lambda\in\Lambda$. Then, by Lemma 3.6, there is $y\in X$ such that $x=y\langle y,y\rangle_A=\theta_{y,y}(y)$. From

$$\begin{aligned} p_\lambda\left(T_i^*(x)\right) &= p_\lambda\left(T_i^*\left(\theta_{y,y}(y)\right)\right)\leqslant p_{\lambda,L_A(X,A)}\left(T_i^*\circ\theta_{y,y}\right)p_\lambda^A(y)\\ &= p_{\lambda,L_A(A,X)}\left(\theta_{y,y}\circ T_i\right)p_\lambda^A(y)=p_\lambda^{M(A)}\left(\theta_{y,y}\circ T_i\right)p_\lambda^A(y), \end{aligned}$$

we deduce that the net $\{p_\lambda(T_i^*(x))\}_{i\in I}$ converges to 0.

(2) If $\{p_{(\lambda,a,x)}(T_i)\}_{i\in I}$ converges to 0 for all $a\in A$, $x\in X$ and $\lambda\in\Lambda$, then $\{p_\lambda^A(T_i(a))\}_{i\in I}$ converges to 0 for all $a\in A$ and $\lambda\in\Lambda$. Let $S\in K_A(X)$, $\lambda\in\Lambda$ and $\varepsilon>0$. Then there is $\sum\limits_{k=1}^n\theta_{x_k,y_k}$ such that $p_{\lambda,L_A(X)}\left(S-\sum\limits_{k=1}^n\theta_{x_k,y_k}\right)<\varepsilon$, and since $\{T_i\}_{i\in I}$ is bounded, there is $M_\lambda>0$ such that $p_\lambda^{M(A)}(T_i)\leqslant M_\lambda$ for all $i\in I$. From

$$\begin{aligned} p_\lambda^{M(A)}(S\circ T_i) &\leqslant p_{\lambda,L_A(X)}\left(S-\sum_{k=1}^n\theta_{x_k,y_k}\right)\,p_\lambda^{M(A)}(T_i)+p_\lambda^{M(A)}\left(\sum_{k=1}^n\theta_{x_k,y_k}\circ T_i\right)\\ &\leqslant \varepsilon M_\lambda+p_{\lambda,L_A(A,X)}\left(\sum_{k=1}^n\theta_{x_k,T_i^*(y_k)}\right)\\ &\leqslant \varepsilon M_\lambda+\sum_{k=1}^n p_\lambda^A(x_k)\,p_\lambda\left(T_i^*(y_k)\right) \end{aligned}$$

we deduce that $\{p_\lambda^{M(A)}(S\circ T_i)\}_{i\in I}$ converges to 0. $\square$

Let (X,A) and (Y,B) be two Hilbert pro-C^*-bimodules.

DEFINITION 3.8. *A morphism of Hilbert pro-C^*-bimodules from* (X,A) to (Y,B) is a pair (Φ,φ) consisting of a pro-C^*-morphism $\varphi:A\to B$ and a map $\Phi:X\to Y$ such that:

1. $\Phi(xa)=\Phi(x)\varphi(a)$ for all $x\in X$ and for all $a\in A$;
2. $\Phi(ax)=\varphi(a)\Phi(x)$ for all $x\in X$ and for all $a\in A$;
3. $\langle\Phi(x),\Phi(y)\rangle_B=\varphi(\langle x,y\rangle_A)$ for all $x,y\in X$;
4. ${}_B\langle\Phi(x),\Phi(y)\rangle=\varphi({}_A\langle x,y\rangle)$ for all $x,y\in X$.

The relation (3) implies the relation (1) and the relation (4) implies (2).

If $(\Phi,\varphi):(X,A)\to(Y,B)$ is a morphism of Hilbert pro-C^*-bimodules, then Φ is continuous, since for each $\delta\in\Delta$, there is $\lambda\in\Lambda$ such that

$$q_\delta^B(\Phi(x))^2=q_\delta\left(\langle\Phi(x),\Phi(x)\rangle_B\right)=q_\delta\left(\varphi\left(\langle x,x\rangle_A\right)\right)\leqslant p_\lambda\left(\langle x,x\rangle_A\right)=p_\lambda^A(x)^2$$

for all $x\in X$. It is easy to check that if φ is injective, then Φ is injective, and if (X,A) is full and Φ is injective, then φ is injective.

DEFINITION 3.9. An isomorphism of *Hilbert pro-C^*-bimodules is a morphism of Hilbert pro-C^*-bimodules* (Φ,φ) such that φ is a pro-C^*-isomorphism and the map Φ is bijective.

The Hilbert pro-C^*-bimodules (X,A) *and* (Y,B) are *isomorphic* if there is an isomorphism of Hilbert pro-C^*-bimodules $(\Phi,\varphi):(X,A)\to(Y,B)$.

DEFINITION 3.10. A *morphism of Hilbert pro-C^*-bimodules* $(\Phi,\varphi):(X,A)\to(M(Y),M(B))$ is nondegenerate if φ is nondegenerate and $[\Phi(X)B]=Y$.

REMARK 3.11. If $(\Phi,\varphi):(X,A)\to(M(Y),M(B))$ is nondegenerate and (X,A) is full, then (Φ,φ) is nondegenerate in the sense of [9, Definition 3.1], since

$$\begin{aligned}\left[\Phi(X)^*Y\right] &= \left[\Phi(X)^*\Phi(X)B\right] = \left[\langle\Phi(X),\Phi(X)\rangle_{M(B)}B\right]\\ &= \left[\varphi\left(\langle X,X\rangle_A\right)B\right] = \left[\varphi(A)B\right] = B.\end{aligned}$$

LEMMA 3.12. *Let* (X,A) *be a full Hilbert pro-C^*-bimodule. Then the maps*

$$\left(\chi_\lambda^{L_A(A,X)},\pi_\lambda^{M(A)}\right):(M(X),M(A))\to(M(X_\lambda),M(A_\lambda)),\lambda\in\Lambda,$$

where $\pi_\lambda^{M(A)}=\chi_\lambda^{L_A(A)}$, *and*

$$\left(\chi_{\lambda\mu}^{L_A(A,X)},\pi_{\lambda\mu}^{M(A)}\right):(M(X_\lambda),M(A_\lambda))\to\left(M(X_\mu),M(A_\mu)\right),\lambda,\mu\in\Lambda \text{ with } \lambda\geqslant\mu$$

where $\pi_{\lambda\mu}^{M(A)}=\chi_{\lambda\mu}^{L_A(A)}$, *are all strictly continuous morphisms of Hilbert bimodules.*

Proof. Let $\lambda,\mu\in\Lambda$ with $\lambda\geqslant\mu$. For $T_1,T_2\in M(X_\lambda)$ we have

$$\begin{aligned}\left\langle\chi_{\lambda\mu}^{L_A(A,X)}(T_1),\chi_{\lambda\mu}^{L_A(A,X)}(T_2)\right\rangle_{M(A_\mu)} &= \chi_{\lambda\mu}^{L_A(A,X)}(T_1)^*\circ\chi_{\lambda\mu}^{L_A(A,X)}(T_2)\\ &= \chi_{\lambda\mu}^{L_A(X,A)}(T_1^*)\circ\chi_{\lambda\mu}^{L_A(A,X)}(T_2)\\ &= \chi_{\lambda\mu}^{L_A(A)}(T_1^*\circ T_2) = \pi_{\lambda\mu}^{M(A)}\left(\langle T_1,T_2\rangle_{M(A_\lambda)}\right)\end{aligned}$$

and

$$\begin{aligned}{}_{M(A_\mu)}\left\langle\chi_{\lambda\mu}^{L_A(A,X)}(T_1),\chi_{\lambda\mu}^{L_A(A,X)}(T_2)\right\rangle &= \overline{\Phi_{A_\mu}^{-1}}\left(\chi_{\lambda\mu}^{L_A(A,X)}(T_1)\circ\chi_{\lambda\mu}^{L_A(A,X)}(T_2)^*\right)\\ &= \overline{\Phi_{A_\mu}^{-1}}\left(\chi_{\lambda\mu}^{L_A(A,X)}(T_1)\circ\chi_{\lambda\mu}^{L_A(X,A)}(T_2^*)\right)\\ &= \overline{\Phi_{A_\mu}^{-1}}\left(\chi_{\lambda\mu}^{L_A(X)}(T_1\circ T_2^*)\right)\\ &= \chi_{\lambda\mu}^{L_A(A)}\left(\overline{\Phi_{A_\lambda}^{-1}}(T_1\circ T_2^*)\right)\\ &= \pi_{\lambda\mu}^{M(A)}\left({}_{M(A_\lambda)}\langle T_1,T_2\rangle\right).\end{aligned}$$

Therefore, $\left(\chi_{\lambda\mu}^{L_A(A,X)},\pi_{\lambda\mu}^{M(A)}\right)$ is a morphism of Hilbert C^*-bimodules.

Let $\{T_i\}_{i\in I}$ be a net in $M(X_\lambda)$ which converges strictly to 0. From

$$\begin{aligned}\left\|\chi_{\lambda\mu}^{L_A(A,X)}(T_i)\,\pi_\mu^A(a)\right\|_{X_\mu} &= \left\|\chi_{\lambda\mu}^{L_A(A,X)}(T_i)\,\pi_{\lambda\mu}^{M(A)}\left(\pi_\lambda^A(a)\right)\right\|_{X_\mu}\\ &= \left\|\sigma_{\lambda\mu}^X\left(T_i\left(\pi_\lambda^A(a)\right)\right)\right\|_{X_\mu} \leqslant \left\|T_i\left(\pi_\lambda^A(a)\right)\right\|_{X_\lambda}\end{aligned}$$

for all $a\in A$, and

$$\begin{aligned}\left\|\chi_\mu^{L_A(X)}(S)\circ\chi_{\lambda\mu}^{L_A(A,X)}(T_i)\right\|_{M(X_\mu)} &= \left\|\chi_{\lambda\mu}^{L_A(X)}\left(\chi_\lambda^{L_A(X)}(S)\right)\circ\chi_{\lambda\mu}^{L_A(A,X)}(T_i)\right\|_{L_{A_\mu}(A_\mu,X_\mu)}\\ &= \left\|\chi_{\lambda\mu}^{L_A(A,X)}\left(\chi_\lambda^{L_A(X)}(S)\circ T_i\right)\right\|_{L_{A_\mu}(A_\mu,X_\mu)}\\ &\leqslant \left\|\chi_\lambda^{L_A(X)}(S)\circ T_i\right\|_{L_{A_\lambda}(A_\lambda,X_\lambda)}\end{aligned}$$

for all $S\in K_A(X)$, and taking into account that $K_{A_\mu}(X_\mu)=\chi_\mu^{L_A(X)}(K_A(X))$, we deduce that the net $\{\chi_{\lambda\mu}^{M(X)}(T_i)\}_{i\in I}$ converges strictly to 0.

In a similar way, we show that the maps $\left(\chi_\lambda^{L_A(A,X)},\pi_\lambda^{M(A)}\right):(M(X),M(A))\to(M(X_\lambda),M(A_\lambda)),\lambda\in\Lambda$ are all strictly continuous morphisms of Hilbert bimodules. □

THEOREM 3.13. *Let (X,A) be a full Hilbert pro-C^*-bimodule.*

1. *$M(X)$ is complete with respect to the strict topology;*
2. *$(\iota_X,\iota_A):(X,A)\to(M(X),M(A))$, where $\iota_X(x)(a)=xa$ and $\iota_A(b)(a)=ba$ for all $x\in X$ and $a,b\in A$, is a nondegenerate morphism of Hilbert pro-C^*-bimodules;*
3. *X can be identified with a closed $M(A)-M(A)$ pro-C^*-sub-bimodule of $M(X)$ which is dense in $M(X)$ with respect to the strict topology.*

Proof. (1) For each $\lambda\in\Lambda$, $M(X_\lambda)$ has a structure of Hilbert $M(A_\lambda)-M(A_\lambda)$ C^*-bimodule (see, [5, Proposition 1.10]). It is easy to check that

$$\{M(A_\lambda);M(X_\lambda);\pi_{\lambda\mu}^{M(A)};\chi_{\lambda\mu}^{M(X)};\lambda,\mu\in\Lambda,\lambda\geqslant\mu\},$$

where $\chi_{\lambda\mu}^{M(X)}=\chi_{\lambda\mu}^{L_A(A,X)}$ for all $\lambda,\mu\in\Lambda$ with $\lambda\geqslant\mu$, is an inverse system of Hilbert C^*-bimodules. Then $\lim\limits_{\leftarrow\lambda}M(X_\lambda)$ has a structure of Hilbert $\lim\limits_{\leftarrow\lambda}M(A_\lambda)-\lim\limits_{\leftarrow\lambda}M(A_\lambda)$ pro-C^*-bimodule. Moreover, by Lemma 3.12 the maps $\chi_{\lambda\mu}^{M(X)}:M(X_\lambda)\to M(X_\mu)$, $\lambda,\mu\in\Lambda,\lambda\geqslant\mu$ are all strictly continuous.

Consider, the maps:

$$\Phi:M(X)\to\lim_{\leftarrow\lambda}M(X_\lambda),\Phi(T)=\left(\chi_\lambda^{M(X)}(T)\right)_\lambda$$

and

$$\varphi : M(A) \to \lim_{\leftarrow\lambda} M(A_\lambda), \varphi(m) = \left(\pi_\lambda^{M(A)}(m)\right)_\lambda.$$

It is easy to check that (Φ,φ) is a morphism of Hilbert pro-C^*-bimodules. Moreover, Φ is bijective, and since φ is a pro-C^*-isomorphism, (Φ,φ) is an isomorphism of Hilbert pro-C^*-bimodules. Clearly, a net $\{T_i\}_{i\in I}$ in $M(X)$ converges strictly to 0 in $M(X)$ if and only if the net $\{\Phi(T_i)\}_{i\in I}$ converges strictly to 0 in $\lim_{\leftarrow\lambda} M(X_\lambda)$. Therefore, the strict topology on $M(X)$ can be identified with the inverse limit of the strict topologies on $M(X_\lambda)$, $\lambda\in\Lambda$, and since $M(X_\lambda)$, $\lambda\in\Lambda$, are complete with respect to the strict topology [5, Proposition 1.27], $M(X)$ is complete with respect to the strict topology.

(2) Let $\lambda\in\Lambda$. By [5], $\left(\iota_{X_\lambda},\iota_{A_\lambda}\right):(X_\lambda,A_\lambda)\to(M(X_\lambda),M(A_\lambda))$, where $\iota_{X_\lambda}\left(\sigma_\lambda^X(x)\right)\left(\pi_\lambda^A(a)\right)=\sigma_\lambda^X(xa)$ and $\iota_{A_\lambda}\left(\pi_\lambda^A(b)\right)\pi_\lambda^A(a)=\pi_\lambda^A(ba)$ for all $x\in X$ and $a,b\in A$, is a morphism of Hilbert C^*-bimodules. Since

$$\chi_{\lambda\mu}^{M(X)}\circ\iota_{X_\lambda}=\iota_{X_\mu}\circ\sigma_{\lambda\mu}^X \text{ and } \pi_{\lambda\mu}^{M(A)}\circ\iota_{A_\lambda}=\iota_{A_\mu}\circ\pi_{\lambda\mu}^A$$

for all $\lambda,\mu\in\Lambda$ with $\lambda\geqslant\mu$, there is a morphism of Hilbert pro-C^*-bimodules

$$\left(\lim_{\leftarrow\lambda}\iota_{X_\lambda},\lim_{\leftarrow\lambda}\iota_{A_\lambda}\right):\left(\lim_{\leftarrow\lambda}X_\lambda,\lim_{\leftarrow\lambda}A_\lambda\right)\to\left(\lim_{\leftarrow\lambda}M(X_\lambda),\lim_{\leftarrow\lambda}M(A_\lambda)\right).$$

Identifying X with $\lim_{\leftarrow\lambda}X_\lambda$ and A with $\lim_{\leftarrow\lambda}A_\lambda$, and using (1), we obtain a morphism of Hilbert pro-C^*-bimodules $(\iota_X,\iota_A):(X,A)\to(M(X),M(A))$, where $\iota_X(x)(a)=xa$ and $\iota_A(b)(a)=ba$ for all $x\in X$ and $a,b\in A$. We know that ι_A is nondegenerate and XA is dense in X, therefore (ι_X,ι_A) is nondegenerate.

(3) Since, for each $\lambda\in\Lambda$,

$$p_\lambda^{M(A)}(\iota_X(x))=\left\|\iota_{X_\lambda}\left(\sigma_\lambda^X(x)\right)\right\|_{M(X_\lambda)}=\left\|\sigma_\lambda^X(x)\right\|_{X_\lambda}=p_\lambda^A(x)$$

for all $x\in X$, X can be identified with a closed $M(A)-M(A)$ pro–C^*-sub-bimodule of $M(X)$. Using $(1)-(2)$ and [13, Chapter III, Theorem 3.1], we have

$$\begin{aligned}\overline{\iota_X(X)}^{str}&=\lim_{\leftarrow\lambda}\overline{\chi_\lambda^{M(X)}(\iota_X(X))}^{str}=\lim_{\leftarrow\lambda}\overline{\iota_{X_\lambda}\left(\sigma_\lambda^X(X)\right)}^{str}=\lim_{\leftarrow\lambda}\overline{\iota_{X_\lambda}(X_\lambda)}^{str}\\&=\lim_{\leftarrow\lambda}M(X_\lambda)=M(X),\end{aligned}$$

where $\overline{Z}^{str}$ denotes the closure with respect to the strict topology of the Hilbert sub-bimodule Z of a Hilbert bimodule Y. Therefore, X can be identified with a closed $M(A)-M(A)$ pro–C^*-sub-bimodule of $M(X)$ which is dense in $M(X)$ with respect to the strict topology. $\square$

REMARK 3.14. Let (X,A) be a full Hilbert pro-C^*-bimodule.

1. A net $\{x_i\}_{i\in I}$ in X converges strictly to 0 if and only if the nets $\{p_\lambda^A(x_ia)\}_{i\in I}$ and $\{p_\lambda^A(ax_i)\}_{i\in I}$ converge to 0 for all $a\in A$ and $\lambda\in\Lambda$.

2. The morphism of Hilbert pro-C^*-bimodules $(\iota_X, \iota_A) : (X,A) \to (M(X), M(A))$ is strictly continuous.

LEMMA 3.15. *Let (X,A) be a full Hilbert pro-C^*-bimodule, let $\{e_i\}_{i\in I}$ be an approximate unit for A and $T \in M(X)$. Then the net $\{T \cdot e_i\}_{i\in I}$ converges strictly to T.*

Proof. The net $\{T \cdot e_i\}_{i\in I}$ is bounded, since

$$p_\lambda^{M(A)}(T \cdot e_i) \leqslant p_\lambda^{M(A)}(T)\, p_{\lambda, L_A(A)}(e_i) = p_\lambda^{M(A)}(T)\, p_\lambda(e_i) \leqslant p_\lambda^{M(A)}(T)$$

for all $i \in I$ and for all $\lambda \in \Lambda$. Moreover, we have that

$$p_\lambda^{M(A)}((T \cdot e_i - T)(a)) = p_\lambda^A(T(e_i a - a)) \leqslant p_{\lambda, L_A(A,X)}(T)\, p_\lambda(e_i a - a)$$

for all $a \in A,\ i \in I,\ \lambda \in \Lambda$, and

$$p_\lambda\left(\left((T \cdot e_i)^* - T^*\right)(x)\right) = p_\lambda(e_i T^*(x) - T^*(x))$$

for all $x \in X,\ i \in I,\ \lambda \in \Lambda$. Based on Proposition 3.7 , and taking into account that $\{e_i\}_{i\in I}$ is an approximate unit for A, we conclude that $\{T \cdot e_i\}_{i\in I}$ converges strictly to T. $\square$

In the following theorem we show that any nondegenerate morphism of pro-C^*-bimodules is strictly continuous.

THEOREM 3.16. *Let (X,A) and (Y,B) be two full Hilbert pro-C^*-bimodules and let (Φ,φ) be a nondegenerate morphism of Hilbert pro-C^*-bimodules from (X,A) to $(M(Y), M(B))$. Then (Φ,φ) extends to a unique nondegenerate morphism of Hilbert pro-C^*-bimodules $\left(\overline{\Phi},\overline{\varphi}\right)$ from $(M(X), M(A))$ to $(M(Y), M(B))$. Moreover, $\overline{\Phi}$ is strictly continuous.*

Proof. For each $\delta \in \Delta$, there is $\lambda \in \Lambda$ such that $q_{\delta, M(B)}(\varphi(a)) \leqslant p_\lambda(a)$ for all $a \in A$ and $q_\delta^{M(B)}(\Phi(x)) \leqslant p_\lambda^A(x)$ for all $x \in X$. So there exists a C^*-morphism $\varphi_{(\lambda,\delta)} : A_\lambda \to M(B_\delta)$ such that $\varphi_{(\lambda,\delta)} \circ \pi_\lambda^A = \pi_\delta^{M(B)} \circ \varphi$ and a linear map $\Phi_{(\lambda,\delta)} : X_\lambda \to M(Y_\delta)$ such that $\Phi_{(\lambda,\delta)} \circ \sigma_\lambda^X = \chi_\delta^{M(Y)} \circ \Phi$. It is easy to check that $\left(\Phi_{(\lambda,\delta)}, \varphi_{(\lambda,\delta)}\right)$ is a morphism of Hilbert C^*-bimodules from (X_λ, A_λ) to $(M(Y_\delta), M(B_\delta))$. Moreover, $\left(\Phi_{(\lambda,\delta)}, \varphi_{(\lambda,\delta)}\right)$ is nondegenerate, since

$$\left[\varphi_{(\lambda,\delta)}(A_\lambda) B_\delta\right] = \left[\varphi_{(\lambda,\delta)}\left(\pi_\lambda^A(A)\right) B_\delta\right] = \left[\pi_\delta^{M(B)}(\varphi(A) B)\right] = \left[\pi_\delta^{M(B)}(B)\right] = B_\delta$$

and

$$\begin{aligned}\left[\Phi_{(\lambda,\delta)}(X_\lambda) B_\delta\right] &= \left[\Phi_{(\lambda,\delta)}\left(\sigma_\lambda^X(X)\right) \pi_\delta^{M(B)}(B)\right] = \left[\chi_\delta^{M(Y)}(\Phi(X) B)\right] \\ &= \left[\sigma_\delta^Y(Y)\right] = Y_\delta.\end{aligned}$$

Then, by [5, Theorem 1.30], $\Phi_{(\lambda,\delta)}$ is strictly continuous and $\left(\Phi_{(\lambda,\delta)},\varphi_{(\lambda,\delta)}\right)$ extends to a unique nondegenerate morphism of Hilbert C^*-modules $\left(\overline{\Phi_{(\lambda,\delta)}},\overline{\varphi_{(\lambda,\delta)}}\right)$ from $(M(X_\lambda),M(A_\lambda))$ to $(M(Y_\delta),M(B_\delta))$. Let $\overline{\Phi_\delta}=\overline{\Phi_{(\lambda,\delta)}}\circ\chi_\lambda^{M(X)}$ and $\overline{\varphi_\delta}=\overline{\varphi_{(\lambda,\delta)}}\circ\pi_\lambda^{M(A)}$. Clearly, $\left(\overline{\Phi_\delta},\overline{\varphi_\delta}\right)$ is a morphism of pro-C^*-bimodules from $(M(X),M(A))$ to $(M(Y_\delta),M(B_\delta))$. Moreover, $\overline{\Phi_\delta}$ is strictly continuous, since $\chi_\lambda^{M(X)}$ is strictly continuos (see Lemma 3.12).

Let $\delta_1,\delta_2\in\Delta$ with $\delta_1\geqslant\delta_2$. We have

$$\begin{aligned}\overline{\Phi_{\delta_1}}\left(\iota_X(x)\right) &= \left(\overline{\Phi_{(\lambda_1,\delta_1)}}\circ\chi_{\lambda_1}^{M(X)}\right)\left(\iota_X(x)\right)=\overline{\Phi_{(\lambda_1,\delta_1)}}\left(\iota_{X_{\lambda_1}}\left(\sigma_{\lambda_1}^X(x)\right)\right)\\ &=\Phi_{(\lambda_1,\delta_1)}(\sigma_{\lambda_1}^X(x))=\chi_{\delta_1}^{M(Y)}(\Phi(x))\end{aligned}$$

for some $\lambda_1\in\Lambda$ and for all $x\in X$. Then

$$\left(\chi_{\delta_1\delta_2}^{M(Y)}\circ\overline{\Phi_{\delta_1}}\right)(\iota_X(x))=\chi_{\delta_1\delta_2}^{M(Y)}\left(\chi_{\delta_1}^{M(Y)}(\Phi(x))\right)=\chi_{\delta_2}^{M(Y)}(\Phi(x))=\overline{\Phi_{\delta_2}}(\iota_X(x))$$

for all $x\in X$. From these relations and taking into account that $\chi_{\delta_1\delta_2}^{M(Y)}$, $\overline{\Phi_{\delta_1}}$, $\overline{\Phi_{\delta_2}}$ are strictly continuous and X is dense in $M(X)$ with respect to the strict topology, we conclude that $\chi_{\delta_1\delta_2}^{M(Y)}\circ\overline{\Phi_{\delta_1}}=\overline{\Phi_{\delta_2}}$. Therefore there is a strictly continuous linear map $\overline{\Phi}:M(X)\to M(Y)$ such that $\chi_\delta^{M(Y)}\circ\overline{\Phi}=\overline{\Phi_\delta}$ for all $\delta\in\Delta$, and $\overline{\Phi}\circ\iota_X=\Phi$.

By [14, Proposition 3.15], there is a pro-C^*-morphism $\overline{\varphi}:M(A)\to M(B)$ such that $\pi_\delta^{M(B)}\circ\overline{\varphi}=\overline{\varphi_\delta}$ for all $\delta\in\Delta$ and $\overline{\varphi}\circ\iota_A=\varphi$.

It is easy to check that $\left(\overline{\Phi},\overline{\varphi}\right)$ is a morphism of Hilbert pro-C^*-bimodules. Since $\overline{\varphi}$ is nondegenerate [7, Proposition 6.1.4] and

$$\begin{aligned}\left[\overline{\Phi}(M(X))B\right] &= \left[\overline{\Phi}(M(X))\varphi(A)B\right]=\left[\overline{\Phi}(M(X)A))B\right]\\ &=[\Phi(X))B]=Y\end{aligned}$$

the morphism of Hilbert pro-C^*-bimodule $\left(\overline{\Phi},\overline{\varphi}\right)$ is nondegenerate.

Suppose that there is another morphism of Hilbert pro-C^*-bimodules (Φ_1,φ_1) : $(M(X),M(A))\to(M(Y),M(B))$ such that $\Phi_1(\iota_X(x))=\Phi(x)$ for all $x\in X$ and $\varphi_1(\iota_A(a))=\varphi(a)$ for all $a\in A$. Let $\{e_i\}_{i\in I}$ be a approximate unit for A. Then, by Lemma 3.15 for each $T\in M(X)$ and $m\in M(A)$, the nets $\{T\cdot e_i\}_{i\in I}$ and $\{m\cdot e_i\}_{i\in I}$ are strictly convergent to T respectively m. Thus we have

$$\Phi_1(T)=\text{str-}\lim_i\Phi_1(T\cdot e_i)=\text{str-}\lim_i\Phi(T\cdot e_i)=\overline{\Phi}(T)$$

for all $T\in M(X)$ and

$$\varphi_1(m)=\text{str-}\lim_i\varphi_1(m\cdot e_i)=\text{str-}\lim_i\varphi(m\cdot e_i)=\overline{\varphi}(m)$$

for all $m\in M(A)$. $\square$

Let X be a Hilbert $A-A$ pro-C^*-bimodule. For a closed two sided ideal $\mathscr{I}$ of A we put $\mathscr{I}X=\operatorname{span}\{ax/a\in\mathscr{I},x\in X\}$ and $X\mathscr{I}=\operatorname{span}\{xa/a\in\mathscr{I},x\in X\}$. By [12, Lemma 3.7], $\mathscr{I}X$ and $X\mathscr{I}$ are closed Hilbert pro-C^*-sub-bimodules of X.

DEFINITION 3.17. Let (X,A) and (Y,C) be two Hilbert pro-C^*-bimodules. We say that (Y,C) is an extension of (X,A) if the following conditions are satisfied:

1. C contains A as an ideal;
2. there exists a morphism (φ_X, φ_A) of Hilbert pro-C^*-bimodules from (X,A) to (Y,C), such that $\varphi_A : A \to C$ is just the inclusion map;
3. $\varphi_X(X) = \varphi_A(A)Y = Y\varphi_A(A)$.

REMARK 3.18. If (Y,C) is an extension of (X,A), and if the topology on C is given by the family of C^*-seminorms $\{p_\lambda;\lambda \in \Lambda\}$, then the topology on A is given by $\{p_\lambda|_A;\lambda \in \Lambda\}$, and $p_\lambda(\varphi_A(a)) = p_\lambda(a)$ for all $a \in A$ and for all $\lambda \in \Lambda$. Therefore, $p_\lambda^C(\varphi_X(x)) = p_\lambda^A(x)$ for all $x \in X$ and for all $\lambda \in \Lambda$, and so, for each $\lambda \in \Lambda$, there is a linear map $\varphi_{X_\lambda} : X_\lambda \to Y_\lambda$ such that $\sigma_\lambda^Y \circ \varphi_X = \varphi_{X_\lambda} \circ \sigma_\lambda^X$. Then $\varphi_X = \lim\limits_{\leftarrow\lambda} \varphi_{X_\lambda}$, and for each $\lambda \in \Lambda$, (Y_λ, C_λ) is an extension of (X_λ, A_λ) via the morphism $\left(\varphi_{X_\lambda}, \varphi_{A_\lambda}\right)$, where φ_{A_λ} is the inclusion of A_λ into C_λ.

In the following proposition, we show that $(M(X),M(A))$ is a maximal extension of (X,A) in the sense that if (Y,C) is another extension of (X,A) via a morphism (ψ_X, ψ_A), then there is a morphism of Hilbert pro-C^*-bimodules $(\vartheta_Y, \vartheta_C) : (Y,C) \to (M(X),M(A))$ such that $\vartheta_Y \circ \psi_X = \iota_X$ and $\vartheta_C \circ \psi_A = \iota_A$ (for the case of Hilbert C^*-modules, see [3,4]).

PROPOSITION 3.19. *Let X be a full Hilbert pro-C^*-bimodule over A. Then $(M(X),M(A))$ is a maximal extension of (X,A).*

Proof. Let (ι_X, ι_A) be the morphism of Theorem 3.13(2) between (X,A) and $(M(X),M(A))$, where $\iota_X(x)(a) = xa$, $\iota_A(a)(b) = ab$, for $x \in X$, $a,b \in A$. From [15, Corollary 3.3] we have that for every $\lambda \in \Lambda$, $M(X_\lambda)\iota_{A_\lambda}(A_\lambda) = \iota_{X_\lambda}(X_\lambda) = \iota_{A_\lambda}(A_\lambda)M(X_\lambda)$. Therefore, since from Theorem 3.13, we have that $M(X) = \lim\limits_{\leftarrow\lambda} M(X_\lambda)$, $\iota_X = \lim\limits_{\leftarrow\lambda} \iota_{X_\lambda}$, $\iota_A = \lim\limits_{\leftarrow\lambda} \iota_{A_\lambda}$, and since both $\iota_A(A)M(X), M(X)\iota_A(A)$ and $\iota_X(X)$ are closed submodules of $M(X)$, we deduce that $\iota_A(A)M(X) = \iota_X(X) = M(X)\iota_A(A)$. Hence $(M(X),M(A))$ is an extension of (X,A).

To show that $(M(X),M(A))$ is a maximal extension, let (Y,C) be another extension of (X,A) via a morphism (ψ_X, ψ_A). Then, by Remark 3.18, $\psi_X = \lim\limits_{\leftarrow\lambda} \psi_{X_\lambda}$, $\psi_A = \lim\limits_{\leftarrow\lambda} \psi_{A_\lambda}$, and for each $\lambda \in \Lambda$, (Y_λ, C_λ) is an extension of (X_λ, A_λ) via the morphism $\left(\psi_{X_\lambda}, \psi_{A_\lambda}\right)$. By [15, Proposition 3.4], there exists a unique morphism $(\vartheta_{Y_\lambda}, \vartheta_{C_\lambda}) : (Y_\lambda, C_\lambda) \to (M(X_\lambda), M(A_\lambda))$ such that $\vartheta_{Y_\lambda} \circ \psi_{X_\lambda} = \iota_{X_\lambda}$ and $\vartheta_{C_\lambda} \circ \psi_{A_\lambda} = \iota_{A_\lambda}$. Moreover,

$$\vartheta_{Y_\lambda}\left(\sigma_\lambda^Y(y)\right)\left(\pi_\lambda^A(a)\right) = \psi_{X_\lambda}^{-1}\left(\sigma_\lambda^Y(y)\,\psi_{A_\lambda}\left(\pi_\lambda^A(a)\right)\right)$$

and

$$\vartheta_{C_\lambda}\left(\pi_\lambda^C(c)\right)\left(\pi_\lambda^A(a)\right) = \psi_{A_\lambda}^{-1}\left(\pi_\lambda^C(c)\,\psi_{A_\lambda}\left(\pi_\lambda^A(a)\right)\right)$$

for all $a\in A$, for all $c\in C$ and for all $y\in Y$. It is easy to check that $\left(\vartheta_{Y_\lambda}\right)_\lambda$ is an inverse system of linear maps, $\left(\vartheta_{C_\lambda}\right)_\lambda$ is an inverse system of C^*-morphisms, and $(\vartheta_Y,\vartheta_C):(Y,C)\to(M(X),M(A))$, where $\vartheta_Y=\lim\limits_{\leftarrow\lambda}\vartheta_{Y_\lambda}$ and $\vartheta_C=\lim\limits_{\leftarrow\lambda}\vartheta_{C_\lambda}$, is a morphism of Hilbert pro-C^*-bimodules such that $\vartheta_Y\circ\psi_X=\iota_X$ and $\vartheta_C\circ\psi_A=\iota_A$. □

4. Crossed products by Hilbert pro-C^*-modules

A covariant representation of a Hilbert pro-C^*-bimodule (X,A) on a pro-C^*-algebra B is a morphism of Hilbert pro-C^*-bimodules from (X,A) to the Hilbert pro-C^*-bimodule (B,B).

The crossed product of A by a Hilbert pro-C^*-bimodule (X,A) is a pro-C^*-algebra, denoted by $A\times_X\mathbb{Z}$, and a covariant representation (i_X,i_A) of (X,A) on $A\times_X\mathbb{Z}$ with the property that for any covariant representation (φ_X,φ_A) of (X,A) on a pro-C^*-algebra B, there is a unique pro-C^*-morphism $\Phi:A\times_X\mathbb{Z}\to B$ such that $\Phi\circ i_X=\varphi_X$ and $\Phi\circ i_A=\varphi_A$ [11, Definition 3.3].

REMARK 4.1. If (Φ,φ) is a morphism of Hilbert pro-C^*-bimodules from (X,A) to (Y,B), then $(i_Y\circ\Phi,i_B\circ\varphi)$ is a covariant representation of X on $B\times_Y\mathbb{Z}$ and by the universal property of $A\times_X\mathbb{Z}$ there is a unique pro-C^*-morphism $\Phi\times\varphi$ from $A\times_X\mathbb{Z}$ to $B\times_Y\mathbb{Z}$ such that $(\Phi\times\varphi)\circ i_A=i_B\circ\varphi$ and $(\Phi\times\varphi)\circ i_X=i_Y\circ\Phi$.

LEMMA 4.2. *Let (Φ,φ) be a a morphism of Hilbert pro-C^*-bimodules from (X,A) to (Y,B). If Γ and Γ' have the same index set and $\varphi=\lim\limits_{\leftarrow\lambda}\varphi_\lambda$, then $\Phi=\lim\limits_{\leftarrow\lambda}\Phi_\lambda$, for each $\lambda\in\Lambda,(\Phi_\lambda,\varphi_\lambda)$ is a morphism of Hilbert C^*-bimodules, $(\Phi_\lambda\times\varphi_\lambda)_\lambda$ is an inverse system of C^*-morphisms and $\Phi\times\varphi=\lim\limits_{\leftarrow\lambda}\Phi_\lambda\times\varphi_\lambda$. Moreover, if (Φ,φ) is an isomorphism of Hilbert pro-C^*-bimodules and $\varphi_\lambda,\lambda\in\Lambda$ are C^*-isomorphisms, then $(\Phi_\lambda,\varphi_\lambda)$, $\lambda\in\Lambda$ are isomorphisms of Hilbert C^*-bimodules.*

Proof. Let $\lambda\in\Lambda$. From

$$q_\lambda^B(\Phi(x))^2=q_\lambda(\varphi(\langle x,x\rangle))\leqslant p_\lambda(\langle x,x\rangle)=p_\lambda^A(x)^2$$

for all $x\in X$, we deduce that there is a linear map $\Phi_\lambda:X_\lambda\to Y_\lambda$ such that $\Phi_\lambda\circ\sigma_\lambda^X=\sigma_\lambda^Y\circ\Phi$. It is easy to verify that $(\Phi_\lambda)_\lambda$ is an inverse system of linear maps and $\Phi=\lim\limits_{\leftarrow\lambda}\Phi_\lambda$. Moreover, for each $\lambda\in\Lambda$, $(\Phi_\lambda,\varphi_\lambda)$ is a morphism of Hilbert C^*-bimodules. Let $\Phi_\lambda\times\varphi_\lambda$ be the C^*-morphism from $A_\lambda\times_{X_\lambda}\mathbb{Z}$ to $B_\lambda\times_{Y_\lambda}\mathbb{Z}$ induced by $(\Phi_\lambda,\varphi_\lambda)$. From

$$\begin{aligned}\pi_{\lambda\mu}^{B\times_Y\mathbb{Z}}\circ(\Phi_\lambda\times\varphi_\lambda)\circ i_{A_\lambda}&=\pi_{\lambda\mu}^{B\times_Y\mathbb{Z}}\circ i_{B_\lambda}\circ\varphi_\lambda=i_{B_\mu}\circ\pi_{\lambda\mu}^B\circ\varphi_\lambda\\&=i_{B_\mu}\circ\varphi_\mu\circ\pi_{\lambda\mu}^A=\left(\Phi_\mu\times\varphi_\mu\right)\circ\pi_{\lambda\mu}^{A\times_X\mathbb{Z}}\circ i_{A_\lambda}\end{aligned}$$

and

$$\pi_{\lambda\mu}^{B\times_Y\mathbb{Z}}\circ(\Phi_\lambda\times\varphi_\lambda)\circ i_{X_\lambda}=\left(\Phi_\mu\times\varphi_\mu\right)\circ\pi_{\lambda\mu}^{A\times_X\mathbb{Z}}\circ i_{X_\lambda}$$

for all $\lambda,\mu \in \Lambda$ with $\lambda \geqslant \mu$ and taking into account that $i_{A_\lambda}(A_\lambda)$ and $i_{X_\lambda}(X_\lambda)$ generate $A_\lambda \times_{X_\lambda} \mathbb{Z}$, we deduce that $(\Phi_\lambda \times \varphi_\lambda)_\lambda$ is an inverse system of C^*-morphisms. Moreover, since

$$\lim_{\leftarrow\lambda}(\Phi_\lambda \times \varphi_\lambda)\circ \lim_{\leftarrow\lambda} i_{A_\lambda} = \lim_{\leftarrow\lambda}(\Phi_\lambda \times \varphi_\lambda)\circ i_{A_\lambda} = \lim_{\leftarrow\lambda} i_{B_\lambda}\circ\varphi_\lambda = \lim_{\leftarrow\lambda} i_{B_\lambda}\circ \lim_{\leftarrow\lambda}\varphi_\lambda$$

and

$$\lim_{\leftarrow\lambda}(\Phi_\lambda \times \varphi_\lambda)\circ \lim_{\leftarrow\lambda} i_{X_\lambda} = \lim_{\leftarrow\lambda}(\Phi_\lambda \times \varphi_\lambda)\circ i_{X_\lambda} = \lim_{\leftarrow\lambda} i_{Y_\lambda}\circ\Phi_\lambda = \lim_{\leftarrow\lambda} i_{Y_\lambda}\circ \lim_{\leftarrow\lambda}\Phi_\lambda,$$

we obtain $\Phi \times \varphi = \lim\limits_{\leftarrow\lambda}\Phi_\lambda \times \varphi_\lambda$.

Suppose that (Φ,φ) is an isomorphism of Hilbert pro-C^*-bimodules and $\varphi_\lambda, \lambda \in \Lambda$ are C^*-isomorphisms. Then, since $\varphi^{-1} = \lim\limits_{\leftarrow\lambda}\varphi_\lambda^{-1}$, by the first part of the proof, $\Phi^{-1} = \lim\limits_{\leftarrow\lambda}\psi_\lambda$ and $\left(\psi_\lambda, \varphi_\lambda^{-1}\right)$ is a morphism of Hilbert C^*-bimodules for all $\lambda \in \Lambda$. Let $\lambda \in \Lambda$. From

$$\psi_\lambda \circ \Phi_\lambda \circ \sigma_\lambda^X = \psi_\lambda \circ \sigma_\lambda^Y \circ \Phi = \sigma_\lambda^X \circ \Phi^{-1}\circ\Phi = \sigma_\lambda^X$$

and

$$\Phi_\lambda \circ \psi_\lambda \circ \sigma_\lambda^Y = \Phi_\lambda \circ \sigma_\lambda^X \circ \Phi^{-1} = \sigma_\lambda^Y \circ \Phi \circ \Phi^{-1} = \sigma_\lambda^Y$$

and taking into account that σ_λ^X and σ_λ^Y are surjective, we deduce that $\psi_\lambda = \Phi_\lambda^{-1}$. $\square$

The following proposition gives the relation between the crossed product of A by X and the crossed product of $M(A)$ by $M(X)$.

PROPOSITION 4.3. *Let (X,A) be full Hilbert pro-C^*-bimodule. Then $A \times_X \mathbb{Z}$ can be embedded into $M(A)\times_{M(X)}\mathbb{Z}$.*

Proof. Let ι_A be the embedding of A in $M(A)$ and ι_X the embedding of X in $M(X)$. Then (ι_X, ι_A) is a morphism of Hilbert pro-C^*-bimodules, and since $\iota_A = \lim\limits_{\leftarrow\lambda}\iota_{A_\lambda}$, by Lemma 4.2, $\iota_X \times \iota_A = \lim\limits_{\leftarrow\lambda}\iota_{X_\lambda}\times\iota_{A_\lambda}$ is a pro-C^*-morphism from $A\times_X\mathbb{Z}$ to $M(A)\times_{M(X)}\mathbb{Z}$. Moreover, since

$$p_{\lambda,M(A)\times_{M(X)}\mathbb{Z}}(\iota_X\times\iota_A(c)) = \left\|\iota_{X_\lambda}\times\iota_{A_\lambda}\left(\pi_\lambda^{A\times_X\mathbb{Z}}(c)\right)\right\|_{M(A_\lambda)\times_{M(X_\lambda)}\mathbb{Z}}$$

$$\overset{[1,\text{ Remark }2.2]}{=} \left\|\pi_\lambda^{A\times_X\mathbb{Z}}(c)\right\|_{A_\lambda\times_{X_\lambda}\mathbb{Z}} = p_{\lambda,A\times_X\mathbb{Z}}(c)$$

for all $c \in A\times_X\mathbb{Z}$ and for all $\lambda\in\Lambda$, $A\times_X\mathbb{Z}$ can be identified with a pro-C^*-subalgebra of $M(A)\times_{M(X)}\mathbb{Z}$. $\square$

The following proposition is a generalization of [15, Proposition 4.7].

PROPOSITION 4.4. *Let (X,A) be a full Hilbert pro-C^*-bimodule. Then $M(A)\times_{M(X)}\mathbb{Z}$ can be identified with a pro-C^*-subalgebra of $M(A\times_X\mathbb{Z})$.*

Proof. Since X is full, (i_X,i_A) is nondegenerate and $i_A=\lim\limits_{\leftarrow\lambda} i_{A_\lambda}$ and $i_X=\lim\limits_{\leftarrow\lambda} i_{X_\lambda}$ [11, Propositions 3.4 and 3.5]. Then, by Theorem 3.16, (i_X,i_A) extends to a covariant representation $\left(\overline{i_X},\overline{i_A}\right)$ of $(M(X),M(A))$ on $M(A\times_X\mathbb{Z})$, and moreover, $\overline{i_A}=\lim\limits_{\leftarrow\lambda}\overline{i_{A_\lambda}}$ and $\overline{i_X}=\lim\limits_{\leftarrow\lambda}\overline{i_{X_\lambda}}$. It is easy to check $\left(\overline{i_{X_\lambda}},\overline{i_{A_\lambda}}\right)$ is a covariant representation of $(M(X_\lambda),M(A_\lambda))$ on $M(A_\lambda\times_{X_\lambda}\mathbb{Z})$ for each $\lambda\in\Lambda$. By [15, Proposition 4.7], for each $\lambda\in\Lambda$, there is an injective C^*-morphism $\Phi_\lambda: M(A_\lambda)\times_{M(X_\lambda)}\mathbb{Z}\to M(A_\lambda\times_{X_\lambda}\mathbb{Z})$ such that $\Phi_\lambda\circ i_{M(X_\lambda)}=\overline{i_{X_\lambda}}$ and $\Phi_\lambda\circ i_{M(A_\lambda)}=\overline{i_{A_\lambda}}$. From

$$\begin{aligned}\pi_{\lambda\mu}^{M(A\times_X\mathbb{Z})}\circ\Phi_\lambda\circ i_{M(X_\lambda)} &= \pi_{\lambda\mu}^{M(A\times_X\mathbb{Z})}\circ\overline{i_{X_\lambda}}=\overline{i_{X_\mu}}\circ\chi_{\lambda\mu}^{M(X)}\\ &=\Phi_\mu\circ i_{M(X_\mu)}\circ\chi_{\lambda\mu}^{M(X)}=\Phi_\mu\circ\pi_{\lambda\mu}^{M(A\times_X\mathbb{Z})}\circ i_{M(X_\lambda)}\end{aligned}$$

and

$$\begin{aligned}\pi_{\lambda\mu}^{M(A\times_X\mathbb{Z})}\circ\Phi_\lambda\circ i_{M(A_\lambda)} &= \pi_{\lambda\mu}^{M(A\times_X\mathbb{Z})}\circ\overline{i_{A_\lambda}}=\overline{i_{A_\mu}}\circ\pi_{\lambda\mu}^{M(A)}\\ &=\Phi_\mu\circ i_{M(A_\mu)}\circ\pi_{\lambda\mu}^{M(A)}=\Phi_\mu\circ\pi_{\lambda\mu}^{M(A\times_X\mathbb{Z})}\circ i_{M(A_\lambda)}\end{aligned}$$

for all $\lambda,\mu\in\Lambda$, with $\lambda\geqslant\mu$, and taking into account that $i_{M(X_\lambda)}(M(X_\lambda))$ and $i_{M(A_\lambda)}(M(A_\lambda))$ generate $M(A_\lambda)\ \times_{M(X_\lambda)}\mathbb{Z}$, we deduce that $(\Phi_\lambda)_\lambda$ is an inverse system of isometric C^*-morphisms. Hence $\Phi=\lim\limits_{\leftarrow\lambda}\Phi_\lambda$ is an injective pro-C^*-morphism from $\lim\limits_{\leftarrow\lambda}M(A_\lambda)\ \times_{M(X_\lambda)}\mathbb{Z}$ to $\lim\limits_{\leftarrow\lambda}M(A_\lambda\times_{X_\lambda}\mathbb{Z})$ such that $p_{\lambda,M(A\times_X\mathbb{Z})}(\Phi(c))=p_{\lambda,M(A)\times_{M(X)}\mathbb{Z}}(c)$ for all $c\in M(A)\times_{M(X)}\mathbb{Z}$ and for all $\lambda\in\Lambda$. Therefore, $M(A)\times_{M(X)}\mathbb{Z}$ can be identified with a pro-C^*-subalgebra of $M(A\times_X\mathbb{Z})$. $\square$

An automorphism α of a pro-C^*-algebra A such that $p_\lambda(\alpha(a))=p_\lambda(a)$ for all $a\in A$ and $\lambda\in\Lambda'$, where Λ' is a cofinal subset of Λ, is called an inverse limit automorphism. If α is an inverse limit automorphism of the pro-C^*-algebra A, then $X_\alpha=\{\xi_x;x\in A\}$ is a Hilbert $A-A$ pro-C^*-bimodule with the bimodule structure defined as $\xi_x a=\xi_{xa}$, respectively $a\xi_x=\xi_{\alpha^{-1}(a)x}$, and the inner products are defined as $\left\langle\xi_x,\xi_y\right\rangle_A=x^*y$, respectively ${}_A\left\langle\xi_x,\xi_y\right\rangle=\alpha(xy^*)$. The crossed product $A\times_\alpha\mathbb{Z}$ of A by α is isomorphic to the crossed product of A by X_α [11].

COROLLARY 4.5. *If α is an inverse limit automorphism of a non unital pro-C^*-algebra A, then $M(A)\times_{\overline{\alpha}}\mathbb{Z}$ can be identified with a pro-C^*-subalgebra of $M(A\times_\alpha\mathbb{Z})$.*

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Maria Joiţa
Department of Mathematics, Faculty of Applied Sciences
University Politehnica of Bucharest
313 Spl. Independentei Street, 060042, Bucharest, Romania
and
Simion Stoilow Institute of Mathematics of the Roumanian Academy
21 Calea Grivitei Street, 010702, Bucharest, Romania
e-mail: joita@fmi.unibuc.ro
http://sites.google.com/a/g.unibuc.ro/maria-joita/

Radu-B. Munteanu
Department of Mathematics, University of Bucharest
14 Academiei St., 010014, Bucharest, Romania
e-mail: radu-bogdan.munteanu@g.unibuc.ro

Ioannis Zarakas
Department of Mathematics, University of Athens
Panepistimiopolis, Athens 15784, Greece
e-mail: gzarak@math.uoa.gr